Graph and Order Theory
of N-Body Problem's Eigenclustering Networks

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Abstract

For N-body configurations of points-or particles, eigenclustering vectors are alias (relative) Jacobi vectors. $N = 4$ supports 2 distinct eigenclustering networks of these: H and K . This ambiguity growingly persists for subsequent N . The arena of eigenclusterings $\mathfrak{EN}(N)$ is $\mathfrak{Tree}^*_2(2N - 1)$. Or, equivalently $\mathfrak{Tree}^*_{\leq 2}(N - 1)$. Where in these arenas of tree graphs the $*$ stands for rooted, the 2 for binary and the ≤ 2 for at-most binary. These trees are realized by the eigenclustering's nontrivial centres of mass (CoMs), respectively with and without the constituent points-or-particles themselves. These trees can furthermore be viewed as posets under the 'add leaf' operation with (excess-)subsystem mass acting as depth function.

The count $e(N) := |\mathfrak{EN}(N)|$ returns the Wedderburn–Etherington numbers $w(N)$ for $N \neq 0$.

We provide a collection of minimum properties for $N = 5$ to 10 . $N = 5$ for $\mathfrak{EN}[N] \neq \mathfrak{Tree}[N]$: the unlabelled trees. Where the square bracket denotes cumulative arena: $\leq N$. $N = 6$ for $\mathfrak{EN}[N]$ to produce a new graph. $N = 7$ for $\mathfrak{EN}[N]$ a nonplanar graph and a new cycle system. $N = 8$ for $e(N) < h(N)$: the half-Catalan numbers. And $N = 9$ for $e(N) < t(N) := |\mathfrak{Tree}(N)|$: the unlabelled trees. We also apply the asymptotics worked out for the $w(N)$ to the $e(N)$.

We next place an algebra on $\mathfrak{EN} = \mathfrak{EN}[\infty]$. This and its poset nature are compatible and explain what 'hierarchical eigenclustering' consists of. We finally consider the subarena of nontrivial eigenclusterings. Each element of which supports one Eigenclustering Length-Exchange Theorem [51, 52, 53]. The smallest case of which has long been known as Apollonius' Median-Length Theorem. And one of the 2 next smallest as Euler's Quadrilateral Theorem, but extends to a 4-body Theorem: dimension-independent.

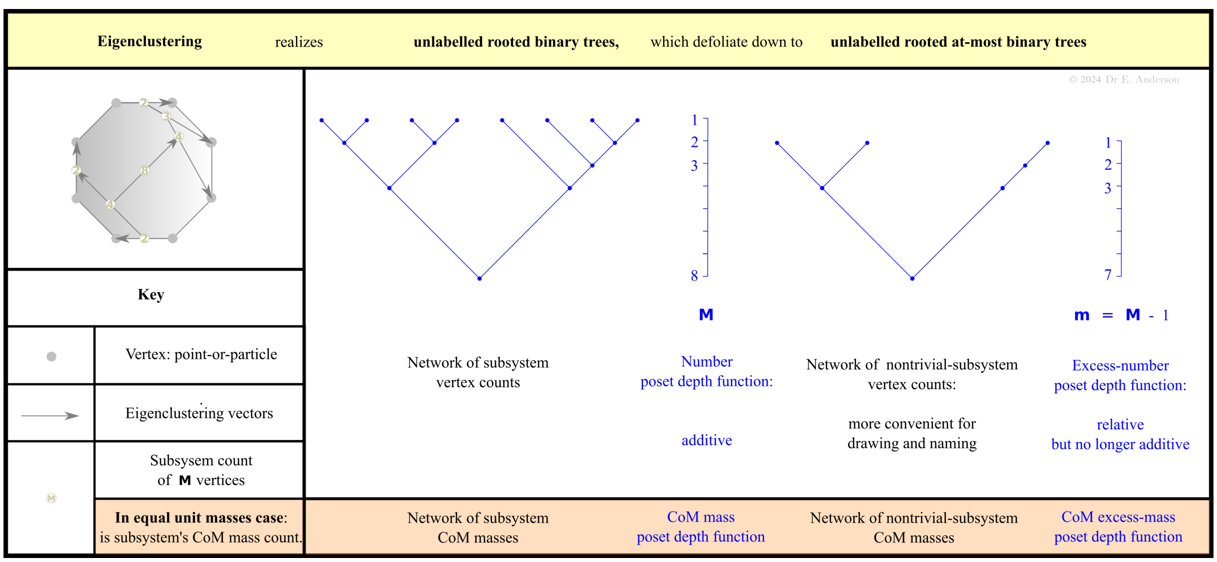

Figure 1:

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Date-stamps v1: 07-09-2024. v2: 04-02-2025. v3: 07-02-2026.

1 Introduction

1.1 Preamble

N-body configurations play major roles in Dynamics [15, 16, 17, 25, 32, 33, 47], Molecular Physics [34, 37] and Shape Statistics [27, 36, 38, 40]. Their natural home is however Flat Geometry [57]. To cover all of these applications at once, let us conceptualize in terms of points-or-particles [41]. Where the latter are classical nonrelativistic point particles. For now, we take these to be of equal mass, and elsewise indistinguishable. And where in Statistics, the former are location data points in some $\mathbb{R}^d$. Some well-known complexities arising as N increases are as follows.

$N = 0$ is the empty case: it does not even have any points-or-particles.

$N = 1$ has 1 : the minimum for material content. But it supports no separations.

$N = 2$ possesses a single separation.

$N = 3$ has 3 separations, but now [41] only 2 are linearly independent (LI). This 2 becomes

$$n = N - 1 \tag{1}$$

in the general case. Which plays the role of the dimension of relative space [41].

Furthermore, the inertia quadric is no longer diagonal in terms of these [41]. This is resolved by basis change from independent separation vectors to independent *eigenclustering vectors*. The nontrivial such, no longer run between two points-or-particles, involving rather ≥ 1 nontrivial centre of mass (CoM). Eigenclustering vectors [41, 49] are elsewhere alias *(relative) Jacobi vectors* [16, 34, 37, 39, 47]. The names *eigenclustering coordinates* and *eigenclustering magnitudes* follow suit. $N = 3$ is also minimum for this nontriviality.

For $N = 3$, there are 3 labelling choices for these if the points-or-particles are themselves labelled. For equal point-or-particle masses, these consist of each of the 3 side vectors paired with its corresponding median vector. These collapse to a single 'T' shape (row 4 of Fig 7) if the points-or-particles are furthermore indistinguishable.

So we subsequently conceptualize in terms of

$$N \in \mathbb{N}_0 \ : \quad \text{the } \textit{maximum range} \ . \tag{2}$$

$$N \in \mathbb{N} \ : \quad \text{the } \textit{material range} \ . \tag{3}$$

$$N \geq 2 \ : \quad \text{the } \textit{separational range} \ . \tag{4}$$

$$N \geq 3 \ : \quad \text{the } \textit{nontrivial-eigenclustering range} \ . \tag{5}$$

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$N = 4$ is minimum for 2 distinct eigenclustering networks to be realized: the first H and K in column 1 of Fig 20.

Such eigenclustering network ambiguity is independent of from the also-present labelling ambiguity. And growingly persists for subsequent N . The arena of eigenclustering networks on N points-or-particles is

$$\mathfrak{EN}(N) \ . \tag{6}$$

It is isomorphic to $\mathfrak{Tree}^*_2(2N - 1)$. Or, equivalently to $\mathfrak{Tree}^*_{\leq 2}(n)$ [50]. Where in these arenas of tree graphs ([110] and Sec A) the $*$ stands for rooted, the 2 for binary, and the ≤ 2 for at-most binary (AMB). These trees are realized by the eigenclustering's nontrivial CoMs, with and without the constituent points-or-particles themselves respectively. For $N \neq 0$, the count

$$e(N) := |\, \mathfrak{EN}(N)\, | \tag{7}$$

returns [50], the *Wedderburn–Etherington (WE)* numbers [61, 64, 65, 82, 114], $w(N)$.

1.2 Outline of our new work

We depict eigenclustering networks and 6 presentations for these in Secs 2 and 3.

Key point 1 Eigenclustering networks can furthermore be viewed as posets (Appendix B) with subsystem mass acting as depth function. With total CoM as top element and root. To points-or-particles as bottom elements in the binary presentation or binary CoMs in the AMB presentation.

Key point 2 We furthermore place a distinct poset structure on the *cumulative arena of eigenclustering networks*,

$$\mathfrak{EN}[N] \;=\; \mathfrak{Tree}^*_2[N] \;. \tag{8}$$

Where 'cumulative arena' [110] means $\leq N$, as denoted by square brackets. And the poset structure's operation is 'add leaf' [110] at the level of trees. Corresponding to 'add point-or-particle' at the N-body level. For each N, the eigenclusterings run from a bottom K-like eigenclustering to a top H-like one. These are however secondary notions of bottom and top within $\mathfrak{EN}[N]$. The remaining 'middling elements' are then discussed in Sec 3 for $N = 5$ to 8.

While now

$$\mathfrak{Tree}^*_{\leq\,2}[N] \;\neq\; \mathfrak{Tree}^*_2[N] \;\;\text{for}\;\; N \;\geq\; 2 \;, \tag{9}$$

the structural difference is fixed and slight, so we work throughout with the smaller AMB presentations.

We provide a string of minimum properties for $N = 5$ to 10 in Sec 4.

$N = 5$ for Order-Theoretically distinguishing between $\mathfrak{EN}[N]$ and $\mathfrak{Tree}[N]$: the arena of unlabelled trees.

$N = 6$ for $\mathfrak{EN}[N]$ to produce a new graph as its skeleton.

$N = 7$ for $\mathfrak{EN}[N]$ to produce a new nonplanar graph and a new cycle system.

$N = 8$ for

$$e(N) \;<\; h(N) \;\; : \tag{10}$$

the half-Catalan numbers (see Appendix B).

$N = 9$ for

$$e(N) \;<\; t(N) \;:=\; |\,\mathfrak{Tree}(N)\,| \;. \tag{11}$$

$N = 10$ has minimum combinatorial and symmetry features.

We finally apply the asymptotics worked out for [77] the $w(N)$ to the $e(N)$.

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Key point 3 We additionally use WE's work to place a nonassociative commutative algebra on a deleted-point version of

$$\mathfrak{EN} \;=\; \mathfrak{EN}[\infty] \tag{12}$$

in Sec 5. Or a slightly more structured such without the deleted point (the untree). This algebra has a simple and natural Physical interpretation in terms of adjoining pairs of systems' total CoMs to form a larger system's total CoM.

Key point 4 The algebraic and poset aspects of $\mathfrak{EN}$ are moreover compatible and explain what 'hierarchical eigenclustering' consists of. By this compatibility, and by its occurring in a number of further foundational areas of the Theory of STEM, $\mathfrak{T}\mathrm{ree}^*_2$ is a citizen of Kallista [43, 108, 110, 120].

We finally consider the subarena of nontrivial eigenclusterings in Sec 6. Each element of which supports one Eigenclustering Length-Exchange Theorem. The smallest case of which has long been known as Apollonius' Median-Length Theorem [1, 13, 57]. And one of the 2 next smallest as Euler's Quadrilateral Theorem [3, 9, 10, 11, 51, 57] (more generally Euler's 4-Body Theorem: dimension-independent). Interestingly, Algebra tells us to extend by 2 or 3 trivial identity equations, but not 1 or ≥ 4 .

2 Bottom and top eigenclusterings: K's and H's

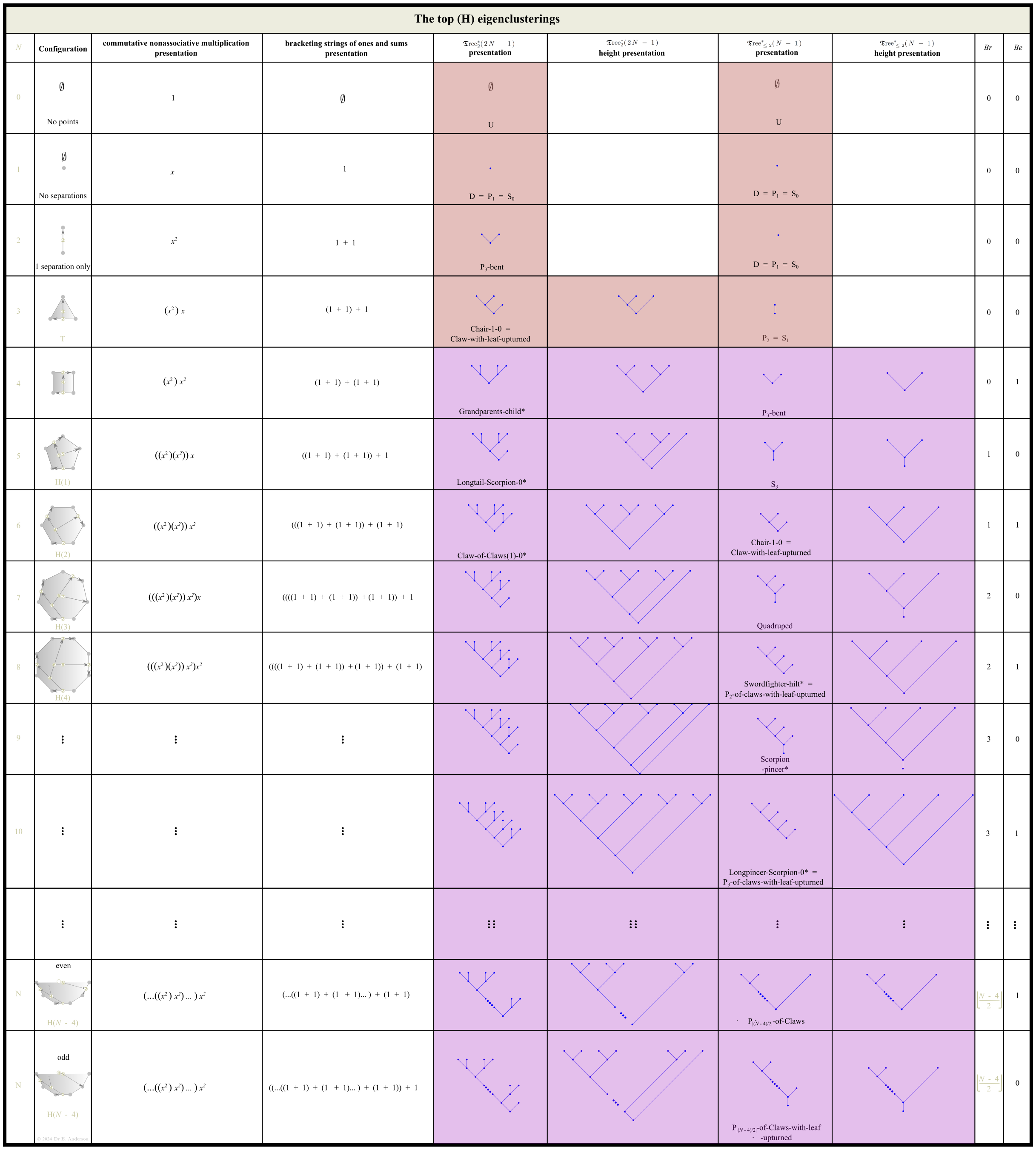
2.1 Introduction

Remark 1 Picking an eigenclustering amounts to forming a network of CoMs for the arbitrary N-body configuration, by coarse-graining two subsystems into one at each step.

Structure 1 For now we consider the K- and H-eigenclustering networks (Figs 2 and 3 respectively). Which can be viewed as making 'minimum and maximum use of pairs'. In fact, maximum use of pairs is ambiguous for odd $N \geq 5$. So we need to further specify that we make maximum use of pairs *and then* bring in the last point-or-particle by itself in those cases in which there is one left over.

The bottom (K) eigenclusterings						
N	Configuration	Nonassociative commutative multiplication presentation	Bracketing strings of ones and sums presentation	$\mathfrak{Tree}^*_2(2N-1)$ prsentation	$\mathfrak{Tree}^*_2(2N-1)$ height presentation	$\mathfrak{Tree}^*_{\leq 2}(N-1)$ presentation
0	$\emptyset$ No points	1	0	$\emptyset$ U	$\emptyset$	$\emptyset$ U
1	$\emptyset$ No separations	x	1	D = P_1 = S_0		D = P_1 = S_0
2	1 separation only	x^2	1 + 1	P_3-bent		D = P_1 = S_0
3	T	$(x^2)\,x$	(1 + 1) + 1	Chair-1-0 = Claw-with-leaf-upturned		P_2 = S_1
4	K	$((x^2)\,x)\,x$	((1 + 1) + 1) + 1	Swordfighter-hilt* = P_2-of-claws-with-leaf-upturned	M 1 2 3 4	m 1 2 3 P_3-straight
5	K(1)	$(((x^2)\,x)\,x)\,x$	(((1 + 1) + 1) + 1) + 1	Longpincer-Scorpion-0* = P_3-of-claws-with-leaf-upturned		P_4-straight
6	K(2)	$((((x^2)\,x)\,x)\,x)\,x$	((((1 + 1) + 1) + 1) + 1) + 1	P_4-of-claws-with-leaf-upturned		P_5-straight
7	K(3)	$(((((x^2)\,x)\,x)\,x)\,x)\,x$	(((((1 + 1) + 1) + 1) + 1) + 1) + 1	P_5-of-claws-with-leaf-upturned		P_6-straight
8	K(4)	$((((((x^2)\,x)\,x)\,x)\,x)\,x)$	((((((1 + 1) + 1) + 1) + 1) + 1) + 1) + 1	P_6-of-claws-with-leaf-upturned		P_7-straight
	⋮	⋮	⋮	⋮		⋮
N	K(N - 4)	$(((x^2)\,x)\,x)\,x$ © 2024 Dr E. Anderson	(((1 + 1) + ...) + 1) + 1	$P_{(N-3)/2}$ -of-claws-with-leaf-upturned		P_n-straight

Figure 2:



The top (H) eigenclusterings									
N	Configuration	commutative nonassociative multiplication presentation	bracketing strings of ones and sums presentation	$\mathfrak{Tree}_2^*(2N-1)$ presentation	$\mathfrak{Tree}_2^*(2N-1)$ height presentation	$\mathfrak{Tree}^*_{\leq 2}(N-1)$ presentation	$\mathfrak{Tree}^*_{\leq 2}(N-1)$ height presentation	*Br*	*Be*
0	∅ No points	1	∅	∅ U		∅ U		0	0
1	∅ No separations	x	1	D = P_1 = S_0		D = P_1 = S_0		0	0
2	1 separation only	x^2	1 + 1	P_3-bent		D = P_1 = S_0		0	0
3	T	$(x^2)\,x$	(1 + 1) + 1	Chair-1-0 = Claw-with-leaf-upturned		P_2 = S_1		0	0
4		$(x^2)\,x^2$	(1 + 1) + (1 + 1)	Grandparents-child*		P_3-bent		0	1
5	H(1)	$((x^2)(x^2))\,x$	((1 + 1) + (1 + 1)) + 1	Longtail-Scorpion-0*		S_3		1	0
6	H(2)	$((x^2)(x^2))\,x^2$	(((1 + 1) + (1 + 1)) + (1 + 1)	Claw-of-Claws(1)-0*		Chair-1-0 = Claw-with-leaf-upturned		1	1
7	H(3)	$(((x^2)(x^2))\,x^2)x$	((((1 + 1) + (1 + 1)) +(1 + 1)) + 1			Quadruped		2	0
8	H(4)	$(((x^2)(x^2))\,x^2)x^2$	((((1 + 1) + (1 + 1)) +(1 + 1)) + (1 + 1)			Swordfighter-hilt* = P_3-of-claws-with-leaf-upturned		2	1
9	⋮	⋮	⋮			Scorpion -pincer*		3	0
10	⋮	⋮	⋮			Longpincer-Scorpion-0* = P_3-of-claws-with-leaf-upturned		3	1
	⋮	⋮	⋮	⁝⁝	⁝⁝	⋮	⋮	⋮	⋮
N	even H(N - 4)	$(\ldots((x^2)\,x^2)\ldots)\,x^2$	(...((1 + 1) + (1 + 1)...) + (1 + 1)			$P_{\lfloor(N-4)/2\rfloor}$-of-Claws		$\left\lfloor\frac{N-4}{2}\right\rfloor$	1
N	odd H(N - 4)	$(\ldots((x^2)\,x^2)\ldots)\,x^2$	((...((1 + 1) + (1 + 1)...) + (1 + 1)) + 1			$P_{\lfloor(N-4)/2\rfloor}$-of-Claws-with-leaf -upturned		$\left\lfloor\frac{N-4}{2}\right\rfloor$	0

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Figure 3:

Remark 2 The regular polygon presentation of eigenclusterings with subsystem labels on the CoMs is new to the current Article. It is more general than previous conventions. It is also useful as regards immediate visualization of which N and which eigenclustering one is considering, at least for small N. It is additionally more efficient to draw. This is since many of a fixed N's higher CoMs share plotting position over the set of eigenclusterings supported.

Naming Remark 1 One of us previously used *eigenclustering shape* [49, 51]. 'Shape' was also used of the non-associativity presentation of the WE numbers [64]. But shape is overbooked by eigenclustering considerations being naturally followed by [41] larger quotients in Kendall-type Shape Theory [27, 36, 38, 40, 41] . On these grounds, we have now coined the more distinctive and accurate term *eigenclustering networks*. These cover the variety in indistinguishable point-or-particle eigenbases, which form networks of CoMs as per above. We thus arrive at a truer name for what were originally called 'clustering ambiguities' in assigning 'relative Jacobi vectors'.

Remark 3 The unifying feature of the following six presentations on display is what they count out as. Which is [50] the WE numbers, $w(N)$ [61, 64, 65, 114]. Though in standard conventions, $w(0) = 0$, while $e(0) = 1$. See Sec 5 for further discussion of this discrepancy.

2.2 A first two presentations

Presentation 1 This is by the elements of the commutative but non-associative algebra with sole generator x . This is how WE [61, 64, 65] arrived at the $w(N)$.

Presentation 2 This is by the distinct ways of binarily bracketing the sum of N 1 's . Which is a priori a Combinatorial consideration.

Remark 1 Presentation 2 is more tailored to our eigenclustering application [44]. For its 1 's model equal-mass points-or-particles. While its plusses encode which subsystems are being paired to make larger subsystems. Whose diversity then provides a need for the role that the brackets play. The sum of 1 's inside each bracket reads off the corresponding subsystem's mass.

Remark 2 Presentation 1's further Algebraic connotations shall however also subsequently play a role. Presentation 1 also uses less of the page.

Remark 3 These differences are furthermore conceptual and aesthetic rather than mathematical. For there is an algebraic isomorphism between these two presentations. Namely between x and 1 , multiplication and addition, and, where relevant, between 1 and 0 .

2.3 Two Graph Theory presentations

Presentation 3 The binary trees on $2N - 1$ vertices, or equivalently, with precisely N leaves [50] are well-known to give [114] equivalent counts to the WE numbers.

Presentation 4 Subject to Remark 2's caveat, these are equivalent to the AMB trees with n vertices.

Remark 1 Presentation 3 is additionally realized by the N points-or-particles and the nontrivial CoMs used by the eigenclustering in question [50]. While Presentation 4 corresponds to dropping the points-or-particles themselves, thus solely summarizing the eigenclustering's nontrivial CoM information.

Remark 2 Presentations 3 and 4 are related by defoliation and refoliation. For $N \geq 2$, this constitutes an isomorphism. Below this, $N = 1$ defoliates to the same tree as $N = 2$ does. By which injectivity is lost if $N = 1$ is included. $N = 0$ is not however pathological. Thus

$$\mathfrak{Tree}^*_{\leq 2}[P, 2N - 1] \cong \mathfrak{Tree}^*_{\leq 2}[P, n]$$

for $P \geq 2$. This isomorphism is useful in our application since nontrivial eigenclustering starts for $N = 3$.

Remark 3 A third graph presentation often to be found in the literature instead of Presentation 4 is Presentation 4 with an extra root-labelling vertex appended. This is less natural in our Physical context, however, and is not on display in our tables. It plays a small role in our sequel paper [58].

2.4 Branching and bending numbers

Structure 1 The *branching number* Br quantifies the amount of branching in a graph. We need no consideration of excess branching strength, since AMB graphs are 3-subregular. By which the count of degree-3 vertices suffices for the current Article's purposes

Structure 2 The *bending number* Be quantifies the minimum amount of bending in presenting a rooted graph. See Fig 2 for a first few nontrivial instances of bending number.

2.5 Two final presentations: posets with depth functions

Structure 1 The *subsystem mass function* M runs from 1 to N. And the *subsystem excess-mass function*

$$m := M - M_{\text{min}} = M - 1 \tag{13}$$

runs from 1 to n. These serve furthermore as poset depth functions on the *elements of* the $\mathfrak{T}\text{ree}_2^*[2N - 1]$ and $\mathfrak{T}\text{ree}^*_{\leq}[n]$ presentations respectively. I display these in row 4 of Fig 2.

Notation 1 We use M and m for subsystem masses, but M and m for CoM strengths.

Presentations 5 and 6 These build in the ones-sums-and-brackets presentation into the binary trees, as per columns 5 and 7 respectively. These unified presentations of Sánchez [108] come at the cost of occupying somewhat more space on the paper. They do however always have enough space to be presented rectangularly. Though we choose to display any leading edges symmetrically.

Remark 1 For reasons of efficiency, we pick Presentations 4 and 6 as the most useful ones for the rest of the current Article. 6 for maximal manifest encodement and conceptual work, and 4 for Graph Theory level processing.

Remark 2 In forming cumulative arenas, we however need to patch up that our presentation goes bad within the elsewise easily understood bottom chain leading into the rest of the poset.

Remark 3 Once dealing with infinite cumulative arenas, however, that the AMB trees are smaller for fixed N ceases to be significant. Then evoking the $\mathfrak{T}\text{ree}_2^*$-valued presentation is simpler.

2.6 Overall picture of our 6 reps

Remark 1 m is an example of a relative-difference variable. While M being an absolute variable.

Remark 2 The counting interpretation of the above height function holds irrspective of whether any masses are assigned. Also for arbitrary masses, the counting version stays the same, while the mass version takes a distinct form.

Remark 3 To move left from the regular polygon eigenclustering diagram, we observe that most of the other presentations have closer ties to the absolute version. Present the indistinguishable points-or-particles by 1 (which can be viewed as assigning equal unit masses). And the eigenclustering CoMs by plusses. Then straightening out the ensuing network of 1's, plusses and brackets returns the Combinatorial presentation. Finish off with our Algebraic isomorphism for $N \geq 2$ to arrive at the WE Algebraic presentation.

Remark 4 To move right instead, we form a stick model [119]. And deform it into a tree rooted at the total CoM (the black peg). We then further deform it so as to respect the depth function M and view it as a poset.

Remark 5 Superposing the leftmost and rightmost absolute presentations, on the one hand, M counts the number of uses of the generator in the algebra element. On the other hand, m counts the *excess* number of uses of the generator.

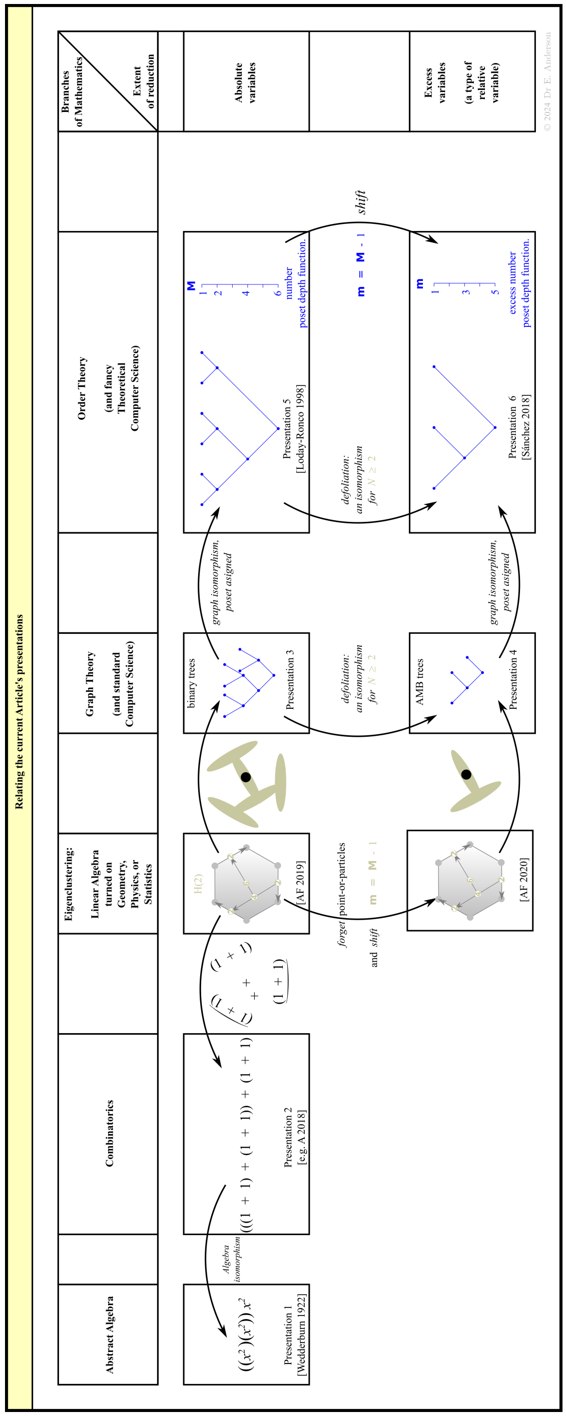


Figure 4:

Remark 6 The Figure also gives down arrows to relative versions.

Remark 7 M is additive but m is not. This gives a direct conversion between M and the number of 1 's in the Combinatorial presentation. Or of the number of uses of the generator x in the Algebraic presentation. See Sec 5 for further comment.

3 Middling eigenclusterings

3.1 Introduction

Remark 1 This Section's networks – for $N = 5$ to 8 – are 'middling' as regards their use of pairs. The underlying ranking's need to specify use of 'all pairs first' is minimally demonstrated for $N = 5$ (row 1 of Fig 5). Where the sole middle element uses one pair, then a loose point-or-particle and then another pair.

Remark 2 We next consider arenas of eigenclustering networks, $\mathfrak{EN}$. This Section's networks are also 'middling' in the sense of forming the the middles of the $\mathfrak{EN}(N)$ posets.

3.2 $N = 5$ and 6

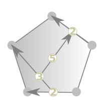
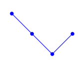
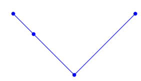
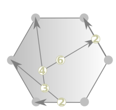
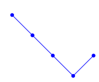
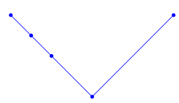
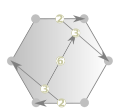
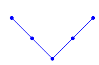
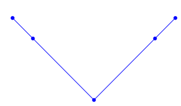
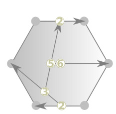
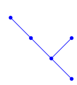
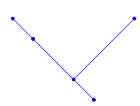
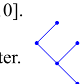
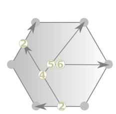
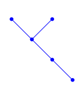
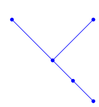
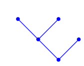
The middling eigenclusterings for $N = 5, 6$						
N	**Configuration**	$\mathfrak{Tree}^*_{\leq 2}(N-1)$ **presentation**	$\mathfrak{Tree}^*_{\leq 2}(N-1)$ **height presentation**	Br	Be	**Notes**
5	 K(1)	 P_4-bent		0	1	Called P_4-hockey in [110]
6		 P_5-bent		0	1	
	 T-dimer	 P_5-v		0	1	
		 S(1) Chair-0*		1	0	Or Chair-back-leg*. Called baton-up in [110]. wrench-up is useful later. 
		 S(1) Chair-1*		1	0	Or Chair-front-leg*. Or baton-down. Or wrench-down 

Figure 5:

Remark 1 See Fig 5.

Remark 2 $N = 6$ is minimum to support a nontrivial-eigenclustering composite. The T-dimer corresponding to P_{5-v} ; see row 3 of the Figure.

Remark 3 $N = 6$ is minimum for Br and Be to cease to serve as a full classification (the last 4 rows of the Figure).

3.3 $N = 7$ and 8

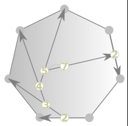
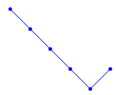
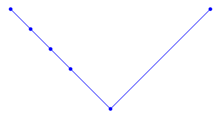
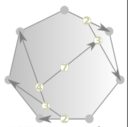
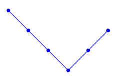
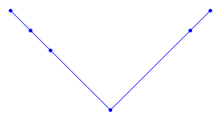
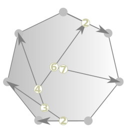
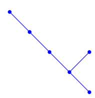
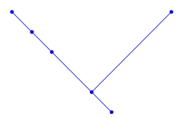
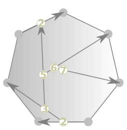
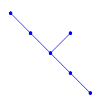
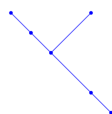
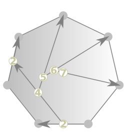
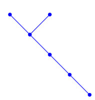
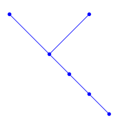
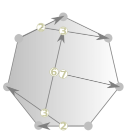
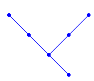
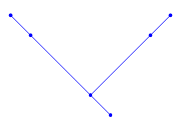
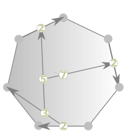
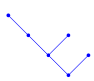
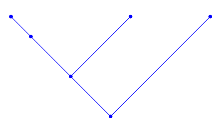
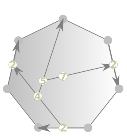
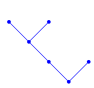
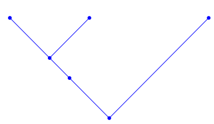
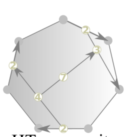
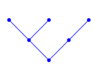
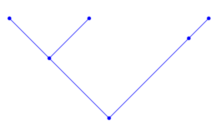
The middling eigenclusterings for $N = 7$						
N	**Configuration**	$\mathfrak{T}\mathrm{ree}^*_{\leq 2}(N-1)$ **presentation**	$\mathfrak{T}\mathrm{ree}^*_{\leq 2}(N-1)$ **presentation**	*Br*	*Be*	**Notes**
		 P_5-1		0	1	P_6-hockey
	 KT-composite	 P_5-2		0	1	Called P_6-ice(hockey) in [110]
		 S(3) Longchair-0*		1	0	Upturned baton
		 S(1^2 0) Bird-wingtip*		1	0	Bird-1* Upturned wrench
7		 S(3) Longchair-2*		1	0	Downturned baton
		 S(1^2 0) Bird-leg*		1	0	Bird-0*
		 S(1^2 0) Bird-midwing*		1	1	Downturned wrench Bird-1-0*
		 S(3) Longchair-1*		1	1	-longstalk-λ -front-of-seat
	 HT-composite	 S(3) Longchair-2-0*		1	1	-longtail-λ -mid-seat

Figure 6:

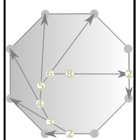
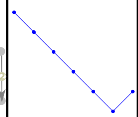
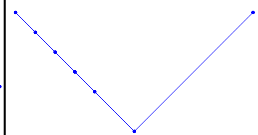
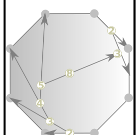
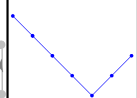
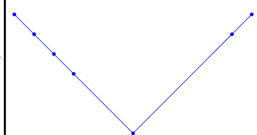
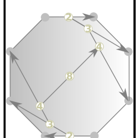
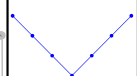
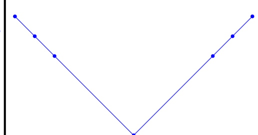
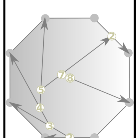
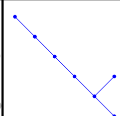
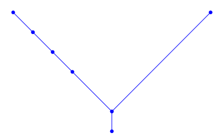
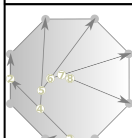
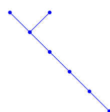
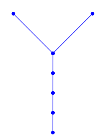
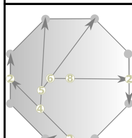
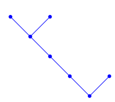
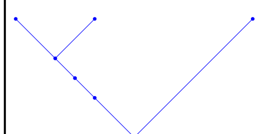
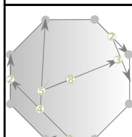
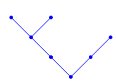
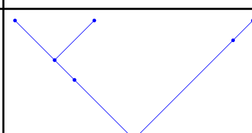
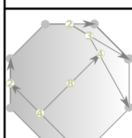
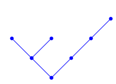
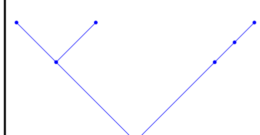
The middling eigenclusterings for *N* = 8					
N	**Configuration**	$\mathfrak{T}\mathrm{ree}^*_{\leq 2}(N-1)$ **presentation**	$\mathfrak{T}\mathrm{ree}^*_{\leq 2}(N-1)$ **height presentation**	*Br*	*Be*
		 P_7-1		0	1
	 K-dimer	 P_7-2		0	1
		 P_7-v		0	1
		 S(4) Longchair-0*		1	0
8		 S(4) Longchair-3*		1	0
		 S(4) Longchair-2*		1	1
		 S(4) Longchair-1*		1	1
	 HK-composite	 S(4) Longchair-3-0*		1	1

Figure 7:

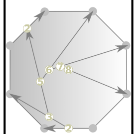
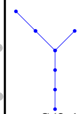
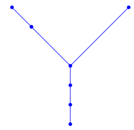
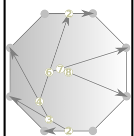
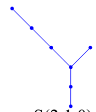
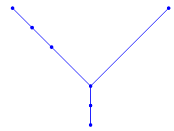
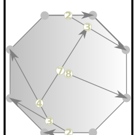
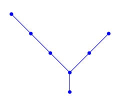
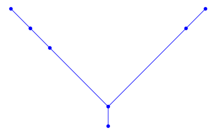
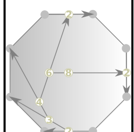
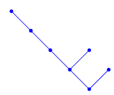
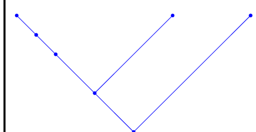
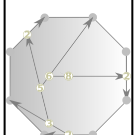
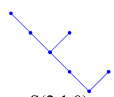
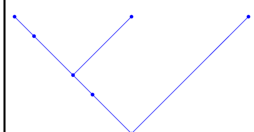
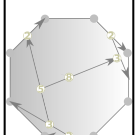
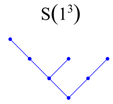
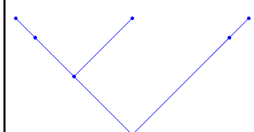
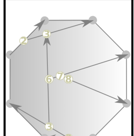
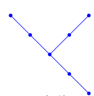
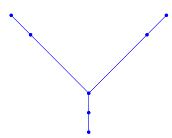
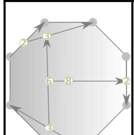
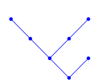
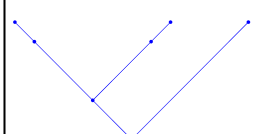
N	Configuration	$\mathfrak{T}ree^*_{\leq 2}(N-1)$ presentation	$\mathfrak{T}ree^*_{\leq 2}(N-1)$ height presentation	*Br*	*Be*
8		 S(2 1 0) SAT-2*		1	0
		 S(2 1 0) SAT-1*		1	0
		 S(2 1 0) SAT-0*		1	0
		 S(2 1 0) SAT-1-0*		1	1
		 S(2 1 0) SAT-2-1*		1	1
		S(1^3)  S(2 1 0) SAT-2-0*		1	1
		 S(1^3) Lobster-1		1	0
		 S(1^3) Lobster-0		1	1

Figure 8:

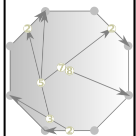
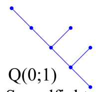
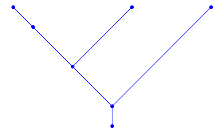
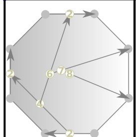
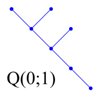
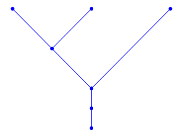
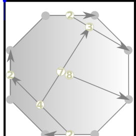
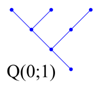
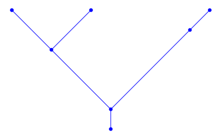
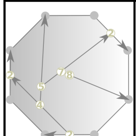
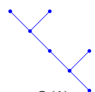
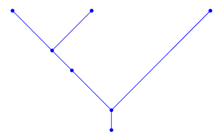
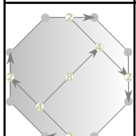
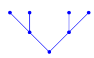
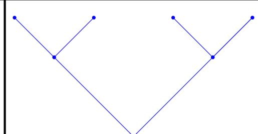
N	Configuration	$\mathfrak{Tree}^*_{\leq 2}(N-1)$ presentation	$\mathfrak{Tree}^*_{\leq 2}(N-1)$ height presentation	*Br*	*Be*
8		 Q(0;1) Swordfighter -leg*		2	0
		 Q(0;1) Swordfighter-point*		1	0
		 Q(0;1) Swordfighter -arm*		2	0
		 Q(1) Grandparents*		2	0
	 H-dimer	 Q(1) Grandparents -child*		2	1

Figure 9:

Remark 1 See Figs 6–9.

Remark 2 $N = 8$ is minimum as regards supporting ≥ 1 nontrivial eigenclustering dimer. This is a knock-on effect of the H, K ambiguity for $N = 4$. Yielding the K-dimer $P_{7\text{-}v}$ (row 3 of Fig 7). And the H-dimer $\text{Grandparents}_{\text{child}}$ (the bottom row of Fig 9).

Remark 3 It is now also possible to have an equal-parts partition with distinct equal parts. I.e. the H-K composite corresponding to S(3)-3-0* . (the bottom row of Fig 7). This plays a distinguished role below.

4 Some minimum special features for $N = 5$ to 10

4.1 $N \leq 5$ 'add leaf' suffices to discern between eigenclusterings and trees

Proposition 1 $N = 5$ is minimum for $\mathfrak{EN}[N] \neq \mathfrak{Tree}[N]$

<u>Proof</u> Compare Figs 21 and 10. The latter has 1 extra edge: $\text{P}_{3\text{-straight}}$ to S_3 . By which these two posets are not even isomorphic as graphs. □

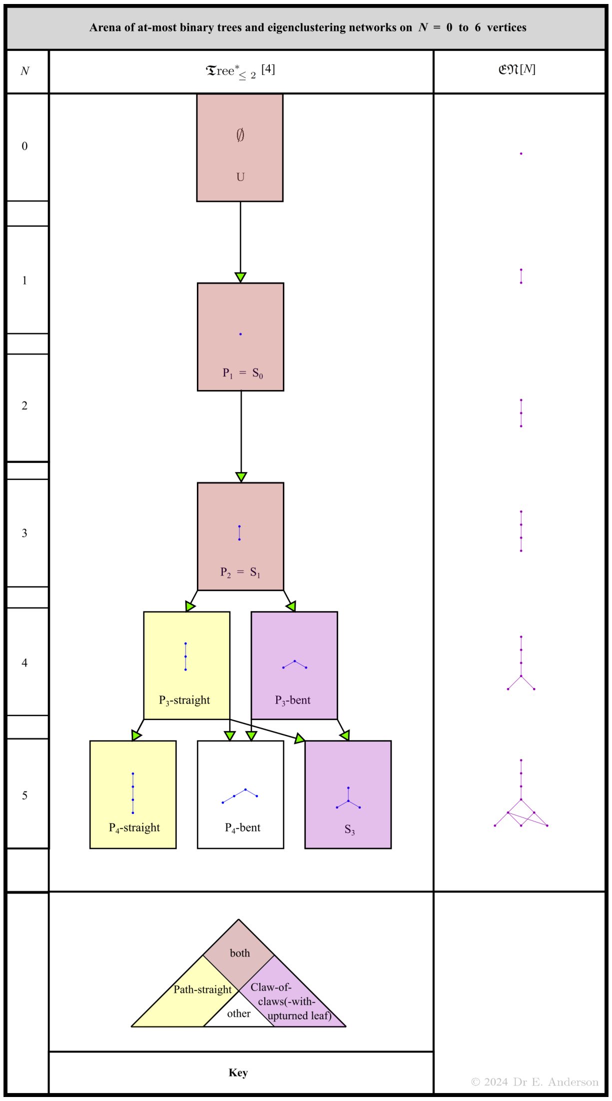


Figure 10:

Remark 1 This is furthermore Order-Theoretically consequential. For the latter forms a crossed C_4 (4-cycle subposet). Which is a forbidden subposet if our poset is furthermore to be a lattice (defined in Appendix B) [89, 110]. By which $\mathfrak{Tree}[5]$ manages to be a lattice, while $\mathfrak{EN}[N]$ does not.

Remark 2 Observe that Remark 2 of Subsec 2.5's patching has been made in passing from the first column to the second.

Remark 3 All of the graph skeletons formed by $\mathfrak{EN}[N]$ and $\mathfrak{Tree}[N]$ for $N \leq 5$ are however standard graphs, or homeomorphs of standard graphs. See Fig 11. This gives us an immediate reason for considering slightly higher N .

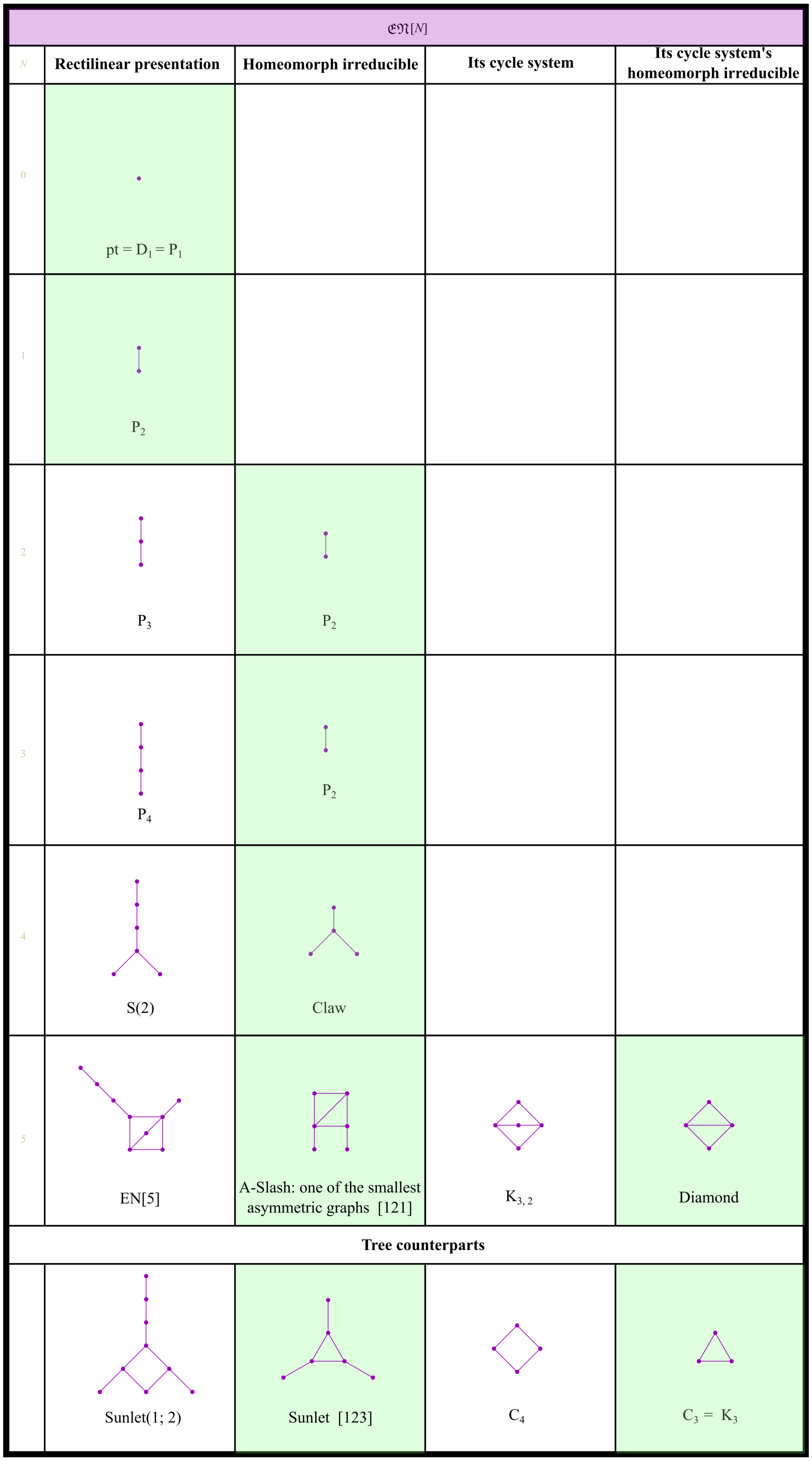


Figure 11:

4.2 $N \leq 6$ poset of eigenclusterings

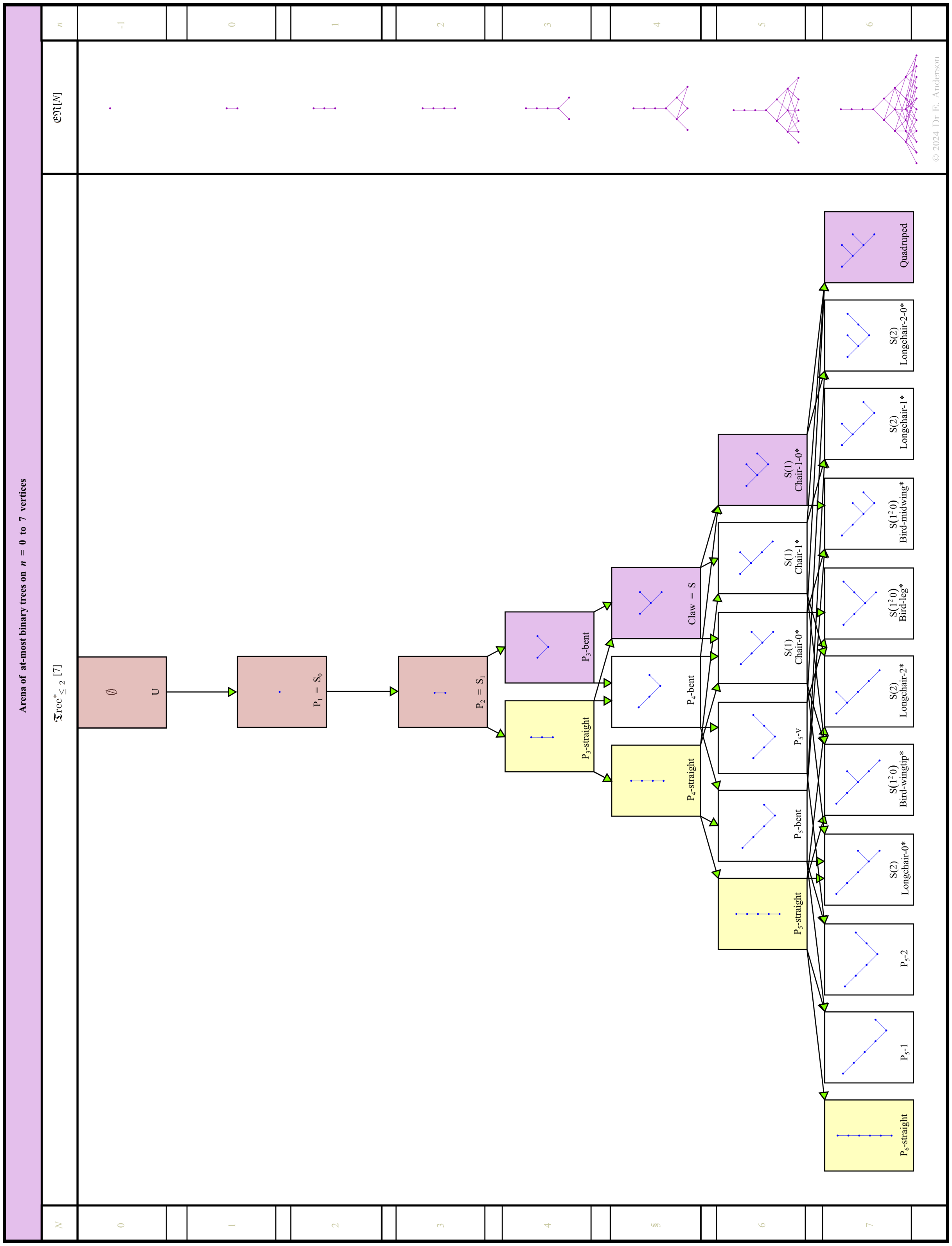


Figure 12:

Remark 1 We set up the posets $\mathfrak{EN}[6]$ in Fig 12.

Remark 2 $N = 6$ is minimum for $\mathfrak{EN}[N]$ to yield new graph skeletons, as per Fig 13. See Appendix B for the notions in the last column. And [121, 122] for more about this particular graph.

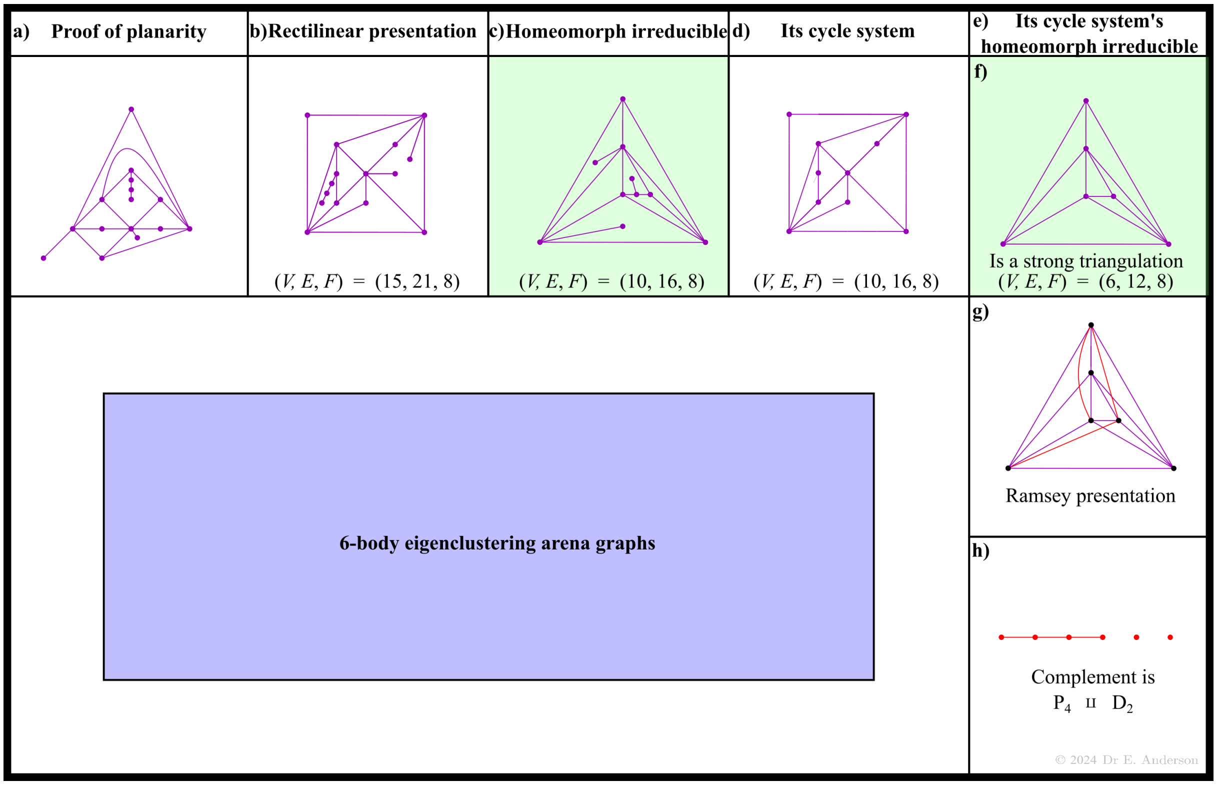


Figure 13:

Remark 4 But no new cycle systems arise.

4.3 $N \leq 7$ poset of eigenclusterings

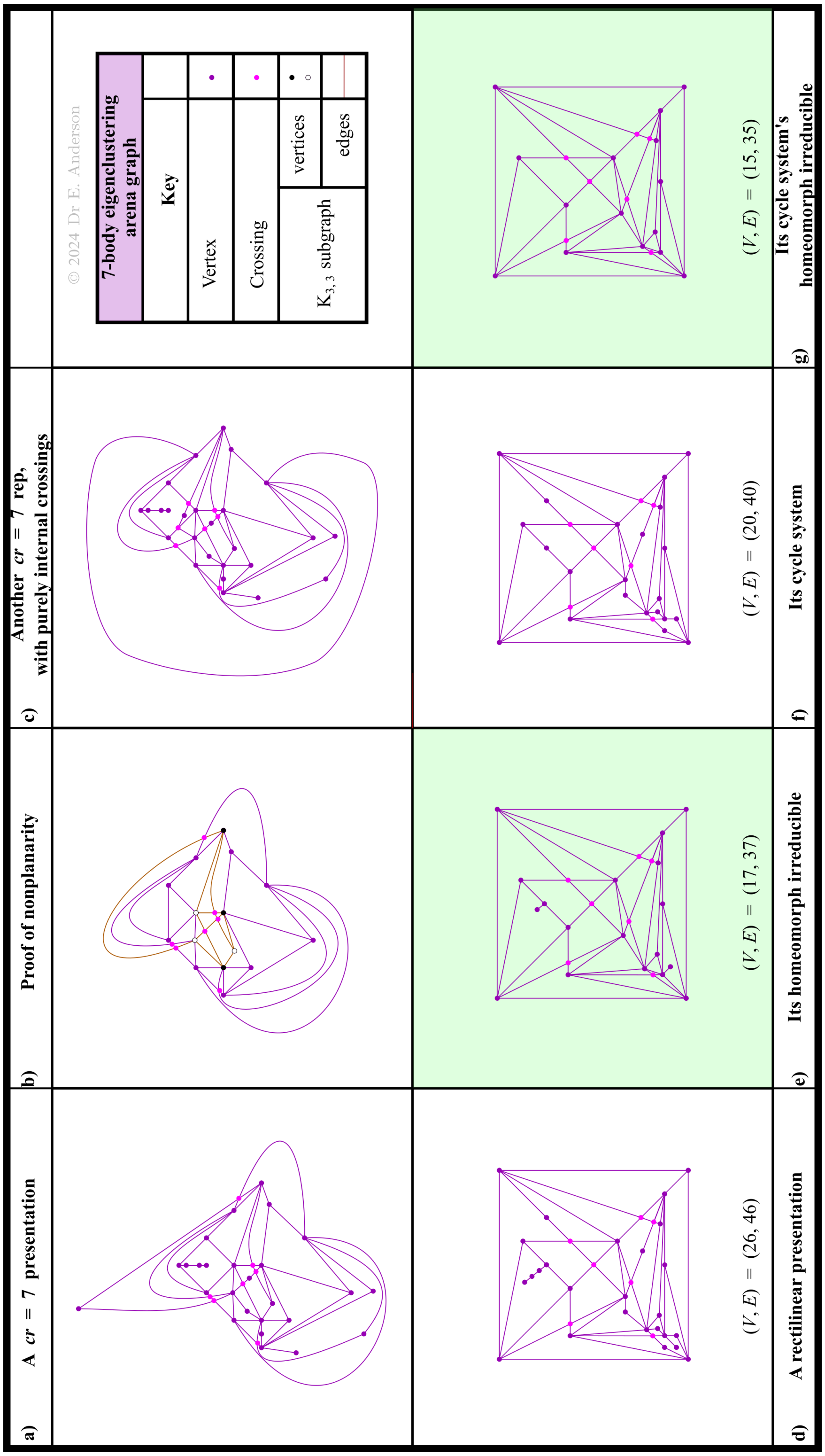


Figure 14:

Remark 1 $N = 7$ is minimum for $\mathfrak{EN}[N]$ to yield nonplanar graph skeletons. And a new homeomorph irreducible, cycle system and homeomorph irreducible cycle system. All of which are nonplanar by containing a $K_{3,3}$ 'Utilities' subgraph (see Appendix B).

Remark 2 See [121] for more about cycle systems. And for more about the 6-graph depicted in various ways in Subfig e).

Open Question 1 While we have provided a graph presentation manifesting a crossing count $cr = 7$, we have not shown that the crossing number Cr is itself 7 . It is tidier to consider this question for g) rather than a). See also Appendix B for definitions of cr and Cr .

Aside 1 $\mathfrak{Tree}[7]$ also picks up a crossed C_4 subposet [110]. By which it joins $\mathfrak{EN}[7]$ in not being a lattice.

4.4 $N = 8$ is minimum to show that $e(N) \neq h(N)$

Remark 1 Some context here is that so far counts for these coincide.

$$e(N) = w(N) = h(N) = t(N) \text{ for } N \leq 7 . \tag{14}$$

Remark 2 But then

$$e(8) = w(8) = t(8) < h(8) . \tag{15}$$

For the first three are 23 while the fourth is 24 .

This is a consequence of the following. The $w(N)$ recurrence relation modifies that for $h(N)$ to not count this and subsequent equal-partition-size joinings of distinct isomers as distinct. Both of these convolution-type recurrence relations are in Appendix B, and were also contrasted in [50]. The first place where this distinction appears is that a K joined to an H and vice versa are the same eigenclustering. I.e. the composite picked out in Remark 3 of Subsec 3.3. But if the half-Catalan numbers counted the eigenclustering networks, then these 2 copies of the same network would have to be treated as distinct.

Remark 3 Subsequently

$$e(N) = w(N) < h(N) . \tag{16}$$

as a ready consequence of each's recurrence relation.

.

4.5 $N = 9$ is minimum to show that $e(N) \neq t(N)$

Remark 1 Up to $N = 8$,

$$e(N) = w(N) = t(N) : \tag{17}$$

the number of unlabelled trees on N vertices [82, 105, 111].

Remark 2 But

$$e(9) = w(9) < t(9) : \tag{18}$$

For the first three of these give 46 while the last yields 47 .

Remark 3 However exceptionally

$$h(9) = t(9) . \tag{19}$$

4.6 Hedges

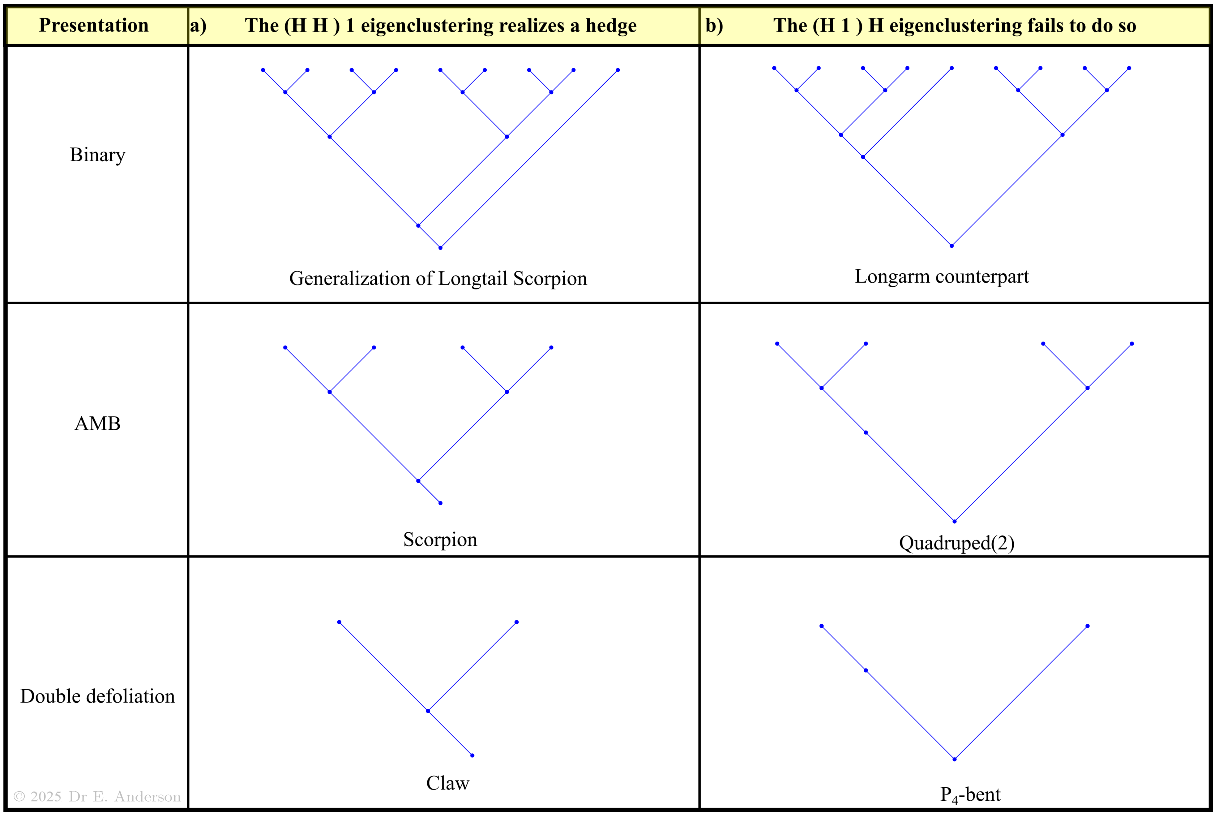


Figure 15:

Remark 1 These are ≥ 1 defoliation out from being lobsters.

4.7 A start on $N = 10$

Remark 1 Subsequently

$$e(N) < h(N) < t(N) . \tag{20}$$

Where the first inequality follows from two Subsecs back. While we leave the second as observed in small-N tables. This suffices for our current purposes, since the current Article only covers $N = 0$ to 10. While its sequel [58] extends this up to 33.

Remark 2 $N = 10$ is also minimum for symmetry to pick a distinct secondary top from our definition of H. This is rooted in Claw-of-Claws now being available in addition to the P_4-of-Claws that our H progress through. Discerning the subsequent pattern of symmetry groups and series supported by them is in fact the main feature of our sequel.

4.8 Asymptotic count of eigenclusterings

Structure 1

$$e(N) = w(N) \sim A\, N^{-3/2} \exp(B\, N) . \tag{21}$$

Where $A \simeq 0.3188$ and $B \simeq 0.9095$.

Structure 2 This is much tighter than using

$$t(N) \sim C\, N^{-5/2} \exp(D\, N) . \tag{22}$$

Where $C \simeq 0.5349$ and $D \simeq 1.0836$.

Remark 1 The second of these asymptotic bounds is in [69]. While the first follows by rearranging [77] to match the second's parametrization.

Remark 2 We did not find any result in the literature for the asymptotic count of half-Catalan numbers. So we comment that if the second inequality in (20) could be established $\forall \ N$ large enough (≥ 10 at least), then these would be trapped between the above two exponential-times-power expressions. See [97] for more about asymptotic Combinatorics.

5 Algebraic structure of eigenclusterings

5.1 Two algebras

Structure 1 The WE Algebraic interpretation can be viewed as placing a binary multiplication operation on $\mathfrak{Tree}_2^*[1, \infty]$. That combines two of these trees by taking them to be subtrees emanating from a new root vertex. It equips this to be *free commutative magma* on the single generator, x .

Naming Remark 2 *Magmas* (a term coined by Serre [125]) parallel groups but with only the closure axiom holding [126, 127]. *Free* is used here in the usual algebraic sense of not having any relations.

Structure 2 $\mathfrak{Tree}_2^*$ itself is obtained by appending a single identity element to the preceding. By which the *free unital commutative magma* on the single generator, x , is obtained.

Theorem 1 The following arenas of eigenclustering networks admit the stated Algebraic interpretations.

i) $\mathfrak{EN}[1, \infty]$ is the free commutative magma with a single generator.

ii) $\mathfrak{EN}$ is the free unital commutative magma with a single generator.

Remark 1 We can here take the magma's generator to be 1 . Each use of which amounts to bringing in one further equal-mass and elsewise indistinguishable point-or-particle. The magma's operation is then $+$, i.e. overall our Combinatorial Presentation 1. Each use of which has a natural and foundational interpretation, as follows. Forming a new system by taking two systems, reinterpreting each as a subsystem. And bolting them together by their formerly-total CoMs to the new total CoM.

5.2 The above structures are compatibly both posets and algebras

Remark 1 So the above binary tree spaces carry both poset structure and free (unital) commutative magma structure. Thus they belong to the interface between Order Theory and Algebra.

Remark 2 Furthermore, our natural depth function M on the *poset elements* of our poset arena [1] is compatible with this Algebra's operation. It adds up the hierarchical subsystem masses in the sum of 1 's presentation. Or the number of uses of the generator x in the free commutative non-associative Algebra presentation. On the poset arena, N (or some shifted version thereof) serves as depth function. Which can furthermore be viewed as

$$N = \max(M) . \tag{23}$$

So the depth function on the arena poset is (a possibly shifted version of) the maximum of the depth function of its constituent poset elements.

Remark 3 In the WE Algebraic Presentation 2, the equivalent of the poset element depth function is the count of each bracket's number of powers of the generator x . While the equivalent of the poset arena depth function is each element's total number of powers of x . Which can also be construed as the max over all brackets of each bracket's number of powers.

[1] So far, it would appear to be quite rare for the arena of some type of mathematical object to itself be the same kind of mathematical object. See e.g. [110] for other examples of this.

5.3 Two citizens of Kallista

Application 2 The above arenas of binary trees also arise naturally in Theoretical Computer Science, as suitable arenas in which to formulate Syntax.

Remark 1 The current Article's main Application (1) − eigenclustering − furthermore arises naturally from applying Linear Algebra to whichever of Geometry, Dynamics or Statistics. Thus the same structure arises in all of these, and Algebra, and Graph Theory, and Theoretical Computer Science.

Remark 2 By Subsec 5.2's compatibility. And by how we can arrive at them by many a priori conceptually distinct routes as per Remark 1. These two structures meet the criteria for being citizens of Kallista [43, 108, 110, 120].

Naming Remark 3 In one sense, any of the following are truer names for the WE (Wedderburn–Etherington numbers.

1) *Eigenclustering network counts.*

2) *Syntax counts.*

3) *Free commutative magma element counts.*

4) *Bracketing-of-sums-of-ones counts. Binary tree counts.*

Or (given that none of these are counting cumulatively),

6) *AMB tree counts.*

In another sense, these are however but truer names of *facets of* whatever the WE numbers may be as a whole. Since we cannot be sure that we have yet encountered all of their facets in basic and widely-applicable Mathematics, for now we consider it premature to attempt to give these numbers an overall truer name.

5.4 Arenizing our presentation inter-relation Figure

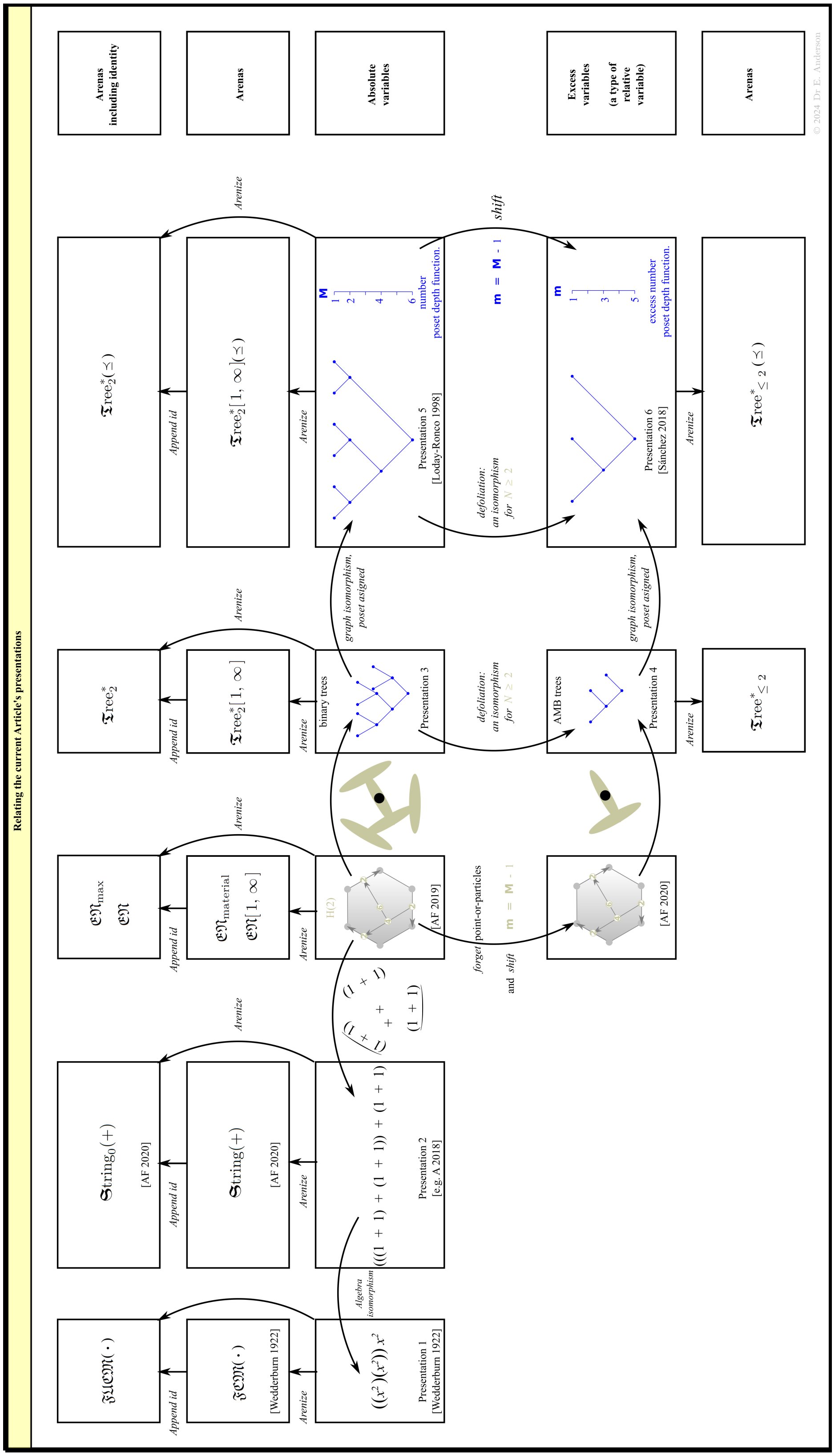
Remark 1 We expand here on Fig 16 to include the corresponding arenas.

Remark 2 The $N \geq 1$ version, thus avoiding $w(0) = 0$, is in the second row. While the extension to include $N = 0$ is in the first row. This is at the expense of requiring a free *unital* commutative magma.

Remark 3 In each of these rows, we have an Algebra leftmost and a poset rightmost. We can furthermore view everything in its row as equipped with both its algebra and its poset.

Remark 4 Compare with the power set [110]. This is both a Boolean algebra (Order Theory [75]) and a commutative group. Constituting a far better known citizen of Kallista.

Remark 5 The bottom row exhibits the corresponding AMB areas. These do not benefit from being an algebra for the reason given in Sec 7.5.



Relating the current Article's presentations
Arenas including identity
Arenas
Absolute variables
Excess variables (a type of relative variable)
Arenas
Presentation 1 [Wedderburn 1922]
Presentation 2 [e.g. A 2018]
Presentation 3
Presentation 4
Presentation 5 [Loday-Ronco 1998]
Presentation 6 [Sánchez 2018]
[AF 2019]
[AF 2020]
binary trees
AMB trees
Arenize
Append id
Algebra isomorphism
graph isomorphism, poset asigned
defoliation: an isomorphism for N ≥ 2
forget point-or-particles and shift
shift
m = M - 1
number poset depth function.
excess number poset depth function.
[Wedderburn 1922]
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Figure 16:

6 Eigenclustering Length-Exchange Theorems (ELETs)

6.1 Motivation

Remark 1 Our current reason for considering eigenclustering networks is that the following has recently been established. That Apollonius' Median-Length Theorem [1, 8, 13, 48, 57, 54] and Euler's Quadrilateral Theorem [3, 9, 10, 11, 51, 46, 57] extend to [52, 53] 1 ELET per eigenclustering network. In Apollonius' case, median length is exchanged for side length information. In Euler's case, Newton line interval [14, 4, 5, 6, 11] length is exchanged for separation length information; this corresponds to the H-eigenclustering. [52] provides the K counterpart. While [53] cover the K and H series as well as the $N = 5$ and 6 middles.
Remark 2 Arbitrary vertex-mass counterparts – AMELETs – generalizing Stewart's Cevian-length Theorem [2, 7, 8, 12, 13, 57], have been developed in [55, 56].

6.2 The number of nontrivial ELETs

Remark 1 To have nontrivial such Theorems, one needs nontrivial eigenclusterings, meaning not just separations. For then they have other eigenclustering lengths to exchange for separations. Nontrivial eigenclusterings occur for every $N \geq 3$.

Remark 2 There is furthermore one nontrivial such Theorem per eigenclustering network supported by N .

Remark 2 Thus for each $N \geq 3$, the number of nontrivial ELETs supported $n(N)$ is the number of eigenclustering networks $e(N)$, and thus the corresponding WE number $w(N)$ [50].

Remark 3 But also

$$n(N) = 0 \text{ for } N \leq 2 . \tag{24}$$

So we have a slight truncation.

6.3 Order Theory of nontrivial ELETs

Remark 1 This can be viewed as a replay of Fig 10 and 21's three cases, with two further cases to compare. Four of these have a priori conceptual meaning as per the Introduction's list of types of range.

Remark 2 The nontrivial ELETs form

$$\mathfrak{Elet}_{\mathrm{nontrivial}} = \mathfrak{EN}[\,3, \infty\,] = \mathfrak{Tree}^*_2[\,3, \infty\,] = \mathfrak{Tree}^*_{\leq\, 2}[\,3, \infty\,] . \tag{25}$$

By Remark 2 of Subsec 2.5, the binary tree presentation is the simplest to state.

Remark 3 This is straightforwardly a subposet of

$$\mathfrak{EN} = \mathfrak{Tree}^*_2 . \tag{26}$$

Its bottom element is Apollonius' Median-Length Theorem. Since it is an upper-semi-infinite poset, it has no top element.

6.4 Diversity of notions of ELET. i) Interplay with Algebra

Remark 1 $\mathfrak{Elet}_{\text{nontrivial}}$ also corresponds to those elements of the free commutative magma that are *long enough to need a bracket*. The Apollonius case is long enough for this and yet not long enough to carry a *bracketing ambiguity*. While this gives a sharp algebraic characterization of $\mathfrak{Elet}_{\text{nontrivial}}$, it does not however pick out a subalgebra. That this has no x available to turn $x^2 \cdot x$ into $x^2 \cdot x^2$ suffices to see this.

Structure 1 To place upon the ELETs the simplest algebraic structure – the free commutative magma – we need to include $N = 1$ and 2 : the material conceptualization. This corresponds to appending 2 '$0 = 0$' trivial identities to our nontrivial ELETs. Since this completion is Algebraically motivated, let us denote this space of ELETs by

$$\mathfrak{Elet}_{\text{alg}} .$$

But also by

$$\mathfrak{Elet}_{\text{material}} ,$$

for using $N \geq 1$ corresponds to the presence of 'physical material'. See row 2 of Fig 17 for various equivalent formulations. Observe furthermore that, while $\mathfrak{Tree}^*_{\leq 2}$ is mathematically isomorphic to this, it is interpretationally different by its vertices' meaning rather what is indicated in row 3).

Structure 2 We can furthermore include $N = 0$ at the cost of involving rather the free unital commutative magma. This corresponds to appending not 2 but 3 trivial identities. See row 1 for notation and various equivalent formulations.

6.5 ii) Discussion

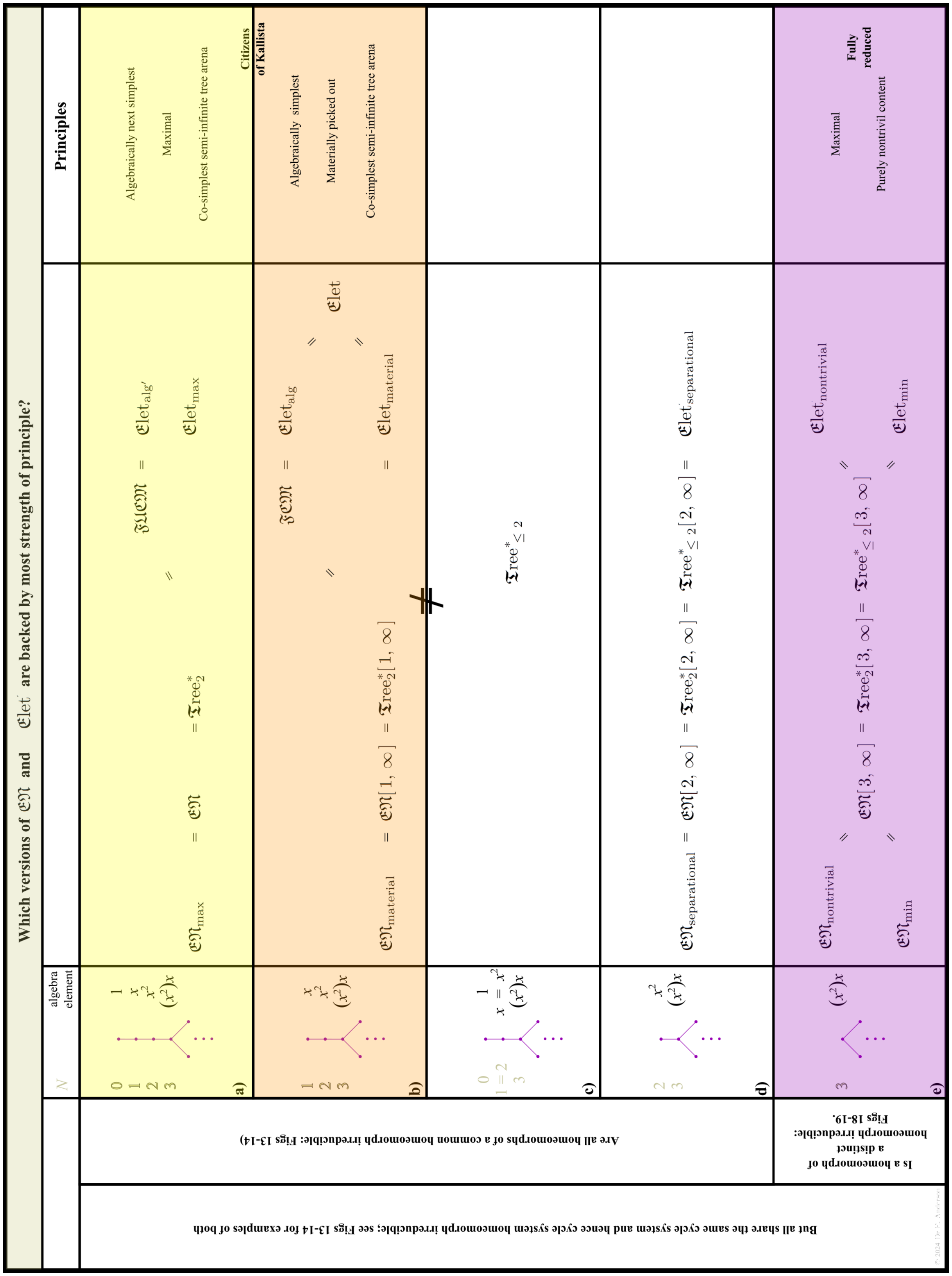
Remark 1 Interestingly, appending 2 or 3 trivial identities gives an algebra, while appending 1 does not help. This corresponds to Algebra not acommodating the separational point of view. 3 is furthermore the maximal amount of trivial identities that can be appended. By which this case picks up the 'max' subscript.

Remark 2 A choice between two citizens of Kallista is thus possible in arenizing the ELETs. For now we argue for the material $=$ Algebraically-simplest arena over the maximal extension arena which is Algebraically-next-simplest.

Remark 3 The nontrivial ELETs' advantage is in containing no redundancy. It is also interesting that algebraic simplicity selects a *middlingly*-redundant arena.

Remark 4 For Subfig c), the identification $x = x^2$ spoils the algebra.

Remark 5 For Subfig d), x is not available. While for Supbfig e), neither x nor x^2 are available.



Which versions of 𝔈𝔐 and 𝔈𝔩𝔢𝔱 are backed by most strength of principle?
N
algebra element
Principles
Algebraically next simplest
Maximal
Co-simplest semi-infinite tree arena
Citizens of Kallista
Algebraically simplest
Materially picked out
Co-simplest semi-infinite tree arena
Maximal
Purely nontrivil content
Fully reduced
Are all homeomorphs of a common homeomorph irreducible: Figs 13-14)
Is a homeomorph of a distinct homeomorph irreducible: Figs 18-19.
But all share the same cycle system and hence cycle system homeomorph irreducible; see Figs 13-14 for examples of both

Figure 17:

6.6 Further specific posets and graphs for ELETs

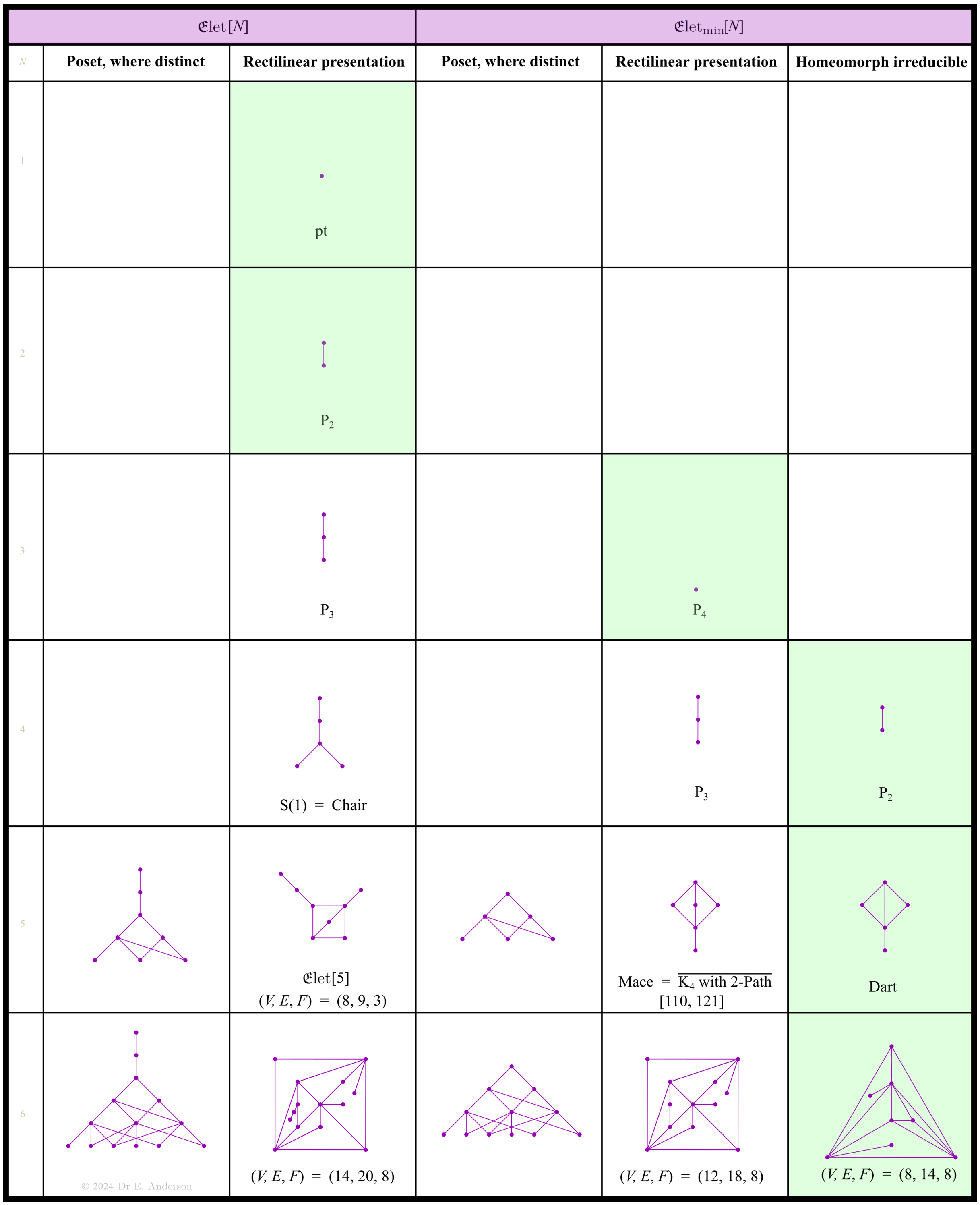


Figure 18:

Remark 1 The previous subsection also motivates our depicting posets and underlying graph skeletons for the following. $\mathfrak{C}\mathrm{let}_{\mathrm{material}} = \mathfrak{C}\mathrm{let}_{\mathrm{alg}}$ and $\mathfrak{C}\mathrm{let}_{\mathrm{nontrivial}}$. [$\mathfrak{C}\mathrm{let}_{\mathrm{max}} = \mathfrak{C}\mathrm{let}_{\mathrm{alg}'}$ just coincides with the previously depicted $\mathfrak{EN}$.] See Figs 18 and 19 for $N \leq 6$ and $N = 7$ respectively as regards what new structures arise. Observe that the nontrivial case manages to produce new homeomorph irreducibles, but neither new case has any capacity to produce new cycle systems.

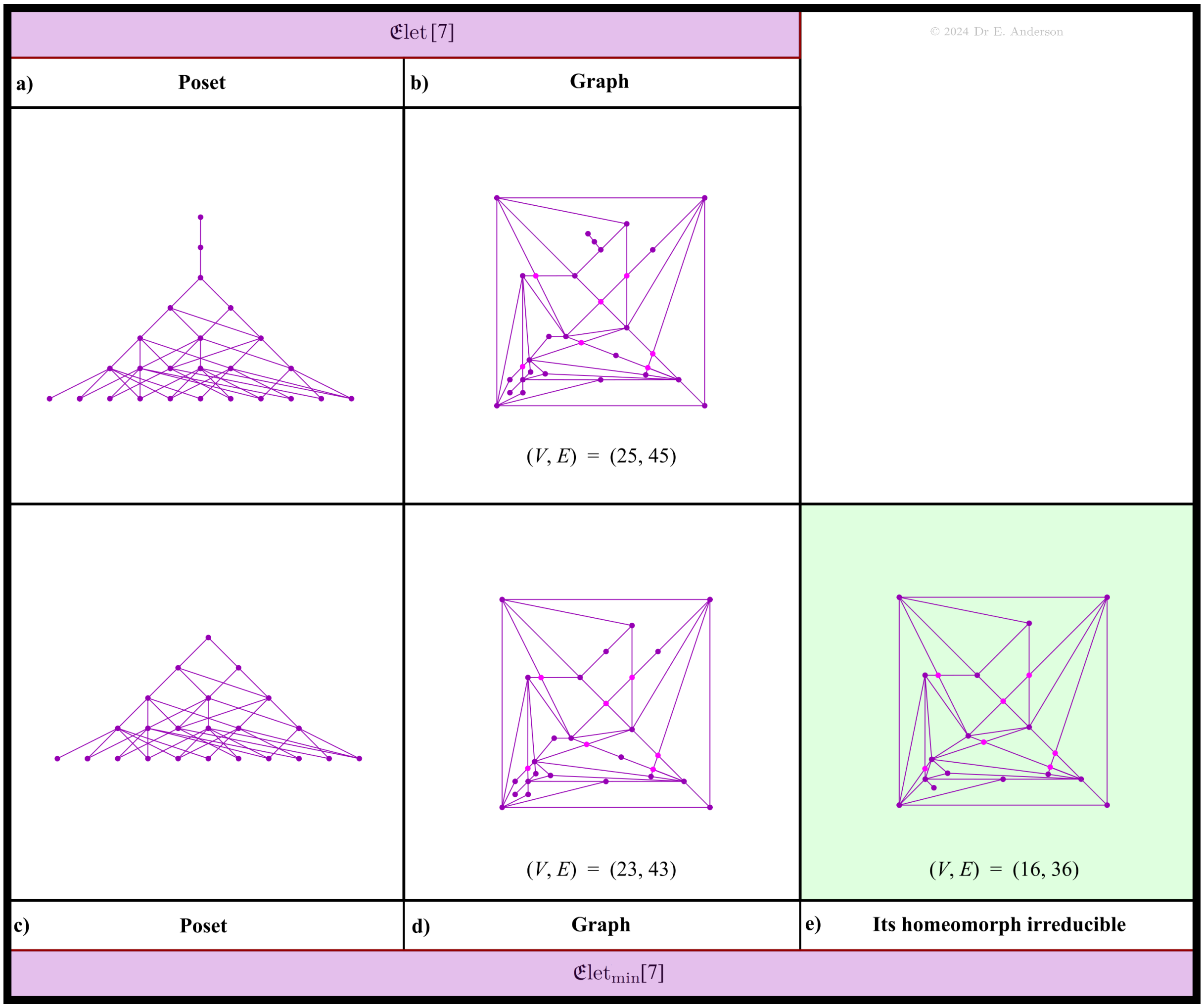


Figure 19:

Acknowledgments This Article is dedicated to S. Sánchez. E.A. and A.F. thank S and K for previous discussions. And the other participants at the Institute of the Theory of STEM's "Linear Algebra of Quadrilaterals" Summer School 2024. And the Referees for input as regards Appendix C and the subsections on lobsters and hedges. E.A. also thanks C, Malcolm MacCallum, Reza Tavakol, Jeremy Butterfield and Enrique Alvarez for career support.

A The smallest trees

A.1 Preliminary definitions

Definition 1 *Graphs* G [99, 79, 83, 110, 87, 107] consist of a set of vertices $\mathfrak{V}$ and a set of edges $\mathfrak{E}$ between them. For the simple graphs considered here, there is ≤ 1 edge between any vertex pair. Nor are there any loops: edges from a vertex to itself. The *order* of a graph is $V = |\mathfrak{V}|$ and the *size* of a graph is $E = |\mathfrak{E}|$.

Definition 2 A *tree* is a connected acyclic graph.

A.2 The smallest examples of tree

Example 0 The *ungraph* $\emptyset$ – no vertices and thus no edges – serves more specifically as the *untree* U.

Example 1 The 1-*component discrete graph* D_1 – one vertex and thus again no edges – is the smallest tree. This can also be thought of less structuredly as the 1-point *point cloud* pt , in which vertices are postulated but not edges. The ungraph is thus furthermore the unpoint.

Example 2 The smallest nontrivial graph is the 2-*path* P_2 : two vertices and the edge between them. This is the smallest nontrivial tree. It is not the only 2-graph, for there is also the 2-point cloud, pt^2. P_2 is however the only connected 2-graph. See row 2 of Fig 20.

Example 3 3 vertices also supports one tree: the 3-*path* P_3 . There is now a connected graph that is not a tree: the 3-cycle alias complete 3-graph. See row 3 of Fig 20.

A.3 Stars, starlike graphs and the quadruped

Example 4 4 vertices is minimum so as to have more than 1 tree: the 4-path P_4 and the 3-star S_3 alias Claw . See row 4 of Fig 20.

Example 5 5 vertices supports 3 trees . P_5 . The 4-star S_4 alias Plus . And a first tree graph that is neither a path nor a star: the chair graph Chair . Which is however *nontrivially starlike*. Meaning a single nexus from which at least 3 rays emanate, at least 1 of which is of length ≥ 2 . See row 5 of Fig 20.

Notation 1 $S(1)$ means the minimum sized star – $\mathrm{S}_3 = \mathrm{S}$ – with 1 extra vertex inserted. We shall develop this notation further in considering larger startlike graphs.

Example 6 6 vertices has 4 middling cases aside from its path and star. 3 are starlike: Longchair $=$ S(2) , Bird $= \mathrm{S}\left(1^2\,0\right)$ and Cross $= \mathrm{S}\left(1\,0^3\right)$. A first nonstarlike graph is additionally realized: Quadruped $=$ Q . See row 6 of Fig 20.

A.4 Arenas of unlabelled trees

Definition 1 Let

$$\mathfrak{T}\mathrm{ree}(N)$$

denote the *arena of unlabelled trees* supported by precisely N vertices. Also let

$$\mathfrak{T}\mathrm{ree}[N]$$

denote the *cumulative arena of unlabelled trees* supported by $\leq N$ vertices. Finally let

$$\mathfrak{T}\mathrm{ree} = \coprod_{N=0}^{\infty} \mathfrak{T}\mathrm{ree}(N)$$

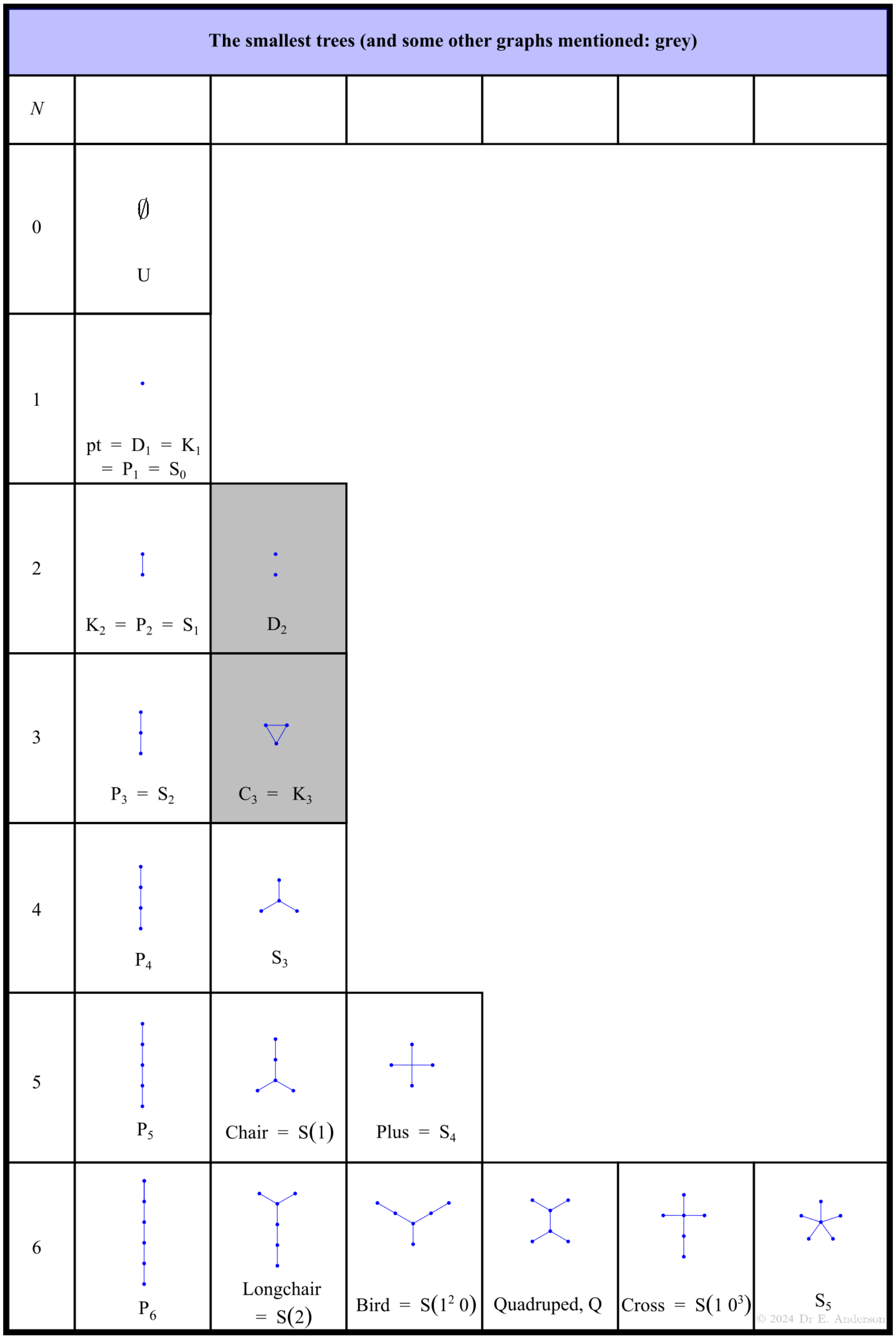


Figure 20:

denote the *arena of all unlabelled trees* [110].

Remark 1 The $\mathfrak{T}\mathrm{ree}[N]$ can be viewed as successive approximations to $\mathfrak{T}\mathrm{ree}$.

Remark 2 To cover the eventuality of needing to exclude some of the smallest trees, we finally introduce the following. Let

$$\mathfrak{T}\mathrm{ree}[\,P,\,Q\,]$$

denote *truncated cumulative arena of unlabelled trees* $\forall\,N$ such that $P \leq N \leq Q$.

Remark 3 We let the (N), $[N]$, blank and $[\,P,\,Q\,]$ source of variety pervade all other arenas considered in the current Article without further comment.

A.5 The 'add leaf' operation, and the order it places on $\mathfrak{T}\mathbf{ree}[N]$

Structure 1 The 'add leaf' operation is illustrated by the green arrow relations in Fig 21.

Notation 1 The cumulative-N posets (defined in Appendix B) thus formed are drawn in purple, so as to keep arenas visually distinct from their constituent objects: drawn in blue.

Notation 2 We pick out bottom and top elements by highlighting their backgrounds in yellow and purple respectively. When both of these coincide, red ensues.

A.6 Trees of claws

Remark 1 Quadruped can be decomposed into the P_2-of-Claws . A common alias for P_3-of-Claws is the Scorpion graph: see Fig 22.b).

Remark 2 For $N = 10$, the P_4-of-Claws (Subfig c) is accompanied by the Claw-of-Claws (Subfig d).

Remark 3 Claw groupings give the larger vertex symbols [110]. While defoliation interprets these as being the same as our standard blue vertices.

A.7 Caterpillars

Naming Remark 4 C.C. Lim calls the K(N)-eigenclusterings' binary-tree presentations 'caterpillar trees'.

Our AMB presentation name P_n-straight is however far clearer and requires rather less knowledge of Graph Theory to understand.

We have also been asked to comment on how 'caterpillar' has been used for a rather more general notion of tree in Graph Theory [76, 115]. So let us refer to C.C. Lim's notion as 'katerpillars', and to Graph Theory's as *caterpillars*.

Definition 1 *Caterpillars* are trees whose defoliations are paths.

Naming Remark 5 The central path is here envisaged as the caterpillar's body, while the remaining vertices – necessarily leaves – are its hairs.

Our own name for katerpillars is *straight paths-of-claws*. Which is a binary name since claws are the building blocks of binary trees. Binary caterpillars are then also *paths* of claws.

Remark 2 All trees on ≤ 6 vertices are caterpillars (Fig 20). The first counterexample is $V = 7$'s Lobster (Fig 23.c), which is the minimum nontrivial lobster (Appendix A.12).

The minimum caterpillar that is not a katerpillar is P_2. This is however rather trivial since it can be taken to be all body and no hair (or vice versa). The following is one of the minimum caterpillars to admit a nontrivial path-hair decomposition that is not also a katerpillar. $V = 7$'s Monkey (Bottom right in Fig 26). This is however also non-binary.

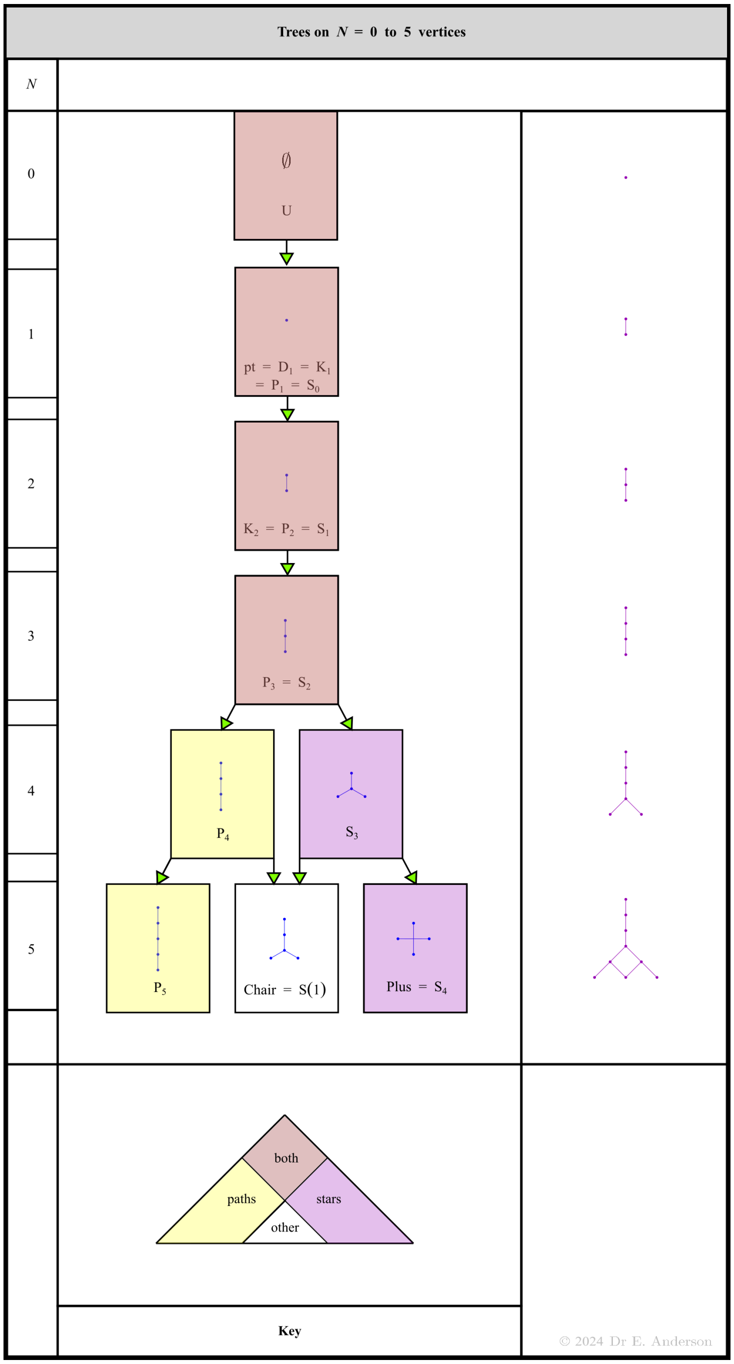


Figure 21:

So as to have a binary caterpillar that is not a katerpillar, take $V = 7$'s Grandparents (Fig 24.d). In other words, even $N = 4$'s H-eigenclustering manages to be a binary caterpillar that is not a katerpillar. $N = 5$'s eigenclusterings correspond to 1 katerpillar, modelling K(1) = P_4-straight (First entry of 5 row in Fig 12). A binary caterpillar that is not a katerpillar, modelling M(1) = P_4-bent (Second entry in the previous). And a lobster (Fig 24.c), modelling H(1) = S_3 (Row 5 of Fig 3). This is the longtail-Scorpion graph Scorpion(1) .

The minimum non-AMB lobster is the anticross (Bottom left in Fig 26). I.e. with 1 shor and 3 long prongs,

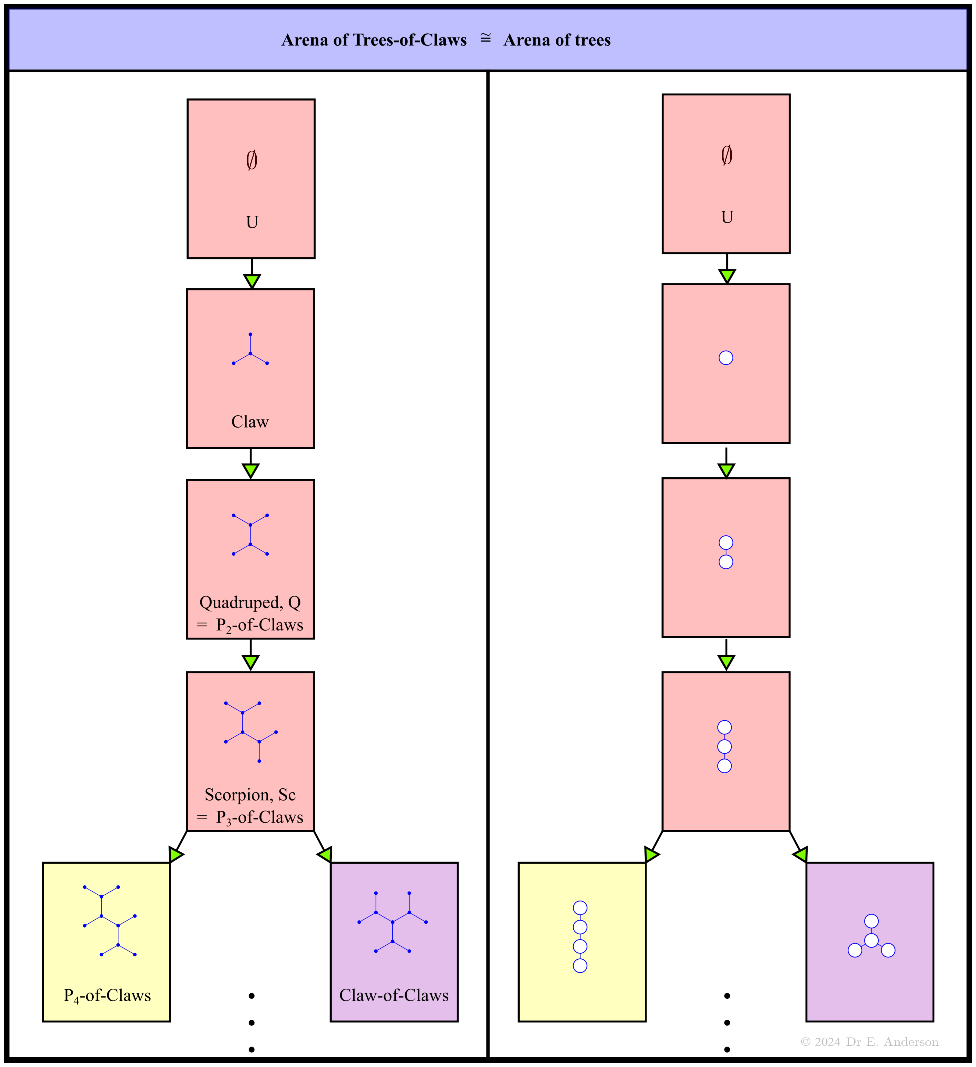


Figure 22:

to the cross having these the other way around. While the smallest tree that is neither a caterpillar nor starlike is the Orangutan (second left Fig 26).

The AMB presentations that are paths correspond to the binary caterpillars. No matter where a single bend is placed in them, about the vertex thus designated as root. Also

$$\#\,(\text{ distinct bends of } \mathrm{P}_n\,)\left\lceil \frac{n}{2} \right\rceil$$

distinct such bends The first bending ambiguity is P_3-straight versus P_3-bent for $n = 3$ and thus $N = 4$. I.e. to K- versus H-eigenclusterings. $N = 5$ also has $2:$ K(5) and M(5) . While $N = 6$ has 3 .

This counting function grows linearly with large N , while $w(N)$ obeys a positive exponential times power law. By which the proportion of binary trees that are caterpillars becomes very small: in power law times negative exponential proportion.

♝ ♙ ♞

A.8 Homeomorphs

Definition 1 Two graphs G_1, G_2 are *homeomorphs* [110, 121, 119, 93, 103] if one of them can be formed from the other by inserting vertices into some of its edges.

Definition 2 A '*homeomorph irreducible*' is a graph that is not the homeomorph of any smaller graph. Thoughout the current Article, we pick these out by highlighting their backgrounds in emerald.

Naming Remark 6 Quadruped, Swordfighter and Scorpion are examples of *stick figure names*. These are truer names than ones that only refer to particular-posture presentations. See [119] for discussion and further examples.

♠ ◇ ♣

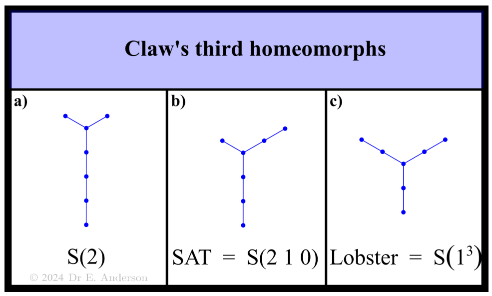


Figure 23:

Example 0 The 2-path graph $P_2 = P$ is a homeomorph irreducible. Whose homeomorphs are the path graphs P_N , $N \geq 3$.

Example 1 The star graphs S_n are homeomorph irreducibles. With the exception of $S_2 = P_3$, which is rather the first homeomorph of $P_2 = P$. The starlike graphs are then their homeomorphs.

Claw $=$ S_3 $=$ S has as first homeomorph $S(1)$ $=$ Chair . As second homeomorphs, $S(2)$ $=$ Longchair and $S(1^2\,0)$ $=$ Bird . And as third homeomorphs, $S(3)$ (Fig 23.a). $S(2\,1\,0)$ $=$ SAT : the smallest asymmetric tree (Subfig b). And $S\left(1^3\right)$ $=$ Lobster (Subfig c).

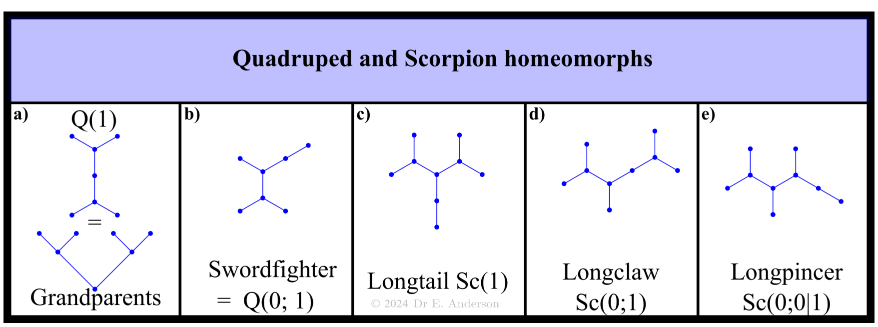


Figure 24:

Example 2 Quadruped is a homeomorph irreducible. The first homeomorphs it supports are Grandparents $=$ $Q(1)$ and Swordfighter $=$ $Q(0;1)$; see Fig 24.a-b). Fig 6.b)'s presentations is chosen to clearly explain the swordfighter name. See also the last row of Fig 15 for an explanation of the grandparents name.

Notational Remark 1 We use P, S, Q as 'homeomorph-irreducible' notation for P_2, S_3, Quadruped respectively.

Example 3 Scorpion is a homeomorph irreducible. The first homeomorphs it supports are Longtail = Sc(1) , Longclaw = Sc(0; 1) and Longpincer Sc(0; 0 | 1) ; see Fig 24.c-d).

Pointer 1 See e.g. [110, 93, 103] for more about graph homeomorphs.

A.9 Rooted trees

Definition 1 A *rooted tree* [87, 109, 103, 104] is a tree in which some particular vertex – called the *root* – has been assigned a distinguished role.

Notation Remark 1 We depict rooted trees with the root as the unique lowest vertex. And denote it by pinning a * on the descriptor of the rooted vertex.

Rooted trees are a type of *directed tree*, with its arrows all pointing away form the root. Up flow or down flow depictions save us from having to actually draw any arrows. Its being placed lowest can also be given extremal and Order-Theoretic conotations.

Example 1 There are 2 ways in which Claw can be rooted, as per Fig 25.a)-b). More botanical names are -stem* and -leaf* And more topological name for the first of these is -nexus* or -starpoint* . More systematic names are -3* and -1* with reference to vertex degree. Or -1* and -0* with reference to foliation to layering.

Example 2 As can Quadruped (Subfigs c-d). These have the same number aliases as the previous.

Example 3 Chair can however be rooted in 4 different ways (Subfigs e-h). -0* denotes that the ray of length 1 and thus excess length 0 is used as root. -1* that the ray of excess length 1 's end point is used as root. And 1-0* that the latter ray's interior point plays the role of root. Chair already serves as a minimum counterexample to degree labelling being unambiguous.

Example 4 Scorpion can also be rooted in 4 different ways (Subfigs i-l). Foliation-depth labels still work here: -2* , -1* , -1-0* and -0* .

Remark 1 We are not trying to be exhaustively systematic. Just to find enough names and notations for eigenclusterings on 0 to 8 points-or-particles used by the current Article.

A.10 (At-most-)binary trees

Definition 1 A directed tree is *binary* [87, 109, 96, 110, 103, 104] if every vertex has 0 , or 2 direct descendants. And *at-most binary (AMB)* if every vertex has 0 , 1 or 2 direct descendants.

Remark 1 A tree is binary if it has precisely 1 degree-2 vertex. N degree-0 vertices. and $N - 2$ degree-3 vertices. With the exception that *pt* is a binary tree consisting solely of a degree-0 vertex. In all the other cases, the degree-2 vertex serves as root. Almost-binary trees have no degrees > 3 ; in other words, they are 3-subregular [110]. Each corresponds to the defoliation of a binary tree [110], with the sole exception of the untree.

Remark 2 Rooting binary and AMB trees has more limited options.

Example 0 The P_N are AMB but not binary for $N \neq 3$, which is binary.

Example 1 Claw = S_3 is AMB. Plus = S_4 is not, for it has a vertex of too high a degree. This extends to $S_{\leq 4}$ and its homeomorphs.

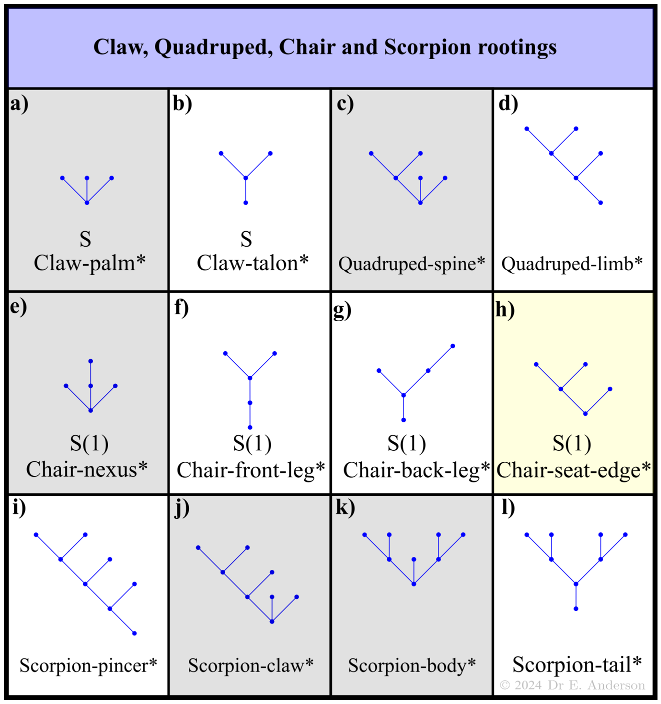


Figure 25:

Example 2 Quadruped and Scorpion are AMB.

Example 3 Every homeomorph formed from an irreducible AMB tree is itself AMB.

Example 4 In Fig 25, rootings that fail to be even AMB have backgrounds shaded in grey. While ones that manage to be binary have backgrounds highlighted in ivory.

A.11 Some corresponding arenas

Definition 1 Let

$$\mathfrak{T}\text{ree}_2^*$$

denote arenas of binary (directed rooted) trees. Also let

$$\mathfrak{T}\text{ree}^*_{\leq\ 2}$$

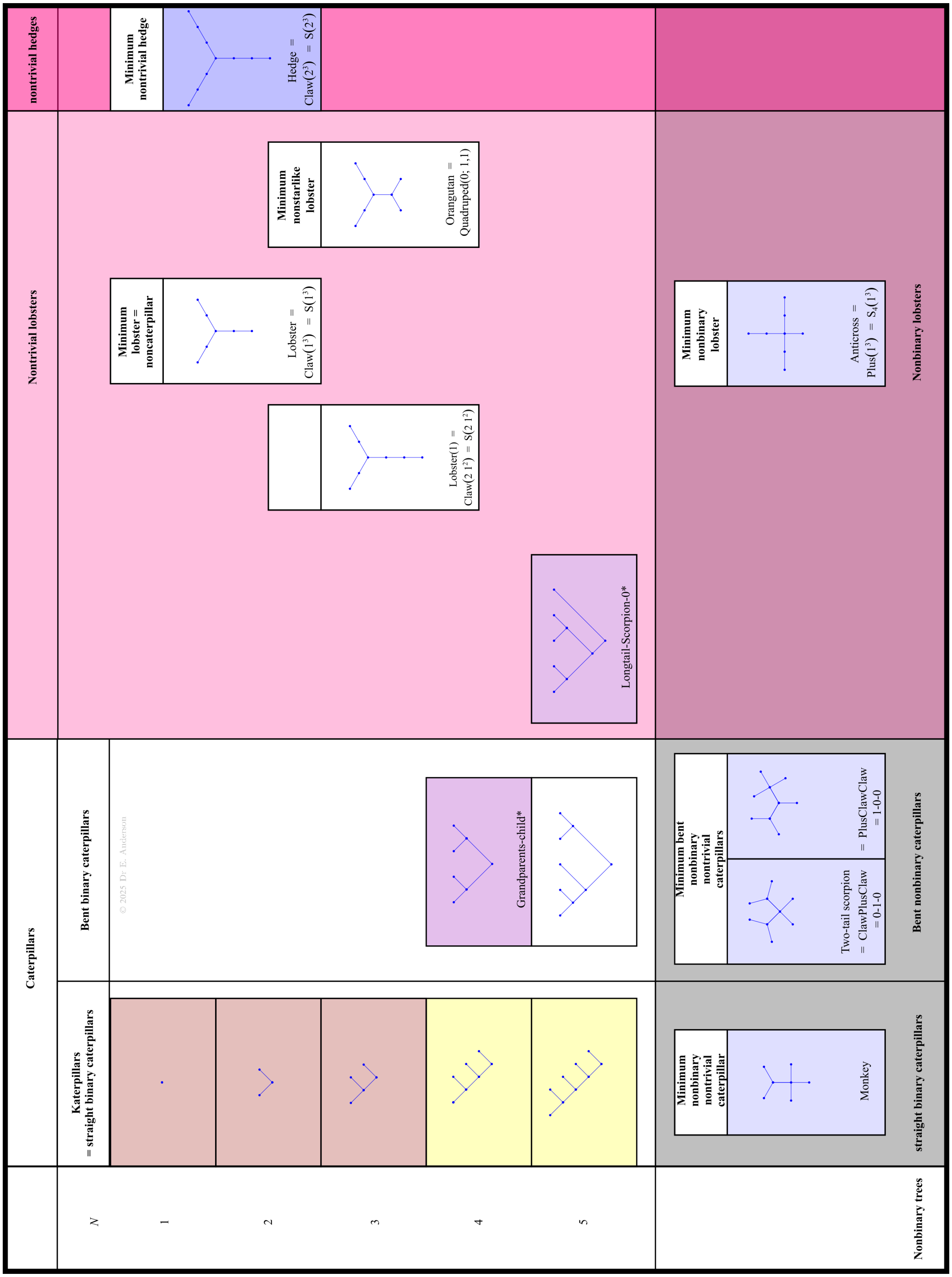
denote arenas of AMB (unlabelled directed rooted) trees.

A.12 Lobsters

Definition 1 A *lobster* is a tree such that all vertices are within distance 2 of some central path subgraph.

Remark 1 Equivalently, it is a tree such that its first defoliation is a caterpillar. Whose own first defoliation is a path graph. Concatenating, a lobster is a tree whose second defoliation is a path.

Definition 2 A *nontrivial lobster* [117] is then a lobster that is not itself a caterpillar. I.e. it now takes precisely 2 defoliations to first encounter a path.



nontrivial hedges
Nontrivial lobsters
Caterpillars
Katerpillars
= straight binary caterpillars
Bent binary caterpillars
N
1
2
3
4
5
Minimum nontrivial hedge
Hedge =
Claw(2^3) = S(2^3)
Minimum nonstarlike lobster
Orangutan =
Quadruped(0; 1,1)
Minimum lobster = noncaterpillar
Lobster =
Claw(1^3) = S(1^3)
Lobster(1) =
Claw(2 1^2) = S(2 1^2)
Longtail-Scorpion-0*
Grandparents-child*
© 2025 Dr. E. Anderson
Minimum nonbinary lobster
Anticross =
Plus(1^3) = S_4(1^3)
Nonbinary lobsters
Minimum bent nonbinary nontrivial caterpillars
Two-tail scorpion
= ClawPlusClaw
= 0-1-0
= PlusClawClaw
= 1-0-0
Bent nonbinary caterpillars
Minimum nonbinary nontrivial caterpillar
Monkey
straight binary caterpillars
Nonbinary trees

Figure 26:

B Combinatorial Miscellany

B.1 Three common Combinatorial counts

Definition 1 The *floor function*

$$\lfloor\ \rfloor \ : \ \mathbb{R} \longrightarrow \mathbb{Z}$$
$$x \longrightarrow \big(\text{the greatest integer} \ \leq x\big) \ .$$

Definition 2 The *Catalan numbers* c_n correspond to the following convolution recurrence relation.

$$c(N) = \sum_{K=1}^{N-1} c(K-1)\,c(N-K) \ . \tag{27}$$

The *half-Catalan numbers* h_n [82, 113], to

$$h(N) = \sum_{K=1}^{\lfloor \frac{N}{2} \rfloor} h(K)\,h(N-K) \ . \tag{28}$$

Remark 1 The *Wedderburn–Etherington (WE) numbers*' counterpart is as follows.

$$w(N) = \sum_{i=1}^{\lfloor \frac{N-1}{2} \rfloor} w(N)\,w(N-K) + \frac{1+\sigma(N)}{4}\, w\left(\left\lfloor \frac{N}{2} \right\rfloor\right)\left(1 + w\left(\left\lfloor \frac{N}{2} \right\rfloor\right)\right) \ . \tag{29}$$

For *parity function* $\sigma(N) = (-1)^N$ is -1 for N odd and 1 for N even. Our $e(N) = w(N)$ for $N \neq 0$.

Remark 2 In each case, the accompanying initial condition is that the $N = 1$ case is 1 .

Remark 3 Each of these can be defined in a number of other equivalent ways [105, 109, 102, 61, 65, 82, 112, 113, 114]. The Catalan numbers occur particularly often in basic Mathematics [106, 112]; they are one of the most widely known citizens of Kallista.

B.2 Graph complements

Definition 3 The *complement* [110, 121] of a simple graph is the result of replacing its edge set with its non-edge set. The *Ramsey presentation* [107, 87, 109, 110] treats edges and non-edges on an equal footing, in the distinguishable-but-meaningless-label sense.

Notation 1 The Ramsey presentation is often depicted with blue edges and red non-edges superposed on the same vertex set (most nicely picked out in a third, and thus neutral, colour).

B.3 Bipartite graphs

Definition 4 A graph is *bi-partite* if its vertex set can be partitioned into two parts, neither of which have any internal edges.

Remark 4 So the only possible edges are mutual ones: from a vertex in one part to a vertex in the other.

Definition 5 A bi-partite graph is *complete* if all such mutual edges feature. Let its parts be of order p and q . Then the standard notation for this graph is $\mathrm{K}_{p,\,q}$.

Example 1 and Naming Remark 7 $\mathrm{K}_{3,3}$ (Subfig 27.a) is the complete bipartite graph with both parts of size 3 [99]. *Utilities graph* [60, 78, 110] is a common alias. For which the following provides Subfig b)'s 'modern' interpretation. Three new houses need connecting to the water mains, power grid and internet. Can you find a way of doing this such that the edges (pipes and cables) do not cross over? This graph first appeared in an attempt to identify the structure of the benzene molecule [59], giving another alias: *Thomsen graph*. $\mathrm{K}_{3,3}$ became very theoretically significant due to the following result.

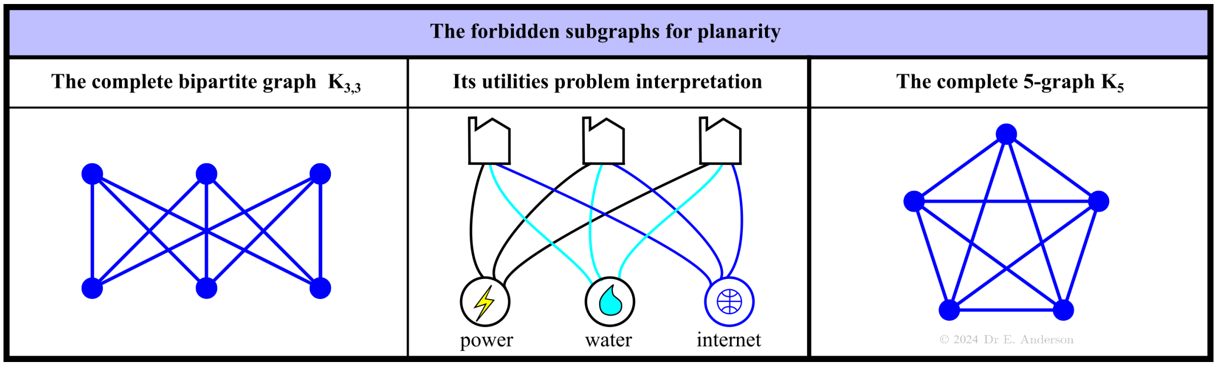


Figure 27:

B.4 Graph (non-)planarity

Theorem 1 (Kuratowski's Planarity Theorem) [62, 79, 83, 107]. A graph is planar iff it does not contain a homeomorph of the complete graph K_5 (Subfig c) or of the complete bi-partite graph $\mathrm{K}_{3,3}$.

Theorem 2 (Wagner–Fáry–Stein Rectilinear-Representation Theorem) [63, 68, 70, 79, 83]. Every planar graph admits a rectilinear edge presentation.

Remark 5 Thereby, the rectilinear presentations that the current Article employs are guaranteed (albeit highly ununique).

Definition 6 A *crossing* in the embedding of a graph is an intersection of edges at other than a vertex.

Definition 7 The *crossing count* of a graph G 's presentation by embedding $\mathfrak{E}$ into $\mathbb{R}^2$ is as follows.

$$\#cr(\mathrm{G}, \mathfrak{E}) \ := \ \#\,(\text{ crossings in embedding } \mathfrak{E} \text{ of graph } G\,) \,.$$

The *crossing number* [87, 109, 79, 67, 103, 104] of a graph G is

$$Cr(\mathrm{G}) \ := \ \min_{\mathfrak{E} \,\in\, \mathfrak{emb}(\mathrm{G})} \#cr(\mathrm{G}, \mathfrak{E}) \,. \qquad (30)$$

Where $\mathfrak{emb}(\mathrm{G})$ is the arena of the embeddings $\mathfrak{E}$ involved in presenting the graph G in $\mathbb{R}^2$.

B.5 Triangulations

Definition 8 A *triangulation* of a polygon is a subdivision of it into triangles. It is a *strong triangulation* if the polygon is itself a triangle.

Remark 6 For then not only the inner faces but also the outer face are triangles. So it is in the sense of all of the faces being triangles. See e.g. [73, 95, 122] for more about triangulations.

B.6 Order Theory

Definition 9 A relation $\preceq$ is a *partial ordering* if it is reflexive, antisymmetric and transitive. A set equipped with a partial ordering is termed a *poset* [84, 75, 89, 110, 80, 118, 102, 100].

Definition 10 A *lattice* [66, 89, 110, 118, 100, 90, 98, 85] is a poset for which each pair of elements a, b possesses both of the following.

i) A *join* (least upper bound), denoted by

$$a \ \wedge \ b \,.$$

ii) A *meet* (greatest lower bound), denoted by

$$a \ \vee \ b \,.$$

C Survey of the Literature

C.1 Aquilanti, Cavalli and Grossi in Molecular Physics

Remark 1 We were asked to start this with the 1986 review [28] – "Method of Trees" for K-sphere harmonics in the N-body context, which is quite well-known in the Molecular Physics literature. So let us start with it.

Remark 2 In Kendall's [27, 38] conceptualization of configuration spaces for for N points in flat d-dimensional space, these K-sphere harmonics can readily be identified as being defined over preshape space. Which is both Topologically and Geometrically a sphere

$$\mathbb{S}^{d\,n\,-\,1}\,.$$

Where

$$n \;:=\; N \;-\; 1\;.$$

And also

$$K \;:=\; n\,d \;-\; 1 \tag{31}$$

specifies which N-body problem in which dimension d corresponds to which K-sphere harmonics. This is very closely related to eigenclustering coordinates. In that these as normalized by the configuration space radius [2] provide valuable coordinatizations of the K-sphere.

Remark 3 The K-spherical harmonics correspond to placing a Laplacian operator on the K-sphere. With further $SO(\,K\;+\;1\,)$ group representation connotations. This also increases the scope of applications of eigenclustering coordinates to Physics and continuum Applied Mathematics more generally.

Remark 4 [28] point to an accompanying Pruning Lemma. This is used in reducing [128] from $SO(\,K\;+\;1\,)$ representations to $SO(K)$ ones. So one is not only calculating K-sphere harmonics but also finding means of inter-relating higher and lower-dimensional such.

Remark 5 [28] use the binary-tree presentation, and yet not the AMB one. They also point to circumstances under which a parallel ternary tree model is useful; this lies outside the context of our own study.

C.2 Smith and the Russian literature's precursors

Remark 1 The above method of trees rests upon two earlier strands of work. Firstly, F.T. Smith's [18] parametrization of internal coordinates, also within the Molecular Physics literature. Secondly, some Russian Methods of Mathematical Physics and Nuclear Physics literature.

Remark 2 One of us was already familiar with Smith's work, from previous 4-body collaborations. By which we knew that the material we were writing up in v1 was not in this work on 'generalized angular momentum'.

With reference to settings in which other than angular momentum in 3-d space is being modelled. Which the previous paragraph's collaborations used for dilational momenta for 4 particles on a line. And for a mixture of these and physical angular momentum in 2-d . This work's scope also extends to $SO(P)$ representations for arbitrary P . This mathematical name is thereby clearer than 'generalized angular momentum'.

Remark 3 As far as we know, the Russian literature on this started with Vilenkin's 1965 book [19] on Methods of Mathematical Physics. This covers the general ternary case, corresponding to the 'polyspherical harmonics' While also explaining that the usual hyperspherical coordinates correspond to what we here call K(N) or P_n-straight. The middling specialization to the binary trees starts with [20]. Which

[2] Alias *hyperradius* elsewhere in the literature [16, 123, 43, 41].

is further developed in e.g. [21], and in [22] specifically for $\mathrm{K}(N)$. And was then further applied to Nuclear Physics in e.g. [23, 24]. [26] subsequently reviewed this work.

Naming Remark 8 We suggest 'ternary-tree harmonics' in place of 'polyspherical harmonics'. 'Binary-tree harmonics' for the current Article's middling case of interest. And then using the third sentence of Remark 3, alongside the clarification that the usual hyperspherical coordinates are tied to a single type of eigenclustering. By which these should not be confused with K-spherical harmonics, or similar: p-harmonics, $\mathbb{S}^p$-harmonics... They are *standard* p-harmonics, resting on every N admitting a unique $\mathrm{K}(N)$ alias P_n-straight eigenclustering network.

C.3 Lim's work in Classical Mechanics

Remark 1 Wikipedia cites Cabral and Diacu [39] in this regard. This actually turns out to refers to a review Article by D. Schmidt in the collection *edited by* Cabral and Diacu. Which in turn rests upon an earlier work by C.C. Lim. Whose earliest work on this topic is however from 1989, i.e. (well) after the work in the previous two subsections. Noting also the year of translation of Vilenkin into English: 1968.

Remark 2 C.C. Lim presents interesting results from the point of view of phase space and applications to Celestial Mechanics. Including presenting specifics for all 3 eigenclustering shapes for $N = 5$. And the $\mathrm{K}(N)$ series, with $\mathrm{K}(6)$ case worked out in further detail.

Remark 3 See Appendix A.7 for a comparison of [45]'s notion of caterpillar with the more general use from the Graph Theory literature.

C.4 The poset height function presentation

Remark 1 None of the above works use the poset height function presentation. S. Sánchez [43] presented figures using just this in 2017.[3]. While Daily [42] wrote a routine to implement review [26]'s analysis at around the same time.

Remark 2 In the Order Theory literature, however, such a presentation has been in use at least since the work of Loday and Ronco [86, 88] around the turn of the millennium; seee.g. [94, 101] for more recent related articles. Some work that this approach builds upon is a lattice on the poset of bracketings (presentation 2), as set up by D. Tamari in the 1950s and 60s [71, 72]. Some of this approach's arenas [124, 29, 91] have come to be quite widely known under the name of *associahedra*. Observe that presentation 2's counting up 1 's already has a natural role for this poset height function. By which Combinatorics – with no reference to mass – can already exhibit this. Thereby, literature that makes no mention of eigenclusterings (or relative Jacobi coordinates) was indeed able to develop this notion first.

It is furthermore an objects-level poset, to the Tamari lattices and associahaedra constituting an arena-level use of Order Theory. With Tamari viewing the objects in question in the Wedderburn–Etherington way.

Naming Remark 9 With literature following Tamari having developed this notion and notation first, let us call it the *Tamari poset height function*. Or, for a truer name, the *additive poset-height function*. Which property the current Review's other height function – a quotient of the preceding – fails to inherit. Giving technical reason in this case to stick with the more absolute of the two.

End Note For equal masses, we have a contributing vertex counter that is aligned with a mass counter. For arbitrary masses, the contributing-vertex counter stays the same, but the mass counter becomes unaligned. For the symmetries by which various groupings of entries share a floor break down. So now generically nothing is on the same floor as anything else.

[3] And pointed out that the diagrammatic presentations in previous Physical literature were not being used to maximal effect

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