Linear Algebra of Cyclic Quadrilaterals:
Ptolemy, Diagonal, Area, and Circumradius Formulae

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Abstract

We give an overview of recent work on the Linear Algebra of small-N body problems. Some of which has given new Flat Geometry Theorems and proofs. For triangles, this yields multiple nicely-compatible 3-d matrices. But for quadrilaterals, their counterparts turn out to be everything between 3- and 9-d. Quadrilaterals are additionally minimum for various further kinds of Flat Geometry matrix to arise.

The current Article contributes a systematic Linear Algebra treatment of cyclic quadrilaterals. By abstracting matrices from Ptolemy's Theorems and Brahmagupta's formula, which then also enter diagonal-length and circumradius formulae. And studying their properties and interactions.

All Ptolemy matrices are involutions, while all Brahmagupta matrices are not quite. The 3 Ptolemy sides matrices commute with each other. The 2 Brahmagupta factor matrices commute with each other, and with one Ptolemy sides matrix. While the original Brahmagupta matrix commutes with all Ptolemy sides matrices and with the Lagrange projector.

Each of the 3 Ptolemy sides matrices can be given an eigenbasis that is aligned with that of the underlying Lagrange projector matrix. Or with that of a Brahmagupta factor matrix. But not with both at once. This is related to the corresponding eigenvalues partitioning each matrix's eigenspace in a different way. By which eigenspace alignment does not occur either. Also the vectors entering our overall Brahmagupta quadratic form are identified to be Ptolemy side matrix eigenvectors.

We provide a supporting account of eigentheory, including some new notions and names. Since the above interplay leads to commutator algebra posets containing competing lattices of zero commutator algebras, we also include some supporting Order Theory notions and examples.

The above compatibilities are rather more limited than those between the triangle's matrices. A subsequent work shows that the Bretschneider sides-matrix at the core of Bretschneider's second convex-quadrilateral area formula has better properties and compatibilities than the Brahmagupta matrices.

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1 Introduction

1.1 Applying Linear Algebra *in detail* to Euclidean Geometry

We have set up a program [108, 113, 125, 128, 105, 106, 109, 107, 111, 122, 123, 124, 126, 127, 129, 130, 145, 131, 132, 133, 149, 139, 140, 143, 144, 147, 134, 135, 138, 136, 137, 146, 150, 148, 151] of systematically applying Linear Algebra in detail to small-N body problems [11, 12, 16, 26, 33, 37, 43, 48, 51, 52, 50, 56, 57, 56, 60, 56, 64, 65, 67, 68, 69, 102, 118]. Some of which has given new Flat Geometry Theorems and proofs concerning triangles and quadrilaterals [15, 19, 20, 28, 30, 34, 36, 46, 59, 70, 100, 150]. We however say 3- and 4-body problem when results turn out to be independent of spatial dimension [125, 128, 105, 129, 131, 132, 133, 3, 140, 145] is preferable.

The below account renders clear that *in detail* entails a distinct and rather more fruitful venture than the following. The past few decades' common practice of using Linear Algebra as a *general framework for* Euclidean Geometry [28, 83, 70, 84]. I.e. we are in the business of finding *specific* matrix (or other Multi-Linear) formulations for individual foundationally-useful Theorems. And then finding and exploiting *specific* Linear-Algebraic properties of these matrices (and other Multi-Linear objects). So as to reach further Geometrical, Topological, and Shape-Theoretic [37, 42, 48, 51, 62, 67, 103, 105, 107, 111, 150, 153] conclusions.

So for instance [123] studies what had hitherto been known as the triangle's *Heron(–Euler–Buchholz) matrix* [2, 10, 53]. Revealing it to be the same matrix as in the cycle of cosine rules and the cycle of triangle inequalities. By which we term it the *fundamental triangle matrix*,

$$\boldsymbol{F} := \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} . \tag{1}$$

A name that is further vindicated by its eigentheory yielding [108, 113, 126], Hopf's little map

$$\mathit{H} : \mathbb{S}^3 \longrightarrow \mathbb{S}^2 \tag{2}$$

[21, 77, 79, 144]. And by $\boldsymbol{F}$ admitting various further Algebraic and Representation-Theoretic interpretations [127].

[125, 105, 126] study the N-body problem's *Lagrange matrix* $\boldsymbol{L}$. And its sharpening to the *positions-to-separations-basis projector*, $\boldsymbol{P}$ (for equal point-or-particle masses, the two coincide). With detailed consideration of the equal-masses 3-body case,

$$\boldsymbol{L} = \boldsymbol{P} = \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} . \tag{3}$$

While [108, 126, 127] also introduce the *Apollonius matrix.* This encodes the cycle of Apollonius Median-Length Theorems. And rescales to the *sides–medians involutions matrix*,

$$\boldsymbol{J} := \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} . \tag{4}$$

For the simplest nontrivial case of the triangle with equal vertex masses, $\boldsymbol{F}$, $\boldsymbol{J}$ and $\boldsymbol{P}$ furthermore turn out to commute with each other [108, 126]. This is in part underlied by the linear dependency (LD)

$$\boldsymbol{F} = \boldsymbol{J} - \boldsymbol{P} . \tag{5}$$

[127]. The involutivity and $\boldsymbol{F}$-commutativity properties of $\boldsymbol{J}$ furthermore conspire to give a new proof of Heron's formula for medians [108, 113, 134, 126].

Overall, quite a lot of the theory of the triangle with equal vertex masses is controlled by the above 3 matrices. (Or one's pick of [127] an independent pair of them.) Obtaining Hopf's little map furthermore compounds to giving a new derivation of Kendall's Little Theorem [42, 48, 67]. I.e. that that the shape space of triangles is topologically a sphere, as equipped with the standard spherical metric. In fact, Hopf's little map arises by *double*-diagonalizing $\boldsymbol{F}$. Meaning furthermore expressing its eigenvectors in terms of the coordinate vectors arising from diagonalizing $\boldsymbol{P} = \boldsymbol{L}$. Which have been widely called *relative Jacobi vectors*, though a truer name [110] for 'relative Jacobi' is '*eigenclustering*' [125, 105]. In contrast, Smale's Little Theorem [37] – the earlier topological part of Kendall's Little Theorem – can be rederived just from diagonalizing $\boldsymbol{F}$ without evoking eigenclustering vectors.

This means that everything in the previous paragraph can be reproven as following from just Heron's formula. So this 2000-year-old formula in fact contains some quite deep secrets that Linear Algebra *in detail* reveals. That this body of results arises from so small and long-known an input as Heron's formula is interesting enough for the following three robustness tests to have been designed for it [110].

1.2 Three robustness tests for the new matrix theory of the triangle

S.1) What happens if the triangle's vertices are ascribed non-equal masses? Now the nontrivial eigenclustering vector passes from median to arbitrary Cevian [5, 7, 30, 34, 70, 100, 135]. The Apollonius matrix correspondingly becomes the *Stewart matrix* [134]. I.e. the matrix encoding the cycle of Stewart Cevian-Length Theorems [8, 36, 100, 134].[1]

So far, this analysis has given [134, 137] 2 new 1-parameter families of strongly Heron-like formulae, meaning that they manifest $\boldsymbol{F}$ itself. Alongside various even larger families that are weakly Heron-like: manifesting matrices *tensorially-related to* $\boldsymbol{F}$ [134, 135]. With special roles played by the 'equi-Cevian', 'altilarity' and concurrent-Cevian families of triples of triangle co-transversals. In which the plain, medians [15, 59, 75] and altitudes [46, 82] Heron's formulae play distinguished-point roles (bounding, extremal...)

S.2) What happens to flat space's matrix theory of triangles upon passing to spherical or hyperbolic triangles? This shall be described shortly in [150].

S.3) Is there a comparable matrix theory of quadrilaterals? It quickly becomes apparent that there is not [113]. One reason for this is that triangles involve mostly 3-indexed quantities. While all of 2- to 9-indexed quantities crop up for quadrilaterals. By which the corresponding matrices (or higher arrays) encoding these cease to be multiplicatively compatible with each other.

This is underlied by, for instance, sides and separations no longer coinciding for quadrilaterals; there are 4 and 6 of these respectively. By the shape space of quadrilaterals' isometry group being a quotient [76] of $SU(3)$ [98]. Giving an 8-fold [80, 31] of shape quantities [98]. Plus [60] what is identified in [113] to be a Casimir [39] shape quantity, so for some purposes one has 9 shape quantities. While separation-pairs enter some results, usefully halving the 6 to a 3 [131].

The previous three paragraphs also hold for tetrahaedrons in place of quadrilaterals. Though many of the matrices do not coincide for these two cases.

1.3 Pivoting to a systematic study of the Linear Algebra of quadrilaterals

We thus find ourselves in a much more generic situation in considering quadrilaterals (or tetrahaedrons, or any subsequent N-body problems). As such, we shift focus from our outset position [113] in treating these. Which was understanding why common quadrilateral area formulae do not give a route to the first nontrivial case of Hopf's generalized map [22]

$$H \ : \ \mathbb{S}^{2p+1} \longrightarrow \mathbb{CP}^p \ . \tag{6}$$

Namely,

$$H \ : \ \mathbb{S}^5 \longrightarrow \mathbb{CP}^2 \ . \tag{7}$$

[1] Though we have also argued that [110, 135] a truer name for 'Cevian' itself is 'triangle co-transversal'. So an even truer name for Stewart's Theorem is *Triangle Co-transversal-Length Theorem*!

Or the generalized Smale and Kendall results that the shape space of N-a-gons is $\mathbb{CP}^{N-2}$. To the position of trying to systematically understand the much richer Linear Algebra structure of quadrilateral Theorems in their own right.

So far, we have studied quadrilateral eigenclusters [105] (certainly previously known). The Linear Algebra of [131] Euler's Quadrilateral Theorem (better Euler's 4-Body Theorem) [9, 72, 85, 96]. Revealing it to be part of a binary-tree valued set of Theorems [131, 132, 133] via the classification of eigenclustering networks for each N [130]. Starting by giving a new tie between Apollonius Theorem and Euler's 4-Body Theorem [131], as the 2-path P_2 and the $\mathrm{P}_{3\text{-bent}}$ alias H-eigencluster cases. Alongside a new $\mathrm{P}_{3\text{-straight}}$ alias K-eigencluster counterpart [132]. The separations-level Linear Algebra [139] of Ptolemy's Theorem and inequality [3, 30, 34, 46, 81, 59, 70, 100, 150]. And the Linear Algebra [113, 147, 143, 150] of Brahmagupta's [4, 34, 46, 59, 88] and one of Bretschneider's [13, 14, 18, 24, 46, 59] area formulae.

We shall gradually uncover reasons why the following is a useful order in which to start a systematic sweep through quadrilaterals' Linear Algebra.

0) The Lagrange projector [125, 105].

1) Eigenclusterings arising from the network ambiguities in picking eigenvectors for the previous [105, 129, 130].

2) Euler's 4-Body Theorem, corresponding to the H-eigenclustering, and its K-eigenclustering counterpart: ELETs (Eigenclustering-Length Exchange Theorems).

3) Ptolemy's Theorem, the second form of Ptolemy's Theorem, and the diagonal-length formulae, all for cyclic quadrilaterals. And Ptolemy's 4-body inequality, for arbitrary 4-body configurations. These are covered in [139] and the current Article, with interplay with 1) in [141] and with 2) in [140].

4) Brahmagupta's cyclic-quadrilateral area formula, and cyclic-quadrilateral circumradius formulae (including Parameshvara's). These are also covered in the current Article.

5) Bretschneider's second[2] convex-quadrilateral area formulae [143, 150].

6) A generalized Ptolemy Theorem [150].

Many other results remiain – such as less widely publicized area formulae – [18, 114]. Useful Linear Algebra analysis of some of these shall be covered in [147]. And more eventually, the outcome of applying similar techniques to the hitherto essentially unstudied quadrilateral Casimir [60, 113, 148]. For this *square root of the sum of the squares of the constituent 2-eigenclustering subsystems' areas* is likely more theoretically significant than [65, 110, 113] formulae for the *overall area* of quadrilaterals!

1.4 Inter-relation of some of the program's works so far

Remark 1 See Fig 1 for some of the current program's main lines. We omit S.2) since it is not Flat Geometry, and also restrict to $d \leq 2$ and $N \leq 4$. These (and various sidelines also omitted) are collected into [146]'s larger overview. Fig 2 lists the corresponding Articles, now including also higher- and arbitrary-N.

Notational Remark 1 Blue backgrounds pick out eigenclustering. Yellow, Kendall's Shape Theory. And orange, nontrivial Representation Theory

[2] Bretschneider's first such includes angle data, rendering it unamenable to Linear-Algebraic formulation. While Bretschneider's second area formula is purely in terms of separations, and Linear-Algebraically identical to [113] Coolidge's [24].

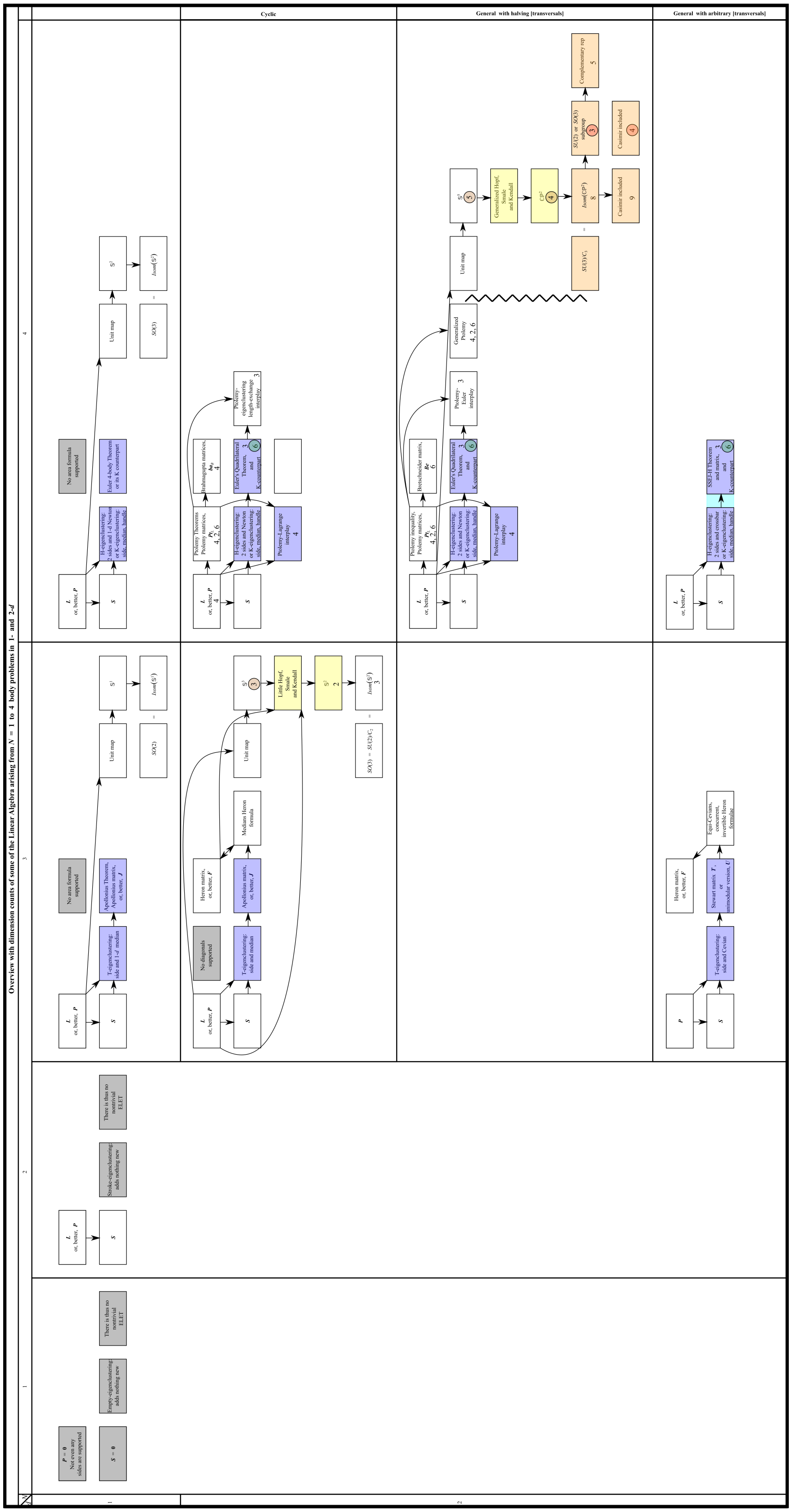

Overview with dimension counts of some of the Linear Algebra arising from N = 1 to 4 body problems in 1- and 2-d
Cyclic
General with halving [transversals]
General with arbitrary [transversals]

Figure 1:

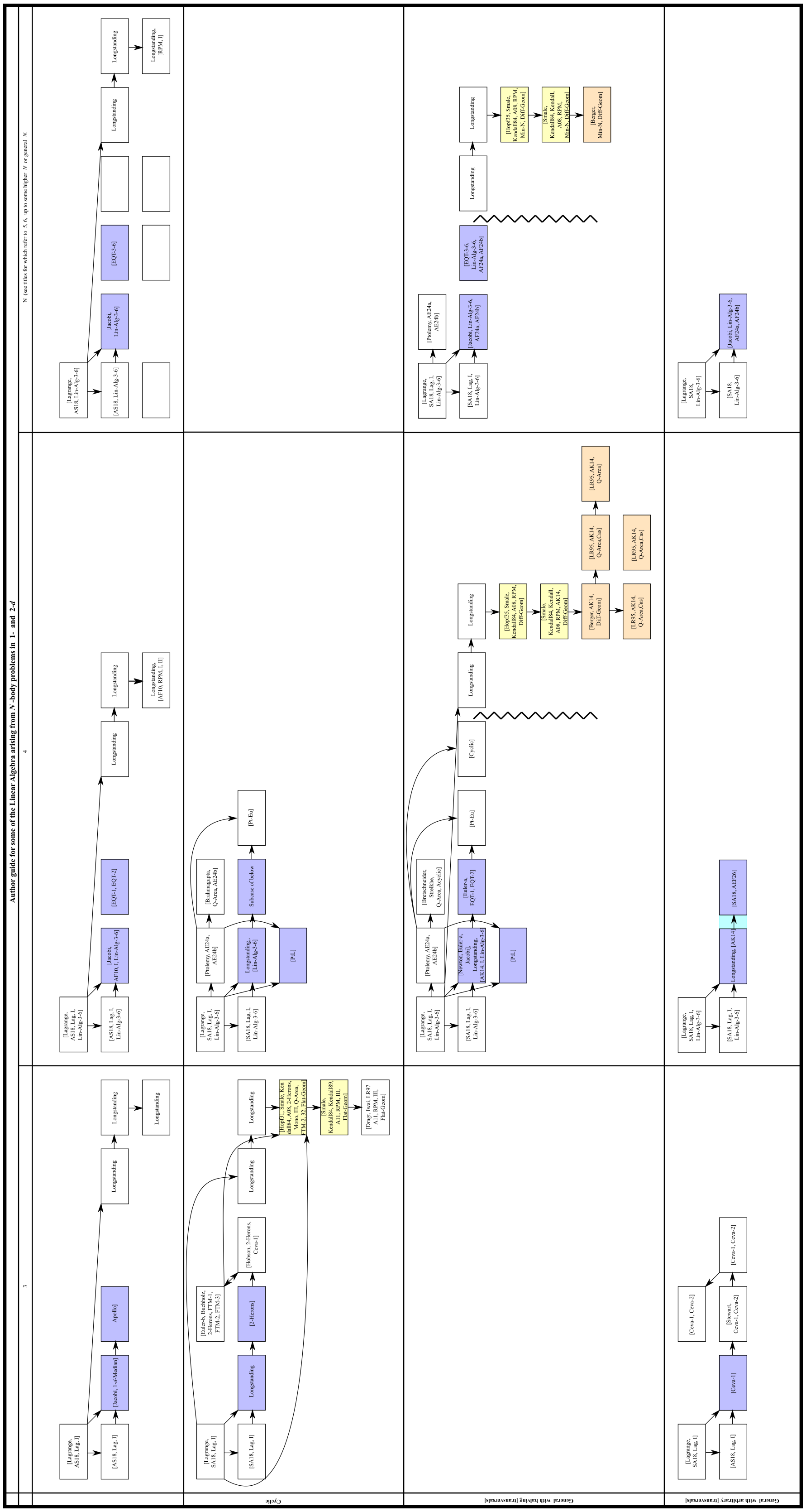

Author guide for some of the Linear Algebra arising from N-body problems in 1- and 2-d

Figure 2:

1.5 Outline of the current Article

Sec 2 serves to preliminarily introduce more traditional Geometry formulations of the cyclic quadrilateral formulae listed in 3) and 4) above.

In Sec 3 we consider Linear Algebra underpinnings for Ptolemy's results. Yielding 3 *Ptolemy sides matrices* $\boldsymbol{Pt}_L \, : \, 4 \times 4$, and 1 *Ptolemy diagonals matrix* $\boldsymbol{Pt}_{\mathrm{d}} \, : \, 2 \times 2$. For all that [139] itself considered just the *Ptolemy separations matrix* $\boldsymbol{Pt} \, : \, 6 \times 6$. Ptolemy's Theorem and inequality use $\boldsymbol{Pt}$, or $\boldsymbol{Pt}_1$ and $\boldsymbol{Pt}_{\mathrm{d}}$. While the second form of Ptolemy's Theorem uses $\boldsymbol{Pt}_2$ and $\boldsymbol{Pt}_3$. All 5 Ptolemy matrices are involutions. And all 3 Ptolemy sides matrices commute with each other and with the Lagrange projector $\boldsymbol{P}$ [141] (Sec 4).

In Sec 5, we consider the Brahmagupta quadratic form, Ba , at whose core lies the *Brahmagupta matrix* $\boldsymbol{Ba}$. We explain how Ptolemy matrix eigentheory accounts for the a priori strange vector previously used [113] in formulating Ba . This builds in our program's second instance of a double-diagonalization. We also bring in a new factorized formulation

$$Ba \; = \; ba_1 \, ba_2 \tag{8}$$

with constituent *Brahmagupta factor matrices* $\boldsymbol{ba}_1$ and $\boldsymbol{ba}_2$. One benefit of this is eigenbasis alignment between $\boldsymbol{Pt}_3$ and $\boldsymbol{ba}_1, \boldsymbol{ba}_2$.

In Sec 6, we find that $\boldsymbol{Ba}$ commutes with all of the $\boldsymbol{Pt}_L$ and $\boldsymbol{P}$. And yet $\boldsymbol{ba}_1$ and $\boldsymbol{ba}_2$ only commute with $\boldsymbol{Pt}_1$ and with each other. This further motivates the unfactorized version of Brahmagupta's formula. We apply some of the above results to give Linear Algebra formulations for circumradius formulae.

In Appendix A, we provide supporting – and yet partly novel – conceptual development of the Linear Algebra of eigenvalues and eigenvectors. Alongside development of the corresponding arenas: eigenspectra and eigenspaces. We use this in our detailed tabulation of the current Article's matrices' eigentheory in Appendix B. Finally Appendix C provides supporting material from Order Theory [74, 120] as regards the posets of competing lattices found in the current Article.

2 Some traditional Geometry of cyclic quadrilaterals

2.1 Preliminary notation

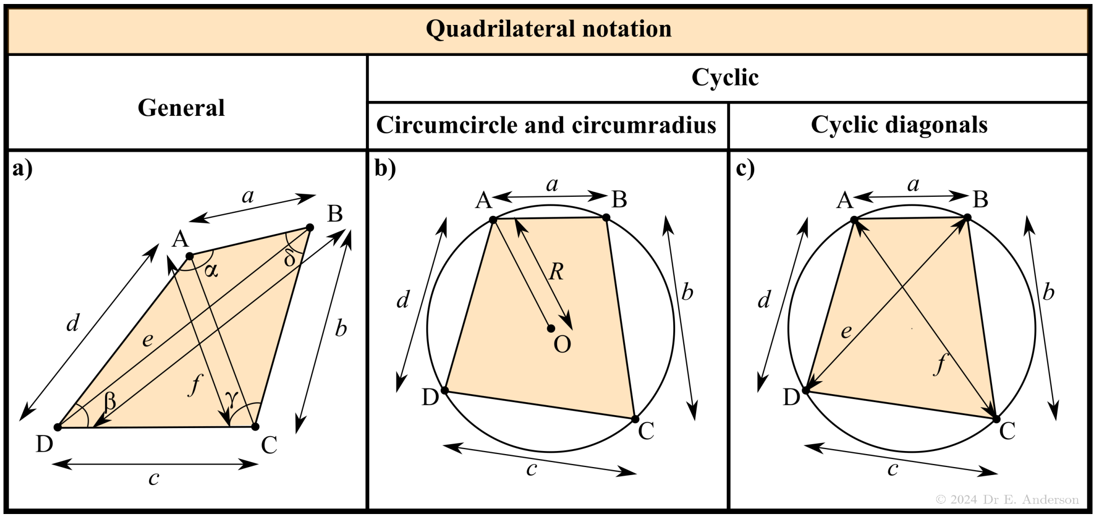


Figure 3:

Notational Remark 1 Let us denote our quadrilateral's separations as per Fig 3.a). With the following extra collective notations. Separations: s_S , $S = 1$ to 6 . Sides: a_I , $I = 1$ to 4 . Diagonals: d_D , $D = 1$ to 2 .

Notational Remark 2 The following separation sums-and-differences turn out to be useful variables.

$$s_\pm := a \pm b \, , \tag{9}$$

$$t_\pm := c \pm d \, , \tag{10}$$

$$d_\pm := e \pm f \, . \tag{11}$$

Alongside their squares

$$S_\pm := {s_\pm}^2 \, , \tag{12}$$

$$T_\pm := {t_\pm}^2 \, . \tag{13}$$

$$D_\pm := {d_\pm}^2 \, . \tag{14}$$

The *semi-perimeter*

$$s := \frac{1}{2} \sum_{I = 1}^{4} a_I \, , \quad .$$

The *product of side-lengths*

$$x := \prod_{I = 1}^{4} a_I \, .$$

The *product of diagonal-lengths*

$$p := \prod_{D = 1}^{2} d_D = e f \, .$$

The *ratio of diagonal-lengths*

$$q := \frac{e}{f} \, .$$

Structure 2 The following 3-cycle of useful variables correspond instead to pairing the sides in all 3 possible ways. I.e.

$$l \ := \ a\,c \ + \ b\,d \ , \tag{15}$$

$$m \ := \ a\,d \ + \ b\,c \ , \tag{16}$$

$$n \ := \ a\,b \ + \ c\,d \ . \tag{17}$$

For which we introduce the cyclic notation l_L , $L = 1$ to 3 . These quantities turn out to be 'adapted variables': in the sense of recurring in major results about cyclic quadrilaterals. Let us introduce also

$$y \ := \ \prod_{L\,=\,1}^{3} l_L \ = \ l\,m\,n \ . \tag{18}$$

And

$$l_{\pm} \ = \ a\,c \ \pm \ b\,d \ . \tag{19}$$

2.2 Ptolemy

Ptolemy's Theorem (1) [3] For a cyclic quadrilateral,

$$p \ = \ l \ . \tag{20}$$

Remark 1 Assuming that the quadrilaterals are nondegenerate, this Theorem's converse also holds.

Ptolemy's inequality [3] For an arbitrary quadrilateral (in fact 4-body configuration: dimension independent)

$$p \ \leq \ l \ . \tag{21}$$

Second form of Ptolemy's Theorem (2) [19, 20, 87] For a cyclic quadrilateral,

$$q \ = \ \frac{m}{n} \ . \tag{22}$$

Exercise 1 Prove this by a combination of area sums and the triangle circumradius formula.

Remark 2 Both of the above express a piece of diagonal-length information in terms of sides-length information. So a first motivation for Structure 1's cycle is their naturality in simplifying the two Ptolemy Theorems.

2.3 Pair of diagonal lengths

Theorem 3 (Cyclic-Quadrilateral Diagonal-Length Formulae) [19, 20, 87]

$$e \ = \ \sqrt{\frac{l\,m}{n}} \ , \tag{23}$$

$$f \ = \ \sqrt{\frac{l\,n}{m}} \ . \tag{24}$$

Remark 1 We have now isolated each diagonal as an expression in terms of sides data. Thus cyclic quadrilaterals have no distinction between licit separations data and licit sides data. For the sides data fully controls the diagonals data, thus leaving no room for licit separations data to be larger than licit sides data.

Exercise 2 a$^-$) Show that the two forms of Ptolemy's Theorem together imply the two diagonal-length formulae and vice versa.

b) Find a proof for the latter, in order to be able to use a) non-circularly, so as to give a further proof of each of the former.

2.4 3-cycle of diagonal lengths

Remark 1 The rearrangement of (24) to

$$f \;=\; \sqrt{\frac{n\,l}{m}}$$

turns out to be nontrivial. For then it and (23) form the first two members of a 3-cycle. Prompting contemplation of the third member of this cycle,

$$g \;:=\; \sqrt{\frac{m\,n}{l}} \; .$$

Remark 2 Treating the 3 of them together corresponds to ordering a given set of side lengths in the various possible ways aroung the circle. Let us also introduce the collective notation e_L , $L = 1$ to 3 for g, f, e . And

$$w \;:=\; \prod_{L\,=\,1}^{3} e_L \; . \tag{25}$$

The below then readily follows.

Theorem 3′ (Cyclic-Quadrilateral Extended Diagonal-Length Formulae)

$$e_L \;=\; \frac{\sqrt{y}}{{l_L}^2} \; . \tag{26}$$

Exercise 3 Prove the inequalities in Fig 4

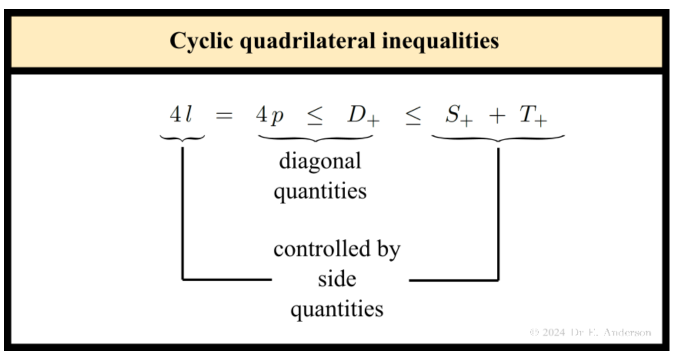


Figure 4:

2.5 Area formula

Theorem 4 (Brahmagupta's Sides-data Cyclic-Quadrilateral Area Formula)

$$Area \;=\; \sqrt{\prod_{I\,=\,1}^{4} (\, s \;-\; a_I \,)} \;\; . \tag{27}$$

Remark 1 In this most commonly encountered square-root formulation, Brahmagupta's formula very closely resembles Heron's.

Exercise 4 Deduce this formula from Heron's and vice versa. Also provide a trigonometric proof of it.

2.6 Cyclic-quadrilateral circumradius formulae

Theorem 5 (Cyclic-Quadrilateral Circumradius Formula)

$$r_{\mathrm{c}} \;=\; \frac{\sqrt{y}}{T} \;=\; \frac{1}{4}\sqrt{\frac{y}{\prod_{I\,=\,1}^{4} (\, s \;-\; a_I \,)}} \;\; . \tag{28}$$

Where r_{c} is the circumradius (Fig 3.b) T is the useful tetra-area variable,

$$T \;:=\; 4 \; Area \; .$$

Remark 1 The second expression here is a condensed presentation of *Parameshvara's formula* [6]. As obtained from the first form by substituting in Brahmagupta's area formula.

Exercise 5 Prove Parameshvara's formula.

Remark 2 Furthermore, using $R_{\mathrm{c}} \;:=\; {r_{\mathrm{c}}}^2$,

$$R_{\mathrm{c}} \, T^2 \;=\; y \; . \tag{29}$$

This is a relation of physical dimension $[L]^6$. Where we use the convention that capital letters are the squares of the corresponding lower-case length variables.

Remark 3 Also

$$r_c \, T \;=\; w \; . \tag{30}$$

Which has the advantage of being just third-order, and partly in terms of more primary variables: e and f .

3 Linear Algebra of Ptolemy Theorems and Diagonal-Length Formulae

3.1 Ptolemy quadratic forms and matrices

Structure 1 The *Ptolemy diagonal quadratic form* is

$$Pt_{\mathrm{d}} \;=\; \frac{1}{2}\,\overline{\boldsymbol{d}} \cdot \underline{\underline{\boldsymbol{Pt}}}_{\mathrm{d}} \cdot \overline{\boldsymbol{d}}\;. \tag{31}$$

For *diagonal lengths* 2-*vector* for the quadrilateral,

$$\boldsymbol{d} \;:=\; \begin{pmatrix} a \\ c \end{pmatrix}\;.$$

And *Ptolemy diagonal matrix*

$$\boldsymbol{Pt}_{\mathrm{d}} \;=\; \mathcal{T}\;. \tag{32}$$

Where in turn

$$\mathcal{T} \;:=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;: \tag{33}$$

the sole *transposition matrix* supported in 2-d .

Structure 2 The *Ptolemy sides quadratic forms* [141]

$$Pt_{L} \;=\; \frac{1}{2}\,\overline{\boldsymbol{r}} \cdot \underline{\underline{\boldsymbol{Pt}}}_{L} \cdot \overline{\boldsymbol{r}}\;. \tag{34}$$

For quadrilateral *side lengths* 4-*vector*

$$\boldsymbol{r} \;:=\; \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \;=:\; \begin{pmatrix} \boldsymbol{r}_1 \\ \hline \boldsymbol{r}_2 \end{pmatrix} \tag{35}$$

in standard basis. Or

$$\boldsymbol{r} \;:=\; \begin{pmatrix} a \\ c \\ b \\ d \end{pmatrix} \;=:\; \begin{pmatrix} \boldsymbol{o}_1 \\ \hline \boldsymbol{o}_2 \end{pmatrix} \tag{36}$$

in *opposite-side pairs* basis. Where the second expressions in each case are 2-vector blocks.

And *Ptolemy matrices*

$$\boldsymbol{Pt}_1 \;:=\; \left(\begin{array}{c|c} \mathbb{0} & \mathbb{I} \\ \hline \mathbb{I} & \mathbb{0} \end{array}\right)\;,\;\; \boldsymbol{Pt}_2 \;:=\; \left(\begin{array}{c|c} \mathbb{0} & \mathcal{T} \\ \hline \mathcal{T} & \mathbb{0} \end{array}\right)\;,\;\; \boldsymbol{Pt}_3 \;:=\; \left(\begin{array}{c|c} \mathcal{T} & \mathbb{0} \\ \hline \mathbb{0} & \mathcal{T} \end{array}\right) \tag{37}$$

in standard basis.

Or

$$\boldsymbol{Pt}_1 \;:=\; \left(\begin{array}{c|c} \mathcal{T} & \mathbb{0} \\ \hline \mathbb{0} & \mathcal{T} \end{array}\right)\;,\;\; \boldsymbol{Pt}_2 \;:=\; \left(\begin{array}{c|c} \mathbb{0} & \mathcal{T} \\ \hline \mathcal{T} & \mathbb{0} \end{array}\right)\;,\;\; \boldsymbol{Pt}_3 \;:=\; \left(\begin{array}{c|c} \mathbb{0} & \mathbb{I} \\ \hline \mathbb{I} & \mathbb{0} \end{array}\right) \tag{38}$$

in the opposite-sides basis. Where in turn $\mathbb{I}$ is the 2-d identity matrix.

Remark 1 Observe that $\boldsymbol{Pt}_2$ is invariant under this change of basis. While the standard basis is $\boldsymbol{Pt}_3$-adapted, the opposite-sides basis is $\boldsymbol{Pt}_1$-adapted.

Remark 2 The above 3 quadratic forms provide a truer notation for l, m, n respectively. We can summarize this by

$$\boldsymbol{Pt}_L \;=\; \boldsymbol{l}_L \; .$$

Structure 3 The *Ptolemy separations quadratic form* [139] is

$$Pt \;:=\; \frac{1}{2}\,\overline{\boldsymbol{s}} \cdot \underline{\underline{\boldsymbol{Pt}}} \cdot \overline{\boldsymbol{s}} \; . \tag{39}$$

For *separation lengths* 6-*vector* for the quadrilateral

$$\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \;=\; \left(\begin{array}{c} \boldsymbol{r}_1 \\ \hline \boldsymbol{r}_2 \\ \hline \boldsymbol{d} \end{array}\right) \;=\; \left(\begin{array}{c} \boldsymbol{r} \\ \hline \boldsymbol{d} \end{array}\right)$$

in the standard basis.
Or

$$\begin{pmatrix} a \\ c \\ b \\ d \\ e \\ f \end{pmatrix} \;=\; \left(\begin{array}{c} \boldsymbol{o}_1 \\ \hline \boldsymbol{o}_2 \\ \hline \boldsymbol{d} \end{array}\right) \;=\; \left(\begin{array}{c} \boldsymbol{o} \\ \hline \boldsymbol{d} \end{array}\right)$$

in the *opposite-side pairs and then diagonal-pair basis.*

And *Ptolemy separations matrix*

$$\boldsymbol{Pt} \;:=\; \left(\begin{array}{c|c|c} \mathbb{0} & \mathbb{I} & \mathbb{0} \\ \hline \mathbb{I} & \mathbb{0} & \mathbb{0} \\ \hline \mathbb{0} & \mathbb{0} & -\mathcal{T} \end{array}\right) \tag{40}$$

in the standard basis. Or

$$\boldsymbol{Pt} \;:=\; \left(\begin{array}{c|c|c} \mathcal{T} & \mathbb{0} & \mathbb{0} \\ \hline \mathbb{0} & \mathcal{T} & \mathbb{0} \\ \hline \mathbb{0} & \mathbb{0} & -\mathcal{T} \end{array}\right) \tag{41}$$

in the opposite-side pairs and then diagonal-pair basis. In either case, additionally,

$$\boldsymbol{Pt} \;:=\; \left(\begin{array}{c|c} \boldsymbol{Pt}_1 & \mathbb{0} \\ \hline \mathbb{0} & \mathcal{T} \end{array}\right) \; . \tag{42}$$

3.2 Their eigentheory

Remark 1 An eigenvalue-level analysis is tabulated in rows 2 to 4 of Fig 7. With eigenvector-level analysis in Fig 9 for $\boldsymbol{Pt}_{\mathrm{d}}$. In row 2 of Fig 8 for the $\boldsymbol{Pt}_L$. In particular, eigenbasis can here be shared with $\boldsymbol{P}$'s H-eigenclustering choice [141]. And in Fig 10 for $\boldsymbol{Pt}$ [139].

3.3 Applications

Remark 1 In terms of separations, Ptolemy's Theorem reads

$$Pt \ = \ 0 \ . \tag{43}$$

And Ptolemy's inequality reads

$$Pt \ \geq \ 0 \ . \tag{44}$$

Remark 2 While in sides–diagonals split form, Ptolemy's Theorem reads

$$Pt_1 \ = \ \mathrm{Pt_d} \ . \tag{45}$$

And Ptolemy's inequality reads

$$Pt_1 \ \geq \ \mathrm{Pt_d} \ . \tag{46}$$

Remark 3 The second form of Ptolemy's Theorem now also reads

$$y \ = \ \frac{Pt_2}{Pt_3} \ . \tag{47}$$

This can be rearranged into the form

$$\boldsymbol{r} \cdot (\ e \, \boldsymbol{Pt}_3 \ + \ f \, \boldsymbol{Pt}_2 \ ) \cdot \boldsymbol{r} \ = \ 0 \ . \tag{48}$$

For which a common-factor matrix can be abstracted:

$$\boldsymbol{Se} \ := \ \begin{pmatrix} 0 & 0 & e & f \\ 0 & 0 & f & e \\ e & f & 0 & 0 \\ f & e & 0 & 0 \end{pmatrix} \ . \tag{49}$$

And yet for cyclic quadrilaterals e and f can be viewed as dependent variables, so this is of limited use.

This dependency also gives a reason for using sides rather than separations in formulating Ptolemy matrices.

Remark 4 The diagonal formulae can also now be recast as follows.

$$e_M^2 \, {Pt_M}^2 \ = \ \Pi \ .$$

For

$$\Pi \ := \ \prod_{L \, = \, 1}^{3} Pt_L \ .$$

3.4 All of our Ptolemy matrices are involutions

Lemma 1 $\boldsymbol{Pt}$, the $\boldsymbol{Pt}_L$ and $\boldsymbol{Pt}_{\mathrm{d}}$ are all involutions.

Exercise 6$^-$ Prove this.

4 Commutativity properties of Lagrange and Ptolemy-sides matrices

Lemma 2 [141] i)

$$[\, \boldsymbol{Pt}_L \, , \, \boldsymbol{Pt}_{L'} \,] \;=\; \mathbb{0} \; . \tag{50}$$

ii)

$$[\, \boldsymbol{P} \, , \, \boldsymbol{Pt}_L \,] \;=\; \mathbb{0} \; . \tag{51}$$

Remark 1 This is now with reference to the $N = 4$ Lagrange projector,

$$\boldsymbol{P} \;=\; \frac{1}{4} \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix} \; . \tag{52}$$

Remark 2 ii) is not an independent statement for the following reason [141]. The identity matrix trivially commutes with all other matrices of the same size. And the $\boldsymbol{Pt}_L$, $\boldsymbol{P}$ and $\mathbb{I}$ are linearly dependent (LD). According to

$$\boldsymbol{P} \;=\; \mathbb{I} \;-\; \langle \boldsymbol{Pt} \rangle \; . \tag{53}$$

In words,

(4-body Lagrange matrix) $=$ (identity) $-$ (the 4-average Ptolemy matrix) .

Where the identity matrix has been 'adjoined as a 4th Ptolemy matrix'.

5 Linear Algebra of Brahmagupta's cyclic-quadrilateral area formula

5.1 The overall Brahmagupta matrix revisited

Structure 1 As in treating Heron's formula, we approach Brahmagupta's from a Linear- Algebraic point of view by expanding out its square. From the definition of semi-perimeter and tetra-area,

$$\begin{aligned} T^2 &= (a + b + c - d)(a + b - c + d)(a - b + c + d)(-a + b + c + d) \\ &= \left[(a + b)^2 - (c - d)^2 \right] \left[(c + d)^2 - (a - b)^2 \right] \\ &= \left({s_+}^2 - {t_-}^2 \right) \left({t_+}^2 - {s_-}^2 \right) = (S_+ - T_-)(T_+ - S_-) \ . \end{aligned} \tag{54}$$

Where the second step makes 2 uses of differences of 2 squares. And the last step brings in (9, 10)'s adapted variables.

Thus

$$T^2 = Ba = \frac{1}{2} \overline{\boldsymbol{Y}} \cdot \underline{\underline{\boldsymbol{Ba}}} \cdot \overline{\boldsymbol{Y}} \ .$$

For

$$\boldsymbol{Y} := \begin{pmatrix} S_+ \\ S_- \\ T_+ \\ T_- \end{pmatrix} = \begin{pmatrix} \mathbf{S} \\ \hline \mathbf{T} \end{pmatrix} \ . \tag{55}$$

And *Brahmagupta matrix*

$$\boldsymbol{Ba} := \left(\begin{array}{c|c} -\mathcal{T} & \mathbb{I} \\ \hline \mathbb{I} & -\mathcal{T} \end{array} \right) \ . \tag{56}$$

Remark 1 This was essentially already presented in [113]. Which however used a different basis and different notation for what we are now calling adapted variables. We now account for $\boldsymbol{Y}$ as being built out of the standard eigenbasis of Ptolemy vectors. Thus further justifying its use, as well as ordering our study so that Brahmagupta postcedes Ptolemy.

In this way, we have found some compatibility between sides matrices for quadrilaterals.

Notational Remark 1 We use Ba since one of us will shortly be presenting [143] the counterpart for Bretschneider's second formula, with quadratic form denoted by Be .

5.2 Its eigentheory

Remark 1 See the bottom row of Fig 7 for an eigenvalue-level analysis. In particular, it is twofold degenerate. And the bottom row of Fig 8 for an eigenvector-level analysis.

5.3 Two new Brahmagupta matrices: factors of the previous

Structure 1 Next introduce

$$ba_1 = \left({s_+}^2 - {t_-}^2 \right) = S_+ - T_- \ , \tag{57}$$

$$ba_2 = \left({t_+}^2 - {s_-}^2 \right) = T_+ - S_- \ . \tag{58}$$

Such that

$$Ba = ba_1 \, ba_2 \ . \tag{59}$$

Index these by $g = 1, 2$. As quadratic forms,

$$ba_g := \frac{1}{2} \overline{\boldsymbol{r}} \cdot \underline{\underline{\boldsymbol{ba}}}_g \cdot \overline{\boldsymbol{r}} \ . \tag{60}$$

For *Brahmagupta factor matrices*

$$\boldsymbol{ba}_1 \;:=\; \left(\frac{\mathbb{1} \mid \quad \mathbb{0}}{\mathbb{0} \mid \mathcal{T} \;-\; \mathbb{I}} \right) \;,\;\; \boldsymbol{ba}_2 \;:=\; \left(\frac{\mathcal{T} \;-\; \mathbb{I} \mid \mathbb{0}}{\mathbb{0} \quad \mid \mathbb{I}} \right) \;. \tag{61}$$

These are presented in standard basis

5.4 Their eigentheory

Remark 1 See the bottom row of Fig 7 for an eigenvalue-level analysis. In particular, spotting the two factors fails to lift the twofold degeneracy. And the third row of Fig 8 for an eigenvector-level analysis. Observe here that the eigenbasis is aligned with the second one given for $\boldsymbol{Pt}_3$. This, on top of the natural action on $\boldsymbol{r}$ gives two advantages of the $\boldsymbol{ba}_g$ over $\boldsymbol{Ba}$. As regards setting up a coherent matrix theory for cyclic quadrilaterals.

Remark 1 We use 'the $\boldsymbol{Ba}$ ' to refer to all three of the above matrices together.

5.5 The $\boldsymbol{Ba}$ are not quite involutions

Lemma 3

$$\boldsymbol{Ba}^3 \;=\; \boldsymbol{Ba} \;, \;\; \text{but} \;\; \boldsymbol{Ba}^2 \;\neq\; \mathbb{I} \;. \tag{62}$$

$$\boldsymbol{ba}_g{}^3 \;=\; \boldsymbol{ba}_g \;, \;\; \text{but} \;\; \boldsymbol{ba}_g{}^2 \;\neq\; \mathbb{I} \;. \tag{63}$$

Exercise 7 Prove these statements. Also relate these results to differences between these matrices' minimal and characteristic polynomials. In the process, derive row 4 of Fig 7.

5.6 Application to cyclic quadrilaterals' circumradius formula

Theorem 5′ i)

$$r_{\rm c} \;=\; \frac{\sqrt{\Pi}}{T} \;=\; \sqrt{\frac{\Pi}{Ba}} \;. \tag{64}$$

ii)

$$Pt_1\, Pt_2\, Pt_3 \;=\; \Pi \;=\; R_{\rm c}\, T^2 \;=\; R_{\rm c}\, Ba \;\;=\;\; R_{\rm c}\, ba_1\, ba_2 \;. \tag{65}$$

Remark 1 For intended further use in Shape Theory [48, 62, 67, 103], we also normalize the circumradius to 1 , thus obtaining an algebraic relation between the sides, as follows.

Theorem 5″ (Adapted-variables Shape-Theoretic Parameshvara constraint

$$Pt_1\, Pt_2\, Pt_3 \;=\; \Pi \;=\; T^2 \;=\; Ba \;=\; ba_1\, ba_2 \;. \tag{66}$$

Remark 2 It is also plausible to use area and the Pt_L as 4 variables, which are simply related by the 1 unit-circumradius condition. Or area and the extended 3-cycle version of the diagonals as the 4 variables, also simply related by the same condition.

6 Brahmagupta and mutual commutator algebras

6.1 (Non)zero Commutators

Lemma 4 i)

$$[\, \boldsymbol{Pt}_L \,,\, \boldsymbol{Ba} \,] \;=\; \mathbb{0} \;. \tag{67}$$

ii)

$$[\, \boldsymbol{P} \,,\, \boldsymbol{Ba} \,] \;=\; \mathbb{0} \;. \tag{68}$$

Lemma 5 (Brahmagupta-factors' self commutation)

$$[\, \boldsymbol{ba}_g \,,\, \boldsymbol{ba}_{g'} \,] \;=\; \mathbb{0} \;. \tag{69}$$

A fortiori,

$$\boldsymbol{ba}_1 \, \boldsymbol{ba}_2 \;=\; \mathbb{0} \;=\; \boldsymbol{ba}_2 \, \boldsymbol{ba}_1 \;. \tag{70}$$

Lemma 6 (Brahmagupta-factors–Ptolemy partial mutual commutation) i)

$$[\, \boldsymbol{Pt}_1 \,,\, \boldsymbol{ba}_g \,] \;=\; \mathbb{0} \;. \tag{71}$$

ii) But

$$[\, \boldsymbol{Pt}_2 \,,\, \boldsymbol{ba}_g \,] \;\neq\; \mathbb{0} \;, \tag{72}$$

$$[\, \boldsymbol{Pt}_3 \,,\, \boldsymbol{ba}_g \,] \;\neq\; \mathbb{0} \;. \tag{73}$$

iii) Also

$$[\, \boldsymbol{P} \,,\, \boldsymbol{ba}_g \,] \;\neq\; \mathbb{0} \;. \tag{74}$$

Remark 1 i) is not however an independent equation, by the LD

$$\boldsymbol{Pt}_3 \;=\; \langle \boldsymbol{ba} \rangle \;. \tag{75}$$

In words,

(third Ptolemy matrix) = (2-average of Brahmagupta factor matrices)

Exercise 8$^-$ Verify Lemmas 4 to 6, computing all the nonzero right-hand-side terms. What are the individual products $\boldsymbol{Pt}_2 \cdot \boldsymbol{ba}_g$, $\boldsymbol{ba}_g \cdot \boldsymbol{Pt}_2$, $\boldsymbol{Pt}_3 \cdot \boldsymbol{ba}_g$, $\boldsymbol{ba}_g \cdot \boldsymbol{Pt}_3$?

Project 1 Does an interesting multiplicative algebra ensue? An interesting brackets algebra?

6.2 Zero-commutativity forms competing posets

Remark 1 See Fig 5 for which independent zero commutativity algebras can be formed. Alongside the consequences following from those pairs of matrices which do not commute. The underlying notions are provided in Appendix B.

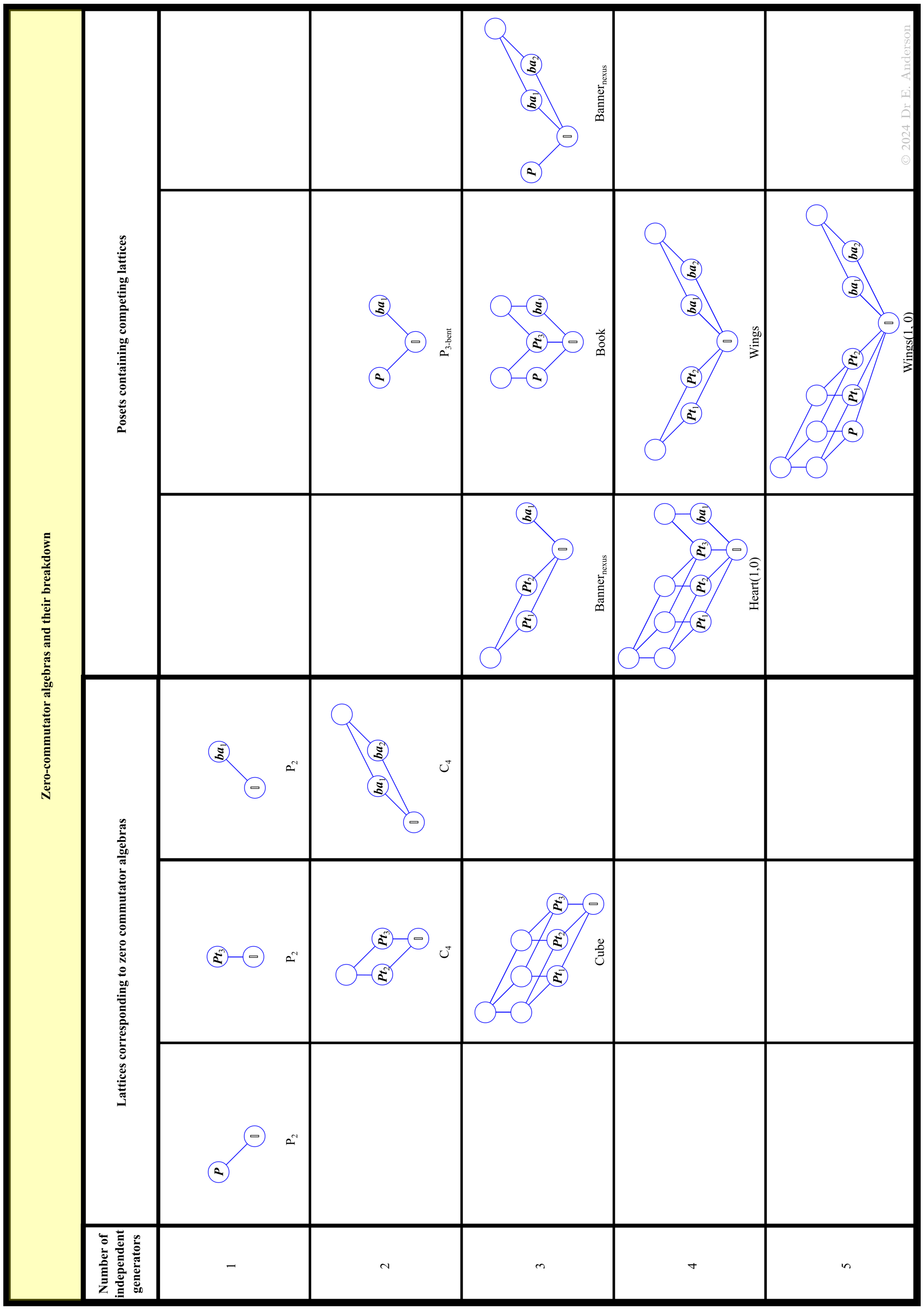


Figure 5:

Project 2 We leave working out which new matrices are discovered as nonzero commutators for another occasion. Alongside which commutator algebras are realized in the process.

7 Conclusion

7.1 Scholium

Remark 1 On the one hand, for the triangle, 3 particularly significant 3×3 matrices arise [123, 126, 127]. Namely, the Apollonius involution $\boldsymbol{J}$, the Lagrange projector $\boldsymbol{P}$ and the fundamental triangle matrix $\boldsymbol{F}$. Which is the difference of the other two as well as the core of Heron's area formula and the cycles of cosine rules and of triangle inequalities. These all commute with each other, can be assigned a common eigenbasis and have aligned eigenspaces. On the other hand, we have now worked out the following.

Remark 2 For quadrilaterals, many more matrices arise, with 2- through to 9-indices in play: so life is more complicated [110, 113]. Any flat-space N-body problem admits a projector P encoding the Physics of being able to decouple relative motion from centre of mass motion. For equal masses, this projector coincides with [125] the Lagrange matrix; ours here is 4×4 . For $N = 4$, Apollonius' Medians-Length Theorem most naturally becomes two Eigenclustering Length-Exchange Theorems [130]. One is Euler's [9, 72, 85, 96, 131] while the other is new [132]. But the matrices encoding these carry 3- and 6-indices – not compatible with 4-indices – and are not involutions either.

Remark 3 It however turns out that a subtlety first realized for $N = 4$ – sides–diagonals distinction – begets Ptolemy's inequality which provides a separate source of involutions. Be this at the level of separations: 6×6 [139]. Or of the sides–diagonals split – 4×4 and 2×2 . The 4×4 case consists of a 3 cycle [141] $\boldsymbol{Pt}_L = \boldsymbol{J}_L$, all of which feature within Ptolemy's two Theorems. These are relevant in the current Article's particular simplified setting of cyclic quadrilaterals.

Remark 4 Therein, Brahmagupta's area formula is a natural generalization of Heron's area formula, at least prima facie with both in square-root form. So we abstract a Brahmagupta matrix $\boldsymbol{Ba}$ [113]. But this turns out to be doubly-degenerate, unlike the nondegenerate $\boldsymbol{F}$. Nor does our new working building on the Brahmagupta quadratic form's factorization manage to ameliorate this double degeneracy. Nor is it immediately clear what function this double degeneracy plays. Brahmagupta matrices do not quite manage to be involutions either, in the sense that they obey

$$\boldsymbol{M}^3 = \boldsymbol{M} \text{ without cancellation to } \boldsymbol{M}^2 = \mathbb{I} . \tag{76}$$

In this way, they are less Algebraically simple than most of the other matrices in the current account.

Remark 5 The $\boldsymbol{Pt}_L = \boldsymbol{J}_L$ turn out to commute with each other and with $\boldsymbol{P}$ [141]. The last of which turns out to be unsurprising [141] since the following are LD. The $\boldsymbol{Pt}$, $\boldsymbol{P}$ and the trivially commuting $\mathbb{I}$. All of the $\boldsymbol{Pt}$, nor $\boldsymbol{P}$ commute with $\boldsymbol{Ba}$. While only $\boldsymbol{Pt}_3$ commutes with our 2 Brahmagupta factor matrices $\boldsymbol{ba}_g$, which additionally commute with each other, for all that the 3 of them are LD. This gives a further reason to work with the $\boldsymbol{Ba}$ rather than the $\boldsymbol{ba}_g$. [150]'s Exercises shall outline an underlying Algebraic explanation, generalizing [142]'s for triangle matrices.

Remark 6 We have now identified the vectors entering the overall Brahmagupta quadratic form to be Ptolemy side matrix eigenvectors. This gives a particular way in which it is beneficial to study Ptolemy before Brahmagupta.

Remark 7 Each of the $\boldsymbol{Pt}_L$ can be given an eigenbasis that is aligned with an H-eigenclustering basis of $\boldsymbol{P}$. Or with that of a $\boldsymbol{ba}_g$. But not with both at once. This is related to the corresponding eigenvalues partitioning each matrix's eigenspace in a different way. Respectively, $3\,|\,1$, $2\,|\,2$ and $2\,|\,1\,|\,1$. By which eigenspace alignment is not manifested either in our cyclic quadrilateral study.

Remark 8 The Ptolemy separations matrix $\boldsymbol{Pt}$ plays a similar role to $\boldsymbol{F}$ as follows. Firstly, each encodes an inequality: a disguised version of the triangle inequality versus Ptolemy's inequality. Secondly, both of these encodements can be rooted in indefiniteness. For Heron's formula, sides data violating the triangle inequality is timelike, corresponding to negative area [123]. While separations data violating Ptolemy's inequality is also 'timelike'.

With the following caveat. $\boldsymbol{F}$ is hyperbolic while $\boldsymbol{Pt}$ is ultrahyperbolic. In the Heron case, the corresponding null statement – that degenerate triangles have zero area – separates a spacelike cone interior from a timelike exterior. In the Ptolemy case, the corresponding null statement is that cyclic quadrilaterals obey Ptolemy's Theorem. But the topological inter-relations between null, spacelike and 'timelike' regimes are more nuanced.[3]

More precisely, these are $++-$ and $+++---$ respectively. So the realized ultrahyperbolicity is furthermore balanced (Appendix A.9): invariant under exchange of signs. Thus an ultrahyperbolic counterpart of how $+-$ wave equations and spacetimes are simpler than [29] any higher-d wave equations or spacetimes is in play here.

7.2 Outlook

Pointer 1 A subsequent work [143] shows that the *Bretschneider sides-matrix* at the core of Bretschneider's second convex-quadrilateral area formula has better properties and compatibilities than the Brahmagupta matrices. So convex turns out to be more satisfactory than cyclic, with both falling short of the triangle case's results. An overall moral is as follows [110].

In generalizing away from an application of Hopf's little map, we should not expect all features to be persistent. Especially if Hopf's generalized maps only partly embody the generalized case's features.

Pointer 2 Another sequel of the current Article is our comparative study of notions of departure from cyclicness for quadrilaterals [150].

Pointer 3 Some of the current Article's sets of matrices turn out to be useful (Appendix B.3) in critically re-examining the QM literature's account of CSCO (complete sets of commuting observables).

[3] As part of this, the quotation marks are themselves nuanced.

A (Multi-)Linear Algebra entering our tabulation

A.1 Eigentheory's basic objects

Remark 1 Throughout, let $\boldsymbol{M}$ be a $\mathbb{R}$-valued square $p \times p$ matrix for some finite $p \in \mathbb{N}$. For use in studying quadrilaterals, indices taking $p = 2$ to 9 values are relevant. The current Article's are $p \leq 6$. We thus concentrate specific examples on such small p . All of the current Article's matrices are furthermore symmetric.

Definition 1 An *eigenvalue* of $\boldsymbol{M}$ is any number λ solving

$$\det(\boldsymbol{M} - \lambda \mathbb{I}) = 0 . \tag{77}$$

Naming Remark 1 Eigenvalues are alias *proper values* and *characteristic values.* (77) is most usually called the *characteristic equation.* Though, to match, it can also be called *eigenvalue equation.* And is also alias *secular equation.* Since p is finite, this equation is of polynomial form. Hence the further aliases *characteristic polynomial*, *eigenvalue polynomial* and *secular polynomial.*

Definition 2 An *eigenvector* is any vector solution $\boldsymbol{v}$ of the *eigenvector equation*

$$\overline{\underline{\boldsymbol{M}}} \cdot \overline{\boldsymbol{v}} = \lambda \overline{\boldsymbol{v}} . \tag{78}$$

For whichever particular eigenvalue λ .

Remark 1 In general, since $\mathbb{R}$ is not Algebraically closed, its Algebraic closure $\mathbb{C}$ is needed to accommodate eigenvalues.

Lemma A For real symmetric matrices, however,

i) all eigenvalues are real.

ii) The eigenvectors can be taken to form an orthonormal set.

A.2 Eigentheory's corresponding arenas

Definition 1 The *eigenspectrum* of $\boldsymbol{M}$ is the set of distinct values realized as eigenvalues of $\boldsymbol{M}$. I.e.

$$\mathfrak{E}\text{spec}(\boldsymbol{M}) = \{\lambda_e , \ e = 1 \text{ to } E\} \in \mathbb{R}^E . \tag{79}$$

Definition 2 Let

$$\mathfrak{E}\text{ig}_e(\boldsymbol{M})$$

denote the *eigenspace* spanned by the eigenvectors corresponding to eigenvalue λ_e . And

$$\mathfrak{E}\text{ig}(\boldsymbol{M})$$

denote the *total eigenspace* of $\boldsymbol{M}$.

Lemma B (Eigenspace Decomposition)

$$\mathfrak{E}\text{ig}(\boldsymbol{M}) = \bigoplus_{e \,\in\, \mathfrak{E}\text{spec}} \mathfrak{E}\text{ig}_e(\boldsymbol{M}) = \bigoplus_{e=1}^{E} \mathfrak{E}\text{ig}_e(\boldsymbol{M}) . \tag{80}$$

A.3 Eigenvalue multiplicities

Remark 1 Eigenvalues are often stated with multiplicity. There are however 3 distinct widely useful notions of multiplicity, so we need to specify which we mean.

Definition 1 The *algebraic multiplicity* α_e of λ_e is the number of times that this occurs as a root of the characteristic polynomial. I.e. the powers featuring in

$$C(\boldsymbol{M}; x) = \prod_{e=1}^{E} (x - \lambda_e)^{\alpha_e} . \tag{81}$$

Definition 2 The *geometrical multiplicity* γ_e of λ_e is the dimension of its corresponding eigenspace,

$$\gamma_e := \dim(\mathfrak{Eig}_e(\boldsymbol{M})) . \tag{82}$$

Definition 0 The *minimal polynomial*

$$M(\boldsymbol{M}; x)$$

is defined as follows.

i) It is *monic*: it has been scaled so that its lead coefficient is standardized to 1 .

ii) It is equal to zero:

$$M(\boldsymbol{M}; x) = 0 . \tag{83}$$

iii) Suppose that some other polynomial $N(\boldsymbol{M}; x) = 0$. Then $M \,|\, N$ (i.e. M divides N). This is the *minimality property* itself.

Definition 4 Suppose that we express the minimal polynomial as

$$M(\boldsymbol{M}; x) = \prod_{e=1}^{E} (x - \lambda_e)^{\mu_e} , \tag{84}$$

Then the powers μ_e are each λ_e 's corresponding *minimal multiplicity.*

A.4 The eigenvalue multiplicity inequality, and ensuing adjectives

Lemma C A square $p \times p$ matrix eigenvalue multiplicities obey the poset of inequalities in Fig 6.

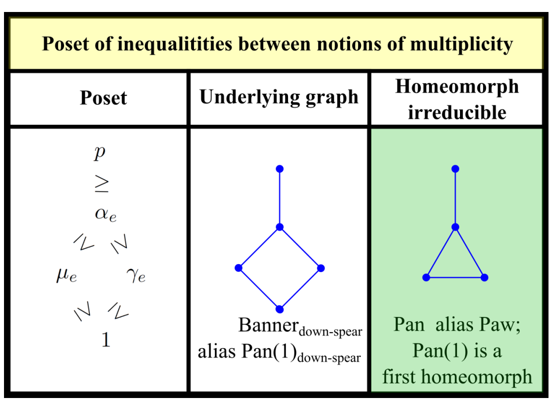


Figure 6:

Exercise 9 Prove this.

♠ ◇ ♣

Remark 1 Suppose that some

$$\alpha_e \; = \; p \; . \tag{85}$$

Then there is no room left for $\boldsymbol{M}$ to have any further eigenvalues. So we have a *single-eigenvalue matrix.* I.e. $\mathfrak{E}\mbox{spec}(\boldsymbol{M})$ is topologically a point.

Remark 2 If not, so that

$$\alpha_e \; < \; p \; , \tag{86}$$

then we have a *multi-eigenvalue matrix.*

Remark 3 The opposite extreme to Remark 1 is as follows.

Definition 1 λ_e is *simple* if

$$\alpha_e \; = \; 1 \; . \tag{87}$$

$\boldsymbol{M}$ is simple if all of its eigenvalues are.

Remark 4 Sandwiching then also fixes that

$$\alpha_e \; = \; \gamma_e \; = \; \mu_e \; = \; 1 \; . \tag{88}$$

Remark 5 Suppose that all of $\boldsymbol{M}$'s eigenvalues are simple. Then it has *all eigenvalues distinct.*

Definition 2

$$\alpha_e \; > \; 1 \tag{89}$$

defines by exclusion *nonsimple eigenvalues.*

♠ ◇ ♣

Definition 3 λ_e is *semisimple* [99] if

$$\alpha_e \; = \; \gamma_e \; . \tag{90}$$

$\boldsymbol{M}$ is *diagonalizable* iff all of its eigenvalues are semisimple. Suppose not, so that

$$\alpha_e \; > \; \gamma_e \tag{91}$$

for ≥ 1 λ_e . Then $\boldsymbol{M}$ is *non-diagonalizable* alias *defective* [99].

Definition 4 An eigenvalue has *trivial eigenspace* if

$$\gamma_e \; = \; 1 \; . \tag{92}$$

And *nontrivial eigenspace* if

$$\gamma_e \; > \; 1 \; . \tag{93}$$

♠ ◇ ♣

Definition 5 λ_e is *minimally-nontrivial* iff

$$\gamma_e \; > \; \mu_e \; . \tag{94}$$

And *minimally-trivial* iff

$$\gamma_e \; = \; \mu_e \; . \tag{95}$$

$\boldsymbol{M}$ is *minimally-trivial* iff all of its eigenvalues are. And *minimally-nontrivial* iff ≥ 1 of its eigenvalues is.

Definition 6 λ_e is *minimally-minimal* iff

$$\mu_e \; = \; 1 \; . \tag{96}$$

And *minimally-nonminimal* iff

$$\mu_e \; > \; 1 \; . \tag{97}$$

$\boldsymbol{M}$ is *minimally-minimal* iff all of its eigenvalues are. $\boldsymbol{M}$ is *minimally-nonminimal* iff ≥ 1 of its eigenvalues is.

Remark 7 The minimal polynomial M is a minimal algebraic object with opposing maximal algebraic object the characteristic polynomial C . Geometrical multiplicity is however geometrical rather than meaningfully associated with some algebraic polynomial. As such, algebraic minimality need not necessarily bound geometrical multiplicity, nor vice versa. Examples suffice to establish that all three of the following cases are realized.

Definition 7 $\boldsymbol{M}$ is *mindominant* if

$$\gamma_e \; < \; \mu_e \; . \tag{98}$$

Geominimal if

$$\gamma_e \; = \; \mu_e \; . \tag{99}$$

And *geomdominant* if

$$\gamma_e \; > \; \mu_e \; . \tag{100}$$

Definition 8 We say that λ_e is of *unified multiplicity* iff

$$\alpha_e \; = \; \gamma_e \; = \; \mu_e \; . \tag{101}$$

And that $\boldsymbol{M}$ is *Order-Theoretically trivial* [152] iff all of its eigenvalues are of unified multiplicity. Definitions 4 to 7 are our own.

Remark 8 Unified multiplicity 1 is a longer phrasing for simple. Unified multiplicity p is a shorter one for single-eigenvalued Order-Theoretically trivial.

A.5 (Non-)degeneracy

Remark 1 Let us now also bring in quadratic-form, PDE classification and Special Relativity notions of degeneracy and indefiniteness

Definition 1 A quadratic form

$$Q(\boldsymbol{x}) \ := \ ||\boldsymbol{x}||_{\boldsymbol{M}}{}^2 \ = \ \overline{\boldsymbol{x}} \cdot \underline{\underline{\boldsymbol{M}}} \cdot \overline{\boldsymbol{x}} \ . \tag{102}$$

viewed as a bi-linear form is *degenerate* [41, 44] if > 1 of the following hold.

i)

$$\exists \quad \text{some} \quad \boldsymbol{u} \neq 0 \in \mathfrak{V}_p \quad \text{such that} \quad \overline{\boldsymbol{u}} \cdot \underline{\underline{\boldsymbol{M}}} \cdot \overline{\boldsymbol{v}} \ = \ 0 \quad \forall \, \boldsymbol{v} \in \mathfrak{V}_p \ . \tag{103}$$

ii)

$$\exists \quad \text{some} \quad \boldsymbol{v} \neq 0 \in \mathfrak{V}_p \quad \text{such that} \quad \overline{\boldsymbol{u}} \cdot \underline{\underline{\boldsymbol{M}}} \cdot \overline{\boldsymbol{v}} \ = \ 0 \quad \forall \, \boldsymbol{u} \in \mathfrak{V}_p \ . \tag{104}$$

If not, then $\boldsymbol{M}$ is said to be *nondegenerate*. Where $\mathfrak{V}_p$ are, for symmetric matrices, $\mathbb{R}$-valued p-dimensional vector spaces.

Lemma D a) $\boldsymbol{M}$ nondegenerate iff $\boldsymbol{M}$ nonsingular.

b) $\boldsymbol{M}$ degenerate iff $\boldsymbol{M}$ singular

Lemma E Degenerate matrices possess ≥ 1 zero eigenvalue.

Exercise 2 Prove Lemmas D and E.

Structure 1 A matrix $\boldsymbol{M}$ is *nondegenerate* if

$$N_0(\boldsymbol{M}) \ = \ 0 \ . \tag{105}$$

It is *degenerate* if

$$N_0(\boldsymbol{M}) \ \geq \ 1 \ . \tag{106}$$

Structure 2 The count $N_0(\boldsymbol{M})$ of zero eigenvalues, including algebraic multiplicity, constitutes a spectral quantifier of the *extent of degeneracy*. While the count $N_{\neq 0}(\boldsymbol{M})$ of nonzero eigenvalues, including algebraic multiplicity, constitutes a spectral quantifier of the *extent of the nondegenerate sector*.

A.6 Rank and nullity

Definition 1 The *rank* of a matrix $\boldsymbol{M}$ is the dimension of its image,

$$r(\boldsymbol{M}) \ = \ \dim\left(\mathfrak{Im}_{\boldsymbol{M}}\right) \ . \tag{107}$$

Definition 2 The *nullity* of $\boldsymbol{M}$ is the dimension of its kernel,

$$n(\boldsymbol{M}) \ = \ \dim\left(\mathfrak{Ker}_{\boldsymbol{M}}\right) \ . \tag{108}$$

Rank–Nullity Theorem [27, 83, 101]

$$r(\boldsymbol{M}) \ + \ n(\boldsymbol{M}) \ = \ \dim(\boldsymbol{M}) \ = \ p \ . \tag{109}$$

Remark 2 Each of $r(\boldsymbol{M})$, $n(\boldsymbol{M})$ can take whichever integer value from 0 to p .

$$r(\boldsymbol{M}) \ = \ p \tag{110}$$

for a matrix of *full rank*. Here

$$n(\boldsymbol{M}) \ = \ 0 \ . \tag{111}$$

While

$$r(\boldsymbol{M}) \ = \ 0 \tag{112}$$

for a *null matrix*: the *zero matrix*. Here

$$n(\boldsymbol{M}) \ = \ p \ . \tag{113}$$

All other cases are of *partial rank* and *partial nullity*. By the Rank–Nullity Theorem, these form 2 topologically equal and yet oppositely-labelled chains. I.e.

$$\{ \ 1, \, 2, \, ..., \, p \ - \ 1, \, p \ \} \quad \text{and} \quad \{ \ p, \, p \ - \ 1, \, ..., \, 2, \, 1 \ \} \ .$$

Remark 1 $n(\boldsymbol{M})$ and $N_0(\boldsymbol{M})$ are numerically coincident notions. As are $r(\boldsymbol{M})$ and $N_{\neq 0}(\boldsymbol{M})$. This dichotomy amounts to plain versus specifically spectral realizations of the same Linear Algebra notions.

Structure 1 At the level of arenas, each matrix acting on a vector space's vectors splits it up as follows.

$$\mathfrak{V}_p(\boldsymbol{M}) \ \ = \ \ \mathfrak{Im}(\boldsymbol{M}) \ \ \oplus \ \ \mathfrak{Ker}(\boldsymbol{M}) \ . \tag{114}$$

The Rank–Nullity Theorem is then the corresponding dimensional count's 2-piece partition.

A.7 Notions of signature

Structure 1 Let $\boldsymbol{M}$ be a symmetric matrix. Then by Lemma A.i), all of its eigenvalues are real. Thus, in addition to the above zero versus nonzero distinction, a meaningful split by eigenvalue sign is furthermore supported. Let us denote these positive and negative eigenvalue counts, including algebraic multiplicity, by $N_{\pm}(\boldsymbol{M})$ respectively. So

$$N_{\neq 0}(\boldsymbol{M}) \ = \ N_+(\boldsymbol{M}) \ + \ N_-(\boldsymbol{M}) \ . \tag{115}$$

The current Subsection is a spectral version of [123]'s notions and names.

Definition 1 The *Mathematicians' signature* is

$$s_{\text{Math}}(\boldsymbol{M}) \ = \ \max \left(\, N_+(\boldsymbol{M}), \, N_-(\boldsymbol{M}) \, \right) \ . \tag{116}$$

Definition 2 The *Physicists' signature in detail* can be taken to be an ordered list of the signs of the eigenvalues. I.e.

$$s_{\text{Phys-detail}}(\boldsymbol{M}) \ = \ + \ ... \ + \ - \ ... \ - \ 0 \ ... \ 0 \ . \tag{117}$$

Somewhat more efficient notation for this is as a triple

$$s_{\text{Phys-detail}}(\boldsymbol{M}) \ = \ (\ N_+(\boldsymbol{M}) \ , \ \ N_-(\boldsymbol{M}) \ , \ \ N_0(\boldsymbol{M}) \ ) \ . \tag{118}$$

Definition 3 The *Physicists' signature in summary* is

$$s_{\text{Phys}}(\boldsymbol{M}) \ = \ N_+(\boldsymbol{M}) \ - \ N_-(\boldsymbol{M}) \ = \ \Delta N(\boldsymbol{M}) \ . \tag{119}$$

Or possibly the opposite sign of this by change of sign convention (see the next item below).

A.8 Notions of (in)definiteness

Definition 0 A quadratic form $Q(x)$ with matrix $\boldsymbol{M}$ at its core is *definite* if

$$N_+(\boldsymbol{M}) \ = \ p \ \text{ and } \ N_-(\boldsymbol{M}) \ = \ 0 \ \text{ or vice versa } \ . \tag{120}$$

$Q(\boldsymbol{x})$ is *positive-definite* in the first case, and *negative-definite* in the second. Differences between these are a matter of convention. I.e. positive-definite $Q(\boldsymbol{x}) \ = \ k$ is equivalent to negative-definite $-Q(\boldsymbol{x}) \ = \ -k$.

It is *indefinite* if both of the following hold.

$$N_{\pm}(\boldsymbol{M}) \;\geq\; 1 \; . \tag{121}$$

Definition 1 A quadratic form $Q(\boldsymbol{x})$ is *elliptic* iff the following hold.

1) *Nondegeneracy* alias *full rank*.

2) *Definiteness.*

Definition 2 It is *hyperbolic* if 1) and the following hold.

3) *Indefiniteness*

4) Precisely 1 eigenvalue has the opposite sign to all the others. I.e. ≥ 1 of the following holds.

$$N_{+} \;=\; 1 \;\; \text{or} \;\;\; N_{-} \;=\; 1 \tag{122}$$

This can be viewed as *minimal realization of indefiniteness.*

Definition 3 It is *ultrahyperbolic* [29] if 1), 3) and the following hold.

5)

$$N_{\pm}(\boldsymbol{M}) \;\geq\; 2 \; . \tag{123}$$

I.e. *non-minimal realization of indefiniteness.*

A.9 Discussion with further structures and adjectives

Remark 1 The above are but a spectral reformulation of 'garden variety' notions of ellipticity and hyperbolicity; there are far more subtle notions of each in PDE Theory [47, 90, 89].

$p = 2$ is minimum for hyperbolicity to be realized.

$p = 4$ is minimum for ultrahyperbolicity to be realized.

Structure 1 In the hyperbolic case, indefiniteness partitions up the arena of values of our quadratic form into 3 qualitatively-distinct cases.

$Q > 0$ is 'spacelike'.

$Q < 0$ is 'timelike'.

$Q = 0$ is 'lightlike' alias null.

This Special Relativity type nomenclature [71] is here aligned with the following convention. That there is 1 minus sign and $p - 1$ plus signs.

The null values form a cone.

Separating a spacelike interior region.

From a generally topologically distinct timelike exterior region.

Remark 2 For $p = 2$, the cone is a pair of intersecting lines. And exceptionally, the interior and exterior regions are topologically the same: wedges. By which there is a space-time exchange symmetry. This case is technically easier in a number of ways [29].

Remark 3 $p = 3$ is minimum for no such exchange to hold. By which 'space' and 'time' are realized distinctly.

Now at the level of PDEs, prescribing data on a spacelike surface works (Cauchy problem from a Cauchy surface).

While prescribing data on a timelike surface does not (the sideways Cauchy problem is ill-posed [66]).

For even $p \geq 4$, the maximally ultrahyperbolic case has

$$N_{\pm}(\boldsymbol{M}) = \frac{p}{2} \, . \tag{124}$$

We term the $p \geq 0$ version of this notion *balanced*. This extends the domain in which 'space'-'time' exchange symmetry applies, where the quotation marks around time are especially nuanced.

For odd $p \geq 5$, the maximally ultrahyperbolic case has the following.

$$N_{+}(\boldsymbol{M}) = \frac{p + 1}{2} \quad \text{and} \quad N_{-}(\boldsymbol{M}) = \frac{p - 1}{2} \quad \text{or vice versa} \, . \tag{125}$$

We term the $p \geq 1$ cases of this *almost-balanced*.

Remark 4 We term any non-minimal-or-maximal ultrahyperbolic matrix *partially ultrahyperbolic*. I.e. within the convention that $N_{+}(\boldsymbol{M}) \geq N_{-}(\boldsymbol{M})$,

$$N_{+}(\boldsymbol{M}) - 2 \geq N_{-}(\boldsymbol{M}) \geq 2 \, . \tag{126}$$

$p = 5$ is minimum for an almost-balanced hyperbolic matrix.

$p = 6$ for a nontrivially partially ultrahyperbolic matrix.

Remark 6 $Q > 0$, $Q < 0$ and $Q = 0$ remain distinguished for ultrahyperbolic matrices. The topology of the corresponding null arena is not however a cone. Nor as straightforward to handle as the cone [29], including as regards separating the other two arenas. *Much nice and simple theory is confined to at-most hyperbolic indefiniteness.*

A.10 Non-degenerate sectors

Structure 1 Given $\boldsymbol{M}$, place it into Jordan normal form with all the nonzero-eigenvalue blocks first and the zero-eigenvalue ones last. Then these nonzero eigenvalue blocks form the matrix's non-degenerate sector. For some purposes – if not others – the zero-eigenvector blocks can be struck off. Leaving us with a non-degenerate matrix for the non-degenerate sector. This can then be analyzed as per the previous two Subsecs.

$p = 2$ is minimum to have an nontrivial elliptic degenerate sector.

$p = 3$ for a hyperbolic one.

$p = 4$ for a hyperbolic one free from space-time exchange symmetry.

$p = 5$ for an ultrahyperbolic one.

$p = 6$ for an almost-balanced one.

$p = 7$ for a nontrivially partially ultrahyperbolic one.

Exercise 3 (Long) Find a minimum-dimensional square matrix exemplifying every qualifier in this Appendix. And every minimum property.

B More on eigenarenas

B.1 Split eigenarenas

Definition 1 The *nonzero eigenspectrum*

$$\mathfrak{E}\text{spec}_*(\boldsymbol{M}) \subseteq \mathfrak{E}\text{spec}(\boldsymbol{M}) , \tag{127}$$

by excision of 0 if present.

Structure 1 The *degenerate-sector eigenspace* is

$$\mathfrak{E}\text{ig}_0(\boldsymbol{M}) = \mathfrak{K}\text{er}(\boldsymbol{M}) . \tag{128}$$

While the *nondegenerate-sector eigenspace* is

$$\mathfrak{E}\text{ig}_*(\boldsymbol{M}) = \mathfrak{I}\text{m}(\boldsymbol{M}) . \tag{129}$$

This is additionally

$$\mathfrak{E}\text{ig}_*(\boldsymbol{M}) = \bigoplus_{e\,=\,\mathfrak{E}\text{spec}_*} \mathfrak{E}\text{ig}_e(\boldsymbol{M}) = \bigoplus_{e\,=\,1}^{E_*} \mathfrak{E}\text{ig}_e(\boldsymbol{M}) . \tag{130}$$

Thus the whole eigenspace admits the decomposition

$$\mathfrak{E}\text{ig}(\boldsymbol{M}) = \mathfrak{E}\text{ig}_*(\boldsymbol{M}) \oplus \mathfrak{E}\text{ig}_0(\boldsymbol{M}) . \tag{131}$$

Which is just a spectral reformulation of (114).

Definition 2 For a symmetric matrix $\boldsymbol{M}$, the *positive* and *negative eigenspectra* are as follows.

$$\mathfrak{E}\text{spec}_\pm(\boldsymbol{M}) \subseteq \mathfrak{E}\text{spec}_*(\boldsymbol{M}) \tag{132}$$

now by excision of the negative and positive eigenvalues respectively.

Structure 2 The *positive* and *negative eigenspaces* are as follows.

$$\mathfrak{E}\text{ig}_\pm(\boldsymbol{M}) = \bigoplus_{e_\pm\,\in\,\mathfrak{E}\text{spec}_\pm(\boldsymbol{M})} \mathfrak{E}\text{ig}_{e_\pm}(\boldsymbol{M}) = \bigoplus_{e_\pm\,=\,1}^{E_\pm} \mathfrak{E}\text{ig}_{e_\pm}(\boldsymbol{M}) . \tag{133}$$

Structure 3 The *nondegenerate-sector eigenspace* furthermore fine-splits into

$$\mathfrak{E}\text{ig}_*(\boldsymbol{M}) = \mathfrak{E}\text{ig}_+(\boldsymbol{M}) \oplus \mathfrak{E}\text{ig}_-(\boldsymbol{M}) . \tag{134}$$

Remark 1 Thus

$$\mathfrak{E}\text{ig}(\boldsymbol{M}) = \mathfrak{E}\text{ig}_+(\boldsymbol{M}) \amalg \mathfrak{E}\text{ig}_-(\boldsymbol{M}) \amalg \mathfrak{E}\text{ig}_0(\boldsymbol{M}) \tag{135}$$

is the full split of the eigenspectrum by sign.

While the eigenspace split by sign is the first of the following.

$$\mathfrak{V}_p = \mathfrak{E}\text{ig}(\boldsymbol{M}) = \mathfrak{E}\text{ig}_{e_+}(\boldsymbol{M}) \oplus \mathfrak{E}\text{ig}_{e_-}(\boldsymbol{M}) \oplus \mathfrak{E}\text{ig}_0(\boldsymbol{M})$$

$$= \bigoplus_{e_+ \,\in\, \mathfrak{E}\mathrm{spec}_+(\boldsymbol{M})} \mathfrak{E}\mathrm{ig}_{e_+}(\boldsymbol{M}) \;\oplus\; \bigoplus_{e_- \,\in\, \mathfrak{E}\mathrm{spec}_-(\boldsymbol{M})} \mathfrak{E}\mathrm{ig}_{e_-}(\boldsymbol{M}) \;\oplus\; \mathfrak{E}\mathrm{ig}_0(\boldsymbol{M})$$

$$= \bigoplus_{e_+ \,=\, 1}^{E_+} \mathfrak{E}\mathrm{ig}_{e_+}(\boldsymbol{M}) \;\oplus\; \bigoplus_{e_- \,=\, 1}^{E_-} \mathfrak{E}\mathrm{ig}_{e_-}(\boldsymbol{M}) \;\oplus\; \mathfrak{E}\mathrm{ig}_0(\boldsymbol{M}) \quad . \tag{136}$$

Finally, the second of these is the signed-grouping version of the finest graining of the vector space by eigenspaces.

♝ ♙ ♞

B.2 Eigenvectors and eigenspaces for a single matrix

Remark 1 A matrix's overall eigenbasis carries a partition by the eigenvalues to which each eigenvector belongs.

Remark 2 Eigenspaces have a $(\gamma_e - 1)$-parameter freedom in choosing eigenvectors up to sign. Which is nontrivially realized whenever the eigenspace is nontrivial. And 2^{γ_e} sign-choice freedoms.

Remark 3 An individual eigenvalue's eigenspace is a direct sum of Jordan blocks,

$$\mathfrak{E}\mathrm{ig}_e(\boldsymbol{M}) \quad = \quad \bigoplus_{j\,=\,1}^{J_e} (\boldsymbol{M}) \ . \tag{137}$$

The *trivial Jordan block* is just the eigenvalue. The *minimally nontrivial Jordan block* is

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \ . \tag{138}$$

Larger nontrivial Jordan blocks are sparse. With the eigenvalue in the principal diagonal and 1's in its first upper parallel comprising the sole nonzero values.

Naming Remark 1 The previously mentioned *non-diagonalizable* alias *defective* is also alias *Jordan nontrivial.*

Remark 4 Jordan block matrices possess $< p$ LI eigenvectors [99]. I.e. there is less than a basis amount of them. By which eigenvectors fail to span the eigenspace.

Remark 2 Symmetric $\Rightarrow$ diagonalizable leaves no room for nontrivial Jordan blocks. So we have no more to say in the current Article about Jordan-nontrivial blocks; or effects dependent on these.

B.3 Eigenvectors and eigenspaces for sets of commuting matrices

Remark 1 We observe the following, even for symmetric matrices.

They can share eigenbasis, whether or not they share eigenspaces.

They can fail to share eigenbasis, whether or not they share eigenspaces.

On some occasions, the previous subsection's freedom has the flexibility to align eigenbases. While on others, it does not. Including by eigenvectors that would need to be linearly combined being partitioned off in distinct eigenspaces for ≥ 1 of our matrices.

Remark 2 Let us also mention *unique specification.* In the sense that picking enough matrices that their collection of eigenvalues uniquely specifies each eigenvector in a shared eigenbasis.

Remark 3 We leave these as informal observations for now. For one of us is currently considering the following. Whether the (often mathematically non-rigourous) literature on interplay between commutation, shared eigenbases and unique specification actually covers all possibilites afforded by 'eigenvalue degeneracy'. In the light that 'eigenvalue degeneracy' = 'eigenspace nontriviality' is a Combinatorial matter. And that (at least many accounts in the literature and in courses) have not made systematic use of Cominatorially-adroit Linear Algebra. The literature in question uses '*CSCO*' as its keyword (*complete sets of commuting observables*). And is part of the QM literature, where there is interest in using eigenvalues to label eigenstates.

End Remark 1 Commuting sets of matrices from the current Article are among those entering the ensuing critical reappraisal of this topic.

C Tabulation of eigentheory for P, the Pt and the Ba

C.1 Eigenvalue-level analysis

This is provided in Fig 7.

Some Linear-Algebraic properties of current Article's cyclic quadrilateral matrices

Matrix	Eigenvalues	Rank	Nullity	s_{Math}	s_{Phys}	$s_{Phys\text{-}detail}$	Multiplicities		Degeneracy and indefiniteness		Further Algebra	
Lagrange projection $\boldsymbol{L = P}$ 4×4	0, 1 1, 3 1, 3 1, 1	3	1	3	2	+ + +0	Diagonalizable. Minimally-minimal.		Degenerate. Nondegenerate sector is elliptic.		Commutes with the $\boldsymbol{Pt}$. LD with them. Commutes with the $\boldsymbol{Ba}$ but only partly with the $\boldsymbol{ba}$.	Projection $\boldsymbol{P}^2 = \boldsymbol{P}$.
Ptolemy separations matrix $\boldsymbol{Pt}$ 6×6	1, -1 3, 3 3, 3 1, 1	6	0	3	0	+ + + - - -		Semisimple. Minimally-nontrivial. Geomdominant.	Nondegenerate. Ultra-hyperbolic. Balanced.	Null is cyclic. Elsewise must be spacelike by Ptolemy's inequality.		Are all involutions: $\boldsymbol{M}^2 = \mathbb{I}$
Ptolemy sides matrix $\boldsymbol{Pt}_L$ 4×4	1, -1 2, 2 2, 2 1, 1	4	0	2	0	+ + - -				Same sign of $\mathbf{Pt_1}$ and $\mathbf{Pt}_d$ is cyclic. Elsewise, both must be spacelike by length axioms and Ptolemy's inequality	Commute with each other and with $\boldsymbol{P}$	
Ptolemy diagonals matrix $\boldsymbol{Pt}_d$ 2×2	1, -1 1, 1 1, 1 1, 1	2	0	1	0	+ -		Simple. Geomdominant.	Nondegenerate. Hyperbolic with space-time exchange symmetry			
Brahmagupta matrices $\boldsymbol{Ba}$ **and** $\boldsymbol{ba}_g$ 4×4	2, -2, 0 1, 1, 2 1, 1, 2 1, 1, 1	2	2	1	0	+ - 0 0		Semisimple. Minimally-nontrivial	Degenerate. Nondegenerate sector is hyperbolic with space-time exchange symmetry	Must be spacelike by Brahmagupta and Area > 0 for cyclic	$\boldsymbol{Ba}$ commutes with the others. $\boldsymbol{ba}$ commute with $\mathbf{Pt_1}$ alone	Are not quite involutions, in that $\boldsymbol{M}^3 = \boldsymbol{M}$ without cancellation.

Key			
	λ_e	Eigenvalue	
	α_e	Algebraic	multiplicities
	γ_e	Geometric	
	μ_e	Minimal	

Figure 7:

C.2 Eigenvectors and eigenspaces

Eigenvectors and eigenspaces for our 4×4 matrices									
Matrix	**Some eigentheory**					**Geometric interpretation**			
Lagrange projector $L = P$	$\frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$	$\frac{1}{\sqrt{12}}\begin{pmatrix}1\\1\\1\\-3\end{pmatrix}$	$\frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\-2\\0\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}$	K-eigenclustering. Not aligned with any of our other matrices' eigenvectors.				
	LC ↕								
	$\frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\-1\end{pmatrix}$	H-eigenclustering. Eigenvectors aligned with $\boldsymbol{Pt}_1$ but not with $\boldsymbol{ba}_1$.	$P = \sum a$ perimeter	$\not\sum a$ alternating sum: opposite sides contribute with opposite sign	$a - b$	$c - d$
Eigenvalues	0	1							
Geometric multiplicities	1	+ 3			= 4				
Eigenspaces	$\mathfrak{e}\mathrm{ig}_0(\boldsymbol{P})$	$\oplus$ $\mathfrak{e}\mathrm{ig}_+(\boldsymbol{P})$			$=$ $\mathfrak{e}\mathrm{ig}(\boldsymbol{P})$				
	$\mathbb{R}$	$\oplus$ $\mathbb{R}^3$			$=$ $\mathbb{R}^4$				
Ptolemy sides involution $\boldsymbol{Pt}_3 = \boldsymbol{J}_3$	$\frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\-1\end{pmatrix}$	Eigenvectors aligned with $\boldsymbol{P}$.				
	LC ↕								
	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\1\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\-1\end{pmatrix}$	Sparse. Block-adapted. Eigenvectors aligned with $\boldsymbol{ba}_1$.	$a + b$	$c + d$	$a - b$	$c - d$
Eigenvalues	1		-1						
Geometric multiplicities	2		+ 2		= 4				
Eigenspaces	$\mathfrak{e}\mathrm{ig}_+(\boldsymbol{J}_3)$		$\oplus$ $\mathfrak{e}\mathrm{ig}_-(\boldsymbol{J}_3)$		$=$ $\mathfrak{e}\mathrm{ig}(\boldsymbol{J}_3)$				
	$\mathbb{R}^2$		$\oplus$ $\mathbb{R}^2$		$=$ $\mathbb{R}^4$				
Brahmagupta factor matrix $\boldsymbol{ba}_1$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\1\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\-1\end{pmatrix}$	Sparse. Block-adapted. Eigenvectors aligned with $\boldsymbol{J}_3$ but not with $\boldsymbol{P}$.				
Eigenvalues	2	0		-2					
Geometric multiplicities	1	+ 2		+ 1	= 4				
Eigenspaces	$\mathfrak{e}\mathrm{ig}_2(\boldsymbol{ba}_1)$	$\oplus$ $\mathfrak{e}\mathrm{ig}_0(\boldsymbol{ba}_1)$		$\oplus$ $\mathfrak{e}\mathrm{ig}_{-2}(\boldsymbol{ba}_1)$	$=$ $\mathfrak{e}\mathrm{ig}(\boldsymbol{ba}_1)$				
	$\mathbb{R}$	$\oplus$ $\mathbb{R}^2$		$\oplus$ $\mathbb{R}$	$=$ $\mathbb{R}^4$				
Brahmagupta matrix $\boldsymbol{Ba}$	$\frac{1}{2}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}$	Sparse. Eigenvectors not aligned with any of our other matrices'.	$2l_+ = 2Pt_1$	$\sum A - 2l_-$	$\sum A + 2l_-$	$\not\sum A$
Eigenvalues	2	0		-2					
Geometric multiplicities	1	+ 2		+ 1	= 4				
Eigenspaces	$\mathfrak{e}\mathrm{ig}_2(\boldsymbol{Ba})$	$\oplus$ $\mathfrak{e}\mathrm{ig}_0(\boldsymbol{Ba})$		$\oplus$ $\mathfrak{e}\mathrm{ig}_{-2}(\boldsymbol{Ba})$	$=$ $\mathfrak{e}\mathrm{ig}(\boldsymbol{Ba})$				
	$\mathbb{R}$	$\oplus$ $\mathbb{R}^2$		$\oplus$ $\mathbb{R}$	$=$ $\mathbb{R}^4$				

Figure 8:

Remark 1 This is provided in Fig 8 for 4×4 . Fig 9 for 2×2 . And Fig 10 for 6×6 .

Remark 2 All 3 middling partitions [120] of 4 objects are represented.

$3 \,|\, 1$ by $\boldsymbol{P}$.

$2 \,|\, 2$ by the $\boldsymbol{Pt}$.

And $2 \,|\, 1 \,|\, 1$ by the $\boldsymbol{Ba}$.

Remark 3 Among these, the green and orange eigenbasis choice for $\boldsymbol{Pt}_3$'s $2 \,|\, 2$ shares eigenbasis and yet clearly not eigenspaces with the following. The blue and yellow ' H ' eigenclustering eigenbasis choice for $\boldsymbol{P}$'s $3 \,|\, 1$.

While the teal and pink eigenbasis choice for $\boldsymbol{Pt}_3$'s $2 \,|\, 2$ shares instead eigenbasis and yet not eigenspaces with the following. The beige, brown and red eigenbasis choice for $\boldsymbol{ba}_1$'s $2 \,|\, 1 \,|\, 1$.

Remark 4 Geometrical interpretations are tabulated up to proportion.

Eigenvectors and eigenspaces for our 2×2 matrix			
Geometric interpretation	$e + f$	$e - f$	
Ptolemy diagonals involution $\boldsymbol{Pt}_{\mathrm{d}} = \boldsymbol{J}_{\mathrm{d}}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$	
Eigenvalues	1	-1	
Geometric multiplicities	1 $+$	1 $=$	2
Eigenspaces	$\mathfrak{E}\mathrm{ig}_{+}(\boldsymbol{J}_{\mathrm{d}})$ $\oplus$ $\mathbb{R}$ $\oplus$	$\mathfrak{E}\mathrm{ig}_{-}(\boldsymbol{J}_{\mathrm{d}})$ $=$ $\mathbb{R}$ $=$	$\mathfrak{E}\mathrm{ig}(\boldsymbol{J}_{\mathrm{d}})$ $\mathbb{R}^2$

Figure 9:

Eigenvectors and eigenspace for our 6×6 matrices									
Geometric interpretation	$a+b$	$c+d$	$e-f$		$a-b$	$c-d$	$e+f$		
Ptolemy separations involution $\boldsymbol{Pt} = \boldsymbol{J}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\\0\\0\\0\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\1\\0\\0\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\0\\0\\1\\-1\end{pmatrix}$		$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\\0\\0\\0\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\-1\\0\\0\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\0\\0\\1\\1\end{pmatrix}$		
Eigenvalues		1				-1			
Geometric multiplicities		3		+		3		=	6
Eigenspaces		$\mathfrak{E}\mathrm{ig}_+(\boldsymbol{J})$		$\oplus$		$\mathfrak{E}\mathrm{ig}_-(\boldsymbol{J})$		=	$\mathfrak{E}\mathrm{ig}(\boldsymbol{J})$
		$\mathbb{R}^3$		$\oplus$		$\mathbb{R}^3$		=	$\mathbb{R}^6$

Figure 10:

♠ ◇ ♣

Remark 4 Finally, Fig 11 decomposes separations-Ptolemy's eigenvectors into padded versions of sides-Prolemy and diagonals-Ptolemy.

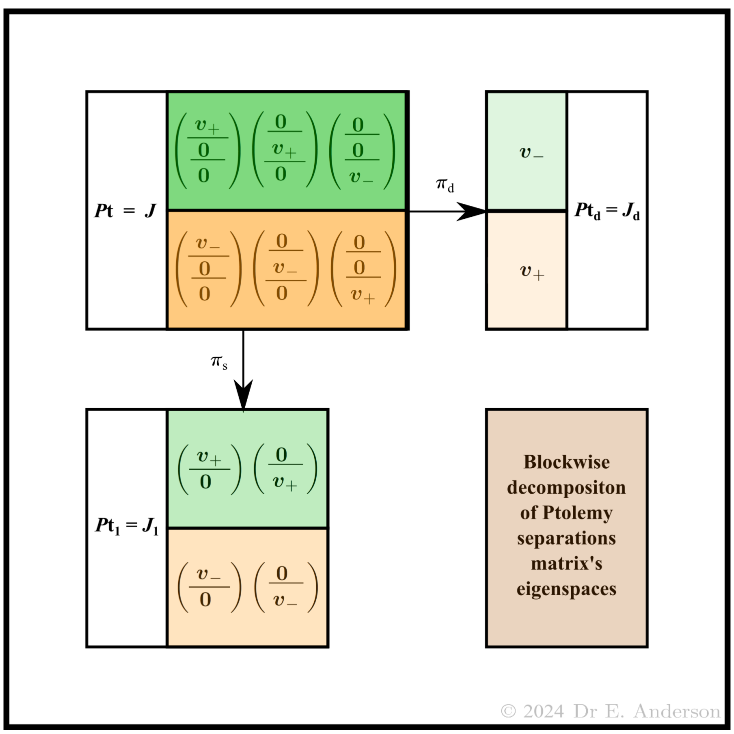


Figure 11:

D Supporting Order Theory

D.1 Posets and lattices

Definition 1 A relation $\preceq$ is a *partial ordering* if it is reflexive, antisymmetric and transitive. A set equipped with a partial ordering is termed a *poset* $\mathfrak{p}$ [74, 97, 49, 120].

Definition 2 A *lattice* $\mathfrak{L}$ [23, 25, 74, 78, 120] is a poset for which each pair of elements a, b possesses both of the following.

i) A *join* (least upper bound), denoted by

$$a \wedge b .$$

ii) A *meet* (greatest lower bound), denoted by

$$a \vee b .$$

Definition 3 An element 1 of a connected poset $\mathfrak{p}$ is a *unit* alias *top element* if the following holds.

$$\forall \; l \in \mathfrak{p} , \; l \preceq 1 . \tag{139}$$

An element 0 of a connected poset $\mathfrak{p}$ is a *zero* alias *bottom element* if the following holds.

$$\forall \; l \in \mathfrak{p} , \; 0 \preceq l . \tag{140}$$

Remark 1 For lattices, these need not exist. But if they do, then they are unique. They need not be unique in posets. A lattice that possesses both of these is termed a *bounded lattice.*

Definition 4 A lattice $\mathfrak{L}$ is *complemented* if for each element $a \in \mathfrak{L}$, there is another element $b \in \mathfrak{L}$ such that the following hold.

$$a \vee b = 0$$

and

$$a \wedge b = 1 .$$

D.2 Power sets as p-cube lattices

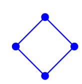
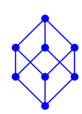
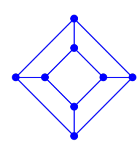
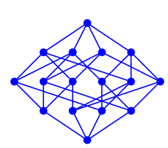
Power set lattices alias free lattices alias *p*-cube lattices		
N	**Lattice**	**Useful rep of graph skeleton**
1	D = Cube(0)	
2	P_2 = Cube(1)	
3	 C_4 = Cube(2)	
4	 Cube(3) = Cube	 Prism(4)
5	 Cube(4)	

Figure 12:

Structure 1 The arena of labelled subsets of an N -element set form the *power set*

$$\mathfrak{P}(N) \ .$$

This concurrently carries the following levels of mathematical structure [120].

Using subset-expansion order they are Fig 12.b)'s free lattices F_N alias N-cube lattices $\mathrm{Cube}(N)$.

They are bounded, with

$$\mathfrak{P}(N)_0 \ = \ \emptyset$$

and

$$\mathfrak{P}(N)_1 \ = \ \mathfrak{X}(N) \ :$$

the N-element universe set.

They are distributive complemented lattices. The previous two sentences combine to establish that power sets $\mathfrak{P}(N)$ are Boolean algebras.

They are also the commutative groups ${C_2}^N$.

Motivation 1 For some purposes, these free lattices are too regular to be interesting. Departures from these are then needed. Consider for instance algebraic structures, such as groups, commutative groups, Lie algebras or Lie algebroids. Here algebraic relations forbid certain combinations of elements (or generators) from forming corresponding subalgebric structures. The Physics and Dynamics keyword associated with these is '*integrability conditions*' [45, 61, 119].

Example A Relational Mechanics' [93, 104, 116] Euclidean $\mathcal{L}$ and chronos $\mathcal{E}$ (equation of time) constraints form blockwise the square lattice of Fig 13.a). Corresponding to no integrabilities being present. By which either of spatial and temporal Relationalism [104, 116] can be realized in the absence of the other.

But General Relativity's (GR's) Hamiltonian constraint $\mathcal{H}$ cannot be entertained in the absence of its momentum contraint. For here the momentum constraint $\mathcal{M}$ is enforced by the Dirac Algorithm [32, 54, 104, 117] as follows. *Teitelboim's integrability condition* [38] from the Poisson bracket of the Hamiltonian constraint with itself. Consequently, the *Dirac algebroid* [32, 40, 94, 104] formed by GR's constraints blockwise is the 3-chain of Fig 13.b) .

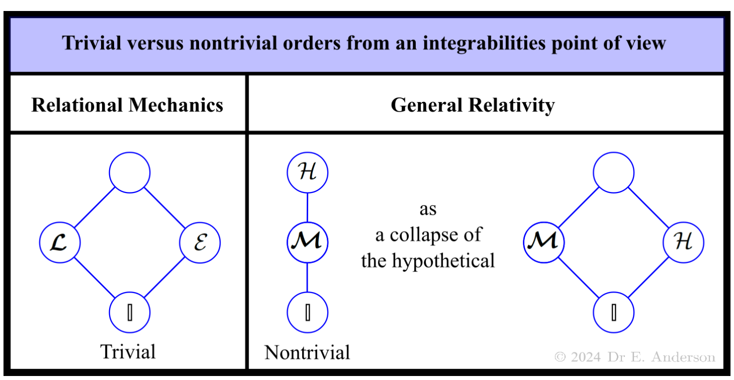


Figure 13:

Now in one sense this constraint algebroid is Order-Theoretically trivial because it is a chain. But in another sense it is nontrivial since the default is that it should be free, i.e. the 2-cube lattice, i.e. the square lattice. And its departure therefrom – 1 missing vertex and 2 missing edges – encodes Teitelboim's integrability condition. Which turns out to be a key feature of GR as a dynamical system, and in canonical attempts to quantize GR [35, 55, 58, 104].

D.3 Posets comprising competing lattices

Structure 2 Such a phenomenon occurs when a lattice (or other order structure [74, 120]) admits multiple incompatible extensions.

Example 1 The minimum example is $P_{3-\mathrm{bent}}$ (row 1 of Fig 14). This is realized in Sec 6.2. All realizations in the current Article come from extending sets of commuting matrices from the theory of quadrilaterals.

Example 2 Another small example realized there is $\mathrm{Banner}_{\mathrm{nexus}}$ (row 2 of Fig 14).

Example 3 The current Article found some larger examples These are in Sec 6.2, and are members of the following specific family of that exhibits competing-lattice behaviour.

Example 4 A well-known case (if not phrased in this way until [112, 121] is as follows. That Conformal Geometry and Projective Geometry are competing top symmetry groups for Flat Geometry.[4] They are incompatible extensions of the similarity group.

D.4 Sánchez' Wings family of competing posets

Structure 1 [115] Take the p-cube and the q-cube lattices. Identify their bottom vertex. This gluing creates $\mathrm{Wings}(\, p - 2, q - 2 \,)$, most of the first few cases of which we tabulate in Fig 14.

Remark 1 The nontrivial wings encountered in the current Article are $\mathrm{Wings}(\, 0, 0 \,)$ and $\mathrm{Wings}(\, 1, 0 \,)$
.

[4]This working is in ≥ 3-d , and to the exclusion of Supersymmetry.

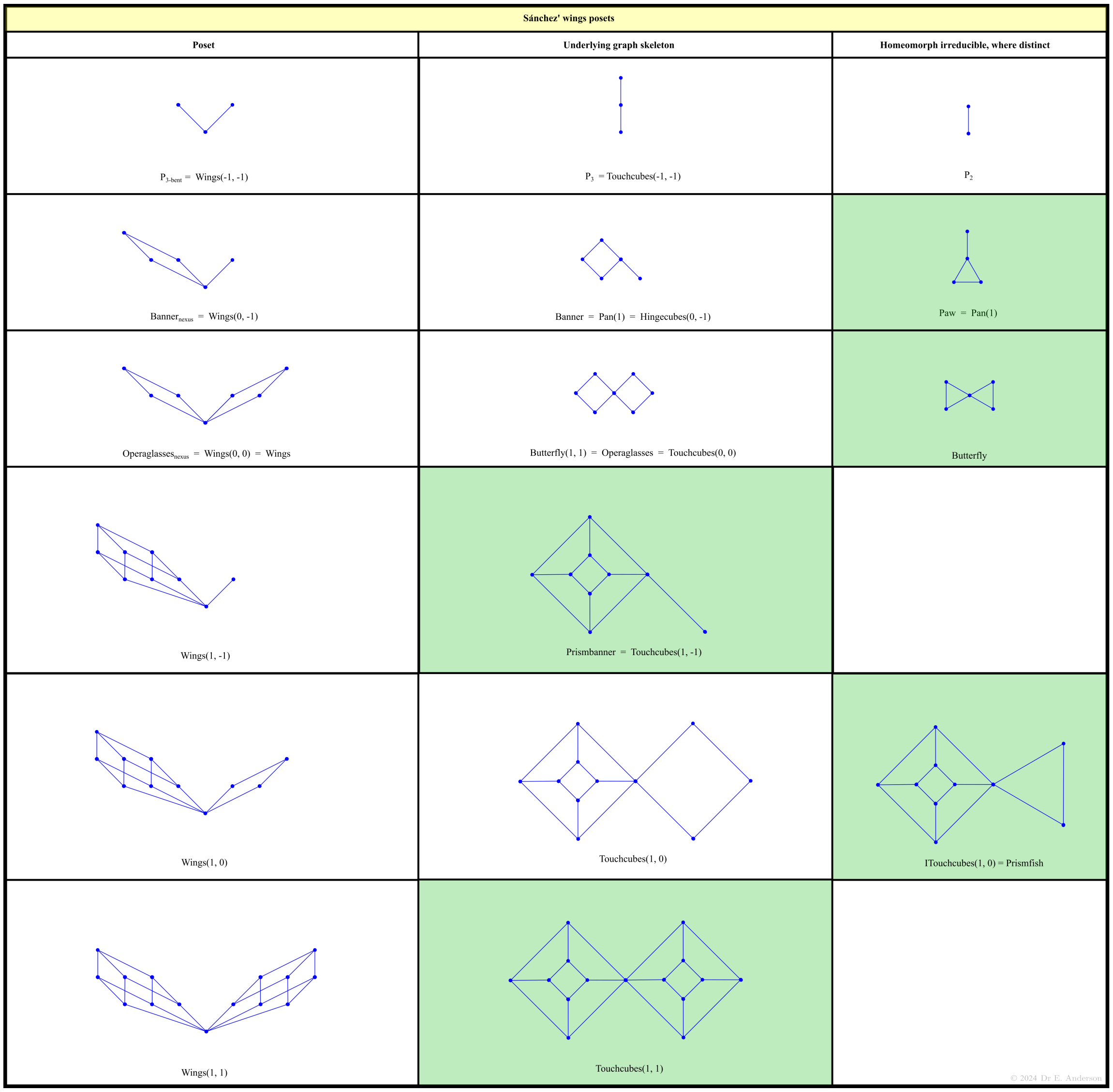


Figure 14:

D.5 Sánchez' Heart family of competing posets

Structure 2 [115] Further identify one bottom-adjacent vertex on each, and each's edge between these two. This gluing creates Heart$(p-2, q-2)$, most of the first few cases of which we tabulate in Fig 15.

Remark 1 The nontrivial heart encountered in the current Article is Heart$(1, 0)$.

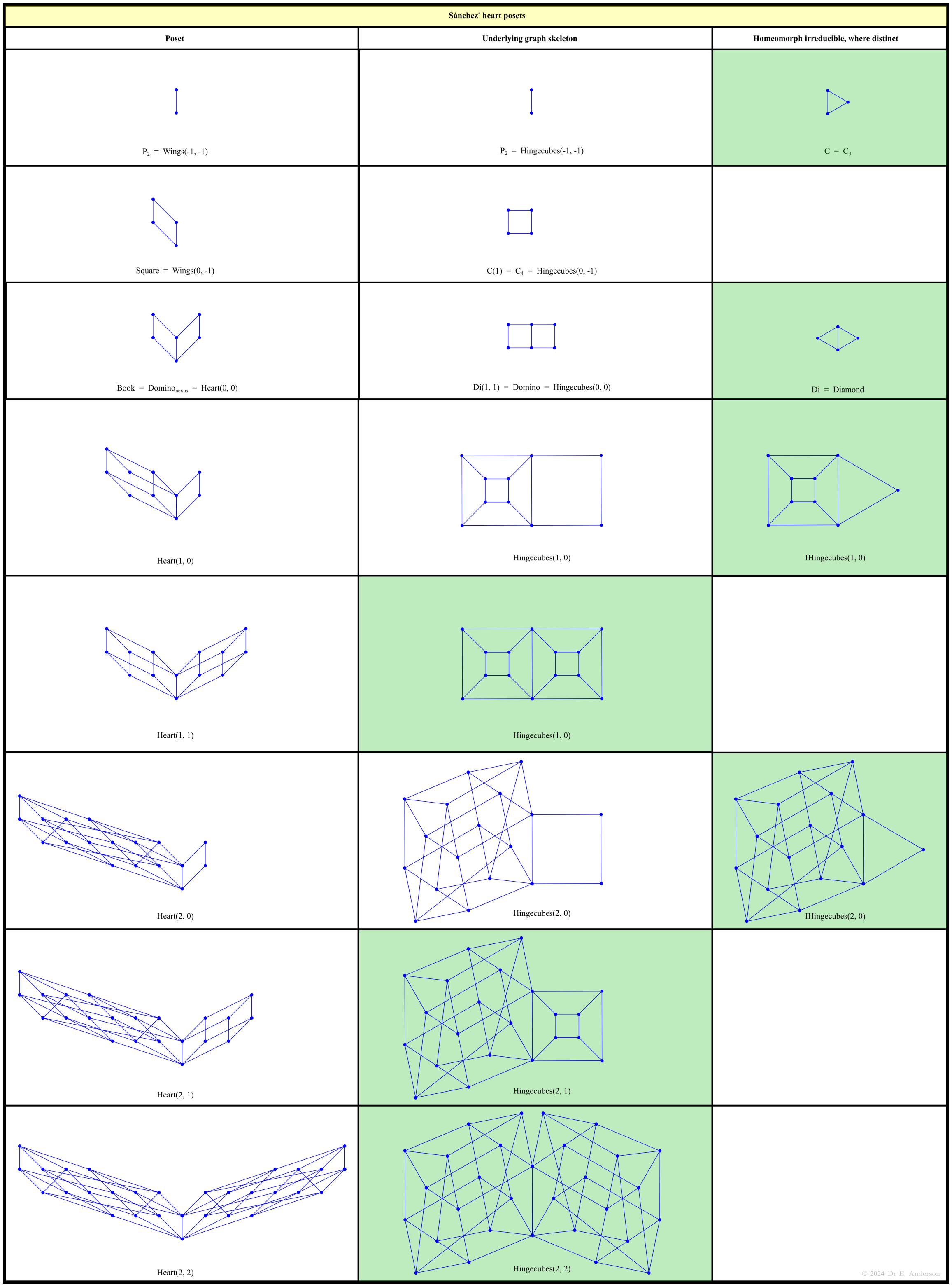


Figure 15:

D.6 Discussion

Remark 1 Wings($p - 2$, $q - 2$) and Heart($p - 2$, $q - 2$) are simple instances of posets realizing competing lattices. Where the individual lattices in question are p-cube lattices (and thus also Boolean algebras and commutative groups [120]).

Notational Remark 1 Our indexing shifts are of the relative or excess type [115, 120]. Calibrated such that the $(1, 0)$ cases are the first to have nontrivial graph skeletons.

D.7 Arenas of wings and hearts

Lemma F

$$\mathfrak{Wings} \cong \mathfrak{Hearts} \cong \mathfrak{p}(2) : \tag{141}$$

the arena of 2-piece partitions. Which forms the trellis-wedge poset of Fig 16.a), whose underlying graph skeleton is the staircase graph of Subfig b).

Remark 1 The number of objects being partitioned is

$$K = p + q .$$

Into pieces $p \,|\, q$, which can be taken to replace the two family labels introduced above. For 2-piece partitions to be realized, $K \geq 2$ is necessary.

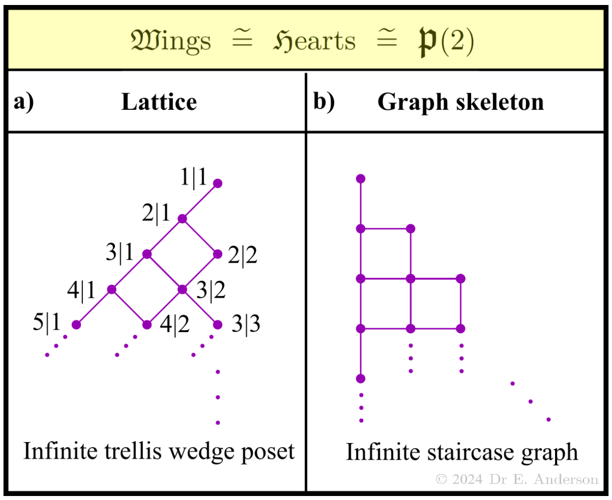


Figure 16:

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