LINEAR MATHEMATICS

Part IX. Multi-Linear Algebra and Tensors

Chapter 1. What is a Tensor?

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This Sample Chapter is ⟨3⟩: accessible to third-year undergraduates.

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1 Cartesian tensors

Sagredo: And so here we are at the second foundational Discussion of the year, on tensors!

Salviati: What do you mean by a tensor, Sagredo?

Sagredo: Tensors are arrays of real numbers transforming under orthogonal transformations $L_{i\,j}$. The 3 smallest cases are the scalar, the vector and the 2-tensor, with respectively the following transformation laws, from an unprimed frame to a primed one.

$$S' \;=\; S \;. \tag{1}$$

$$V'_i \;=\; L_{i\,p}\,V_p \;. \tag{2}$$

$$T'_{i\,j} \;=\; L_{i\,p}\,L_{j\,q}\,T_{p\,q} \;. \tag{3}$$

Salviati: Et voilà the components version of the definition of Cartesian tensors [6, 9, 18] of the three smallest ranks!

To move forward from here, on the one hand, it is often more conceptually clear and presentationally adroit to use instead a coordinate-free formulation. The rank-n tensor is here viewed as a multi-linear machine [29] sending n real-valued vectors to a real number.[1] Though people do have a habit of returning to the components formulation − with respect to a particular linear basis − whenever calculations are required...

There is also an intermediary formulation using underlines and interior products in place of indices and contractions. Where the interior product coincides with the dot product in the case of contractions between vectors, but generalizes to cover contractions between tensors. Passing to this intermediary does not affect the scalar’s transformation law, while the vector’s is now

$$\underline{\boldsymbol{v}}' \;=\; \underline{\underline{\boldsymbol{L}}} \cdot \underline{\boldsymbol{v}} \;. \tag{4}$$

And the 2-tensor’s is

$$\underline{\underline{\boldsymbol{T}}}' \;=\; \underline{\underline{\boldsymbol{L}}} \cdot \underline{\underline{\boldsymbol{T}}} \cdot \underline{\underline{\boldsymbol{L}}}^{\mathrm{T}} \;. \tag{5}$$

Where have you seen this last equation before?

[1] We already called the matrix, and the linear map, ‘machines’. So by calling the general tensor a multi-Linear machine, we are indeed extending this conceptualization.

2 Interplay with matrix transformations and bases

2.1 (Column) vectors

Sagredo (after a pause): My vectors and matrices course had an equation of this form. With notation $\boldsymbol{P}$ in place of $\boldsymbol{L}$, $\boldsymbol{M}$ in place of $\boldsymbol{T}$ and without any dots exhibited:

$$\underline{\underline{\boldsymbol{M}}}' \ = \ \underline{\underline{\boldsymbol{P}}}\,\underline{\underline{\boldsymbol{M}}}\,\underline{\underline{\boldsymbol{P}}}^{\mathrm{T}} \ . \tag{6}$$

Though I do not recall the course pointing out this relation to the tensor transformation law...

Salviati: Just so. But let us next provide a first-principles derivation. A matrix is a linear machine for turning vectors into vectors. Let us for now take our vector to be a column vector $\boldsymbol{c}$ and our matrix to be orthogonal: mapping from Cartesian coordinate system to another.

$$\underline{\boldsymbol{c}}' \ = \ \underline{\underline{\boldsymbol{L}}} \cdot \underline{\boldsymbol{c}} \ . \tag{7}$$

2.2 Scalars

Salviati: A *Cartesian scalar* is a single number whose value is independent of which Cartesian coordinates are in use. So

$$S' \ = \ S \ . \tag{8}$$

2.3 Row vectors

Salviati: A column vector pre-multiplied by a row vector $\underline{\boldsymbol{r}}$ of the same length returns a scalar: this is in one way of envisaging the scalar product. Prompting the question of how does a row vector itself transform for all of this to be consistent?

$$\underline{\boldsymbol{r}} \cdot \underline{\boldsymbol{c}} \ \stackrel{(7)}{=} \ \left(\underline{\boldsymbol{r}} \cdot \underline{\boldsymbol{c}} \right)' \ = \ \underline{\boldsymbol{r}}' \cdot \underline{\boldsymbol{c}}' \ \stackrel{(8)}{=} \ \underline{\boldsymbol{r}}' \cdot \underline{\underline{\boldsymbol{L}}} \cdot \underline{\boldsymbol{c}}$$

$$\Rightarrow \ \left(\underline{\boldsymbol{r}} \ - \ \underline{\boldsymbol{r}}' \cdot \underline{\underline{\boldsymbol{L}}} \right) \cdot \underline{\boldsymbol{c}} \ = \ 0 \ .$$

But taking $\boldsymbol{c}$ to be arbitrary, we can cancel it out,

$$\underline{\boldsymbol{r}} \ = \ \underline{\boldsymbol{r}}' \cdot \underline{\underline{\boldsymbol{L}}} \ . \tag{9}$$

So post-multiplying by $\boldsymbol{L}^{\mathrm{T}}$,

$$\underline{\boldsymbol{r}} \cdot \underline{\underline{\boldsymbol{L}}}^{\mathrm{T}} \ = \ \underline{\boldsymbol{r}}' \cdot \underline{\underline{\boldsymbol{L}}} \cdot \underline{\underline{\boldsymbol{L}}}^{\mathrm{T}} \ = \ \underline{\boldsymbol{r}}' \cdot \underline{\underline{\mathbb{1}}} \ = \ \underline{\boldsymbol{r}}' \ .$$

2.4 2-tensors

Salviati: Next consider a scalar formed by inserting a matrix $\boldsymbol{T}$ within the product.

$$\underline{\boldsymbol{r}} \cdot \underline{\underline{\boldsymbol{T}}} \cdot \underline{\boldsymbol{c}} \ \stackrel{(7)}{=} \ \left(\underline{\boldsymbol{r}} \cdot \underline{\underline{\boldsymbol{T}}} \cdot \underline{\boldsymbol{c}} \right)' \ = \ \underline{\boldsymbol{r}}' \cdot \underline{\underline{\boldsymbol{T}}}' \cdot \underline{\boldsymbol{c}}' \ \stackrel{(8,\,9)}{=} \ \left(\underline{\boldsymbol{r}} \cdot \underline{\underline{\boldsymbol{L}}}^{\mathrm{T}} \right) \cdot \underline{\underline{\boldsymbol{T}}}' \cdot \underline{\boldsymbol{r}}' \cdot \left(\underline{\underline{\boldsymbol{L}}} \cdot \underline{\boldsymbol{c}} \right) \tag{10}$$

$$\Rightarrow \ \underline{\boldsymbol{r}} \cdot \left(\underline{\underline{\boldsymbol{T}}} \ - \ \underline{\underline{\boldsymbol{L}}}^{\mathrm{T}} \cdot \underline{\underline{\boldsymbol{T}}}' \cdot \underline{\underline{\boldsymbol{L}}} \right) \cdot \underline{\boldsymbol{c}} \ = \ 0 \ .$$

But taking $\boldsymbol{c}$ and $\boldsymbol{r}$ to be arbitrary, we can cancel these out,

$$\Rightarrow \ \underline{\underline{\boldsymbol{T}}} \ = \ \underline{\underline{\boldsymbol{L}}}^{\mathrm{T}} \cdot \underline{\underline{\boldsymbol{T}}}' \cdot \underline{\underline{\boldsymbol{L}}} \ .$$

So pre-multiplying by $\boldsymbol{L}$ and post-multiplying by $\boldsymbol{L}^{\mathrm{T}}$,

$$\underline{\underline{\boldsymbol{L}}} \cdot \underline{\underline{\boldsymbol{T}}} \cdot \underline{\underline{\boldsymbol{L}}}^{\mathrm{T}} \ = \ \underline{\underline{\boldsymbol{L}}} \cdot \underline{\underline{\boldsymbol{L}}}^{\mathrm{T}} \cdot \underline{\underline{\boldsymbol{T}}}' \cdot \underline{\underline{\boldsymbol{L}}} \cdot \underline{\underline{\boldsymbol{L}}}^{\mathrm{T}} \ = \ \underline{\underline{\mathbb{1}}} \cdot \underline{\underline{\boldsymbol{T}}}' \cdot \underline{\underline{\mathbb{1}}} \ = \ \underline{\underline{\boldsymbol{T}}}' \ .$$

Sagredo: Does deriving the 3-tensor transformation law involve 2 transposed matrices or 2 untransposed matrices?

Salviati: Either, or all, or none. At the level of Cartesian tensors without upstairs–downstairs distinction, all of these are equivalent. This does also mean that we could get to the corresponding tensor transformation law without introducing separate row vectors. By a working that then unambiguously extends with increase in rank...

2.5 Vector bases

Salviati: Back to vectors, a further little calculation along the above lines reveals that bases transform the other way around from coordinates:

$$\underline{\underline{\boldsymbol{e}}}' \;=\; \underline{\underline{\boldsymbol{e}}} \cdot \underline{\underline{\boldsymbol{L}}}^{\mathrm{T}} \;. \tag{11}$$

Where the underline not used up by the dot runs over the vectors belonging to the basis:

$$\{\underline{\boldsymbol{e}}_i\}_{i\,=\,1}^{\;d} \;.$$

2.6 Bases for tensors

Salviati: Little work is needed to give tensors of higher rank than vectors their own bases... For one can use the *outer product* alias *tensor product* on the vectors' basis. The tensor product has been available since the days of Hassler Whitney [24]. Though rather earlier treatments of tensors already considered strings of products of uncontracted vectors. For instance

$$\underline{\boldsymbol{v}}\,\underline{\boldsymbol{w}}$$

is called a 'dyad' in 19th century and early 20th century sources... In tensor product notation, this is denoted as follows.

$$\underline{\boldsymbol{v}} \;\otimes\; \underline{\boldsymbol{w}} \;.$$

For rank-2 Cartesian tensors, then, a basis is provided by

$$\{\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j\}_{i,\,j\;=\;1}^{\;\;d} \;.$$

While for general rank p, dyads become 'polyads' (in Josiah Willard Gibbs' parlance [2]),

$$\underline{\boldsymbol{v}_1} \dots \underline{\boldsymbol{v}_p} \;\;=\;\; \bigtimes_{i\,=\,1}^{p} \underline{\boldsymbol{v}}_i \;.$$

Tensor product notation for which is

$$\underline{\boldsymbol{v}}_1 \;\otimes\; \dots \;\otimes\; \underline{\boldsymbol{v}}_p \;\;=\;\; \bigotimes_{i\,=\,1}^{p} \underline{\boldsymbol{v}}_i \;.$$

Finally for rank-2 Cartesian tensors, a basis is provided by

$$\left\{\; \bigotimes_{j\,=\,1}^{p} \boldsymbol{e}_{i_j} \;\right\}_{i_1,\,\dots\,,\,i_p\;=\;1}^{\qquad\quad d} \;.$$

3 G-tensors

Salviati: But what is so special about the orthogonal transformations $O(d)$ in the context of defining tensors?

Sagredo: As regards alternatives, I recall you tying pseudo-tensors to the case that excludes the reflections, making the group $SO(d)$. In $\mathbb{R}^d$, $SO(d)$ is the corresponding rotation group, making for a natural match. I also recall you mentioning vectors, and tensors, for arbitrary groups G acting on arbitrary spaces. On which grounds, I think that one would use unitary groups such as $U(n)$ or $SU(n)$ when considering $\mathbb{C}$-valued arrays...

Salviati: Bravo! To be specific, a Cartesian *pseudo-vector*, alias *axial vector* transforms according to

$$\underline{\boldsymbol{a}}' = (-)^r \underline{\underline{\boldsymbol{L}}} \cdot \underline{\boldsymbol{a}} \, . \tag{12}$$

Where r is the number of reflections in $\boldsymbol{L}$, which is in general a product of rotations and reflections. Since the orthogonal group can be described by just 1 rotation with or without just 1 reflection, we can in effect use just $r = 0$ or 1 . This is furthermore often encountered in the form that makes the identification

$$(-)^r = \det \boldsymbol{L} \, .$$

To give concrete examples, the cross product of 2 vectors is a pseudo-vector, so e.g. angular momentum and torque are two such. So is the curl of a vector, by which e.g. vorticity is another. But the curl of a pseudo-vector returns a vector by cancellation of minus factors. As does the cross of a vector and a pseudo-vector...

Salviati: Another prominent example involves taking

$$G = Lor(3, 1) = O(3, 1)$$

– the Lorentz group – in Special Relativity (SR)'s Minkowski spacetime [15]. This indeed returns the Lorentzian analogue of Cartesian tensors. And corresponding to changes between inertial frames, in the SR sense. In the homogeneous-linear case in which our two frames share an origin... its own pseudo-tensors...

A common thread here is that we are taking the homogeneous-linear part of the isometries corresponding to the Euclidean and Minkowskian metric. In each of these cases, the full isometries amount to appending the corresponding notion of translations. Unqualified in Euclidean space, while consisting of space- and time-translations in Minkowski spacetime. With these included, one would have rather the Euclidean group $Eucl(d)$ and the Poincaré group $Poin(3, 1)$ [35] respectively. This common thread generalizes to isometry groups for whichever other metric geometry [35]...

Sagredo: Does this cover General Relativity as well?

Salviati: No and yes. At this stage in the discussion, we do not have enough infrastructure to answer...

Sagredo: What infrastructure?

Salviati : General-linear rather than orthogonal matrices.

Downstairs versus upstairs indices.

General coordinate transformations encoded by Jacobian matrices.

Working patchwise on a manifold rather than in $\mathbb{R}^n$.

4 Infrastructure

4.1 General-linear matrices

Salviati: Let us first set $G = GL(n, \mathbb{R})$: the invertible square $n \times n$ matrices.

And rerun Sec 2 with a more general $\boldsymbol{P}$ in place of the above $\boldsymbol{L}$. For the cancellations to continue to work, we in general require $\boldsymbol{P}^{-1}$ and not $\boldsymbol{P}^{\mathrm{T}}$. This is a consistent extension since for orthogonal matrices the two coincide... Thus now

$$\boldsymbol{r}' = \underline{\underline{\boldsymbol{P}}}^{-1} \cdot \underline{\boldsymbol{r}} \ ,$$

$$\underline{\underline{\boldsymbol{M}}}' = \underline{\underline{\boldsymbol{P}}} \cdot \underline{\underline{\boldsymbol{M}}} \cdot \underline{\underline{\boldsymbol{P}}}^{-1} \ .$$

4.2 The upstairs-downstairs distinction

Sagredo: So far as I know, from online accounts of GR, the downstairs-upstairs index distinction is a curved space phenomenon?

Salviati: Yes, but not just!

Let us first approach this using metrics, since it is often approached in this way in the Physical Natural sciences... The metric $\boldsymbol{g}$ is a 2-tensor encoding various kinds of measurement.

The Euclidean metric's norm leads to the notion of Euclidean distance,

$$|| \underline{\boldsymbol{x}} - \boldsymbol{y} || \, .$$

The corresponding inner product measures angles as well. By rearranging the dot product formula

$$\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{y}} \;=\; ||\underline{\boldsymbol{x}}|| \; ||\underline{\boldsymbol{y}}|| \cos\theta \, .$$

To

$$\theta \;=\; \arccos\left(\frac{\underline{\boldsymbol{x}}}{||\underline{\boldsymbol{x}}||} \cdot \frac{\underline{\boldsymbol{y}}}{||\underline{\boldsymbol{y}}||} \right) \;=\; \arccos\left(\hat{\underline{\boldsymbol{x}}} \cdot \hat{\underline{\boldsymbol{y}}} \right) \, .$$

Where the hat denotes unit vector.

But this is not the only function that a metric performs. It is furthermore a lowerer of indices, while its inverse is a raiser of indices.

Now the Euclidean metric in plane polar coordinates is

$$\underline{\underline{\boldsymbol{g}}}_{\mathrm{Eucl}} \;=\; \mathrm{diag}(\, 1, \, r^2 \,) \, .$$

So if a vector has angular component v^θ , then its lowered counterpart has angular component

$$v_\theta \;=\; g_{\theta\theta} v^\theta \;=\; r^2 \, v^\theta \, .$$

Which is almost always numerically distinct from v^θ . Thus demonstrating that it is already a phenomenon for curvilinear coordinates in flat space. And even for the considerably more specialized case of orthogonal curvilinear coordinates in flat space...

Next consider the Minkowski metric $\boldsymbol{\beta}$ in the $- \quad + \quad + \quad +$ convention, in the most usual inertial-frame rectilinear coordinates. Then

$$v_t \;=\; \eta_{t\,t}\, v^t \;=\; - \, v^t \, .$$

Which is almost always numerically distinct from v^t . So even in a flat space in rectilinear coordinates, indefiniteness forces a downstairs–upstairs index distinction.

And so already flat space (in beyond Cartesian coordinates), and the Minkowski spacetime of SR (now even in its analogue of Cartesian coordinates) exhibit this phenomenon.

Observe furthermore that the name 'downstairs-upstairs' applies just as well to index-free notation. While raised and lowered refers specifically to indices. For all that in the index-free notation that we have deployed so far, there is then an analogous underline-overline distinction...

Finally, to help out those among our Audience who have seen this phenomenon in other sources, this is also alias the co-contra distinction. Which rests upon James Joseph Sylvester's [1] use of covariant vector and contravariant vector (and so on for higher-rank tensors). Whence also the name *covector* for the lowered-index version...

4.3 A start on: what is a covector?

4.3.1 Row vectors

Sagredo: Given that last year's discussion was on 'what is a vector', and pinned many different meanings and aspects on vectors, what is a covector?

Salviati: Some conceptually useful aliases are as follows.

Firstly, in basic 'vectors and matrices' courses, these are the aforementioned row vectors $\boldsymbol{r}$, in contrast to actual vectors being called column vectors $\boldsymbol{c}$... We could indeed have justified the upstairs-downstairs distinction at the level of a vectors and matrices course distinguishing between row and column vectors. With the transpose, or more generally inverse, transformation of the rows justifying their status as a distinct type of mathematical object with its own transformation law...

4.3.2 Dual vectors

Secondly, covectors are known as *dual vectors* in Linear Algebra [4] (in the further presence of an inner product, converting vector spaces to inner product spaces...)

From which one can infer that bases for these take the form of the *dual bases* that one encounters in every course on this subject... I.e.

$$\{\boldsymbol{d}_i\}_{i\,=\,1}^{\;n}$$

such that

$$\boldsymbol{d}_i \cdot \boldsymbol{e}^j \;=\; \delta_i{}^j \;.$$

Where $\boldsymbol{\delta}$ is the Kronecker delta tensor.

4.3.3 1-forms

Thirdly, covectors are known as 1-*forms* [10, 29, 28] in Calculus, Potential Theory and Geometry. In which context they are cast as *differentials*:

$$\sum_i f_i(\boldsymbol{x})\,\mathrm{d}x^i \;.$$

About which questions of closedness and exactness are posed. Eventually leading to Cohomology [14]: a Differential Topology branch of Algebraic Topology...

4.3.4 Linear functionals

Fourthly, they are also known as *linear functionals* [4, 13]: an index-free conceptualization as a linear map from a vector space to the underlying field. I.e.

$$f \;:\; \mathfrak{V} \longrightarrow \mathfrak{F}(\mathbb{R})$$

$$\overline{\boldsymbol{v}} \;\mapsto\; f\,(\overline{\boldsymbol{v}}) \;.$$

For a suitable function space $\mathfrak{F}(\mathbb{R})$. This is the covector's subcase of the above 'linear machine' conceptualization of tensors! And is a point of view that readily extends to infinite spaces, which is a useful asset in such fields as Functional Analysis and Quantum Mechanics.

The linear functional conceptualization covers how while finite vector spaces are isomorphic to their dual vector spaces, infinite vector spaces' duals are always at least as large, and are generally larger...

Two further aliases are unveiled in SSSec 6.1, and yet another in Chapter **??**.

4.4 Tensors of rank (q, p)

Salviati: Let us now return to Sec 2's composition, with an extra distinction – downstairs-upstairs – leading to us having a further input element: the covector. We now view what that section built as contravariant tensors. The bases here are denoted and interpreted as follows.

$$\left\{ \bigotimes_{j=1}^{q} \boldsymbol{e}^{i_j} \right\}_{i_1, \dots i_q = 1}^{d} .$$

Furthermore now also tensoring covectors constructs covariant tensors. With bases

$$\left\{ \bigotimes_{k=1}^{p} \boldsymbol{d}_{i_j} \right\}_{i_1, \dots i_p = 1}^{d} .$$

Finally tensoring a string of q vectors and a string of p covectors produces tensors of mixed rank [32, 28] (q, p) With bases

$$\left\{ \bigotimes_{j=1}^{p} \boldsymbol{e}^{i_j} \otimes \bigotimes_{l=1}^{q} \boldsymbol{f}_{k_l} \right\}_{i_1, \dots i_p, k_1, \dots k_q = 1}^{d} .$$

5 General coordinate transformations and diffeomorphisms

Salviati: Let us now start to go after GR's coordinate systems...

This involves choosing G to be vastly larger that an isometry group, in a way which in fact already has a counterpart in flat space. For we are talking about the general coordinate transformations. Which on $\mathbb{R}^d$ are a vast generalization of Cartesian coordinates and of Sec **??**'s orthogonal curvilinear coordinates. To local use of 'any coordinates'... Modulo the caveat that these coordinates are differentiable enough that the Jacobian matrices inter-relating pairs of such coordinate systems can be defined.

Which in turn usually only holds in portions of $\mathbb{R}^d$: the open domains in Subfig 1.b). This is to be contrasted with Cartesian coordinates being globally defined (Subfig a). While plane polar coordinates already fail to be defined at their choice of origin, since no angular coordinate can be specified there.

Let us next assume that our 2 coordinate systems' open domains intersect. And that the coordinate transformation between them is suitably differentiable therein. So as to support

$$\overline{\underline{\boldsymbol{J}}} := \frac{\partial \underline{\boldsymbol{x}}'}{\partial \underline{\boldsymbol{x}}} . \tag{13}$$

So for starters we are talking about enough differentiablity to be able to have these first partial derivatives... And our truer notation for the coordinate transformation matrix that is to enter vector and tensor transformation laws is $\boldsymbol{J}$, standing for *Jacobian matrix*...
Insisting on having the coordinate transformation both ways round requires also the inverse function to be sufficiently differentiable. So as to also support the inverse transformation's own Jacobian

$$\overline{\underline{\boldsymbol{J}}}^{-1} = \frac{\partial \underline{\boldsymbol{x}}}{\partial \underline{\boldsymbol{x}}'} . \tag{14}$$

One then has the following transformation laws for these generalized coordinate transformations.

$$\overline{\boldsymbol{v}}' = \overline{\underline{\boldsymbol{J}}} \cdot \overline{\boldsymbol{v}} ,$$

$$\underline{\boldsymbol{a}}' = \overline{\boldsymbol{a}} \cdot \overline{\underline{\boldsymbol{J}}}^{-1} ,$$

$$\overline{\underline{\boldsymbol{T}}}' = \overline{\underline{\boldsymbol{J}}} \cdot \overline{\boldsymbol{T}} \cdot \overline{\underline{\boldsymbol{J}}}^{-1} .$$

Which is what many texts [7, 8] going past Cartesian tensors use... And in components?

Sagredo (mostly with a piece of chalk): In components,

$$v^{i\,\prime} = \frac{\partial x'^{\,i}}{\partial x^p} v^p ,$$

$$a'_i = \frac{\partial x^p}{\partial x'^{\,i}} a_p ,$$

$$T_i{}^{j\,\prime} = \frac{\partial x^p}{\partial x^{i\,\prime}} \frac{\partial x'^{\,j}}{\partial x^q} T_p{}^q .$$

Salviati: So general coordinate transformations in fact need to be sufficiently differentiable to support these first derivatives, giving the further alias 'diffeomorphism group.[2] With the identity map as the group identity, the above insistence guaranteeing group inverses, and the chain rule guaranteeing closure. To celebrate, we denote this group by

$$Diff\left(\mathbb{R}^k\right) .$$

And we are indeed here talking about the $Diff\left(\mathbb{R}^k\right)$-tensors corresponding to this group: vastly larger than the orthogonal or general-linear transformation groups...

[2]Though this corresponds to taking an active viewpoint as opposed to coordinate systems' passive viewpoint.

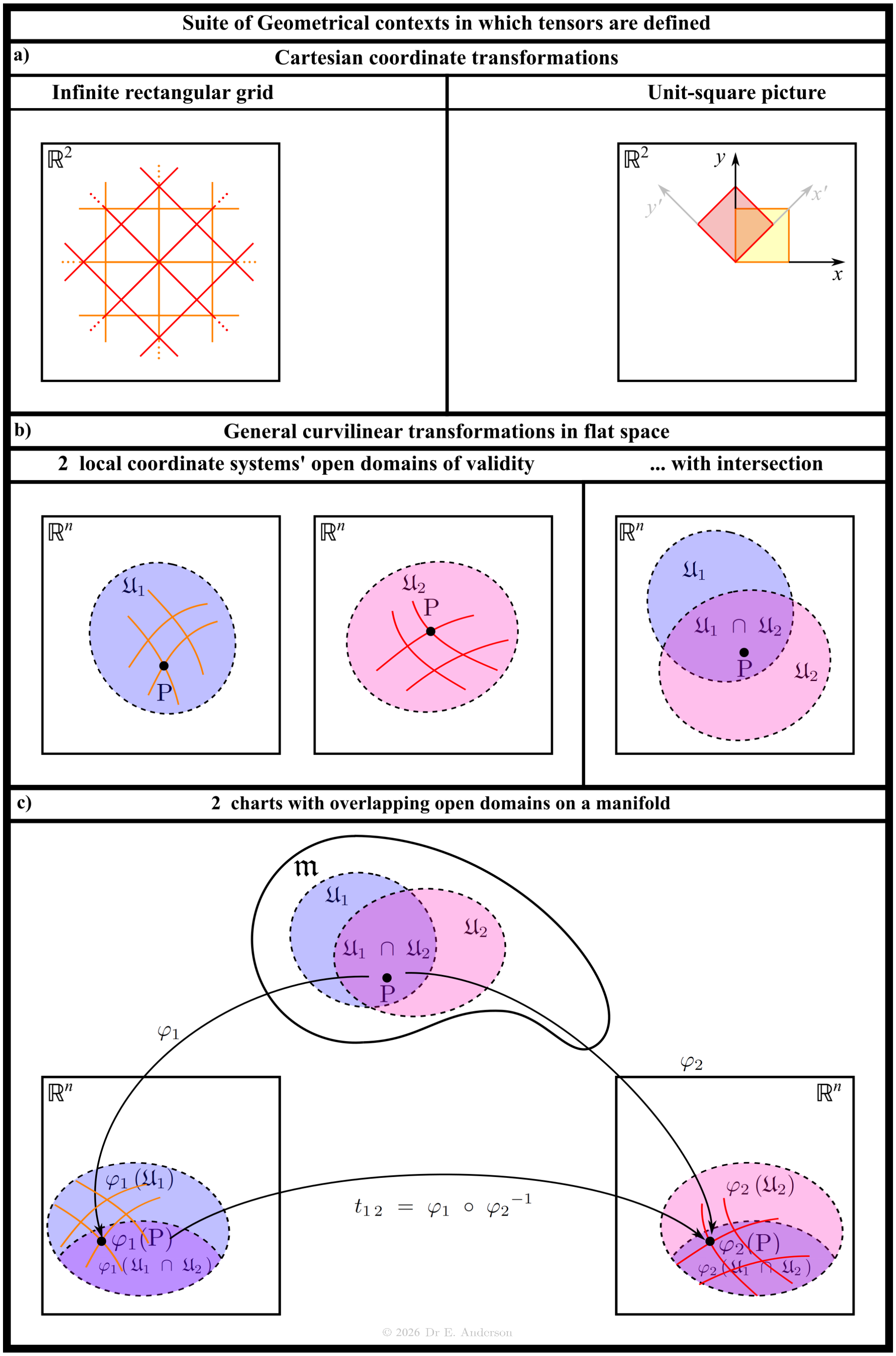
Suite of Geometrical contexts in which tensors are defined
a)
Cartesian coordinate transformations
Infinite rectangular grid
Unit-square picture
ℝ2
ℝ2
y
x
y′
x′
b)
General curvilinear transformations in flat space
2 local coordinate systems' open domains of validity
... with intersection
ℝn
𝔘1
P
ℝn
𝔘2
P
ℝn
𝔘1
𝔘1 ∩ 𝔘2
P
𝔘2
c)
2 charts with overlapping open domains on a manifold
𝔪
𝔘1
𝔘2
𝔘1 ∩ 𝔘2
P
φ1
φ2
ℝn
ℝn
φ1 (𝔘1)
φ1(P)
φ1(𝔘1 ∩ 𝔘2)
φ2 (𝔘2)
φ2(P)
φ2(𝔘1 ∩ 𝔘2)
t12 = φ1 ∘ φ2−1
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Figure 1:

6 Vectors' plot and large subplot revisited

6.1 The subplot

6.1.1 Manifolds

Salviati: Let us now generalize working in $\mathbb{R}^n$ to working on a manifold $\mathfrak{m}^n$.[3] Since these are locally-Euclidean, charts

$$\varphi : \mathfrak{m}^n \longrightarrow \mathbb{R}^n$$

can be defined. Each corresponds to assigning a local coordinate system: the localized orange and red grids in Subfig c).

If the corresponding open domains intersect, then a chart and an inverse chart can be composed. We denote this by $t_{1\,2}$ in this subfigure. Standing for *transition function* from coordinate system 1 to coordinate system 2 . This is a function

$$t_{1\,2} \ : \ \mathbb{R}^n \longrightarrow \mathbb{R}^n \ .$$

Mapping from (the intersection of the two open domains' open images) to itself!

Thus now the standard '$\mathbb{R}^n \longrightarrow \mathbb{R}^m$' Calculus applies. Assuming again suitable differentiablity throughout this intersection, the Jacobian matrix can be defined once more. Assuming this for the opposite-direction map as well, sufficient differentiability of the inverse holds. And so one has a notion of diffeomorphism suitable for manifolds. These now form the *manifold's diffeomorphism group* ,

$$Diff(\mathfrak{m}^n) \ .$$

Now for (suitably differentiable) general coordinate transformations on our manifold $\mathfrak{m}^n$.

6.1.2 1-forms as normal-related and cotangent vectors

Salviati: Recollect at this point that vectors can be defined on manifolds by firstly defining curves thereupon and secondly taking tangents to these curves.

Also recollect at this point the relation between differentials and the field of normals to a surface. That normals are perpendicular to tangents threads together that differentials play an Algebraically-dual [34] role to tangent vectors... Which point of view is 'categorically subsumed' in Differential Geometry by minting a further alias *cotangent vector*.
Normal-related aliases also stem from the previous argument. For all that many of these are (co)dimension-dependent. [Multiple normals for less-than-hypersurfaces, binormals as well as normals and tangents to a curve in $\mathbb{R}^3$ ...]

6.2 The main theme: (co)tangent spaces and bundles

Sagredo: And the main theme?

Salviati: The arena – meaning totality of – vectors based at a point m in a manifold $\mathfrak{m}$ is as follows. The corresponding *tangent space* [34, 40]

$$\mathrm{T}_{\mathrm{m}}(\mathfrak{m}) \ .$$

And the arena of all vectors on a manifold is the corresponding *tangent bundle* [33]

$$\mathfrak{T}(\mathfrak{m}) \ .$$

The arena of covectors at $\mathrm{m} \in \mathfrak{m}$ is the *cotangent space* [40],

$$\mathfrak{T}^*_{\mathrm{m}}(\mathfrak{m}) \ .$$

[3]See [21] for a suitable introductory sketch, or [16, 32, 34, 40, 22] for more advanced development

This is the Linear-Algebraic dual use of $*$; these are in effect the dual spaces to the tangent spaces... Finally, the arena of all covectors on a manifold constitutes the *cotangent bundle*,

$$\mathfrak{T}^*(\mathfrak{m}) \ .$$

For our next trick, tensor products continue to be defined between the above arenas. For purely contravariant tensors – rank $(q, 0)$ – this returns that their arena is as follows.

$$\mathfrak{T}\text{ensor}(q, 0) \ = \ \bigotimes^{q} \mathfrak{T}(\mathfrak{m}) \ .$$

While for purely covariant tensors – rank $(0, p)$ –

$$\mathfrak{T}\text{ensor}(0, p) \ = \ \bigotimes^{p} \mathfrak{T}^*(\mathfrak{m}) \ .$$

Finally, more generally for tensors of some particular mixed rank (p, q)

$$\mathfrak{T}\text{ensor}(p, q) \ = \ \bigotimes^{p} \mathfrak{T}(\mathfrak{m}) \otimes \bigotimes^{q} \mathfrak{T}^*(\mathfrak{m}) \ .$$

6.3 Isometries

Salviati: Here instead of

$$Diff\,(\mathfrak{m}^n) \ ,$$

we employ the much smaller isometry group

$$Isom\,(\mathfrak{m}, \boldsymbol{g}) \ .$$

This is more structured, being for a specific metric $\boldsymbol{g}$ on our manifold.

The example

$$Eucl(d) \ = \ Isom\left(\mathbb{R}^d, \boldsymbol{g}_{\text{Eucl}}\right)$$

ties isometries back to our incipient Cartesian tensors example...
And

$$Lor(3, 1) \ = \ O(3, 1)$$

back to its Minkowski spacetime counterpart...
The first example has not yet been homogeneous-linearized, while the second has... Which groups would we use to present this diversity the other way around?

Sagredo: The orthogonal group and the Poincaré group!

6.4 Isometries versus diffeomorphisms in the study of GR

Salviati: At this point, I am ready to say that your question about GR is a simple question that does not have a simple answer! Isometries turn out to be of some use in studying exact solutions of General Relativity (GR). But not generic GR spacetimes, which are moreover held to play a very major role in GR... Nor does it correspond to GR's vaunted generalization of SR frames!

For generic solutions in GR have no symmetries, so

$$Isom\left(\mathfrak{m}, \boldsymbol{g}_{\text{generic}}\right) \ = \ id \ .$$

And one cannot build a useful Tensor Calculus out of this. While all known exact solutions of GR possess symmetries. With most particularly famous solutions, such as the Schwarzschild black hole, or the homogeneous cosmologies, possessing quite a lot of isometries [32]. Rendering the corresponding isometry groups rather crucial to their study... That generic solutions matter is clear for instance from the Hawking–Penrose singularity theorems [32] being about generic solutions.

In effect, studying exact solutions and studying generic solutions are two separate subjects, each hard in their own way. Legendary difficulties with sailing around Sicily come to mind at this point, with each being associated with a different sea monster... Facing our Scylla of the high symmetry involves some quite subtle Differential-Geometric tools rooted on isometries. While facing our Charybdis of genericity involves sparser tools [32]. With the problem that with sailing toward Scylla, one can slowly work out detailed features of a measure-zero subset of the solution space. In a way that admits no extension to the rest of the solution space. While in sailing toward Charybdis, one is considering the full-measure subset at the cost of losing almost all of one's familiar tools...

6.5 The subplot continues: further Geometrical levels of structure

Salviati: Diffeomorphisms versus isometries is not a dichotomy.

Sagredo: What else is possible?

Salviati: Already in flat space, e.g. the *similarity group* $Sim(d)$ and the *Affine group* $Aff(d)$ are alternatives to $Eucl(d) = Isom\left(\mathbb{R}^d, \boldsymbol{g}_{\text{Eucl}}\right)$ and $Diff\left(\mathbb{R}^d\right)$. See e.g. [21]. Generalizing to $Sim\left(\mathfrak{m}^d\right)$ and $Aff\left(\mathfrak{m}^d\right)$ [23, 25, 22] instead of $Isom\left(\mathfrak{m}^d, \boldsymbol{g}\right)$ and $Diff\left(\mathfrak{m}^d\right)$.

Salviati: Furthermore, the connection, which resolves how to compare at different points on a manifold all Geometrical objects that are not immediately comparable, can already be viewed as an Affine object. More structuredly, any Geometrical metric supports a metric connection. Though Riemannian Geometry enjoys the simplifying compatibility that its metric connection recovers the same affine connection... In this way, we re-encounter [20]'s large subplot concerning the need for parallel transport on manifolds! This juxtaposition of 'Affine' and 'parallel' notions rests upon how Affine Geometry is indeed the great Euler's abstraction of parallelism... For all that Affine connections date to somewhat later (from 19th century Cristoffel to Levi-Civita to mid-20th centrury Ehresmann, even).

Sagredo: What do you mean by not immediately comparable Geometrical objects? And which are immediately comparable?

Salviati: Scalars are. All other tensors are not. And not just.

For instance tensor densities are not. E.g. for a scalar density,

$$S' = |g|^w S .$$

Where g is the determinant of the metric and w is the weight. Applications of densities include the next Section's topic and integration on manifolds.

Finally, our Quantum Mechanics section shall point to a further family of not immediately comparable objects...

7 The alternator

7.1 The lion's share of why $(S)O(3)$ is ubiquitous in basic STEM theory

Salviati: The current Seminar and Book are in no position to cover the general case of the Lie algebra interpretation of tensors. We proceed instead via how the $(S)O(3)$ case's Mathematics enfilades around half of the first- and second-year STEM Theory courses!

Much of this ubiquiety is conceptually rooted in the (special) orthogonal transformations realizing the rotations of Euclidean space. Without and with the reflections. But these are the homogeneous-linear isometries of $\mathbb{R}^3$: an excellent approximate model for physical space. With rather simpler subgroups $(S)O(2)$ playing the corresponding role in the plane $\mathbb{R}^2$. Which features in simplified Physics: the simplest possible setting in which to investigate rotational notions. Such as angular velocity, angular momentum, moment of inertia and torque. And also features as the Euclidean plane of Geometry.

Every cross-product and curl is of this nature. If all the inputs plugged into this are vectors, the output is one of the aforementioned 'axial', 'pseudo-' or ' $SO(d)$'-vector

7.2 Enter the alternator

7.2.1 A little alternator

Salviati: The above can be distilled into the following ubiquitous object. Which, at the point that this Seminar appears in the Book, represents a brief debut for an object that we shall study and apply in rather more detail in various subsequent Chapters... It is the alternator!

This name is a portmanteau of *alternating tensor*. Some common aliases are *permutation tensor*, *Levi-Civita symbol* and *epsilon tensor* [8, 29, 18].

The third of these four names refers to the late 19th and early 20th century Mathematician Tullio Levi-Civita.

The fourth refers to the usual notation for our object

$$\epsilon_{i\,j\,k} \;:=\; \begin{cases} 1 & \text{for } i,\, j,\, k \quad \text{an even perm of} \quad 1,\,2,\,3 \\ \text{-}1 & \qquad\qquad\quad \text{an odd perm of} \end{cases} .$$

The second refers to the relation to the permutation group S_3 acting on 3 objects, of order

$$|S_3| \;=\; 3! \;=\; 6 \;.$$

While the first refers to the right-hand-side of the definition's sign changes. And to the largest proper subgroup of S_3 : A_3 , of order

$$|A_3| \;=\; \frac{3!}{2} \;=\; 3 \;,$$

which is termed the *alternating group*. For all that, in this particular case, it is also just isomorphic to the cyclic group C_3 ...

7.2.2 The general alternator

Salviati: This is a good point at which to provide the generalization

$$\epsilon_{i_1\,...\,i_n} \;:=\; \begin{cases} 1 & \text{for } i_1,\, ...\, i_n \quad \text{an even perm of} \quad 1,\, ...\,,\, n \\ \text{-}1 & \qquad\qquad\quad \text{an odd perm of} \end{cases} .$$

Now the permutation group is the general S_n acting on n objects, of order

$$|S_n| \;=\; n! \;.$$

And the alternating group is A_n , of order

$$|A_n| \;=\; \frac{n!}{2} \;=\; 3\ .$$

Which is sui generis for $n \geq 4$ ...

Both elementary permutations, and these examples of groups, give a second point of entry into various first- and second-year STEM theory courses...

This permits a systematic approach to determinants of every size, giving a third point of entry! All $\boldsymbol{\epsilon}$ are totally antisymmetric, embodying the row and column switch and recombination rules of determinants...

$\boldsymbol{\epsilon}$ joins $\boldsymbol{\delta}$ in being an *isotropic tensor.* Corresponding to looking the same in all directions, as encoded by its entries remaining unchanged upon passing to whichever reference frame...

Though $\boldsymbol{\epsilon}$'s rank runs as the dimension. While $\boldsymbol{\delta}$'s is fixed to be 2 in all dimensions in which distinct ranks of tensor are supported (≥ 2).

This is rooted in $\boldsymbol{\epsilon}$ being the top form supported by the dimension... With the n-d $\boldsymbol{\epsilon}$ encoding the n-volume form, among which our 3-d $\boldsymbol{\epsilon}$ encodes the usual volume form! It is worth mentioning here that 'epsilon' becomes a tensor density

$$\sqrt{|g|}\,\boldsymbol{\epsilon}$$

in the curvilinear and manifold contexts...

Sagredo: For this matches what we know about how volume elements transform under changes of coordinates!

7.2.3 Smallest nontrivial and the trivial.

Salviati: 3-d is not the smallest dimension supporting an $\boldsymbol{\epsilon}$. 2-d already has one, with the unique feature among nontrival dimensions that it has the same rank as the corresponding $\boldsymbol{\delta}$. In 1-d , firstly all tensors degenerate to scalars. Secondly, there is no room for a nontrivial antisymmetric object. This matches the well-known result that?

Sagredo: The zero vector is the only rank-1 isotropic tensor!

Salviati: Just so! While in 0-d , all scalars degenerate to single values (assuming a connected model, rather than a disconnected heap of points). Here any constant is an isotropic tensor, but among these only the zero tensor is antisymmetric.

7.2.4 The alternator as the $so(3)$ structure constants

Salviati: Our specific 3-d $\boldsymbol{\epsilon}$ tied to S_3 and A_3 features furthermore in the following role. As the structure constants in the defining relation of the $so(3)$ Lie algebra.

Lie groups are groups that are also manifolds. Lie algebras embody what happens locally around the origin. This is a continuous venture, involving continuum generators only. As such, rotations feature but reflections do not register. Thus $so(3)$ is the Lie algebra corresponding to both the $SO(3)$ Lie group and the $O(3)$ Lie group...

7.2.5 And $su(2)$ and $sp(2)$'s structure constants

Salviati: It is furthermore a small enough Lie algebra to enjoy several accidental relations. For starters,

$$so(3) \cong su(2) \cong sp(2) .$$

By which we are talking about an early member common to all three series of simple Lie algebras. Corresponding to $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ – quaternions – being the only finite division algebras: an Abstract-Algebraic distinction!

This gives two further points of entry, as follows.

$su(2)$'s manifold is the 3-sphere.

While the symplectic groups $sp(n)$ model aspects of the Hamiltonian formulation of the Principles of Dynamics. Poisson brackets, canonical transformations, phase space [12, 5, 31], and the subsequent harder topics of Symplectic Geometry and Symplectic Topology [36]...

Suppose that we wish to entertain genuinely separated out orthogonal, unitary and symplectic algebras, groups and representations. Then one needs to consider somewhat larger members of each series. This is not however typically done prior to fourth year courses...

7.2.6 $so(3)$ is also a window into $so(4)$

Salviati: A further accidental relation is

$$so(4) = so(3) \times so(3) .$$

7.2.7 Dual tensors

Salviati: Another role played by $\boldsymbol{\epsilon}$ is to send rank q-tensors to rank $(n - q)$-tensors known as the *dual tensors* of the original tensors. In components, the dual tensor

$$\widetilde{T}_{a_1 \, \ldots \, \ldots a_q} = \epsilon_{a_1 \, \ldots \, \ldots a_q \, b_1 \, \ldots \, b_r} T^{b_1 \, \ldots \, \ldots a_r} .$$

See Fig 2 for an index-free rendition. We will eventually be passing to a better tensor networks notation for this (and everything else...)

Among basic courses, this plays a role in SR and Electomagnetism. More generally, dual tensors are a further large input into the theory of forms [10, 29].

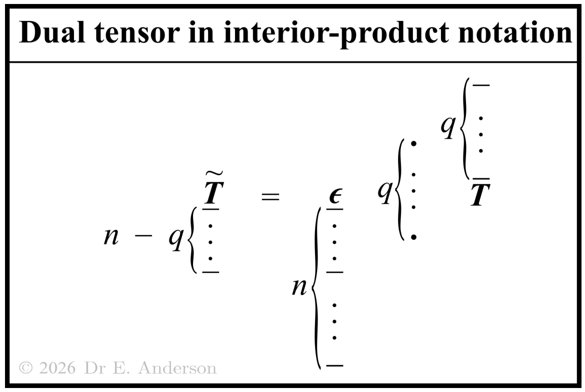

Figure 2:

8 Lie Algebra interpretation of tensors

8.1 Scalars are invariant under Lie brackets

Salviati: Strictly speaking, for now we shall be using classical Poisson brackets [12, 5, 26]. These have one basic Mathematics axiom in excess of Lie brackets [17, 27, 30]: the derivation

$$\{ A , B C \} = B \{ A , C \} + \{ A , B \} C .$$

Then a Poisson G-scalar S is an object that commutes with all of G 's generators $\boldsymbol{G}$. I.e.

$$\{ \underline{\boldsymbol{G}} , S \} = \underline{\boldsymbol{0}} .$$

The Lie algebra definition is exactly the same, just with the Lie bracket

$$[\ \ , \ \ ]$$

in place of the Poisson one...

8.2 The rotation- alias angular-momentum- algebra example

Salviati: Now let $G = (S)O(3)$. Then

$$\underline{\boldsymbol{G}} = \underline{\boldsymbol{L}} .$$

I.e. angular momentum, viewed as a covector, and whose amethyst index happens to coincide with the usual silver index of flat space.

At the infinitesimal level, suitable for Lie algebras, and here corresponding to $so(3)$,

$$\underline{\boldsymbol{p}} = \frac{\partial}{\partial \overline{\boldsymbol{x}}} ,$$

$$\underline{\boldsymbol{L}} = \overline{\boldsymbol{x}} \times \underline{\boldsymbol{p}} = \overline{\boldsymbol{x}} \times \frac{\partial}{\partial \overline{\boldsymbol{x}}} .$$

In terms of the $\boldsymbol{\epsilon}$ symbol, in components,

$$L_i = \epsilon_{i\,j}{}^{k} \, x^j \, \frac{\partial}{\partial x^k} .$$

A coordinate-free rendition of which is

$$\underline{\boldsymbol{L}} = \underline{\underline{\boldsymbol{\epsilon}}}\overline{\ } \cdot \cdot \, \overline{\boldsymbol{x}} \frac{\partial}{\partial \overline{\boldsymbol{x}}} .$$

Where we stagger the overlines and underlines to indicate that the overline is the third index. In a subsequent Chapter, we shall compute, firstly

$$\{ \, \underline{\boldsymbol{L}}, \, \underline{\boldsymbol{L}} \, \} \; = \; \overline{\boldsymbol{\epsilon}}_{=} \cdot \underline{\boldsymbol{L}} \; .$$

Thus confirming that $\boldsymbol{\epsilon}$ indeed supplies the structure constants for $so(3)$.

Secondly, that

$$\{ \, \underline{\boldsymbol{L}}, \, \overline{\boldsymbol{x}} \, \} \; = \; - \, \overline{\boldsymbol{\epsilon}}_{\bar{-}} \cdot \overline{\boldsymbol{x}} \; .$$

Which is the $so(3)$ subcase of the Lie algebra transformation law for a vector.

Thirdly, that

$$\{ \, \underline{\boldsymbol{L}}, \, ||\underline{\boldsymbol{x}}||^2 \, \} \; = \; 0 \; .$$

I.e. that the magnitude of the position is an $so(3)$ scalar.

8.3 The Lorentz algebra example

Salviati: And fourthly how dyads extend the form this takes to rank-2 tensors and so on. Of which the first case encountered in first and second year STEM courses is the Lorentz tensor. I.e. the 4-d Minkowski spacetime tensor of rank 2 that packages together the homogeneous linear isometries of this space. Which, locally around the origin, is the Lorentz algebra

$$lor(\,3,\,1\,) \;\; = \;\; so(\,3,\,1\,) \; .$$

We are now in a position to pedagogically justify Sec 7.2.6. For while $\mathbb{R}^4$ does not feature in basic courses, its indefinite counterpart Minkowski spacetime of course does. And the $so(3, 1)$ group here splits into 2 copies of $so(3)$ as well. This is part of why rotations and boosts [15] can be separately modelled using ‘angular momentum mathematics’.

The other part is the dual tensor construction, in the following combination. We are repackaging a 3-d vector and a dual 3-d vector, each in the Cartesian sense. Into a 4-d 2-tensor, in the Lorentzian sense.

8.4 Pointers

Salviati: In Chapter **??**, we shall give Gauge Theory counterparts of the above treatment of the rotations. Meaning Electromagnetism and Yang–Mills theory. And General Relativity counterparts; the diffeomorphisms remain a Lie algebra, albeit now with an infinite number of generators...

9 Quantum-level applications

Sagredo: And in the quantum world?

9.1 Complex vector and inner-product spaces

Salviati: Firstly, we are to now work over $\mathbb{C}$ in place of $\mathbb{R}$. Initially, this produces initially tightly analogous mathematical structures. Namely the vector spaces over $\mathbb{C}$. As equipped with their own standard inner product, with complex entries in the second slot to complex-conjugate entries in the first. With such *sequilinear* inner products – anti-linear in the first slot – having a very similar theory to real-bilinear forms...

9.2 Bras and kets on Hilbert space

Salviati: Secondly, in the context of Quantum Mechanics, Paul Dirac [3] called covectors *bras*

$$\langle \psi |$$

to vectors' *kets*.

$$| \phi \rangle \ .$$

This conceptualization lies within the vector space and its dual conceptualization of Linear Algebra as equipped with an inner product. Immediately so in the finite vector space case, and extending to the infinite case as the Hilbert space subcase of function space. Wherein the bras can be viewed as linear functionals. As we shall see in Chapter **??**, Hilbert spaces enjoy a protective theorem due to which the dual space is isomorphic (over $\mathbb{R}$) or anti-isomorphic (over $\mathbb{C}$, as here). As opposed to the more general infinite-d function space situation, in which infinite vector spaces' dual spaces are larger than the vector space.

A first consequence of this is that viewing the infinite case's bra-ket pairing as open-brackets close-brackets

$$\langle \Psi | \Phi \rangle \tag{15}$$

indeed emulates the finite inner product. A common notation for which is the syntactically-analgous $(\_ , \_)$.

9.3 Quantum probabilities and overlaps

Salviati: The corresponding norm is

$$| \Psi |^2 \ = \ \langle \Psi | \Psi \rangle \ ,$$

which is interpreted as a quantum probability. To be clear, this is not what a basic course would write as $\Psi^* \Phi$, but rather the integral of this over all space... So, were we talking about a normalized wavefunction, our norm would be just 1 ...

While the corresponding inner product (15) is called a *quantum overlap*.

Upon insertion of a quantum operator,

$$\langle \Psi | \widehat{O} | \Phi \rangle$$

is a (*probability*) *transition amplitude*. This can for instance be viewed as $\widehat{O}$ acting as a machine on the ket vector $| \Phi \rangle$ to produce some other ket vector $| \Phi' \rangle$. Which then overlaps with our other input, the bra dual vector $\langle \Psi |$.

Hermitian matrices are the complex analogues of symmetric matrices. Both have purely-real eigenvalues. This extends to Hermitian and symmetric operators having purely-real eigenvalues as well. In QM, this realness is a pre-requisite for interpreting Hermitian operators as physical operators, whose eigenvalues correspond to what is observed upon making a measurement.

For $\widehat{O}$ Hermitian, one can interpret it just as well as acting backwards on the bra. Thus providing a way in which the bra and ket inputs are placed on the same footing.

9.4 Hilbert spaces defined

Salviati: Indeed the bra-ket can be interpreted as a truer notation for the inner product of a complex Hilbert space. This refers to a crucial feature of Hilbert space, since this is by definition a complete inner product space [13, 38].

9.5 Hilbert space bases in the quantum context

Salviati: Infinite Hilbert spaces admit bases that are countable: with basis members indexed by $\mathbb{N}$. [In fact, the isomorphic shifting to $\mathbb{N}_0$ is usually used, so as to index ground states by 0 .] These can be picked to be orthonormal; c.f. how Fourier modes or Legendre polynomials are...

For a complex infinite Hilbert space in the context of Quantum Mechanics, we can denote such a basis by

$$\{\,|\psi_i\rangle\}_{i\,\in\,\mathbb{N}_0}\ .$$

The corresponding dual basis is then

$$\{\,\langle\psi_i|\}_{i\,\in\,\mathbb{N}_0}\ .$$

So

$$\langle\psi_i|\,\psi_i\rangle \;=\; \delta_{i\,j}\ .$$

With this $\boldsymbol{\delta}$ now providing the components of the countably-infinite identity matrix $\mathbb{I}$.

Just as orthogonal matrices provide Euclidean-inner-product preserving changes of basis, unitary matrices do for our complex inner product. They thus play a prominent role in QM, which is probably part of why you mentioned them in Sec 3, Sagredo...

Sagredo: Indeed!

9.6 From 'quantum dyads' to projectors

Salviati: Let us next espy some quantum tensors that go beyond just vectors and covectors...

The corresponding individual dyad is

$$|\psi_i\rangle \, | \, \langle\psi_j| \ \ .$$

The homogeneous-input version of this is

$$|\psi_i\rangle \, | \, \langle\psi_i| \ \ .$$

If one sums these over the whole orthonormal basis, then one obtains the *completeness relation*

$$\mathbb{I} \ = \ \sum_{i \, \in \, \mathbb{N}} |\psi_i\rangle \, | \, \langle\psi_i| \ \ . \ .$$

If one however only uses some partial sum of these, then we obtain the following. Selecting a finite sum of contiguous indices for presentational convenience...

$$\boldsymbol{P} \ = \ \sum_{i \, = \, 1}^{n} |\psi_i\rangle \, | \, \langle\psi_i| \ \ . \ ,$$

$$\boldsymbol{P}_{\perp} \ = \ \mathbb{I} \ - \ \sum_{i \, \in \, 1}^{n} |\psi_i\rangle \, | \, \langle\psi_i| \ \ . \ \ = \ \sum_{k \, \in \, n \, + \, 1}^{\infty} |\psi_k\rangle \, | \, \langle\psi_k| \ \ . \ .$$

The first form of the second of these definitions is very recognizably a projector... Corresponding to projecting out our selection of basis components $|\psi_i\rangle$, $i \ = \ 1$ to n . The first of these definitions is then recognizable as the orthogonal-complement projector. Corresponding to projecting onto our selection!

The completeness relation then corresponds to projecting onto everything. While its complement $\mathbb{O}$ corresponds to everything being projected out, leaving nothing...

Whichever projector $\boldsymbol{P}$ can furthermore be viewed as providing an answer 'yes' to a question. To which its complement $\boldsymbol{P}_{\perp}$ provides an answer 'no'.

9.7 Quantum: angular momentum, G-tensors, and fundamental fields

Sagredo: Quantum angular momentum is very similar to classical angular momentum. With however factors of $i \, m$ and $\hbar$ all over the place, from the momentum now taking the form

$$\underline{\widehat{\boldsymbol{p}}} \ = \ \frac{\hbar}{i} \frac{\partial}{\partial \underline{\boldsymbol{x}}} \ .$$

And with classical Poisson brackets replaced by quantum commutators!

Salviati: Quantum commutators are moreover a subcase of Lie bracket..

Half-integer spin particles are modelled using spinors. These are the other example that I promised, for which comparison at different points is not immediate, necessitating a connection... Among these particles, the massive and massless spin-1/2 particles of Dirac and of Weyl respectively [35], have observational support.

Finally, as regards G-tensors, Quantum Electromagnetism and Quantum Yang–Mills commutators likewise parallel the corresponding classical brackets. But Quantum GR only more formally so.

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