Linear Mathematics.

Vector spaces, linear methods and tensors throughout STEM.

Volume 1 of

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Improving everything with a pinch of Combinatorics

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Chapter 10

Affine spaces

10.1 Introduction

View $\mathbb{R}^n$ as a vector space. So it has a well-defined $\mathbf{0}$ and thus well-defined position vectors with respect to the corresponding origin. And well-defined addition and scalar multiplication operations, as combine to form

$$\lambda \boldsymbol{a}_0 + (1 - \lambda) \boldsymbol{a}_1 . \tag{10.1}$$

Next consider a region $\mathfrak{a} \subseteq \mathbb{R}^n$. Let the points $A_0 \neq A_1$ with position vectors $\boldsymbol{a}_0, \boldsymbol{a}_1$ lie within $\mathfrak{a}$. The line through these points is

$$\Lambda_{0\,1} := \{ \boldsymbol{a}_\lambda := \lambda \boldsymbol{a}_0 + (1 - \lambda) \boldsymbol{a}_1 \,|\, \lambda \in \mathbb{R} \} .$$

Or, as a set of points,

$$\{ A_\lambda \,|\, \lambda \in \mathbb{R} \} .$$

Definition 1 Then $\mathfrak{a}$ is *Affine* if for all pairs of points $A_0 \neq A_1 \in \mathfrak{a}$ $\mathfrak{L}_{0\,1}$ itself $\subseteq \mathfrak{a}$.

Remark 1 Observe that $\mathfrak{a}$ is a *subset* of $\mathbb{R}^n$, rather than necessarily being a vector subspace. As such, $\mathbf{0}$ may or may not $\in \mathfrak{a}$. It does in Fig 1.1.a) but not in Subfig b).

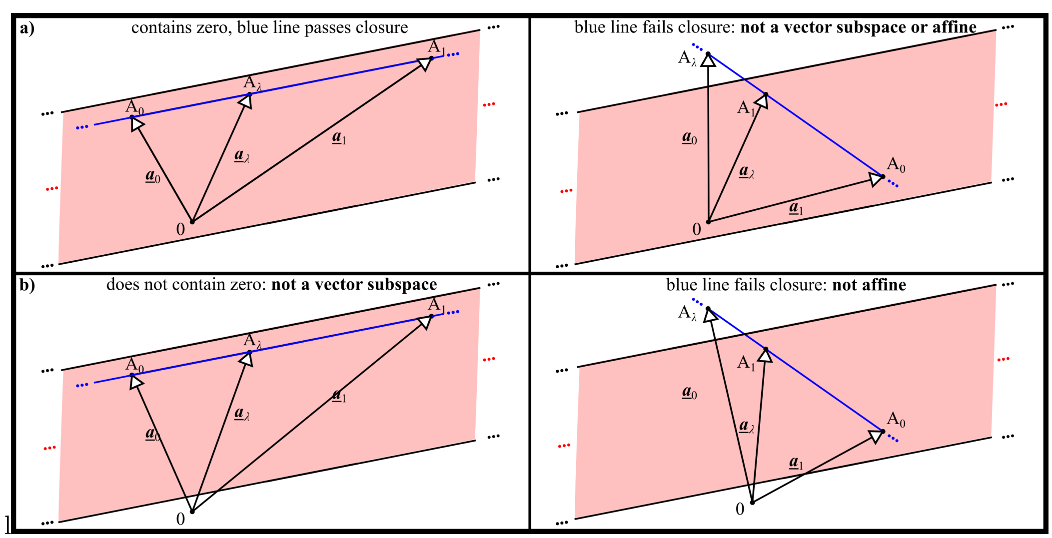

Figure 10.1:

Remark 2 (1.1) rearranges to the more manifestly relative form
$$\boldsymbol{a}_1 + \lambda(\boldsymbol{a}_0 - \boldsymbol{a}_1) \, .$$

Remark 3 Affine spaces can be defined for any vector space $\mathfrak{V}$. In the form
$$\underbrace{\mathfrak{U}}_{\text{vector subspace}} + \underbrace{\boldsymbol{t}}_{\text{offset vector}} \, .$$
I.e. every vector therein takes the form
$$\boldsymbol{f} = \underbrace{\boldsymbol{u}}_{\in\ \mathfrak{U}} + \underbrace{\boldsymbol{t}}_{\text{offset vector}} \, .$$
[For some affine notions to extend, however $\mathbb{R}$-LCs need to remain sufficiently meaningful.]

Remark 4 We can now identify Fig **??**'s 'temporarily unnamed formulation' of lines as the *Affine formulation* of lines!

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Naming Remark 1 'Affine' means 'related', deriving from the Latin 'affinis'. Indeed, this can be taken to mean 'related to vector subspaces by an offset vector' as per above.

Historical Remark 1 Affine Mathematics was pioneered by Euler [1].

Naming Remark 2 The name 'Affine' was however coined by Möbius and Gauss [4].

Naming Remark 3 The action of the offset vector is said to provide a *translate* of the vector space in question. This name and notion is closely related to translations (as in the transformations).

10.2 Exercises and Examples

Exercise 1 Explain why Affineness holds regardless of whether a subset contains the vector $\underline{\mathbf{0}}$.

Exercise 2 Show that scaling, rotating and reflecting regions preserves their Affineness. Show furthermore that in 2-d shears and squeezes do as well.

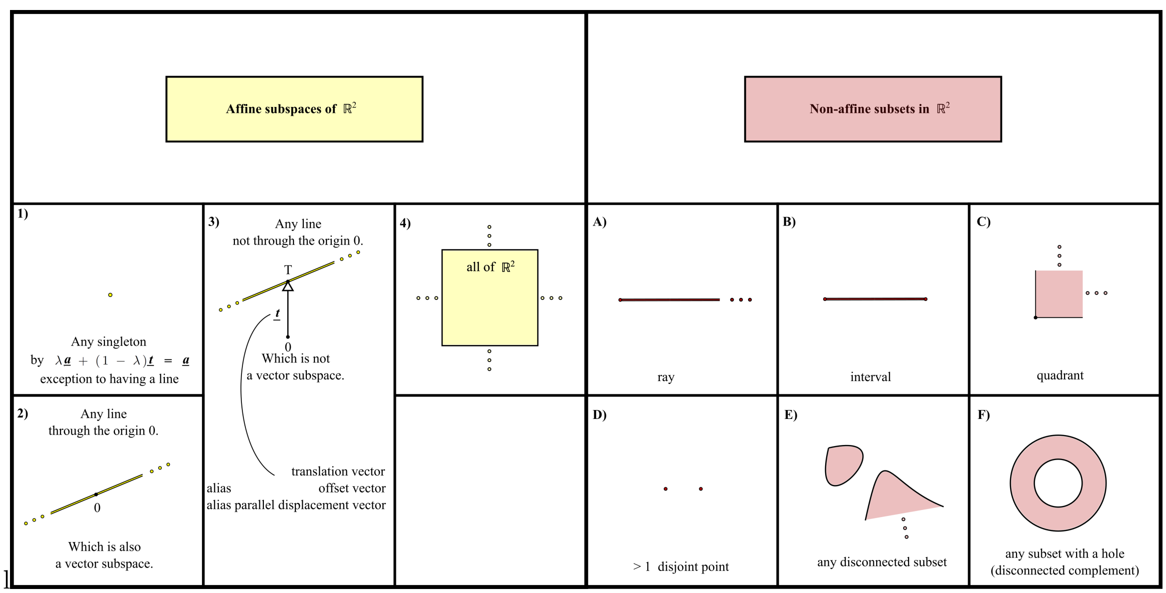

Figure 10.2:

Examples 1 to 4 of affine subsets, and **Examples A to F** of non-affine subsets, all $\subseteq \mathbb{R}^2$, are provided in Fig 1.2.

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Remark 7 The n-vector generalization of (1.1) is as follows.

$$\sum_{i\,=\,1}^{n} \lambda_i\, \boldsymbol{x}_i \tag{10.2}$$

subject to

$$\sum_{i\,=\,1}^{n} \lambda_i \;=\; 1\ . \tag{10.3}$$

Exercise 4 Show that the solution set of a 2-d system of linear equations must be an Affine set.

10.3 Conceptual discussion of flats as an Affine-level notion

Remark 1 In doing Exercise 3, you may have found it useful to recollect that such systems are about intersecting hyperplanes.

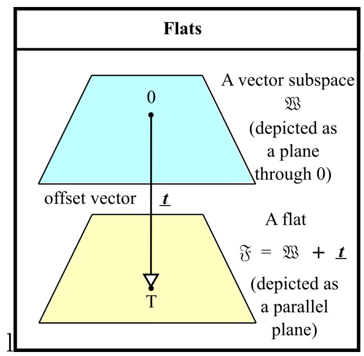

Figure 10.3:

Remark 2 A flat which contains the origin is a vector (sub)space. By which the theory of vector subspaces is nested within that of Affine subspaces.

Remark 3 Going in the reverse direction, non-empty flats are the same thing as all the possible translates of vector subspaces. With the sole exception that the empty set $\emptyset$ is also a flat. [This is not a linear subspace since it has no room to contain $\mathbf{0}$!]

Remark 4 When $\mathbb{R}^d$ is viewed in this Affine way, there ceases to be a meaningful place for absolute notions of origin. For no matter where one imagines an origin to be, the model's Affine structure is equivalent. For instance, in setting up flats in the manner of Figs **??** and **??**, there is no structural difference between cases containing an origin and cases not containing an origin.

Remark 5 Furthermore, the translates of subspaces are the same objects that constitute the vector space version of cosets. I.e. the objects whose totalities form the quotients of vector spaces.[1] So in going on an Affine detour, we find that we have automatically constructed the quotients of vector spaces; see Sec 1.11 for more from this point of view. While, conversely, if we pursued the quotients of vector spaces, we would discover Affine spaces by examining their mathematical structure.

[1] Compare Group Theory's cosets and quotients in Chapters 24 and 26 of Volume 0.

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Naming Remark 4 Another alias of note for offset vector is *parallel displacement vector*. 'Parallel' derives from the Greek $\pi\alpha\rho\alpha$ $\alpha\lambda\lambda\eta\lambda o\iota\varsigma$, meaning 'alongside each other'.

Remark 6 Offset vectors confer a suitably general notion of *parallelism* for vector spaces in general (as opposed to just for Euclidean space). I.e. given flats $\mathfrak{F}\mathrm{lat}$ and $\mathfrak{F}\mathrm{lat}'$, then these are *(strongly) parallel* iff the one is the translate of the other. Though this notion does permit flats to be parallel with themselves (a point made e.g. in [11]). While this notion does not permit statements of parallelism between flats of different dimension (Fig 1.4). Upon being thus equipped with a notion of parallelism, these vector spaces become Affine spaces.

Remark 7 In applications – such as Chapter **??**'s systems – in which coincident and non-coincident parallels need to be distinguished, we can remove the former from contention by stipulating '*proper strong parallelism*. While in applications in which flats containing parallel copies of other flats are to be included, we can define *weak parallelism* to do so (Fig 1.4 again). These 'strong', 'weak' and 'proper' names and notions are however particular to the current Author.

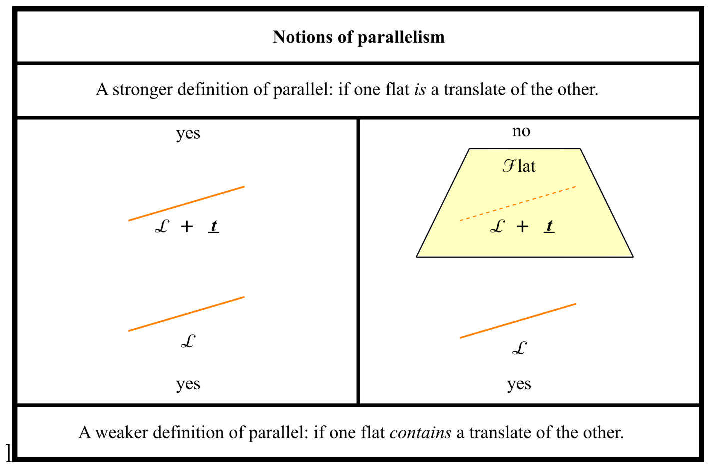

Figure 10.4:

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Pointer A.1 We shall argue in Volume 3, Affine Geometry is the natural home for notions of parallelism [6, 9]. This furthermore turns out to give a less structured Geometry than Euclidean Geometry.

Historical Remark 2 Affine considerations played a foundational role in the passage to Curved Geometry, as outlined in Volume 6.

10.4 Some preliminary Affine arenas

Definition 1 Let us also introduce the arena of subsets of the Affine space $\mathfrak{A}$,

$$\mathfrak{A}\mathfrak{S}\mathrm{ubset}(\mathfrak{A}) \ .$$

Definition 2 And the smaller arena of all of its Affine subspaces, alias flats,

$$\mathfrak{A}\mathfrak{S}\mathrm{ubspace}(\mathfrak{A}) \ = \ \mathfrak{F}\mathrm{lat}(\mathfrak{A}) \ .$$

Exercise 4 What are $\mathfrak{A}\mathfrak{S}\mathrm{ubspace}\left(\mathbb{R}^d\right)$ for $d = 0$ to 2 . a) As sets? b) With the obvious inclusion order?

Exercise 5$^-$ Extend the notion of vector space flags to Affine space flags. Include examples of minimum nontrivial, and maximum, and Affine space flags which are, and are not, vector space flags.

10.5 AC: Affine combinations

Structure 2 Each $\boldsymbol{a}_\lambda$ in Sec 1.1 is an *Affine combination (AC)* of $\boldsymbol{a}_0$ and $\boldsymbol{a}_1$.

Structure $\boldsymbol{n}$ An *Affine combination (AC)* of n vectors is an LC

$$\sum_{i\,=\,1}^{n} \lambda_i \, \boldsymbol{x}_i \tag{10.4}$$

subject furthermore to the *Affine condition*

$$\sum_{i\,=\,1}^{n} \lambda_i \ = \ 1 \ . \tag{10.5}$$

Remark 1 Such cannot possibly be LI. For the condition

$$\lambda_i \ = \ 0 \ \ \forall \ i$$

gives

$$\sum_{i\,=\,1}^{n} \lambda_i \ = \ 0 \ \neq \ 1 \ .$$

Remark 2 One way of formulating *affine independence* (*AI*) is as follows. Given a set of vectors $\mathfrak{Y}$, pick one $\boldsymbol{y}$ and subtract it from all the others. Thus forming a *set of relative vectors* $\boldsymbol{\mathfrak{R}}$ containing 1 vector less than $\mathfrak{Y}$. Then $\mathfrak{Y}$ is AI iff $\boldsymbol{\mathfrak{R}}$ is LI.

Exercise 6$^-$ Show that this is well-defined. I.e. that it is independent of the choice of $\boldsymbol{y}$. Also show that for finite $\mathfrak{Y}$, this is equivalent to no vector in $\mathfrak{Y}$ being expressible as a nontrivial AC of the other vectors in $\mathfrak{Y}$

10.6 AS: Affine span

Definition 1 The *Affine-combinational Affine span* of a subset of points $\boldsymbol{\mathfrak{S}} \subseteq \boldsymbol{\mathfrak{a}}$ is the arena of all possible AC of these points. I.e.

$$\mbox{ACAS}(\boldsymbol{\mathfrak{S}}) \ := \ \left\{ \sum_{i\,=\,1} \lambda_i \, \boldsymbol{s}_i \ \middle| \ \boldsymbol{s}_i \ \in \ \boldsymbol{\mathfrak{S}} \ , \ \ \lambda_i \ \in \ \mathbb{R} \ : \ \ \sum_{i\,=\,1} \lambda_i \ = \ 1 \right\} \ .$$

Definition 1$'$ The *minimal Affine span* of $\boldsymbol{\mathfrak{S}}$ is the smallest Affine space containing $\boldsymbol{\mathfrak{S}}$. I.e.

$$\mbox{MAS}(\boldsymbol{\mathfrak{S}}) \ := \ \min_{\mathfrak{s} \ \subseteq \ \mathfrak{F} \ \leq \ \mathfrak{a}} \mathfrak{F} \ .$$

Definition 1$''$ The *intersectional Affine span* of $\boldsymbol{\mathfrak{S}}$ is the intersection of all the subspaces of $\boldsymbol{\mathfrak{a}}$ that contain $\boldsymbol{\mathfrak{S}}$. I.e.

$$\mbox{IAS}(\boldsymbol{\mathfrak{S}}) \ = \ \bigcap_{\mathfrak{s} \ \subseteq \ \mathfrak{F} \ \leq \ \mathfrak{a}} \mathfrak{F} \ .$$

Remark 1 We are minimizing, and then instead intersecting, over $\mathfrak{F}$ for a fixed subset $\boldsymbol{\mathfrak{S}}$ in a fixed Affine space $\boldsymbol{\mathfrak{a}}$. This can furthermore be identified as minimizing, and then intersecting, *over* the totality of $\mathfrak{F} \ \leq \ \boldsymbol{\mathfrak{a}}$. I.e. over $\boldsymbol{\mathfrak{a}\mathfrak{S}}$ubspace$(\boldsymbol{\mathfrak{a}})$.

Exercise 7$^+$ Establish the following.

i) The last two definitions each uniquely define an Affine space.

ii) The above three definitions coincide. [In the vector space counterpart of this result, $\emptyset$ has to be excluded; does this happen again in the Affine case?]

Naming Remark 5 To celebrate, let us henceforth refer to any of the above three notions as just the *Affine span (AS)*, commonly alias *Affine hull*, which we denote by AS .

Example 0 If $\mathfrak{Y}$ is Affine,

$$\text{ASed}(\mathfrak{Y}) \;=\; \mathfrak{Y} \;\text{ itself} .$$

Exercise 8$^-$ What possible forms can be taken by $\text{ASed}(\,N \text{ points } \subseteq \mathbb{R}^2\,)$ for $N = 0, 1, 2$ and arbitrary?

10.7 AB: Affine bases

Structure 1 We are now free to use the

$$(\,\text{basis}\,) \;=\; (\,\text{minimal spanning set}\,)$$

conceptualization so as to have a notion of Affine basis as well.

Useful Lemma 1 (barycentric coordinates) Given an Affine basis $\boldsymbol{a}_i$, each point in an Affine space can furthermore be !ly expressed as follows.

$$\boldsymbol{x} \;=\; \sum_{i\,=\,0}^{p} \lambda_i\,\boldsymbol{a}_i \;, \quad \sum_{i\,=\,0}^{p} \lambda_i \;=\; 1 \;.$$

The $p + 1$ scalars λ_i constitute the *barycentric coordinates* of $\boldsymbol{x}$ with respect to the points $\boldsymbol{a}_i$.

Exercise 9 Prove this for $n = 1, 2, 3$ and arbitrary.

Application 1 Collinearity and coplanarity can be reformulated as statements of AD. This pre-empts that Collinearity and Coplanarity Theorems are Affine; see Volume 3 for more. For instance, 2 points are collinear if the one is in the AS of the other. And in arbitrary dimension and co-dimension, one has likewise notions of co-flatness, and Co-flat Theorems rooted in AD.

10.8 Affine dimension dim_A

Structure 1 A flat $\mathfrak{F}\text{lat}$ that is the AS of $p + 1$ AI points is of (Affine) dimension p . I.e. we follow the

$$\text{dim}_\text{A}(\mathfrak{F}\text{lat}) \;=\; \text{dim}(\,AS(\mathfrak{F}\text{lat})\,)$$

counterpart of the fourth member of the linear quartet of Fig **??**.

Remark 1 Vector subspaces $\mathfrak{U}$ are a subcase of flats, on which linear dimension is defined. Since AS coincides with LS, for vector subspaces $\mathfrak{U}$ we have the following alignment.

$$\text{dim}_\text{L}(\mathfrak{U}) \;=\; \text{dim}\,(\,\text{LS}(\mathfrak{U})\,) \;=\; \text{dim}\,(\,\text{AS}(\mathfrak{U})\,) \;=\; \text{dim}_\text{A}(\mathfrak{U}) \;.$$

Remark 2 Since translates preserve dimension as mere re-assignments of origin, we also have the following. I.e.

$$\text{dim}_\text{A}(\,\mathfrak{U} + \boldsymbol{t}\,) \;=\; \text{dim}_\text{A}(\mathfrak{U}) \;=\; \text{dim}_\text{L}(\mathfrak{U}) \;.$$

So Affine dimension coincides with linear dimension, while being defined for a wider range of subspaces: Affine rather than just vector.

Remark 3 Within $\mathbb{R}^n$, Euclidean dimension is defined on all flats, since this does not depend on whether the flat contains $\mathbf{0}$. In this context,
$$\mathrm{dim}_{\mathrm{A}}(\mathfrak{U} + \boldsymbol{t}) = \mathrm{dim}_{\mathrm{A}}(\mathfrak{U}) = \mathrm{dim}_{\mathrm{L}}(\mathfrak{U}) = \mathrm{dim}_{\mathrm{Eucl}}(\mathfrak{U}) = \mathrm{dim}_{\mathrm{Eucl}}(\mathfrak{U} + \boldsymbol{t}) .$$

Exercise 10$^-$ Justify each step in Remark 1, 2 and 3's equalities.

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Remark 4 Affine co-dimension $\mathrm{dim}_{\mathrm{A}}$ also coincides with linear co-dimension $\mathrm{dim}_{\mathrm{L}}$. For co-dimension is but a relative notion of dimension: that of the whole space minus that of the subspace in question.

Remark 5 We can now recast our previous 4 Examples of linear co-dimension of subspaces as briefer Examples of Affine co-dimension of flats.

Example 1 For a line in a 2-d Affine space, $\text{co-dim}_{\mathrm{A}} = 2 - 1 = 1$.

Example 2 For a line in a 3-d Affine space, $\text{co-dim}_{\mathrm{A}} = 3 - 1 = 2$.

Example 3 For a plane in a 3-d Affine space, $\text{co-dim}_{\mathrm{A}} = 3 - 2 = 1$.

Example 4 For a hyperplane in an arbitrary finite-d Affine space, $\text{co-dim}_{\mathrm{A}} = n - (n - 1) = 1$. This can be taken to be a more general and even more concise rendition of Sec **??**'s definition. .

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Exercise 11 Prove the *Affine Dimension Theorem.* Namely, that
$$\mathrm{dim}(\mathfrak{a}_1 + \mathfrak{a}_2) + \mathrm{dim}(\mathfrak{a}_1 \cap \mathfrak{a}_2) = \mathrm{dim}\,\mathfrak{a}_1 + \mathrm{dim}\,\mathfrak{a}_2 .$$

Project 1 Pin an Order Theory interpretation on both the Linear and Affine Dimension Theorems.

Exercise 12 Look up what a *manifold* is in Chapter 45 of Volume 0, as supported by Chapter 17. The *manifold dimension* $\mathrm{dim}_{\mathrm{M}}$ is the dimension of the corresponding flat-space charts. By considering a circle, show that $\mathrm{dim}_{\mathrm{A}} = \mathrm{dim}_{\mathrm{L}}$ need not coincide with $\mathrm{dim}_{\mathrm{M}}$.

Remark 6 Also notice that Euclidean dimension is less general than linear dimension, since there are vector spaces that admit no Euclidean interpretation. And less general than manifold dimension $\mathrm{dim}_{\mathrm{M}}$, since flat Euclidean space is but one among the infinity of $\mathrm{dim}_{\mathrm{M}}$-manifolds.

[Provided that $\mathrm{dim}_{\mathrm{M}} \geq 2$, since $\mathrm{dim}_{\mathrm{M}} \leq 1$ has but a very limited repertoire of manifolds. For this distinction, and more about manifolds, see Volume 6 on Differential Geometry.]

10.9 'Linear-Algebraic systems' reinterpreted in Affine terms

Naming Remark 6 We already interpreted 'Linear-Algebraic systems (LASs)' in terms of flats in Chapter **??**. Combining this with the current Chapter's interpretation of flats as Affine subspaces, we reach the following conclusion. 'LASs' in fact concern not vector = Linear subspaces but Affine subspaces. Let us thus truer-name these as *Affine-Algebraic systems* (*AASs*). Or *Algebraic systems of flats.*

Remark 1 Flats have a harder intersection theory than vector subspaces do. This is due to non-coincident parallel flats not intersecting. As such, it is useful to study the bulk of the intersection theory of flats after we have covered its vector subspace counterpart (Sec **??**).

Remark 2 Recollect from Chapter 2 that flats can be formulated as follows.

$$\lambda \boldsymbol{a} + (1 - \lambda) \boldsymbol{b}$$

is in the space whenever $\boldsymbol{a}$ and $\boldsymbol{b}$ are. This can additionally be interpreted as closure under ACs.

Remark 3 We can furthermore identify 'systems of flats' as the well-defined polynomial degree-1 subcase of Affine-Algebraic systems (in general of degree ≥ 1).

10.10 Projects

Project 2 Investigate which of the Restriction, Extension and Replacement Lemmas of Linear Algebra have Affine counterparts.

Project 3 Investigate the smallest few AC, AI, ASing, and AB arenas.

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10.11 Linear Algebra's quotient spaces

Definition 1 Let $\mathfrak{V}$ be a vector space over $\mathbb{F}$. Consider a subspace $\mathfrak{U} \leq \mathfrak{V}$. Then the *vector space cosets* of $\mathfrak{U}$ are the *subsets* $\mathfrak{U} + \boldsymbol{v}$ for whichever $\boldsymbol{v} \in \mathfrak{V}$. In other words, these are the translates or parallels (in our stricter sense) of $\mathfrak{U}$.

Exercise 13 Show that the cosets $\mathfrak{U} + \boldsymbol{v}_1$ and $\mathfrak{U} + \boldsymbol{v}_2$ are equal iff the following holds. The relative quantity

$$\boldsymbol{r}_{12} := \boldsymbol{v}_2 - \boldsymbol{v}_1 \text{ itself } \in \mathfrak{U} .$$

Definition 2 The *quotient of a vector space* $\mathfrak{U}$ by one of its vector subspaces $\mathfrak{U}$ is given by the following.

$$\mathfrak{V} / \mathfrak{U} := \{ \text{the totality of cosets } \mathfrak{U} + \boldsymbol{v} \,|\, \boldsymbol{v} \in \mathfrak{V} \} .$$

Lemma 2 Vector spaces are quotientive. I.e. quotients of vector spaces are themselves vector spaces.

<u>Proof</u> Let $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathfrak{U}$, and $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathfrak{V}$. So reordering

$$(\boldsymbol{u}_1 + \boldsymbol{v}_1) + (\boldsymbol{u}_2 + \boldsymbol{v}_2) = \underbrace{\boldsymbol{u}_1 + \boldsymbol{u}_2}_{\in \mathfrak{U}} + \underbrace{\boldsymbol{v}_1 + \boldsymbol{v}_2}_{\in \mathfrak{V}} .$$

Next define the sum coset

$$\mathfrak{U} + \boldsymbol{v}_1 + \mathfrak{U} + \boldsymbol{v}_2$$

as consisting of

$$(\text{any element of the first coset}) + (\text{any element of the second coset}) .$$

We finally need to specify what the quotient $\mathfrak{V} / \mathfrak{U}$'s zero is. This is $\mathfrak{U}$ itself, since

$$(\boldsymbol{u}_1 + \boldsymbol{0}) + (\boldsymbol{u}_2 + \boldsymbol{v}_2) = (\boldsymbol{u}_1 + \boldsymbol{u}_2) + (\boldsymbol{v}_2 + \boldsymbol{0}) = \underbrace{\boldsymbol{u}_1 + \boldsymbol{u}_2}_{\in \mathfrak{U}} + \boldsymbol{v}_2 . \ \Box$$

Exercise 14 In other words, the vector spaces are a fixed point of the quotientization map. Are groups? Graphs? Trees? Posets?

[Consult Volume 0 if you have not come across some of these mathematical objects before.]

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Example 0 The sole subspace of $\{\boldsymbol{0}\}$ is $\{\boldsymbol{0}\}$. In this case, there is only room for one coset, $\{\boldsymbol{0}\}$, and the quotient space $\{\boldsymbol{0}\} / \{\boldsymbol{0}\}$ is just a point.

Example 1 $\mathbb{R}$ is minimum for the whole vector space and the trivial vector subspace to be distinct.

For the whole space $\mathfrak{V}$, there is always just the one coset, $\mathfrak{V}$, and the quotient space $\mathfrak{V}/\mathfrak{V}$ is just a point.

For the trivial subspace $\{\mathbf{0}\}$, the cosets are $\{\mathbf{0}\} + \boldsymbol{v}$, each consisting of $\mathbf{0} + \boldsymbol{v} = \boldsymbol{v}$: a translated point. The quotient space formed by the totality of such is thus $\mathfrak{V}$. Which, for our current example, is $\mathbb{R}$.

Example 2 $\mathbb{R}^2$ is minimum as regards having proper nontrivial subspaces. Namely, copies of $\mathbb{R}$ passing through the origin. Given such a line, the cosets are its translates to all parallel lines. The totality that these form is conveniently parametrized by the lines' shortest distances to the origin. The corresponding quotient space is thus $\mathbb{R}$ (Fig 1.5.a).

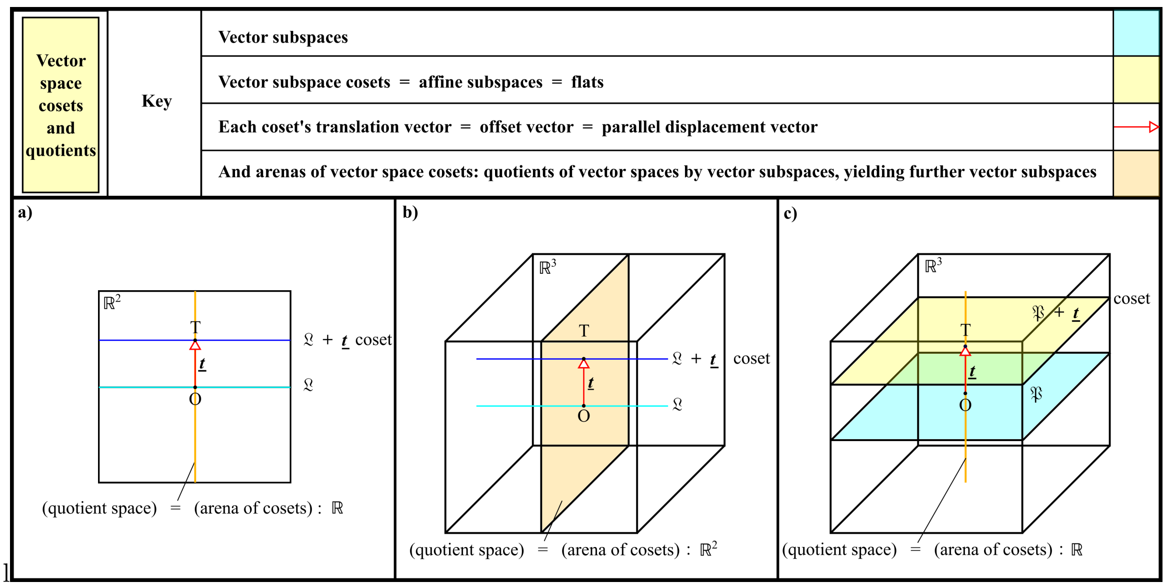

Figure 10.5:

Example 3 $\mathbb{R}^3$ is minimum as regards having two distinct proper nontrivial subspaces: copies of $\mathbb{R}$ and of $\mathbb{R}^2$, in each case passing through the origin. In the first case, we have parallel line cosets forming a $\mathbb{R}^2$ quotient space (Fig 1.5.b). In the second case, we have parallel plane cosets forming a $\mathbb{R}$ quotient space (Fig 1.5.c).

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Pointer A.2 We return to Example 1 and more once armed with orthogonal complement. Our approach to the Dimension Theorem for quotient spaces only becomes possible once this key notion is in play!

Pointers A.3-5 The natural mapping, quotient maps, and quotient spaces from linear maps are left until Part III.

Pointer A.6 Vector space quotients give a way of proving the Jordan Normal Form Theorem stated in Sec **??**.

10.12 Pointers for Affine spaces as a whole

Pointer A.7 Flats *are Affine varieties*: a starting point for Algebraic Geometry [10, 8].

Summary and Pointers A.8-9 By including Linear-Algebraic subspaces, embodying 'Linear-Algebraic systems', modelling parallelism and being the quotients of vector spaces, and Affine varieties, Affine spaces are verily citizens of Kallista. They are furthermore branching stepping stones crossing over to two even more exhalted citizens of Kallista: the *Convex* and *Projective families of concepts* (our next Chapter and Volume 3 respectively). Indeed, one usually swiftly passes from Affine varieties to the more useful *Projective varieties.* Volume 3 furthermore picks out yet further Kallyxtenically-confluent routes to Affine spaces themselves.

10.13 The conic spaces counterpart

Definition 1 A subset $\mathfrak{C}$ of a vector space is a *cone* [7] if the following holds.

$$\text{for every } \boldsymbol{x} \in \mathfrak{C} \text{ and } \lambda \in \mathbb{R}_0 , \ \lambda \boldsymbol{x} \in \mathfrak{C} \text{ as well .}$$

Naming Remark 7 There are also e.g. Metric-Geometrical and Topological notions of cone (the latter including as a subcase the Graph-Theoretical notion in Volume 0). So it makes sense to call the above a *Linear-Algebraic*, or *vector space*, cone so as to keep this distinction clear. Another alias for this notion of cone is

$$\underbrace{\text{non-negative}}_{\lambda \,\in\, \mathbb{R}_0} \ \underbrace{\text{homogeneous}}_{\text{closure under } \lambda\cdot} \ \text{subsets of vector spaces .}$$

The closure in question being of the vectors $\boldsymbol{x} \in \mathfrak{C}$.

Naming Remark 8 'Cone' derives from the Greek *κωνος*, which already had the Geometrical cone among its meanings.

Exercise 15 Show that closed rays are cones, as are lines through 0. Are line segments cones? Describe the cones supported as subsets if $\mathbb{R}^2$.

Exercise 16 How many geometrically-distinct kinds of transformation can you find that send cones to cones?

Definition 2 A *conic combination* of 2 vectors is

$$\lambda \boldsymbol{x} + \mu \boldsymbol{y} \ \text{ with } \ \lambda, \mu \in \mathbb{R}_0 . \tag{10.6}$$

While for n vectors, it is

$$\sum_{i\,=\,1}^{n} \lambda_i \boldsymbol{x}_i \ \text{ with } \ \lambda_i \geq 0 \ \text{ for each } \ i = 1 \ \text{ to } \ n .$$

Exercise 17 What happens to notions of independence (CiI), span (CiS), basis (CiB) and dimension $\dim_{\mathrm{Ci}}$ for conic subsets? Also reinterpret LCs and CiCs as *weighted sums* of vectors.

Project 4 Investigate which of the Restriction, Extension and Replacement Lemmas of Linear Algebra have Ci counterparts.

Project 5 Investigate the smallest few CiC, CiI, CiSing, CiSed and CiB arenas.

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