Linear Mathematics.

With applications throughout STEM

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Widely-Applicable Mathematics Series A: Improving Understanding of Everything with a pinch of Combinatorics

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Chapter 17

The Rank–Nullity Theorem

17.1 Images and kernels

Structure 1 Let $\mathfrak{V}, \mathfrak{V}'$ be v.s.s over $\mathbb{F}$ and $\mu : \mathfrak{V} \longrightarrow \mathfrak{V}'$ be a Linear map.

The *image* $\mathfrak{Im}_\mu(\mathfrak{V})$ is the subset of the target v.s. $\mathfrak{V}'$ that μ maps to from some part of the source v.s. $\mathfrak{V}$. See Fig 17.1.a).

The *kernel* $\mathfrak{Ker}_\mu(\mathfrak{V})$ is the subset of the source v.s. $\mathfrak{V}$ that μ maps to the zero $\mathbf{0}$ of the target v.s. $\mathfrak{V}'$. See Fig 17.1.b).

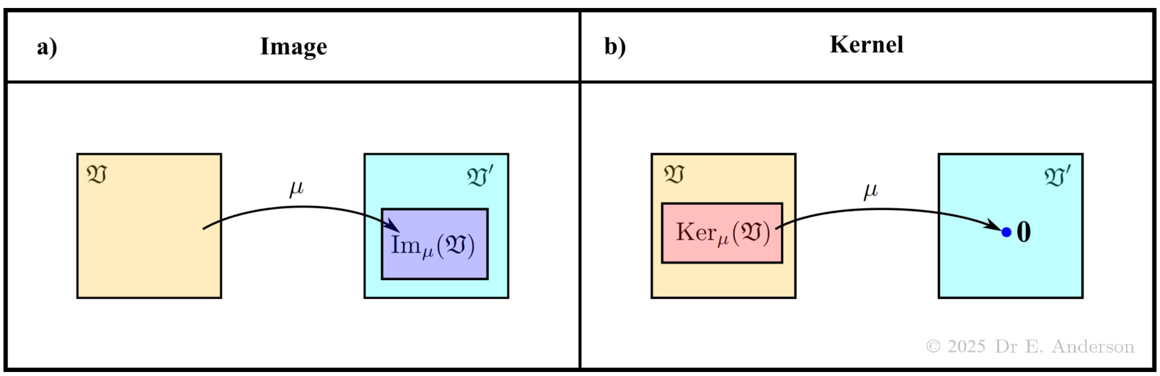

Figure 17.1:

Exercise 1$^-$ Show that

$$\mathfrak{Im}_\mu(\mathfrak{V}) \; < \; \mathfrak{V}' \, , \quad \mathfrak{Ker}_\mu(\mathfrak{V}) \; < \; \mathfrak{V} \, . \tag{17.1}$$

17.2 Rank and nullity

Remark 1 Throughout the current Chapter, dim refers to the Linear notion of dimension, $\mathrm{dim}_{\mathrm{L}}$

Definition 1 The *rank* of μ is

$$\mathrm{rank}(\mu) \; := \; \mathrm{dim}(\, \mathfrak{Im}_\mu(\mathfrak{V}) \,) \, . \tag{17.2}$$

Definition 2 The *nullity* of μ is

$$\mathrm{null}(\mu) \; := \; \mathrm{dim}(\, \mathfrak{Ker}_\mu(\mathfrak{V}) \,) \, . \tag{17.3}$$

Naming Remark 1 'Rank' means a row of soldiers; it was a French term before it became an English one. We know that in its matrix row rank form, rank is a count of the number of LI rows in a matrix.

Naming Remark 2 *Nullity* is another name coined by Sylvester [1]. Its 'null' refers to vectors mapping to zero, now constituting a count of the number of LI vectors in the kernel subspace.

17.3 Examples of rank and nullity

Example 0: The zero maps. Their only matrix representation is the corresponding size of zero matrix, $\mathbb{0}$. Which send all vectors of the source dimensionality n to the zero vector. So their image is just the zero vector. Thus their rank is zero. But the second sentence also means that the kernel is the whole of the source v.s. Thus the nullity of each such map is the corresponding n .

Example 1: The identity maps The $n \times n$ such sends the arbitrary n-vector to itself. Thus the dimension of its image is n . In other words, their rank is the corresponding n . The first sentence also means that the only vector sent to 0 is the zero n-vector, $\mathbf{0}$. Hence each kernel is just $\mathbf{0}$. Thus the nullity of each such map is 0 .

Exercise 2$^-$ a) Describe rank and nullity at the level of matrices.

b) Find the rank and nullity corresponding to each of the following matrices.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} , \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 2 & 2 \end{pmatrix} , \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix} , \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} , \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} , \begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & 2 \end{pmatrix} .$$

Can you spot any pattern in these results?

Exercise 3 Describe rank and nullity at the level of homogeneous LASs and their solutions.

Also explain how, in the inhomogeneous case, $\mathfrak{S}$ol need not be the image under the map μ .

Proposition 1 For square $n \times n$ matrices, zero determinant is a diagnostic for non-maximal rank and vice versa. I.e.

$$\det \boldsymbol{S} = 0 \text{ iff } \operatorname{rank}(\boldsymbol{S}) < n . \tag{17.4}$$

<u>Proof</u> ($\Rightarrow$) det$\boldsymbol{S} = 0$ means LD rows. There are two cases to consider. If > 1 row is nonzero, then we are left with > 1 Linear relations between the target vector's components. So rank$\boldsymbol{S} < n$. But if all the rows are zero, then we have the zero map, whose rank is $0 < n$ also.

($\Leftarrow$) rank $< n$ means that rows are LD. So some row is a LC of other rows, w.l.o.g.

$$a_1 = \sum_{i=2}^{n} \lambda_i a_i . \tag{17.5}$$

But then

$$\det \boldsymbol{A} = \det \left(\sum_{i=2}^{n} \lambda_i a_i, a_2, \ldots, a_n \right) = \sum_{i=2}^{n} \lambda_i \det (a_i, a_2, \ldots, a_n) = \sum_{i=2}^{n} \lambda_i 0 = 0 . \tag{17.6}$$

The first step is by substituting (17.5) in Lagrange's formula for the determinant. The second is by Linearity, and the third uses lemma 3 of Chapter **??**. □

Remark 1 This does not however say by how much the rank is nonmaximum.

17.4 Exercises introducing projection operators

Projectors in general

Naming Remark 3 These [2, 19, 4, 6, 7] are alias *projectors*: a contraction of 'projection operators'.

Exercise 4 Recollect the projection onto a line in 2-d from Chapter **??**.

a) Write this in a coordinate-invariant manner.

b) Show that this is indeed a LM.

c) Show that applying it twice yields the same outcome as applying it once. Give a Geometrical interpretation of this Algebraic property:

$$\boldsymbol{P}^2 = \boldsymbol{P} \quad \text{(idempotency)} \ . \qquad (17.7)$$

Take b) and c) to be defining properties for projectors.

d) Show that this has rank 1, which is nonmaximum for $d > 1$.

What is the corresponding nullity?

Exercise 5 Generalize all of these results to the projection onto
a) a line through the origin in n-d.

b) a p-plane through the origin in n-d for any $p \leq n$.

Exercise 6 Show that each of the current Book's _ *spanning maps* $\mathfrak{Y} \longrightarrow XS(\mathfrak{Y})$ are idempotent. Where X is Linear's L, Affine's A, Convex's C, conic's Ci or Convex-conic's CCi. Are any of these maps Linear?

Exercise 7 Show that LMs do not necessarily preserve LI.

Exercise 8$^-$ Show that the maps sending a Linear product space down to each of its individual factors (posed in Fig **??**.c) are indeed projection maps in the above sense.

Orthogonal projectors

Definition 1 On a v.s. $\mathfrak{V}$ over $\mathbb{R}$, a projector is *orthogonal* [2, 19, 4] if

$$\boldsymbol{P} = \boldsymbol{P}^{\mathrm{T}} \ . \qquad (17.8)$$

Exercise 9 For each orthogonal projector $\boldsymbol{P}$, there is a corresponding orthogonal projector

$$\boldsymbol{P}^{\perp} = 1 - \boldsymbol{P}$$

onto the complement subspace.

a) Check that this is indeed a projector, and an orthogonal one at that.

b) Show that any projection map $\boldsymbol{P}$'s image and kernel are direct complements and orthogonal complements.

Pointer 0 See Exercises **??** for more about projectors in QM, involving the Hermitian counterpart of the above.

17.5 The Rank–Nullity Theorem

Theorem 1 [2, 4, 5, 8] For $\mathfrak{V}, \mathfrak{V}'$ be v.s.s over $\mathbb{F}$ and a LM $\mu : \mathfrak{V} \longrightarrow \mathfrak{V}'$,

$$\operatorname{rank}(\mu) + \operatorname{null}(\mu) = \dim \mathfrak{V} . \tag{17.9}$$

<u>Proof</u> Suppose that $\mu = \mathit{0}$. If μ is injective, then

$$\mathfrak{K} := \mathfrak{K}\mathrm{er}_{\mu}(\mathfrak{V}) = 0$$

and the result holds.

Next suppose that $\mu \neq \mathit{0}$.

LS: μ is an isomorphism. Thus

$$\operatorname{rank}(\mu) = \dim \mathfrak{V} .$$

Suppose that

$$0 < \mathfrak{K} < \mathfrak{V} .$$

Pick a LB

$$\mathfrak{B}_{\mathfrak{K}} : (\boldsymbol{k}_k)_{k=1}^{l} \text{ for } \mathfrak{K} . \tag{17.10}$$

Extend to a LB $\mathfrak{B}_{\mathfrak{V}}$ of $\mathfrak{V}$ by adjoining

$$(\boldsymbol{w}_j)_{j=1}^{r} .$$

Then

$$\mathfrak{V} = \mathrm{LS}\,(\, k_k,\, w_j \,) .$$

So applying μ to both sides,

$$\mu(\mathfrak{V}) = \mu\,(\,\mathrm{LS}\,(\,\boldsymbol{k}_k, \boldsymbol{w}_j)\,) = \mathrm{LS}\,(\,L\,(\boldsymbol{k}_k)\,,\ \mu\,(\boldsymbol{w}_j)\,) = \mathrm{LS}\,(\,\boldsymbol{0},\,\mu\,(\boldsymbol{w}_j)\,) = \mathrm{LS}\,(\,\mu\,(\boldsymbol{w}_j)\,) . \tag{17.11}$$

The penultimate step uses the defining property of kernel. We thus obtain an r-element LSing set.

LI: Let

$$(a_j)_{j=1}^{r} \in \mathbb{F} . \tag{17.12}$$

Also suppose that

$$0 = \sum_{j=1}^{r} a_j\,\mu\,(\boldsymbol{w}_j) = \mu\left(\sum_{j=1}^{r} a_j\,\boldsymbol{w}_j\right) := \mu\,(\boldsymbol{v}) . \tag{17.13}$$

So this $\boldsymbol{v} \in \mathfrak{K}$. Thus the $\boldsymbol{k}_k$ are LSing:

$$\boldsymbol{v} := \sum_{k=1}^{l} b_k\,\boldsymbol{k}_k \tag{17.14}$$

for some coefficients b_k .

So equating (17.13) and (17.14),

$$\sum_{j=1}^{r} \lambda_j \, \boldsymbol{w}_j \; - \; \sum_{k=1}^{l} \nu_k \, \boldsymbol{k}_k \; = \; 0 \; . \tag{17.15}$$

But the $\boldsymbol{k}_k$ and $\boldsymbol{w}_j$ collectively form a LB of $\mathfrak{V}$. Thus they are LI. So all the a_j and b_k must be 0 . Our required LI consequently holds.

Together, LI and LS $\Rightarrow$ an LB for $\mu(\boldsymbol{v})$. $\Box$

Exercise 10 Express the rank–nullity theorem in terms of matrices.

Remark 1 See Fig 17.2 for an underpinning Algebraic diagram.

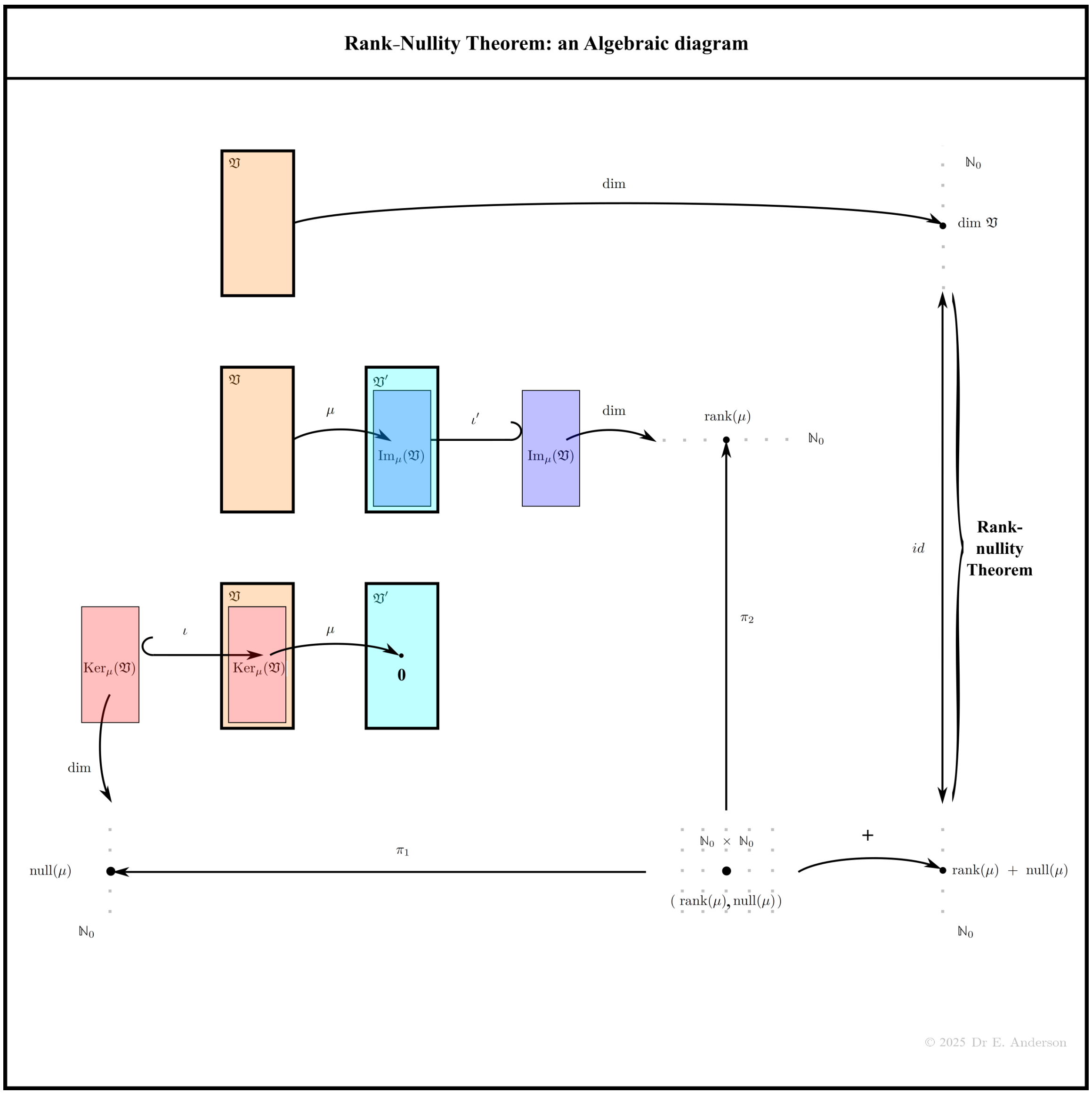

Figure 17.2:

Corollary 1 (Rank inequality)

$$\mbox{rank}(\mu) \;\leq\; \mbox{dim}\,\mathfrak{V}\;. \qquad (17.16)$$

This is saturated only if $\mathfrak{Ker}_{\mu}(\mathfrak{V}) \;=\; \mathbf{0}$: trivial kernel.

Exercise 11$^-$ Prove this.

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Naming Remark 4 The rank–nullity theorem is often called the *fundamental theorem of Linear Algebra*, or the first part thereof [3, 9]. When there is a preceding zeroth part, it is Sec **??**'s dimension theorem. Let us await the introduction of every individual part before listing larger collections in Chapter **??**. I instead elect to call the dimension theorem *FuToLA*, the rank–nullity theorem *FuToLAM*, and more extended versions *FuToLAM+*.

The second of these almost follows Axler [8], who uses 'of Linear Maps'. I say 'of Linear Algebra maps' as part of investigating whether it is tenable [25] to use 'fundamental theorem' solely with reference to subject areas. More precisely, for what intellectually deserve to be subject areas, whether or not other people, communities or institutions treat them as such. The third of these names invites postceding with 'in the sense of' this or that Author, or this or that unified conceptual guiding principle...

Naming Remark 5 Dym [6] calls the rank–nullity theorem the *principle of conservation of dimension*.

For any Linear mapping acts on every LB vector of the source space. This sends the LB either to an LI set or to an LD set. The LI case then enjoys dimensional conservation, in the sense that the *image* has the same dimensionality as the source. N.B. that it is not conservation in the sense of relating the *target* dimension to the source dimension.

To complete the argument, the LD case experiences dimensional suppression, as is clearly visible in every projection map. But nullity is the quantifier of this suppression!

Thus, more precisely, the rank–nullity theorem is *a quantifier of extent of* conservation of dimension *under the map in question*. This gives a unifying conceptual reason to pair the dimension theorem and the rank–nullity theorem. I.e. that they form an (extent of conservation of) dimension theorem pair!

Exercise 12 Show that

$$\mathfrak{V} \;=\; \mathfrak{Ker}(\mu) \;\oplus\; \mathfrak{Im}\,(\mu^*)\;\;.$$

[This is a structural sum whose count returns the rank–nullity theorem in the finite case. It is a further piece of some FuToLAM+.]

Exercise 13 What insights does the rank–nullity theorem afford to Exercises 8 and 9 of Chapter **??**?

17.6 Relative Rank–Nullity Theorem

Theorem 2

$$\mbox{null}(\mu) \;-\; \mbox{rank}(\mu) \;\;=\;\; \mbox{dim}\,\mathfrak{V} \;-\; \mbox{dim}\,\mathfrak{V}'\;. \qquad (17.17)$$

Remark 1 See Fig 17.3 for an underpinning Algebraic diagram. Observe that μ is included upstairs since everything there refers to it. But not downstairs, where nothing does.

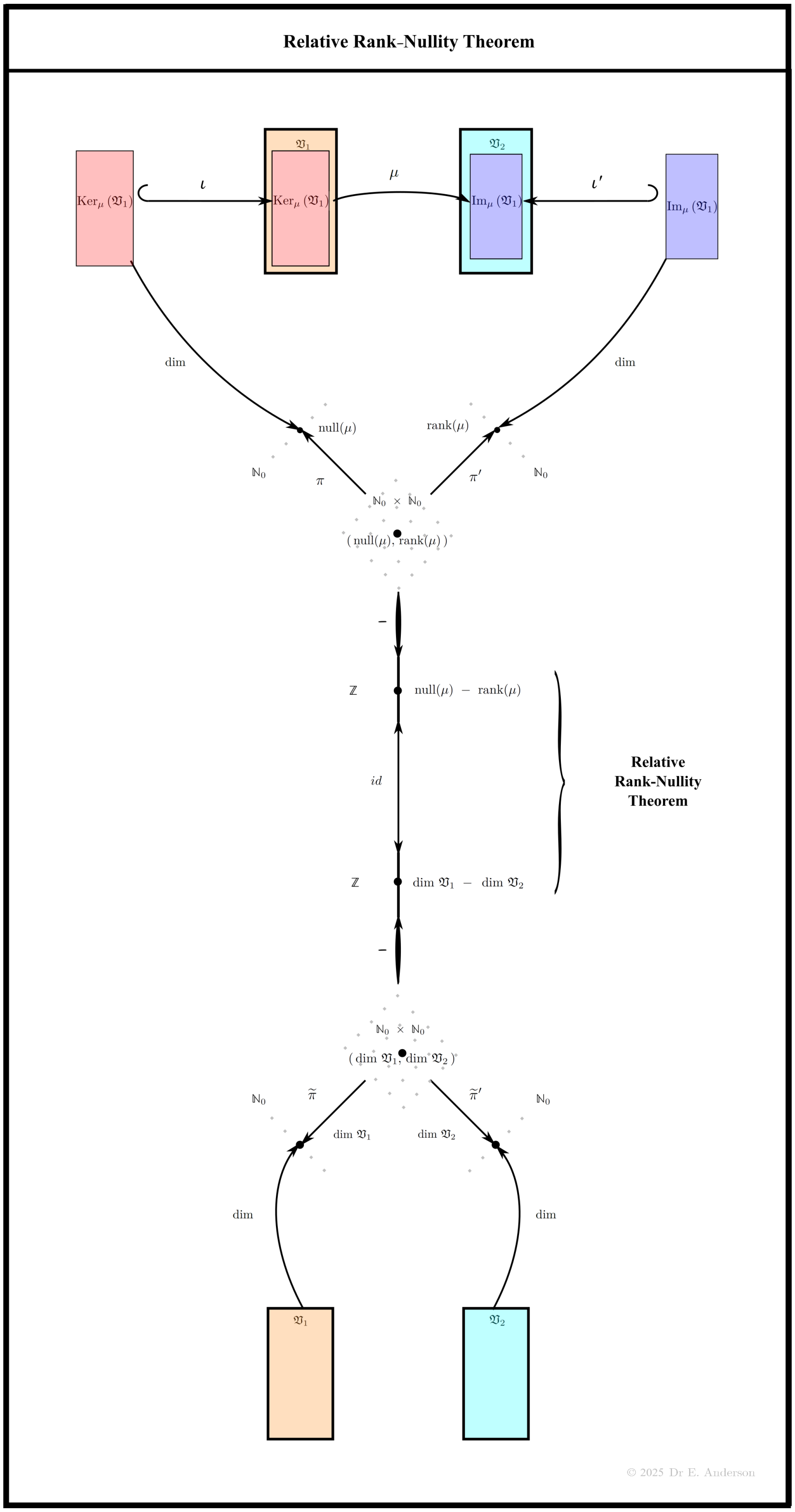
Relative Rank-Nullity Theorem
$\mathrm{Ker}_\mu(\mathfrak{V}_1)$
ι
$\mathfrak{V}_1$
$\mathrm{Ker}_\mu(\mathfrak{V}_1)$
μ
$\mathfrak{V}_2$
$\mathrm{Im}_\mu(\mathfrak{V}_1)$
ι'
$\mathrm{Im}_\mu(\mathfrak{V}_1)$
dim
dim
null(μ)
rank(μ)
$\mathbb{N}_0$
π
π'
$\mathbb{N}_0$
$\mathbb{N}_0 \times \mathbb{N}_0$
(null(μ), rank(μ))
−
$\mathbb{Z}$
null(μ) − rank(μ)
id
Relative Rank-Nullity Theorem
$\mathbb{Z}$
dim $\mathfrak{V}_1$ − dim $\mathfrak{V}_2$
−
$\mathbb{N}_0 \times \mathbb{N}_0$
(dim $\mathfrak{V}_1$, dim $\mathfrak{V}_2$)
$\mathbb{N}_0$
$\widetilde{\pi}$
$\widetilde{\pi}'$
$\mathbb{N}_0$
dim $\mathfrak{V}_1$
dim $\mathfrak{V}_2$
dim
dim
$\mathfrak{V}_1$
$\mathfrak{V}_2$
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Figure 17.3:

17.7 Co-kernel and co-nullity

Exercise 14 Use the LAS perspective to justify that image and kernel are a dual pair of notions. And consequently that rank and nullity are as well.

Naming Remark 6 *Co-kernel* is thus an alias for image. *Co-image* for kernel. *Co-nullity* for rank. And *co-rank* for nullity. These 'co-' prefixes indeed carry further categorical significance...

Notation 1 Redenote our source space $\mathfrak{V}$ by $\mathfrak{V}_1$ and our target space $\mathfrak{V}'$ by $\mathfrak{V}_2$. Follow through by using

$$\mathrm{null}_2(\mu) \ := \ \dim \text{co-ker}_\mu(\mathfrak{V}) \ = \ \dim \mathfrak{Im}_\mu(\mathfrak{V}) \ = \ \mathrm{rank}(\mu) \ ,$$

to

$$\mathrm{null}_1(\mu) \ := \ \dim \mathfrak{Ker}_\mu(\mathfrak{V}) \ = \ \mathrm{null}(\mu) \ .$$

This effectively corresponds to choosing to conceptualize in terms of the 'kernel' half of our dual.

17.8 Manifestly dual form of the relative rank–nullity theorem

Remark 1 Let us now bring to the forefront the midline reflection symmetry that one can espy within Fig 17.3, in the form of a duality: Fig 17.4. Now deploying the notation

$$\mathfrak{V}_+ \ = \ \mathfrak{V}_1 \ = \ \mathfrak{V} \ , \ \ \mathfrak{V}_- \ = \ \mathfrak{V}_2 \ = \ \mathfrak{V} \ .$$

Theorem 3

$$\Delta \, \mathrm{null}(\mu) \ = \ \Delta \dim \mathfrak{V} \ .$$

Where

$$\Delta a \ := \ a_+ \ - \ a_-$$

is the *relative difference* of the quantities $a_\pm$. For $a =$ each of $\mathrm{null}(\mu)$ and $\dim \mathfrak{V}$.

<u>Proof</u> 1) Take the rank–nullity theorem with $+$-subscript.

2) Split up the target space according to

$$\mathrm{rank}_+(\mu) \ + \ \mathrm{null}_-(\mu) \ = \ \dim \mathfrak{V}_- \ . \tag{17.18}$$

Make $\mathrm{rank}_+(\mu)$ the subject of each of 1) and 2), and equate. I.e.

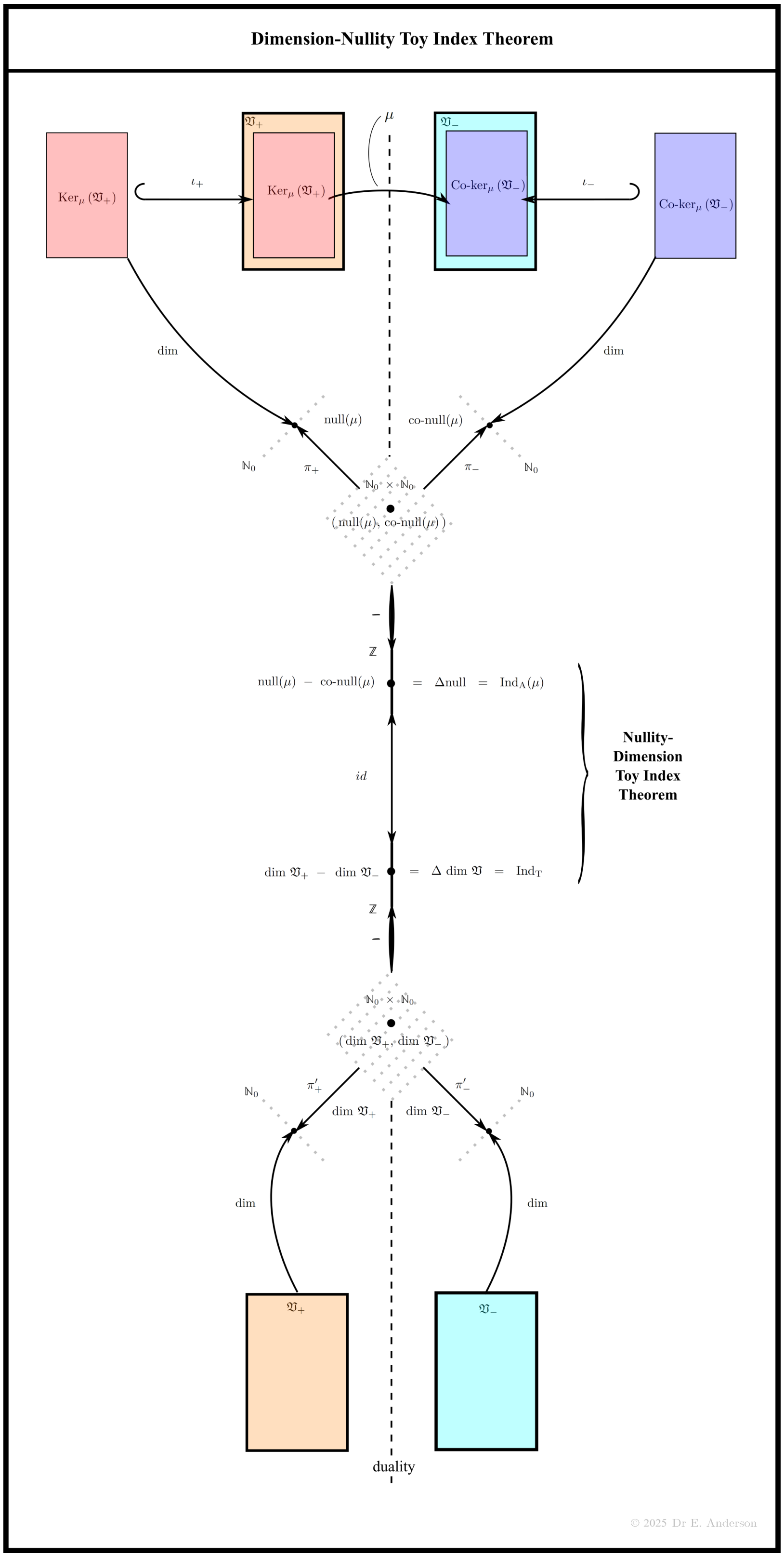
$$\dim \mathfrak{V}_+ \ - \ \mathrm{null}_+(\mu) \ = \ \mathrm{rank}_+(\mu) \ = \ \dim \mathfrak{V}_- \ - \ \mathrm{null}_-(\mu) \ . \tag{17.19}$$

Finally rearrange to form differences of like quantities. □

Exercise 15$^-$ Reprove the relative rank–nullity Theorem directly, i.e. without first reformulating it in terms of index.

Exercise 16 Reprove the rank–nullity theorem using the theory of LASs. Also justify the current Section's proof's 'split as'.

Exercise 17$^-$ Identify the graphs underlying the previous three figures.



Dimension-Nullity Toy Index Theorem
Ker$_\mu$ ($\mathfrak{V}_+$)
ι_+
$\mathfrak{V}_+$
μ
$\mathfrak{V}_-$
Co-ker$_\mu$ ($\mathfrak{V}_-$)
ι_-
dim
null(μ)
co-null(μ)
$\mathbb{N}_0$
π_+
π_-
$\mathbb{N}_0 \times \mathbb{N}_0$
(null(μ), co-null(μ))
$-$
$\mathbb{Z}$
null(μ) $-$ co-null(μ) $=$ Δnull $=$ Ind$_{\mathrm{A}}$(μ)
id
Nullity-Dimension Toy Index Theorem
dim $\mathfrak{V}_+$ $-$ dim $\mathfrak{V}_-$ $=$ Δ dim $\mathfrak{V}$ $=$ Ind$_{\mathrm{T}}$
(dim $\mathfrak{V}_+$, dim $\mathfrak{V}_-$)
π'_+
π'_-
dim $\mathfrak{V}_+$
dim $\mathfrak{V}_-$
duality
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Figure 17.4:

17.9 The Dimension–Nullity Toy Index Theorem

Pointer 1 As already mentioned in Book 0, an *index theorem* is one by which some analytically-computable difference – the analytic index Ind_{A} – coincides with some a priori harder to compute Topologically-significant quantity – the Topological index Ind_{T} . Furthermore, in many interesting cases, Ind_{A} is manifestly $\mathbb{Z}$-valued, while this is not a priori obvious for Ind_{T} . (Theoretical Physicists among the Readers may benefit from Dirac's Topological quantization condition [10, 11] being mentioned at this point.) Theorem 3's relative differences can also be interpreted as indexes, as also marked on Fig 17.4, yielding the following final form for the rank–nullity theorem.

Theorem 4

$$\text{Ind}_{\text{A}}(\mu) \ = \ \text{Ind}_{\text{T}} \ . \tag{17.20}$$

Remark 2 At the level of Linear Algebra,

$$\begin{array}{lllll} \text{null} & = \ \text{null}_+ & = \ \dim \mathfrak{Ker}_\mu(\mathfrak{V}) & & = \ \#(\text{ LI solutions } v \text{ of the equation } \mu(v) \ = \ 0\,) \\ \text{rank} & = \ \text{null}_- & = \ \dim \mathfrak{Co}\text{-}\text{ker}_\mu(\mathfrak{V}) & = & \#(\text{ LI conditions required on } w \text{ for } \mu \ = \ w \text{ to be well-determined}\,) \ . \end{array}$$

Pointer 2 These intuitions in fact carry on somewhat further, into the realms where differential operators, and other more general operators, play the role of μ .

Naming Remark 7 Since we have been conceptualizing in terms of the 'kernel' half of our dual for several Sections now, the root name 'rank–nullity theorem' has ceased to be suitable. Substituting in rank's new name 'co-nullity' has two shortcomings: repetitiveness and failure to identify the index on the other side of the equation. Instead, we identify this theorem to be of the 'index theorem' technical type. And then precede this with 'dimension-nullity', meaning that the theorem equates a dimension index to a nullity index. Within which nullity index the co-nullity, i.e. rank, is subtracted off. Our further qualification as a but a 'toy' index theorem is justified below.

Pointer 3 In the rank–nullity toy index theorem, firstly concatenating definitions following e.g. Ghrist [23],

$$\text{Ind}_{\text{A}}(\mu) \ := \ \dim \mathfrak{Ker}_\mu(\mathfrak{V}) \ - \ \dim \mathfrak{Co}\text{-}\text{ker}_\mu(\mathfrak{V}) \ . \tag{17.21}$$

This combination recurs in further Mathematics along the lines of Pointer 2. It turns out to yield a finite answer even in some instances for which the 2 quantities being subtracted are individually infinite.

Remark 3 Secondly,

$$\text{Ind}_{\text{T}}(\mu) \ := \ \Delta\text{dim}\,\mathfrak{V} \ = \ \dim \mathfrak{V}_1 \ - \ \dim \mathfrak{V}_2 \ = \ \dim \mathfrak{V}_{\text{source}} \ - \ \dim \mathfrak{V}_{\text{target}} \ .$$

Where the second step form renders manifest the Δ . While the third begins to relate the quantity to the direction of the map μ .

In this case, Ind_{T} is in fact easier to compute than $\text{Ind}_{\text{A}}(\mu)$. For $\dim \ = \ \dim_{\text{L}}$ is in principle straightforward to count out for finite v.s.s. Without any need to evoke a map μ between these. Let alone to consider any detailed properties of the associated $\text{LAS}(\mu)$. While both indices manifestly $\in \ \mathbb{Z}$.

Remark 4 These are substantial limitations on the rank–nullity toy index theorem being a useful example of index theorem!

It is however an upgrade on Book 0's use of the *ZIPHoN theorem* as an example. I.e. the *Zero-Index Planar-Hamiltonian Necessity Theorem* reconceptualization and renaming of what has hitherto been referred to as the *Grinberg* (or *Grinberg–Kozyrov*) Theorem [13, 20]. For this equates one index to zero, rather than relating two quantities from different branches of Mathematics

In contrast, the nullity-dimension toy index theorem does equate 2 generally-nonzero indexes. And yet it is such that the usually harder index takes a form so trivial that it is now easier to compute than the usually simpler index. And so trivial that the usually harder quantity can scarcely be said to be Topological: relative Linear dimension can certainly be understood without evoking more than just Linear Algebra. This one is also without a surprising forcing of the harder quantity to take $\mathbb{Z}$-values, since this is already a priori obvious from its Linear-Algebraic interpretation.

Remark 5 As regards e.g. Dym's pairing with a dimension theorem, while the version he evokes (**??**) is not of reflective proto-index form, another version (**??**) is. I.e.

$$0 \ = \ \Delta \, \mathrm{dim}(\, R(\mu) \,) \ . \tag{17.22}$$

Which can be viewed as another null-index theorem equating an analytic object to the trivial Topological object, 0 .

♠ ◇ ♣

Pointer 4 All in all, we are still in need of further ports of call before arriving at index theorems that *function* in the manner that the deep index theorems do. Our next attempts are in Subvolume B's last Chapter, using the full scope afforded by Calculus. These involve the *winding number* and the *Poincaré index*.

Pointer 5 Global Geometry has more, in particular the *Gauss–Bonnet* [16, 12, 21, 24] and *Riemann–Roch* [18] *Theorems* can be viewed in this way.

Pointer 6 The deep *Atiyah–Singer Theorem* [15] is then a unification of these last two, which can be further generalized in multiple ways [17]. Here the Linear-Algebraic map μ has been replaced by a certain class of elliptic differential operators.

17.10 Pointers to other generalizations of the rank–nullity theorem

Remark 1 The rank–nullity theorem can be viewed as the count corresponding to Linear Algebra's realization of the first isomorphism theorem (whose Group Theory counterpart is the theorem in Chapter 27 of Book 0). This carries classification theorem connotations.

Open Exercise 18 Conduct a conceptual analysis of the similarities and diferences between the Linear Algebra and Group Theory versions of this theorem.

[This repertoire will be further extended once Abstract Algebra is in play.]

Pointer 7 The rank–nullity theorem can be recast as an instance of *splitting lemma* for a *short exact sequence*,

$$\mathbf{0} \ \longrightarrow \ \mathfrak{Ker}_\mu(\mathfrak{V}) \ \hookrightarrow \ \mathfrak{V} \ \longrightarrow \ \mathfrak{Im}_\mu(\mathfrak{V}) \ \longrightarrow \ \mathbf{0} \ .$$

Or

$$\mathbf{0} \ \longrightarrow \ \mathfrak{Ker}_\mu(\mathfrak{V}) \ \hookrightarrow \ \mathfrak{V} \ \longrightarrow \ \mathfrak{Co}\text{-}\mathrm{ker}_\mu(\mathfrak{V}) \ \longrightarrow \ \mathbf{0} \ .$$

See Secs **??** and **??** for an Abstract Algebra generalization and this Remark's co-homological significance respectively. This Pointer is indeed the current Section's boundary fence between Algebraic and Topological applications.

Pointers 8 and 9 En route to Topology, the following core topics revisit the RNT.

1) Differential Geometry's *pre-image theorem* alias *submersion level set theorem* [22].

2) Differential Topology's transversality ([14, 22] and Sec **??**), which concerns how hypersurfaces intersect and rests upon 1) and related embeddings.

In both 1) and 2), an associated structural sum plays a part. And the relative version's dimension–nullity theorem takes centre stage.

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[25] Discussions between S. Sánchez and E. Anderson on Geometry and Combinatorics in 2017-2020. Many aspects of these can be found in [26], the current Book, my other forthcoming book *The Structure of Flat Geometry*, and the small online encyclopedias OEAGoT and OEToMA which are being started in 2025-2026.

[26] E. Anderson, *Applied Combinatorics*, *Widely-Applicable Mathematics. A. Improving understanding of everything with a pinch of Combinatorics.* **0**, https://conceptsofshape.space/applied-combinatorics/ (2022). Referred to in the text as Book 0.