Eigentheory of Combinatorial Matrices

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Abstract

The Combinatorial matrices' eigenvalues and eigenvectors are classified. They have ≤ 2 eigenvalues. Other than in the smallest cases, a symmetry-degenerate eigenvalue accompanies a lone eigenvalue. Excepting in the isotropic subcase, which has just 1 eigenvalue. Finer classification by zero-count degeneracy, signature and signs present is also provided. Given any set of Combinatorial matrices of the same size, their eigenbases can be taken to be shared. Up to the exceptions pointed out in the current Article, their eigenspaces are shared as well.

Dynamics' centre of mass (CoM) hierarchies return the Jacobi vectors as eigenvectors. The relative such exhibit a network ambiguity which corresponds to the unlabelled rooted binary trees. At a first glance, this appears to be an instance of Dynamics producing Combinatorial objects. These eigenvectors turn out however to arise for any Combinatorial matrix. By which a more natural perspective is that Combinatorics produces more Combinatorics, with Dynamics then just reproducing a subcase of this.

The triangle matrices, or more generally 2-simplex matrices, had a number of Abstract Algebra properties pointed out that Ford determined to follow solely from their all being Combinatorial matrices of the same size. The current article extends this conclusion to a number of further Linear Algebra and Spectral Theory properties as well.

Finally eigentheory spectral classifications are taken further using Graph Theory and Order Theory.

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1 Introduction

1.1 Combinatorial matrices, with a first few examples

Definition 1 A *Combinatorial matrix* [17] is a $K \times K$ square matrix of the following form.

$$\boldsymbol{C} = \begin{pmatrix} x + y & x & \dots & x \\ x & & & \vdots \\ \vdots & & & x \\ x & \dots & x & x + y \end{pmatrix} . \tag{1}$$

This is symmetric. $x, y \in \mathbb{Z}$ covers many Combinatorial uses. We however extend to $x, y \in \mathbb{R}$ to encompass Dynamics and Linear Algebra use as well.

Naming and Notational Remark 1 Let us term A. Ford's notation [66]

$$\boldsymbol{C} \ = \ (\, x \ + \ y \ , \ \ x\,)_K \tag{2}$$

Ford's symbol of the zeroth kind.

Remark 1 As explained below, this is the notation selected for use in the current Article. Over the course of which we shall illustrate Combinatorial matrices with many examples. A conceptually-primitive first few are as follows; as we develop our main theme – eigentheory – and by pointing to various Algebraic properties and Geometrical applications, more shall materialize.

Example $\mathbb{O}$) Setting

$$x \ = \ 0 \ = \ y \ , \tag{3}$$

we find the *zero matrices* of all sizes K ,

$$\mathbb{O} \ = \ (\,0,\,0\,)_K \ . \tag{4}$$

Example $\mathbb{I}$) Setting

$$x \ = \ 0 \ , \tag{5}$$

$$y \ = \ 1 \ , \tag{6}$$

we arrive at the *identity matrices*

$$\mathbb{I} \ = \ (\,1,\,0\,)_K \ . \tag{7}$$

While relaxing to

$$x \ = \ 0 \ , \ \ y \ \neq \ 0 \tag{8}$$

yields the *matrices proportional to each identity.*

Example $\mathbb{1}$) Setting instead

$$x \ = \ 1 \ , \ \ y \ = \ 0 \ , \tag{9}$$

we encounter the matrices of 1s,

$$\mathbb{1} \ := \ (\,1,\,1\,)_K \ . \tag{10}$$

Each of which we shall refer to as a *block matrix* .

Now relaxing to

$$x \ \neq \ 1 \ , \ \ y \ = \ 0 \tag{11}$$

produces the *matrices proportional to each block.*

Example $\mathbb{T}$) Next setting

$$t \ := \ x \ + \ y \ = \ 0 \ , \tag{12}$$

$$y \ = \ 1 \ , \tag{13}$$

we obtain the *tracefree version of the blocks*,

$$\mathbb{T} \ := \ (\,0,\,1\,)_K \ . \tag{14}$$

t is here the *trace per unit size*: an intensive variable in Physics parlance. The trace itself is the extensive variable counterpart,

$$T \ = \ K\,t \ = \ K\,(\,x \ + \ y\,) \ . \tag{15}$$

Finally relaxing to

$$t \ := \ x \ + \ y \ = \ 0 \ , \ \ y \ \neq \ 0 \tag{16}$$

returns the *matrices proportional to each tracefree block.*

1.2 Explaining some notation

Notational Remark 2 Among these, $\mathbb{I}$ and $\mathbb{T}$ are distinguished as the pieces of $\mathbb{1}$ that are irreducible: Representation-Theoretically [19, 42] significant.[1] We pick Ford's symbol of the zeroth kind as the current Article's notation since this corresponds to expanding each $\boldsymbol{C}$ with respect to the linear basis (LB) consisting of these irreducibles. So e.g. a coordinate-free rendition of (1) is

$$\boldsymbol{C} \;=\; x\,\mathbb{1} \;+\; y\,\mathbb{I}\;. \tag{17}$$

As an incipient foil, *Ford's symbol of the first kind* [66]

$$\boldsymbol{C} \;=\; [\,y\;,\;\;x]_K \tag{18}$$

involves the LB consisting of the identity and the block: $\mathbb{I}$ and $\mathbb{1}$. Indeed, this bracket being square alludes to using the block. Whereas the roundness of the zeroth symbol's bracket amounts to reserving the most commonly encountered bracket for our most commonly used symbol.

Example $\mathbb{V}$) Having encountered trace via tracefree Combinatorial matrices dropping out of the above primitive analysis, the following is also natural to consider. Setting

$$t \;=\; x \;+\; y \;=\; -\,1\;, \tag{19}$$

$$x \;=\; 1 \tag{20}$$

introduces the *trace-reversed blocks*

$$\mathbb{V} \;=\; (\,-1,\,1\,)_K\;. \tag{21}$$

Relaxing to

$$t \;=\; x \;+\; y \;=\; -\,x \;\neq\; 0 \tag{22}$$

leaves us with the *matrices proportional to the trace-reversed blocks.*

Notational Remark 3 The following further foil notation is then espied. *Ford's symbol of the minus-oneth kind* [66]

$$\boldsymbol{C} \;=\; \langle\,y \;+\; 2\,x,\,x\,\rangle_K \tag{23}$$

corresponds to the LB consisting of the identity and the trace-reversed block: $\mathbb{I}$ and $\mathbb{V}$. This is the minus-oneth symbol in the sense of being the reflection about the zeroth symbol of the first symbol. As an 'image' of a type of piecewise-linear bracket – square – it is then denoted by another type of piecewise-linear bracket: the chevron.

Naming Remark 2 *Irreducible*, *block* and *trace-reversed* cases of *Combinatorial matrix symbols* are respectively truer names for the above three symbols.

1.3 Outline of the rest of this Article

We introduce arenas [58, 51, 59] in Sec 2, with Combinatorial matrix [66] and widely-used Linear Algebra [10, 45, 48] examples. We classify Combinatorial matrices' eigenvalues and eigenspectra in Sec 3. And Eigenvectors and eigenspaces in Sec 4. Including the paradigm shift from N-body problem terminology and conceptualization to the general Combinatorial matrix setting. Sec 4 includes pointing out that any Combinatorial matrices of the same size can be taken to share eigenbases. They all need to be isotropic, or all need to be generic, in order to share eigenspaces as well.

In the process, further Algebraic and Geometric examples of specific Combinatorial matrix are pointed out. A. Ford [66] has recently demonstrated that various Abstract Algebra properties that I attributed to [54, 60, 61, 62] the 3 triangle matrices follow purely from this being Combinatorial matrices. The current Article complements this with Linear Algebra and Spectral Theory results. Appendix A includes condensing most of Sec 3 and 4's results. The remainder – the paradigm shift – enters instead the Conclusion (Sec 5)'s comparison table. Finally Appendix B takes the classification of eigenspectra further, using Order Theory [58, 31, 39] and Graph Theory [37] underpinning this.

[1] Matrices proportional to the identity also have particular Representation-Theoretic significance through entering Schur's Lemma.

2 Arenas

Structure 0 Given a type of mathematical object, the corresponding *arena* is the space formed by the totality of mathematical objects of this type. What topologies are natural to each arena is then always a good question. So via arenas, Modern Applied Topology [58, 51, 59] becomes adjacent to every other STEM subject, or indeed to every other subject with at least some objects sharply-defined enough that we can contemplate what their arenas are.

2.1 Arenas of Combinatorial matrices

Definition 0 Focusing this modern Applied Topology line of thought on our main subject matter, Combinatorial matrices form the arenas

$$\mathfrak{CM}_{\mathbb{R}}(K)$$

for each fixed K. The cumulative arenas

$$\mathfrak{CM}_{\mathbb{R}}[K] \ := \ \coprod_{P\,=\,0}^{K} \mathfrak{CM}_{\mathbb{R}}(P)$$

up to whichever fixed K. And the fully cumulative arena

$$\mathfrak{CM}_{\mathbb{R}} \ := \ \mathfrak{CM}_{\mathbb{R}}[\infty] \ .$$

2.2 Vector spaces

Remark 1 Both for the Reader's convenience and as regards developing our subject matter, various examples of arenas that entered widespread use long before the advent of modern Applied Topology are provided in the current Article.

Vector spaces $\mathfrak{V}$ over $\mathbb{R}$ are our first such: the arenas of all $\mathbb{R}$-linear combinations (LCs) of vectors with a given number of components. Or, more generally of whichever objects that are meaningfully represented by vectors. So e.g. polynomials, matrices under addition and multiplication by a scalar, and functions [46] are also covered.

Structure 1 We have the good fortune that the $\mathfrak{CM}_{\mathbb{R}}(K)$ are vector spaces.

And the further good fortune that for $K \geq 2$, they are copies of the same vector space. This occurs via each Combinatorial matrix being described by just 2 parameters − x and y, which are both active for $K \geq 2$. These can each take arbitrary values in $\mathbb{R}$. Yielding the common vector space

$$\mathfrak{CM}_{\mathbb{R}}(K \ \geq \ 2) \ = \ \mathbb{R}^2 \ . \tag{24}$$

This accounts for why the 3 LBs mentioned each have 2 elements.

For $K = 1$, Combinatorial matrices collapse to just numbers. With only 1 active parameter: the one that becomes the trace, $t = x + y$. Yielding the 1-d vector space $\mathbb{R}^1$ over $\mathbb{R}$, i.e. just $\mathbb{R}$ itself:

$$\mathfrak{CM}_{\mathbb{R}}(1) \ = \ \mathbb{R}^1 \ = \ \mathbb{R} \ . \tag{25}$$

Finally $K = 0$ is exceptional in that the sole object supported here is the *unmatrix*, which in many ways is not a matrix at all. While the unmatrix contains just an empty set's amount of information, the set of unmatrices itself constitutes a point. So identifying this as the zero point, it is possible to regard $\mathfrak{CM}_{\mathbb{R}}(K)$ as a vector space, $\mathbb{R}^0 = \{0\}$.

Remark 2 Thus also the $\mathfrak{CM}_{\mathbb{R}}[K]$ are disjoint sums of vector spaces. With ≤ 3 types of vector space present, due to the swift onset of the persistent $\mathbb{R}^2$ vector space.

So we also have

$$\begin{array}{lclcl} \mathfrak{CM}_{\mathbb{R}}[0] & = & \mathbb{R}^0 & = & \{0\} \ , \\ \mathfrak{CM}_{\mathbb{R}}[1] & = & \mathbb{R}^0 \amalg \mathbb{R}^1 & = & \{0\} \amalg \mathbb{R} \ , \\ \mathfrak{CM}_{\mathbb{R}}[K] & = & \mathbb{R}^0 \amalg \mathbb{R}^1 \amalg \coprod_{p\,=\,1}^{k} \mathbb{R}^2 & = & \{0\} \amalg \mathbb{R} \amalg \coprod_{p\,=\,1}^{k} \mathbb{R}^2 \ . \end{array} \tag{26}$$

Where the last result holds for each $K \geq 2$, with

$$k := K - 1 . \tag{27}$$

This immediately extends to describing $\mathfrak{CM}_{\mathbb{R}}$ as well:

$$\mathfrak{CM}_{\mathbb{R}} = \mathbb{R}^0 \amalg \mathbb{R}^1 \amalg \coprod_{p = 1}^{\infty} \mathbb{R}^2 = \{0\} \amalg \mathbb{R} \amalg \coprod_{p = 1}^{\infty} \mathbb{R}^2 . \tag{28}$$

2.3 Eigenspectra, multiplicities, and eigenspaces

Definition 1 [18, 48, 35, 45, 74] Let λ_e denote some eigenvalue of a size-K square matrix $\boldsymbol{M}$. The *algebraic multiplicity* α_e of λ_e is the number of times that it occurs as a root of the characteristic polynomial.

Remark 3 The totality of eigenvalues for our matrix form its *eigenspectrum*, with algebraic multiplicities included, $\mathfrak{E}\text{spec}(\boldsymbol{M})$. With algebraic multiplicities excluded, we have a sparser version $\mathfrak{e}_{\text{spec}}(\boldsymbol{M})$. With however

$$\sum_{\mathfrak{e}\text{spec}(\boldsymbol{M})} \alpha_e = K . \tag{29}$$

Eigenspectra are also arenas, albeit, for finite matrices, they are rather structurally simple ones.

Definition 2 An eigenvalue λ_e's *minimal multiplicity* μ_e is the number of times that it occurs as a root of the minimal polynomial.

Remark 4 Denote eigenvectors corresponding to λ_e by $\boldsymbol{v}_{e_i}$. So as to form a linearly independent (LI) set that is as large as possible. I.e. a LB for the *eigenspace* $\mathfrak{e}\text{ig}_e(\boldsymbol{M})$ corresponding to λ_e. In some cases, this is of dimension α_e, while in others, not as many LI eigenvectors as this can be found. This deficit is measured by the following further multiplicity.

Definition 3 An eigenvalue λ_e's Its *geometrical multiplicity* γ_e is the dimension of its corresponding eigenspace,

$$\gamma_e := \dim (\mathfrak{e}\text{ig}_e(\boldsymbol{M})) . \tag{30}$$

Remark 5 There is also a larger notion of eigenspace:

$$\mathfrak{e}\text{ig}(\boldsymbol{M}) := \bigoplus_{e \in \mathfrak{e}\text{spec}(\boldsymbol{M})} \mathfrak{e}\text{ig}_e . \tag{31}$$

These smaller and larger notions of eigenspace are somewhat more structured simple and widely-used examples of arena. Their structure lies well within basic Linear Algebra, consisting of vector subspaces and direct sums thereof respectively.

3 Combinatorial matrices' eigenspectra

3.1 Eigenvalue degeneracy due to symmetry

Naming Remark 3 This involves one of two unrelated uses of 'degeneracy' used in literature on Eigentheory and its applications. Namely, that which is often found in Quantum Mechanics (QM) [33, 16, 15, 49] and Mathematical Physics [13]. Here, algebraic multiplicity $\alpha_e > 1$ for some eigenvalue λ_e .

Classification Theorem 1 for Combinatorial matrices)

G) A Combinatorial matrix's eigenspectrum consists of the following.

$$z := K\,x + y \text{ with algebraic multiplicity } 1\,, \tag{32}$$

$$y \text{ with algebraic multiplicity } k\,, \tag{33}$$

Unless the Combinatorial matrix is of one of the following exceptional kinds.

I)

$$x = 0\,, \tag{34}$$

for which the sole eigenvalue is

$$y \text{ with algebraic multiplicity } K\,. \tag{35}$$

U)

$$K = 0\,, \tag{36}$$

for which there are no eigenvalues at all.

Notational Remark 4 Let us henceforth index these eigenvalues by their algebraic multiplicities! Let us also denote the $\boldsymbol{C}$ of type G) by $\boldsymbol{G}$ and those of type I) by $\boldsymbol{I}$.

Remark 1 These exceptional cases arise from the following argument. For $K \geq 2$, $k \geq 1$, so both eigenvalues are realized (not necessarily distinctly). The linear equation for equal eigenvalues is then

$$K\,x + y = z = y\,. \tag{37}$$

Which cancels down to

$$K\,x = 0\,. \tag{38}$$

And which of course admits 2 solutions: $x = 0$ and $K = 0$. The first is meaningful: isotropy. While the second is spurious, for our linear system rests on $K \geq 2$.

But one needs to append the 2 cases excluded by the argument.

For the numbers, $K = 1$ so $k = 0$. Thus the eigenvalue y does not occur at all. This gives a rather trivial realization of isotropy: since there is only 1 direction, every direction must be the same!

While $K = 0$ – the unmatrix – has no room for eigenvalues at all. So the unmatrix realizes the *uneigenspectrum*: an incarnation of the empty set $\emptyset$ consisting of no eigenvalues... Also it comes to pass that the above spurious solution coincides with a non-spurious appended case. The unmatrix even manages to be isotropic in the even more trivial sense that *all* directions are the same whenever there are *no* directions. For all that this realization merits the qualifier *unisotropic* [so long as this is not confused with the much more widely used *anisotropic*...]

The generic case G) of the theorem is covered in e.g. [57], without mention however of the exceptional cases I) and U).

Remark 2 That Combinatorial matrices support ≤ 2 distinct eigenvalues is rooted on these being only 2-parameter matrices. So they have no space for > 2 pieces of information. Leaving their number of distinct eigenvalues thus constrained. In this way, the cumulative arenas' swift persistent onset of copies of the same vector space leaves Combinatorial matrices spectrally-truncated as well.

Remark 3 This subsection's notion of degeneracy is indeed underpinned by symmetry, as is particularly clear from the QM literature [33, 16, 15, 49] for the ground-state value of the energy eigenvalue for the Hamiltonian operator. Which are then indexed by the corresponding group. For *real-symmetric* Combinatorial matrices, in case I), we have the total isotropy group – the *orthogonal* group $O(K)$ – hence I) truer-denotes isotropy. While in case G), this is restricted to the next-largest partial isotropy group: $O(k)$. With G) now truer-denoting that among real Combinatorial matrices, this constitutes the generic case. In this way, real Combinatorial matrices are highly non-generic in the space of all (or even all real-symmetric) square matrices... That real-symmetric matrix eigentheory and orthogonal transformations are related is well-known. Let us postpone to Sec 3 commenting further on which groups are involved, and on how to interpret these symmetry degeneracies.

Remark 4 The isotropic condition $x = 0$ revisits a condition present in Sec 1. It corresponds to 2 examples there: $\mathbb{0}$ and $\propto \mathbb{I}$. The current Subsection does not have the means to distinguish between these two cases, while the next does.

Remark 5 In the generic case G), $K \geq 2$ exhibits a $k \,|\, 1$ partition of the underlying vector space by eigenvalue into the corresponding eigenspaces.

$K = 3$ is minimum for this partition to be into a larger piece – size $k = 2$ – and a smaller piece: size 1 (Subfig c). And thus to have a dimensionally-nontrivial eigenspace. These features clearly persist for all subsequent K.

Remark 6 Let us also introduce the 0-*symmetry-degeneracy fractions*

$$\mathcal{A}_e := \frac{\alpha_e}{K} .$$

Clearly for whichever set of partitioning fraction variables[2] the total sum of fractions is unity:

$$\sum_{\mathfrak{E}\mathrm{spec}(\boldsymbol{M})} \mathcal{A}_e = 1 . \tag{39}$$

In case G), this reads

$$\mathcal{A}_1 + \mathcal{A}_k = 1 . \tag{40}$$

While in case I), this just becomes the identity $1 = 1$, via $\alpha_K = K$.

3.2 Zero-eigenvalue count degeneracy

Remark 7 Degeneracy in this sense comes from the study of quadratic forms [32].

Remark 8 Let us also denote a $K \times K$ matrix's counts of zero and nonzero eigenvalues by K_0 and $K_\bullet$ respectively. So

$$K_0 + K_\bullet = K . \tag{41}$$

Also bring in the 0-*eigenvalue degeneracy fraction*

$$\mathcal{K}_0 := \frac{K_0}{K} .$$

And the 0-*eigenvalue nondegeneracy fraction*

$$\mathcal{K}_\bullet := \frac{K_\bullet}{K} .$$

[2] Compare for instance mass fractions in Physics and Dynamics, and partial-pressure fractions in Physics and Chemistry.

Which of course obey

$$\mathcal{K}_0 \ + \ \mathcal{K}_\bullet \ = \ 1 \ . \tag{42}$$

Classification Theorem 2 for Combinatorial matrices The isotropic case I) now splits into 2 subcases as follows.

0)

$$y \ = \ 0 \ , \tag{43}$$

yielding the zero matrix

$$\mathbb{0} \ = \ (\,0,\,0\,)_K \ .$$

For which the eigenvalues are all 0 , i.e.

$$0 \ \text{ with } \ \alpha_K \ = \ K \ : \ \ K_0 \ = \ K \ , \ \ K_\bullet \ = \ 0 \ . \tag{44}$$

And 1)

$$y \ \neq \ 0 \ , \tag{45}$$

for which we have a matrix proportional to the identity,

$$q\,\mathbb{I} \ , \ \ q \ \neq \ 0 \ . \tag{46}$$

Here the eigenvalues are all q , i.e.

$$q \ \text{ with } \ \alpha_K \ = \ K \ , \ \text{ while } \ K_\bullet \ = \ K \ , \ \ K_0 \ = \ 0 \ . \tag{47}$$

The generic case G) also contains 2 subcases that manifest zeros.

k-0)

$$y \ = \ 0 \ , \ \ K\,x \ \neq \ 0 \ . \tag{48}$$

Which are the matrices proportional to the block matrix[3]

$$\mathbb{1} \ := \ (\,1,\,1\,)_K \ .$$

I.e. the matrix whose entries are all 1 . So

$$q\,\mathbb{I} \ , \ \ q \ \neq \ 0 \ . \tag{49}$$

Its eigenvalues are

$$0 \ \text{ with } \ \alpha_k \ = \ k \ : \ \ K_0 \ = \ k \ . \tag{50}$$

And

$$K\,x \ \text{ with } \ \alpha_1 \ = \ 1 \ : \ \ K_0 \ = \ 1 \ . \tag{51}$$

0-1)

$$z \ = \ K\,x \ + \ y \ = \ 0 \ , \ \ y \ \neq \ 0 \ . \tag{52}$$

Which are the matrices proportional to

$$\boldsymbol{P} \ = \ K^{-1}\,(\,k,\,-1\,)_K \ . \tag{53}$$

For which the eigenvalues are

$$0 \ \text{ with } \ \alpha_1 \ = \ 1 \ : \ \ K_0 \ = \ 1 \ . \tag{54}$$

And

$$y \ \text{ with } \ \alpha_k \ = \ k \ . \tag{55}$$

G) contains furthermore a case with no 0s: k-1).

[3]This matrix occurs in Graph Theory and also in the role of identity for the element-wise product of matrices. See Subsec 3.3 for a further special Combinatorial matrix proportional to this one.

In contrast, for case I), the eigenspace is unsplit (a 1-piece partition K). While for case U), the eigenspace is the unpartition of the empty set!

Remark 9 For $K = 0$, the unmatrix has no eigenvalues and thus no capacity to exhibit zero eigenvalues.

Remark 10 The current Subsection partners the linear equation

$$x = 0$$

– shared with Sec 1 and Sec 3.1 – with the following new linear equation.

$$z = K x + y = 0 . \tag{56}$$

These are also the 2 ways in which the determinant of a Combinatorial matrix $\boldsymbol{C}$ can be zero. And thus that $\boldsymbol{C}$ itself can be singular.

Compare (56) with Sec 1's zero-trace equation (12); both determinant and trace are invariants. In fact, for $K = 1$, $t = 0$ and $z = 0$ coincide. This reflects that $K = 1$ are just the numbers, which do not support distinct trace and determinant...

3.3 Combinatorial projectors

Structure 2 At the level of eigenvalues, a nontrivial projector [35, 43, 45, 27, 74] has eigenvalues 0 and 1 . Since this is a specialization of having 2 distinct eigenvalues, it is compatible with class G). And more specifically with zero-count degenerate such.

Remark 11 There are 2 orders in which a Combinatorial matrix can implement such eigenvalues at the level of a linear system of equations. Firstly,

$$z = K x + y = 0 , \quad y = 1 . \tag{57}$$

Which is solved by

$$\boldsymbol{P}(K) = K^{-1} (k, -1)_K . \tag{58}$$

Secondly,

$$z = K x + y = 1 , \quad y = 0 . \tag{59}$$

Which is solved by

$$\boldsymbol{P}_{\perp}(K) = K^{-1} (1, 1)_K = K^{-1} \mathbb{1} . \tag{60}$$

Remark 12 We already encountered $\boldsymbol{P}(K)$ in the previous subsection; we have now established that this is a projector. While $\boldsymbol{P}_{\perp}(K)$ is the special Combinatorial matrix alluded to in footnote 3. Since Combinatorial matrices are symmetric, and we have taken them to be $\mathbb{R}$-valued, projectors in this context are automatically orthogonal [35, 43, 45, 27, 74]. Finally, $\boldsymbol{P}_{\perp}(K)$ is the complement of $\boldsymbol{P}(K)$.

3.4 Combinatorial involutors

Structure 3 At the level of eigenvalues, a nontrivial involutor [12, 71] has eigenvalues ± 1 . Again, this is a specialization of having 2 distinct eigenvalues, which is thus compatible with symmetry-generic Combinatorial matrices. Now more specifically with zero-count nondegenerate such.

Remark 13 There are 2 orders in which a Combinatorial matrix can implement such eigenvalues at the level of a linear system of equations. These can now be jointly posed and solved as follows.

$$z = K x + y = \pm 1 , \quad y = \mp 1 . \tag{61}$$

Which are solved by

$$\pm \boldsymbol{J}(K) = \pm K^{-1} (2 - K, 2)_K . \tag{62}$$

3.5 N-body problem subcase

Notational Remark 5 For this let us use the notation N in place of K .

Structure 4 Here one has a constellation of N points-or-particles in $\mathbb{R}^d$ space. Given a possibly transient absolute origin, each point-or particle has a position vector relative to this. The space of all possible constellations is the configuration space [11, 21, 53] *constellation space*,

$$\mathfrak{q}(\, d; \, N \,) \ = \ \mathbb{R}^{d\,N} \ .$$

Each point-or-particle can also be allotted a label. The space of all possible LCs of point-or-particle labels – position labels – is *constellation label space*,

$$\mathfrak{L}\mathfrak{q}(N) \ = \ \mathbb{R}^N \ .$$

One then passes to separation vectors between points-or-particles. Absolute origin dependence cancels out of these.

Structure 5 The space of all linearly independent (LI) separation vectors is the configuration space *relative space*

$$\mathfrak{R}\mathrm{el}(\, d, \, N \,) \ = \ \mathbb{R}^{d\,N} \ .$$

Naming Remark 4 This name is used in e.g. [53, 71], with reference to an LI set of relative separations or of relative differences.

Remark 14 Each separation in an LI set can be allotted another label, now with values running from 1 to

$$N \ := \ n \ - \ 1 \ . \qquad (63)$$

The totality of LCs of which form in turn *relative label space*,

$$\mathfrak{L}\mathfrak{R}\mathrm{el}(n) \ = \ \mathbb{R}^n \ .$$

[53, 71] explain how the above $\mathfrak{L}$ versions are active factors in the corresponding multi-index tensor product $\mathfrak{L}$ -less versions. Passing to $\mathfrak{R}\mathrm{el}(n)$ and $\mathfrak{L}\mathfrak{R}\mathrm{el}(n)$ amounts to quotienting out translations, $Tr(d)$. Various further simple quotienting procedures [30, 34, 53, 68, 71] permit handling dilations and 2-d rotations.

Remark 15 $\boldsymbol{P}(K)$ is the projector onto relative label space (see Sec 4.1 for further interpretations as a projector). Which takes the form

$$\boldsymbol{P}(K) \ := \ K^{-1} \, (\, k, \, \text{-}1 \,)_K \ . \qquad (64)$$

Another alias for this is *positions-to-relative separations matrix* at the level of the internal labels [53, 71]. Yet another alias is Lagrange matrix [4, 53, 71]. For all that $\boldsymbol{P}$ is numerically, and yet not Physical-dimensionally, equal to this [61, 71]. For $N \ = \ 3$, the above specializes to

$$\boldsymbol{P} \ := \ \frac{1}{3} \, (\, 2, \, \text{-}1 \,) \ . \qquad (65)$$

3.6 N-vertex and n-simplex cases, with triangle or 2-simplex examples

Remark 16 These refer to two ways of describing the Geometrical counterpart of the N-body problem. Within the translation and rotation quotiented setting, the following are natural for $N \ = \ 3$.

A) **Apollonius' theorem** [1, 26, 50, 71] for expressing a triangle's median lengths in terms of its side lengths. The Euler 3-cycle – over sides – of this can be expressed as a matrix equation [54, 60, 71]. Whose matrix turns out to be proportional to an involutor: the 'Apollonius' involutor,

$$\boldsymbol{J} \ := \ \frac{1}{3} \, (\, \text{-}1, \, 2 \,) \ . \qquad (66)$$

A functional alias for this is *sides-medians length-exchange involutor*. Medians, and $\boldsymbol{J}$, can be defined to transcend to [71] arbitrary dimension – whether spatial or configuration-occupying – So $\boldsymbol{J}$ is more generally a 2-simplex matrix. We finally recognize this as the $\boldsymbol{J}(3)$ subcase of (62).

B) **Heron's formula** [2, 12, 44, 52, 71]. The square of this, when viewed as a quadratic form, is built out of the following matrix.

$$\boldsymbol{F} \ := \ (\text{-}1, 1) \ . \tag{67}$$

Which was first written down in linear system form by Euler [3] and explicitly as a matrix by Buchholz [24]. At which level the names *Heron matrix* or, more functionally, *sides-data triangle area formula matrix*, are suitable.

It was however subsequently pointed out to occur in the Euler 3-cycle of cosine rules, and even of triangle inequalities. By which the name *fundamental triangle matrix* [60] and the notation $\boldsymbol{F}$ are more appropriate. Its fundamentality [54, 61] is further warranted by its ties to Hopf's little map [9, 40, 41, 72]. It furthermore transcends to [71] arbitrary dimension – whether spatial or configuration-occupying – sealing our final name for it: *fundamental 2-simplex matrix*. We also recognize this as the $\mathbb{V}(3)$ subcase of the trace-reversed Combinatorial matrices (21).

Structure 6 (65, 66, 67) are the 3 triangle matrices, or more generally by transcending arbotrary dimension, the 3 2-simplex matrices. In each of their equations, we have dropped the K subscripts since they are all 3 's).

Remark 17 All of $\mathbb{T}$, $\boldsymbol{J}$, $\boldsymbol{F}$ and $\boldsymbol{P}$ for $N = K \geq 2$ are generic in sense G). $\boldsymbol{P}$ is of subclass k-0), while the other 3 are of subclass k-1).

Remark 18 $\boldsymbol{P}$ is the $K = 3$ projector of rank 2 .

$\boldsymbol{J}$ is one of the two signs of $K = 3$ involutor.

$\boldsymbol{F}$ is the $K = 3$ trace-reversed matrix.

Thus our purely algebraic considerations manage to find all of these. For all that these considerations do not single them out among various other involutors, projectors and Representation-Theoretically privileged matrices.

3.7 Physicists' signatures

Remark 19 For a real-symmetric matrix, all the eigenvalues are well-known to be real. It is thus meaningful to allot a sign to each. Whether $+$ or $-$ for each nonzero eigenvalue. Or $+$, $-$ or 0 for every eigenvalue.

Notational Remark 6 Let K_+ and K_- denote the counts of positive and negative eigenvalues respectively. In each case with algebraic multiplicity included. Let

$$\mathcal{K}_{\pm} \ := \ \frac{K_{\pm}}{K} \ .$$

So that

$$\mathcal{K}_0 \ + \ \mathcal{K}_+ \ + \ \mathcal{K}_- \ = \ 1 \ .$$

Also for matrices with $K_{\bullet} \neq 0$, define

$$\mathcal{ND}_{\pm} \ := \ \frac{K_{\pm}}{K_{\bullet}} \ .$$

So that

$$\mathcal{ND}_0 \ + \ \mathcal{ND}_+ \ + \ \mathcal{ND}_- \ = \ 1 \ .$$

The $K_{\pm}$ are furthermore reflectively-symmetrically defined. To the extent that which is $+$ and which is $-$ is often taken to be a convention.[4]

Thus

$$\Delta K := K_+ - K_-$$

is in some ways a more meaningful difference than

$$\delta K := K_0 - K_\bullet$$

would be. Hence the difference in notation. Δ is moreover not only reflectively-symmetric but also a proto-index.[5]

ΔK is furthermore (one sign convention choice of) the Physicists' signature in summary. The signature in detail exhibits how many $+$'s, $-$'s and 0 's are present. E.g. $- + + +$ for one sign convention for Minkowski spacetime. Or $+ + + 0$ for the 4-body problem's Lagrange projector $\boldsymbol{P}$ ... This notation is used in Fig 2; with a truncated version of it in Figs 4 and 6. More efficiently especially for much larger examples, it is the Author's sign triple

$$(K_+, K_-, K_0) .$$

Though S. Sánchez' presentation [56]

$$(K_0, K_\bullet, \Delta K)$$

is a more elegant sign-space LB choice. Picked so as to manifest the signature-in-summary proto-index among its LB elements... This kind of parametrization also permits exhibition of beloved cases in which $K_{\pm}$ are infinite and yet ΔK manages to remain finite.

Classification Theorem 3 for Combinatorial Matrices In the generic case G) away from zeros – k-1) – there are for $K \geq 2$ 4 nontrivial cases for signs of eigenvalues.

++)

$$y > 0 , \quad z = K x + y > 0 . \tag{68}$$

is *positive-definite*: all eigenvalues > 0 ,

$$K_+ = K .$$

−+)

$$y < 0 , \quad z = K x + y < 0 \tag{69}$$

is *negative-definite*: all eigenvalues > 0 :

$$K_- = K .$$

+−)

$$y > 0 , \quad z = K x + y < 0 \tag{70}$$

is minimally indefinite with sign convention

$$K_- = 1 , \quad K_+ = k .$$

−−) is also minimally indefinite

$$y < 0 , \quad K x + y > 0 , \tag{71}$$

albeit with the opposite sign convention:

$$K_- = k , \quad K_+ = 1 .$$

[4]So in Special Relativity, spacetime is modelled with $-$ for time and $+$ for space or vice versa!

[5]This is in the sense of index theorems; compare the Poincaré index formula [22]. The rank–nullity [74], Gauss–Bonnet [75], Riemann-Roch [25] and Atiyah–Singer [23] index theorems. And quite a few basic Combinatorics examples in [58] and basic Geometry examples in [71]; see footnote 6 for some examples of each.

Remark 20 So while we have hitherto used linear (systems of) equations, we now pass to linear (systems of) inequalities.

Remark 21 $K = 2$ has both $+ -$) and $-+$) collapse to the *balanced* [65, 71] situation:

$$K_+ = K_- .$$

And indeed are *minimally balanced*:

$$K_+ = 1 = K_- .$$

Balanced entails $+ \leftrightarrow -$ symmetry. Which in the $+$- case has well-documented consequences [14]. For $K \geq 3$, however, there is a larger $+$ or $-$ eigenspace. I.e.

$$K_+ > K_-$$

or

$$K_+ < K_- ,$$

which is a symmetry-degenerate eigenspace.

Remark 22 The *Physicists' signature in summary* provides the following further interpretation. Balanced the corresponding *null proto-index condition* [6]

$$\Delta K = 0 .$$

While the quantifier of departure from balance,

$$Imbalan(\boldsymbol{M}) = \Delta K$$

is the corresponding nontrivial index.

Case k.0) also supports just 2 : its single nonzero eigenvalue can be $-$ or $+$, giving k.0.$\pm$). I.e.

$$y > 0 , \quad z = 0 . \tag{72}$$

And

$$y < 0 , \quad z = 0 . \tag{73}$$

Case 0.1) also supports just 2 : its nonzero eigenspace can be positive- or negative-definite: 0.1.$\pm$). I.e.

$$y = 0 , \quad z > 0 . \tag{74}$$

And

$$y = 0 , \quad z < 0 . \tag{75}$$

For $K = 2$, both k.0) and 1.0) have

$$K_+ = K_0 , \quad K_- = 0 .$$

Or

$$K_- = K_0 , \quad K_+ = 0 .$$

While for $K \geq 3$, 1.0) has

$$K_+ > K_0 , \quad K_- = 0 .$$

[6]This conceptual type covers the Euler characteristic on the circle and the tori, and thus Gauss–Bonnet type theorems thereupon. Grinberg's theorem in planar Hamiltonian Graph Theory [56, 58], Whose index is inside-outside triangulation strength imbalance. And which theorem we thus renamed ZIPHoN: 'zero-index planar Hamiltonian Necessity'. Varignon's theorem and Euler's 3-simplex theorem in Flat Geometry, along with 'smaller' infinite families of generalizations [55, 71]. Whose common index is the left-right child imbalance in unlabelled rooted at-most binary trees.

Or

$$K_- \ > \ K_0 \ , \ \ K_+ \ = \ 0 \ .$$

Case I) supports just 2 sign choices: positive- or negative-definite: 1.$\pm$). I.e.

$$y \ > \ 0 \ , \ \ x \ = \ 0 \ . \tag{76}$$

And

$$y \ < \ 0 \ , \ \ x \ = \ 0 \ . \tag{77}$$

We gather up all cases in Fig 2's end-table.

Remark 23 All of $\mathbb{T}$, $\boldsymbol{J}$ and $\boldsymbol{F}$ are of type G).k-1).$-+$).

Remark 24 Combinatorial matrices' nondegenerate sectors must be elliptic – all of the same sign – or hyperbolic: with a single opposing sign. This helpfully banishes a rather harder [14] and much less understood case – ultrahyperbolic: with ≥ 2 copies of each sign – from being realized in our arena $\mathfrak{CM}_{\mathbb{R}}$.

3.8 Multiplicity equalities

Definition 1 An eigenvalue is *semisimple* [48] if its algebraic multiplicity coincides with its geometrical multiplicity. A matrix is *diagonalizable* iff all of its eigenvalues are semisimple. A matrix is *minimally-minimal* [65, 71] if its minimal polynomial is of the lowest-possible order.

Proposition 1 All the $\boldsymbol{C} \in \mathfrak{CM}_{\mathbb{R}}$ have

$$\alpha_e \ = \ \gamma_e \ \text{ for each } \ \lambda_e \ . \tag{78}$$

And thus are, on the one hand, diagonalizable. On the other hand, they enjoy the arena equation

$$\mathfrak{V}(K) \ = \ \mathfrak{E}\text{ig}(\boldsymbol{C}) \ , \tag{79}$$

Which can be viewed as a completeness relation. I.e. the *spectral completeness relation* that the eigenvectors of $\boldsymbol{C}$ form a LB for the whole K-dimensional vector space that $\boldsymbol{C}$ naturally acts upon.

Proposition 2 All Combinatorial matrices' are minimally-minimal:

$$\mu_e \ = \ 1 \ \text{ for each } \ \lambda_e \ . \tag{80}$$

4 Combinatorial matrices' eigenvectors

4.1 N-body subcase

Remark 1 In this context, the generic G)'s lone eigenvector corresponds to the CoM position vector label R. Which is the normalized version of the vector of 1's. And whose linear span (LS) forms the eigenspace

$$\mathfrak{e}\text{ig}_1\,(\boldsymbol{G}) \;=\; \mathfrak{CoM}(1) \;=\; \mathbb{R}\;.$$

Remark 2 Also in this context, the generic case's symmetry-degenerate eigenspace is relative label space:

$$\mathfrak{e}\text{ig}_n\,(\boldsymbol{G}) \;=\; \mathfrak{LRel}(n) \;=\; \mathbb{R}^n\;. \tag{81}$$

This can be studied by considering a LB of separations between points-or-particles.

Remark 3 These eigenspaces fit together to form

$$\mathbb{R}^N \;=\; \mathfrak{e}\text{ig}(\boldsymbol{G}) \;=\; \mathfrak{e}\text{ig}_1(\boldsymbol{G}) \,\oplus\, \mathfrak{e}\text{ig}_n(\boldsymbol{G}) \;=\; \mathfrak{CoM}(1) \,\oplus\, \mathfrak{D}\text{if}(n)\;. \tag{82}$$

Remark 4 In contrast, in the isotropic case I), there is a single irreducible eigenspace

$$\mathbb{R}^N \;=\; \mathfrak{e}\text{ig}(\boldsymbol{I}) \;=\; \mathfrak{e}\text{ig}_N(\boldsymbol{I})\;. \tag{83}$$

Remark 5 For $N = 1$, this LB is empty.

But for $N \geq 2$, it is not.

For $N \geq 3$, however, this LB is not diagonal. Passing to *point-or-particle cluster separations* – between subsystem CoMs – attains diagonality however. In the Dynamics context, this (non)diagonality is manifested by such as the total moment of inertia and the kinetic energy. The corresponding cluster separation vectors have hitherto been called *relative Jacobi vectors* [5, 29, 30, 53, 68, 71].

Remark 6 For $N = 3$, 3 possible labellings of these are possible. This correspond to the number of ways of making a pair subsystem, or equivalently, of leaving out a single point-or-particle. This ambiguity growingly persists.

Structure 7 For $K = 4$, further network ambiguities appear. The clustering structure can here be H- or K-shaped; these have often been called the Jacobi-H and -K.

Remark 7 The inertia and kinetic quadrics in relative coordinates can be modelled using the Lagrange matrix. In the case of equal particles, this is numerically equal to the projector onto relative space, $\boldsymbol{P}$. In this way, contact is made with Combinatorial matrices and their eigentheory.

Remark 8 Given Remark 1, we can now further qualify on the one hand that $\boldsymbol{P}$ amounts to projecting out the CoM label; hence the name *CoM-removing projector*. Another alias is *relative label space projector*. The first of these bears some relation to the common practise in Physics of passing to the CoM frame. And furthermore explains Montgomery's [68] alias for relative space: *centred configuration space*, with reference to centering about the CoM position. This space featured in e.g. [30, 34] long before the above and Sec 4.1's references.

On the other hand, the orthogonal complement projector $\boldsymbol{P}_\perp$ projects onto the CoM label space.

Naming Remark 5 A truer name for relative Jacobi coordinates is *eigenclusterings* [55, 53, 71].

♠ ◇ ♣

Remark 9 $\boldsymbol{P}$ is generic in sense G), so the full underlying symmetry is $O(n)$. In the N-body problem context, these have been termed *internal rotations*, alias *democracy transformations* [29].

These are internal in the sense that they act not on space but on the internal space of point-or-particle labels.

Remark 10 The network ambiguity also growingly persists, forming [38, 63] the *unlabelled rooted binary trees* (*URBT*) [37, 47, 58] The counts of which are [63] the Wedderburn–Etherington numbers [6, 8, 20, 69]. While the previous ambiguity goes like the sizes of the corresponding orbits of the permutation group S_N acting on the labels.

Structure 8 Each $N \geq 1$ supports a generalized-K LB of relative Jacobi vectors. This corresponds to each N supporting an URBT which, upon defoliating once [63], is the straight path P_n [63]. Aside from P_3 corresponding to K-shaped clustering, P_2 is T-shaped: side and corresponding median. While P_1 just involves the incipient point-or-particle separation.

Remark 11 Take any eigenbasis of relative Jacobi vectors and adjoin R. This forms the corresponding (*absolute*) *Jacobi vectors*, alias *eigenclustering vectors with CoM position adjoined*. Hitherto, in the Dynamics literature, the Jacobi vectors were credited to CoM hierarchies. Which can be reformulated as choices of eigenbasis for the Lagrange matrix, and thus the relative space projector $\boldsymbol{P}$. Which exists for any N-body problem in any $\mathbb{R}^d$.

4.2 Generalization to any Combinatorial matrix

Remark 12 Let us now shift away from the above context to Combinatorial matrices in full generality. The generic G) case's lone eigenvector U corresponds to the sum of the Combinatorial counts acted upon. Or equivalently, given subsequent normalization, the average of the Combinatorial counts. Whose LS forms the eigenspace

$$\mathfrak{E}\mathrm{ig}_1\,(\boldsymbol{G}) \;=\; \mathfrak{A}\mathfrak{o}\mathfrak{C}(1) \;=\; \mathbb{R}\;.$$

Remark 13 Relative separation labels of pairs of point-or-particle positions become differences between our Combinatorial counts. Now forming the generic G)'s symmetry-degenerate eigenspace *difference space*

$$\mathfrak{E}\mathrm{ig}_k\,(\boldsymbol{G}) \;=\; \mathfrak{D}\mathrm{if}(k) \;=\; \mathbb{R}^k\;. \tag{84}$$

This can be studied by considering a LB of differences between our Combinatorial counts.
Remark 14 These eigenspaces fit together to form

$$\mathbb{R}^K \;=\; \mathfrak{E}\mathrm{ig}(\boldsymbol{G}) \;=\; \mathfrak{E}\mathrm{ig}_1(\boldsymbol{G}) \;\oplus\; \mathfrak{E}\mathrm{ig}_k(\boldsymbol{G}) \;=\; \mathfrak{A}\mathfrak{o}\mathfrak{C}(1) \;\oplus\; \mathfrak{D}\mathrm{if}(\boldsymbol{G})\;. \tag{85}$$

Remark 15 In contrast, in the isotropic case I), there is a single irreducible eigenspace

$$\mathbb{R}^K \;=\; \mathfrak{E}\mathrm{ig}(\boldsymbol{I}) \;=\; \mathfrak{E}\mathrm{ig}_K(\boldsymbol{I})\;. \tag{86}$$

Remark 16 For $K \geq 3$, the above LB of differences is not diagonal. Passing to LCs of these – *count subset differences*: between 2 subsets' counts – attains diagonality however.

Naming Remark 6 Given this more general context, *combinatorial-matrix eigenvectors*, or for short *eigencombinatorial vectors*, is in turn a truer name than eigenclustering vectors.

Remark 17 For $K = 3$, 3 possible labellings of these are possible. This correspond to the number of ways of making a pair subset, or equivalently, of leaving out a single count.

Remark 18 $\boldsymbol{P}$ now amounts to projecting out the AoC, giving the more generalized name *AoC-removing projector*. And onto difference space, hence *difference space projector*. While $\boldsymbol{P}_{\perp}$ is the *AoC label space projector*.

♠ ◇ ♣

Remark 19 Take any eigenbasis of eigencombinatorial vectors and adjoin U. This forms the corresponding *extended* basis of *eigencombinatorial vectors* with CoM position adjoined.

Remark 20 The eigencombinatorial vectors present the same network ambiguity of URBT form as described above.

Proposition 3 Any $K \geq 1$ Combinatorial matrix can be equipped with an extended generalized-K eigencombinatorial eigenbasis.

Proof A generalized K is available for all K as the straight P_k URBT. Form the difference between a first pair of objects. Next form the difference between the sum of these and twice a third object. Apply this move recursively between the sum of the first $p - 1$ objects used and $p - 1$ times a pth object. Finally adjoin U . □

Proof of proposition 1. Using this LB, $\gamma_e = \alpha_e$ for each e . □

Exercise 1 Prove proposition 2.

♠ ◇ ♣

Remark 21 In the case of a general network, each step uses rather the difference between left- and right-child sums. Which are the Combinatorial abstraction and generalization of the subsystem CoM labels.

Remark 22 The URBT ambiguity was long known to arise from CoM hierarchies: Mechanics to Combinatorics. A more natural perspective is that general Combinatorial matrices give further standard Combinatorial objects as their eigenvectors. The CoM hierarchy then behaves as it does by being a subcase. In this way, we have passed to a purely Combinatorial explanation.

Remark 23 Eigencombinatorial eigenbases are only a measure-0 subset of the possible eigenbases. This is based upon the relative sizes of the finite permutation subgroup S_K versus the infinite orthogonal group $O(k)$ corresponding to allowing all $\mathbb{R}$-LCs. So far as the Author is aware, the Combinatorial literature has not pinned a name on this generalized setting for what Molecular Physicists call internal rotations or democracy transformations.

4.3 Eigevector classification

Classification theorem 3 for Combinatorial matrices With reference to a cover by some of the above-defined cases, a Combinatorial matrix's normalized eigenvectors take the following corresponding forms.

G) The normalized unit vector. Alongside any normalized LB choice of LCs of relative-difference vectors.

I) Any normalized LB for $\mathbb{R}^K$ will do.

U) The *uneigenbasis* consisting of an empty set's worth of eigenvectors.

Proof G). For the lone eigenvalue, the eigenvector equation is

$$(-k\,x\,, x\,)_K \cdot \boldsymbol{x} = \mathbf{0}\ . \tag{87}$$

Which is solved by

$$x_1 = ... = x_K = 1\ . \tag{88}$$

Finally divide by the corresponding normalization factor

$$\sqrt{\sum_{i\,=\,1}^{K} 1^2} = \sqrt{K}\ . \tag{89}$$

For the other eigenspace's eigenvalue,

$$x\,\mathbb{1} \cdot \boldsymbol{x} = \mathbf{0}\ . \tag{90}$$

Which is solved as claimed.

I) The eigenvector equation now reads

$$(0,0)_K \cdot \boldsymbol{x} = \mathbf{0} . \tag{91}$$

Which places no restrictions on what $\boldsymbol{x}$ can serve as an eigenvector.

U) Now there is no eigenvector equation, but no vectors to restrict either. The restriction of the empty set $\emptyset$ by the (empty set of equations) is of course just $\emptyset$ again. $\Box$

4.4 Sharing eigenbases and eigenspaces

Corollary 1 Any generic set of same-size Combinatorial matrices can be taken to share eigenbasis. If none of them are isotropic, then they additionally share the underlying $k \,|\, 1$ partition into eigenspaces as labelled by eigenvalue (Fig 1.b).
Proof For $K = 0$, all must be copies of the unmatrix, and thus share the same empty set of eigenspaces.

For $K \geq 1$, by theorem 3 any LB for $\mathbb{R}^K$ will do for class I). So pick the extended version of the generalized-K LB so as to match class G). $\Box$

In Fig 3, this alignment is drawn out using green for I)'s single eigenspace and blue and yellow for G)'s pair.

Proposition 4 Suppose that we are given a set of size-K Combinatorial matrices. Then they share eigenspaces iff either of the following hold.

i) $K \leq 1$.

ii) $K \geq 2$ and they are either all generic G) or all isotropic I).

Proof For $K = 0$, each matrix in the set can only be a copy of the unmatrix. All of which share the same eigenspaces: no eigenspaces at all!

For $K = 1$, only 1 eigenspace can be realized and thus must be shared by all.

For $K \geq 2$, two cases work out. Firstly, a set of isotropic matrices with the same K shares the same K-fold eigenspace. I.e. the whole vector space acted upon. Secondly, a set of generic matrices with the same K share the same 1-d eigenspace in each case with the same k-fold complement. The remaining case – sets containing ≥ 1 G) and ≥ 1 I) do not work out, by concurrently realizing both the split and the unsplit eigenspaces. $\Box$

Corollary 2 Our 3 2-simplex matrices

i) possess a shared eigenbasis, which can be taken to be the extended version of T .

ii) They share eigenspaces.

Proof i) All of these matrices are Combinatorial and of the same size $K = N = 3$. So theorem 3 gives that they share eigenbasis. And that this can be taken to be the extended generalized-K eigenbasis. Which for $\mathrm{K} = 3$ is the extended T .

ii) All of these matrices are generic G). So proposition 4 $\Rightarrow$ that they share eigenspaces. $\Box$

Remark 24 This replaces [60, 61, 62]'s piecemeal Geometrical considerations. Paralleling Ford's [66] observations that the 3 2-simplex matrices commuting with each other follows just from their being Combinatorial matrices of the same size. As do their Abstract Algebra properties. Unique specifications of $\boldsymbol{P}$ and $\boldsymbol{J}$ as a particular Combinatorial matrix projector and involutor follow from whichever of Ford's account and the current one.

5 Conclusion

Remark 1 We form a comparison table in Fig 1 for the current Article's paradigm shift from N-body problem use of Combinatorial matrices to general use. A large part of the theory of centres of mass is thereby reduced to Combinatorics. And we have a precise name for what Physicists' 'hierarchies of subsystems' CoMs' are. I.e. one very specific realization of the URBTs; see [63] for the precise correspondence. With each CoM's 2 input subsystems being the right and left children of that CoM as viewed as a node.

Remark 2 We summarize many of the current Article's other results so far in Appendix A. And take our study of eigenspectral classification for Combinatorial matrices further in Appendix B. This is by use of Order Theory alongside more structurally sparse Graph Theory undepinning this.

Pointer 1 Combinatorial matrices are often taken to be $\mathbb{Z}$-valued in Combinatorics. Thus forming the arena $\mathfrak{CM}_{\mathbb{Z}}$. The current Article's analysis extends to $\mathfrak{CM}_{\mathbb{R}}$ for its Dynamics and Geometry significance, and Linear Algebra specifics of our workings. for $\mathbb{R}$-valued models. Let us leave the yet more general $\mathfrak{CM}_{\mathbb{C}}$ for another occasion.

Pointer 2 As regards n-simplex matrices, some remaining open questions are as follows [64, 65, 67]. Which sets of same-size quadrilateral matrices commute, form multiplicative commutative monoids, share eigenspaces and share eigenbases? For here not all of the matrices in any of these sets considered are Combinatorial...

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Wheelerian comparison table		
Notions	***N*-body matrices** (Dynamics); ***N*-vertex matrices**, ***n*-simplex matrices** (Geometry)	**General Combinatorial matrices *C*** (Combinatorics)
Examples	In particular the equal-masses Lagrange matrix, $\boldsymbol{L}$, which is numerically equal to CoM-removing projector $\boldsymbol{P}$. Whose complement is the CoM projector $\boldsymbol{P}_{\perp}$. The Apollonius involutor $\boldsymbol{J}$ and the fundamental 2-simplex matrix $\boldsymbol{F}$ also appear in Geometrical study	Two specific cases among which are the average-count removing projector $\boldsymbol{P}$. And its complement the average-count projector $\boldsymbol{P}_{\perp}$. We do not however need $\boldsymbol{P}$ in order to develop the theory.
Eigenspaces: lone	CoM label direction space $\mathfrak{CoM}(1)$ →	Average of counts direction space $\mathfrak{SoC}(1)$ [A 2025]
symmetry-degenerate	Relative label space $\mathfrak{LRel}(n)$ [1990s?] →	Difference space $\mathfrak{Dif}(k)$ [A 2025]
Totality of eigenvectors: traditional name	Jacobi vectors [19th century]	
Degenerate space's eigenvectors' traditional name	Relative Jacobi vectors [1990s?]	
Degenerate space's eigenvectors: conceptual name	Eigenclustering vectors [SA 2018] →	Eigencombination vectors [A 2025]
Totality of eigenvectors: conceptual name	Eigenclustering vectors extended by CoM vector [SA 2018] →	Eigencombination vectors extended by average-count vector [A2025]
Network ambiguity	H versus K network ambiguity for $N = 4$ [19th century] ↓ Unlabelled rooted binary trees: Mechanics producing Combinatorics [e.g. S 2002]. → Now however explained as a subcase of Combinatorics producing more Combinatorics [A2025]	Unlabelled rooted binary trees: Combinatorics producing more Combinatorics [A2025]
Labelling ambiguity	S_N orbits	S_K orbits
Full ambiguity: generic case G)	$O(n)$, called internal rotations or democracy transformations →	$O(k)$ [A 2025]
Isotropic case I)	$O(N)$ ←	$O(K)$

Figure 1:

A Eigenvalue and eigenvector classifications

Remark 1 We condense many of Sec 3 and 4's other results into tables 2 and 3 respectively.

Classification of Combinatorial matrices'eigenvalues											
Symmetry class	**0 class**	**Eigen-values**		**Rank**	**Nullity**	**Notions of signature** s_{Math}	s_{Phys}	$s_{Phys\text{-}detail}$	**Examples**	**Notes**	
Generic combinatorial matrices *G*	***k*-1)**	*z*	*y*	*K*	0	*K*	*K*	+ ... +	𝕋, ***J*** ⊂ ***J***(*K*), ***F*** ⊂ 𝕍 or proportional	Nondegenerate (no zero eigenvalues)	elliptic
		1	*k*			-*K*	-*K*	− ... −			
		1	*k*			*k*	*K* -2	+ ... + −			hyperbolic
		1	1			-*k*	2− *K*	− ... − +			
	***k*-0)**	0	*y*	*k*	1	*k*	*k*	+ ...+ 0	***P*** ⊂ ***P***(*K*) or proportional	Degenerate (zero eigenvalues)	Nondegenerate sector is elliptic
		1	*k*			-*k*	-*k*	− ... − 0			
		1	*k*								
		1	1								
	0-1)	*z*	0	1	*k*	1	1	+ 0 ... 0	𝟙 or proportional, including $\boldsymbol{P}_\perp$		
		1	*k*			-1	-1	− 0 ... 0			
		1	*k*								
		1	1								
Isotropic matrices *I*	**1)**		0	*K*	0	*K*	*K*	+ ... +	𝕀 or proportional	Nondegenerate (no zero eigenvalues)	elliptic
			K			-*K*	-*K*	− ... −			
			K								
			1								
	0)		0	0	*K*	0	0	0 ... 0	𝕆 is the only example	Fully degenerate (all zero eigenvalues)	
			K								
			K								
			1								
The unmatrix *U*			—	0	0	0	0	—			
			—								
			—								
			—								

Key			
	λ_e	Eigenvalue	
	α_e	Algebraic	multiplicities
	γ_e	Geometric	
	μ_e	Minimal	

See [63,69]
for further explanation of
the notions in these notes

Figure 2:

Classification of Combinatorial matrices by eigenspaces, with a shared K-basis of eigenvectors						
Isotropic I)		$\frac{1}{\sqrt{K}}\begin{pmatrix}1\\ \vdots\\ \vdots\\ \vdots\\ 1\end{pmatrix}$,		$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\\ 0\\ \vdots\\ 0\end{pmatrix}$, ... , $\frac{1}{\sqrt{k\,K}}\begin{pmatrix}1\\ \vdots\\ \vdots\\ 1\\ -k\end{pmatrix}$		
Eigenvalues				0		
Geometric multiplicities				K	$=$	K
Eigenspaces				$\mathfrak{E}\mathrm{ig}(\boldsymbol{I})$ $\mathbb{R}^K$	$=$ $=$	$\mathfrak{E}\mathrm{ig}(\boldsymbol{I})$ $\mathbb{R}^K$
Generic G)		$\frac{1}{\sqrt{K}}\begin{pmatrix}1\\ \vdots\\ \vdots\\ \vdots\\ 1\end{pmatrix}$		$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\\ 0\\ \vdots\\ 0\end{pmatrix}$, ... , $\frac{1}{\sqrt{k\,K}}\begin{pmatrix}1\\ \vdots\\ \vdots\\ 1\\ -k\end{pmatrix}$		
Eigenvalues		z		y		
Geometric multiplicities		1	$+$	k	$=$	K
Eigenspaces		$\mathfrak{E}\mathrm{ig}_z(\boldsymbol{G})$ $\mathbb{R}$	$\oplus$ $\oplus$	$\mathfrak{E}\mathrm{ig}_y(\boldsymbol{G})$ $\mathbb{R}^k$	$=$ $=$	$\mathfrak{E}\mathrm{ig}(\boldsymbol{G})$ $\mathbb{R}^K$
Unmatrix U)				—		
Eigenvalues				—		
Geometric multiplicities				—		© 2025 Dr E. Anderson
Eigenspaces				$\emptyset$	$=$	$\mathfrak{E}\mathrm{ig}(\boldsymbol{U})$

Figure 3:

B Boosting classifications using Order Theory

Spectral classification of Combinatorial matrices

Table presentation

C)											
G)								I)			U)
1-*k*)				0-*k*)		1-0)		1)		0)	
K	-K	K-2	2-K	k	-k	1	-1	K	-K	0	0
++	−+	+−	−−	+	−	+	−	+	−		

Poset presentation

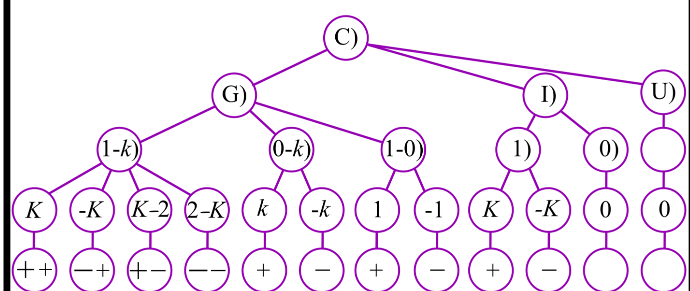

Figure 4:

Structures 7 and 8 Classificatory-table versus classificatory-key depictions are illustrated in Fig 4 for the current Article's cumulative-K eigenspectral classifications. On the one hand, tables can encode some simple patterns of coarse graining. On the other hand, the key diagram can be considered to be a rooted tree, which is a subcase of poset [31, 39]. Rooted trees are not in general preserved under quotients, but more general posets can accommodate these. In this way, key diagrams are stronger when the objects under classification are sharply enough defined to have meaningful quotients.

Remark 1 Fig 5 abstracts posets from the previous figure. Including the following illustrative quotients. b) treating the signs as distinguishable but meaningless. c) Identifying equal signatures. d) Both at once. b) still manages to be a rooted tree, while c) and d) exhibit cycles.

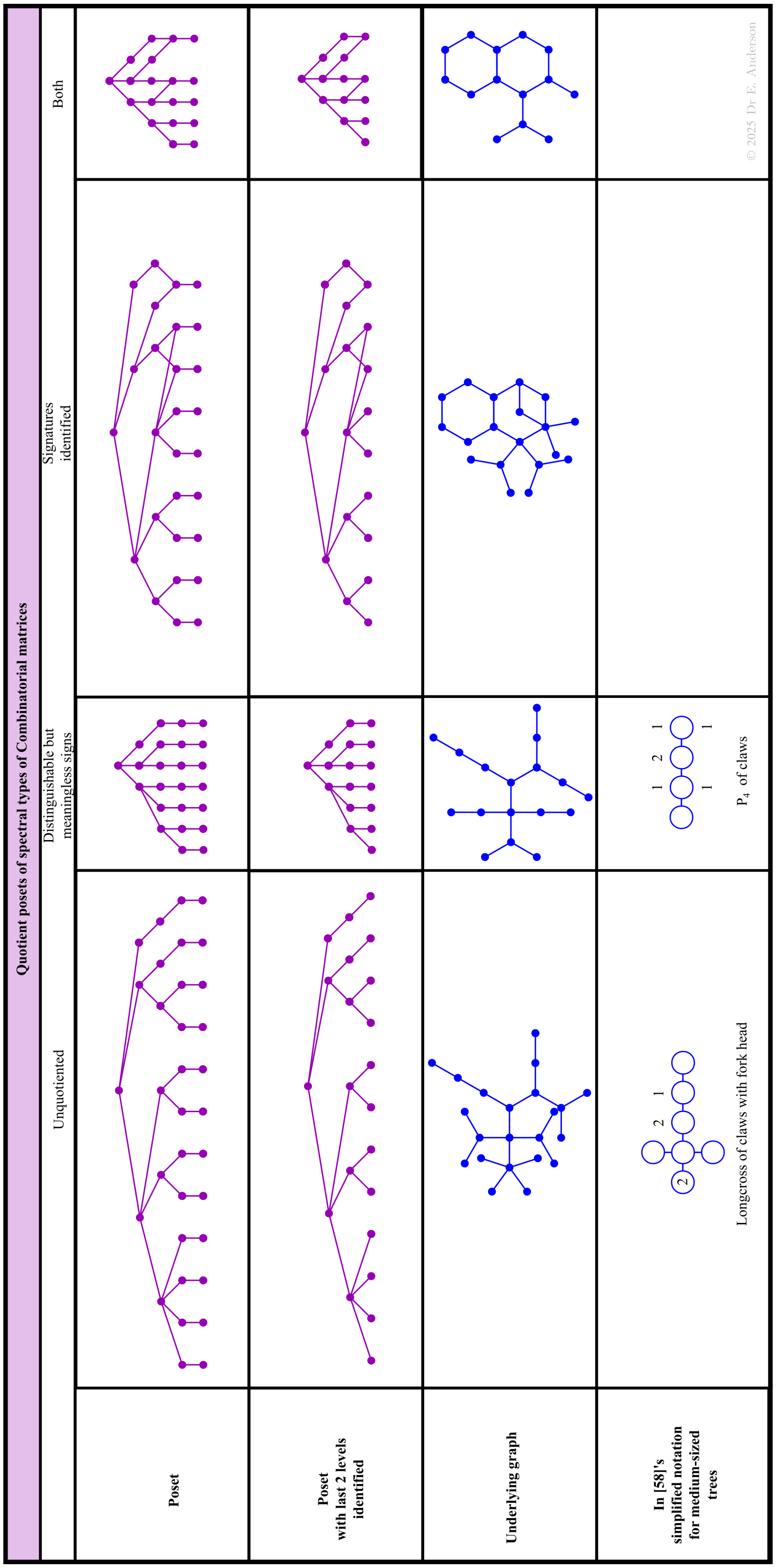
Quotient posets of spectral types of Combinatorial matrices
Unquotiented
Distinguishable but meaningless signs
Signatures identified
Both
Poset
Poset with last 2 levels identified
Underlying graph
In [58]'s simplified notation for medium-sized trees
Longcross of claws with fork head
P_4 of claws
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Figure 5:

♠ ◇ ♣

Remark 2 Graphs underlying these posets are exhibited in rows 3 and 4. See [70] for an explanation of the specific style of these presentations of graphs.

Remark 3 In Fig 6, we split into Combinatorial matrices with each individual value of K.

$K = 0$ forms a disjoint chain: involving objects not present for any subsequent K.

$K = 1$ is also particularly simple, since here the matrices are just numbers, and these support just the one lone eigenspace.

$K = 2, 3$ have extra scope for identifications. $K = 3$ is minimal for the generic case to have a symmetry-degenerate eigenspace.

$K = 4$ is the minimum generic value including our quotienting considerations. With less quotienting, $K = 3$ can play this role. This role corresponds to the totality of Combinatorial matrices $\mathfrak{CM}_{\mathbb{R}}$ being spectrally truncated, which is these matrices' main spectral feature. The Author shall eventually consider size 2 and 3 square matrices' real Jordan normal forms, which are not afflicted by any such truncation.

Remark 4 Underlying graphs for these quotients are provided in Figs 7 and 8. The underlying homeomorph irreducibles are in row 4, cycle systems in row 5, and the homeomorph irreducibles of the cycle systems themselves in row 6. For the first 2 graphs in 5's homeomorph irreducibles, just remove the non-encircled 1's and 2's. All are planar bar Fig 8 column 1 rows 3 to 6, by virtue of the marked $\mathrm{K}_{3,3}$ forbidden subgraph.

Remark 5 All the above planar graphs are furthermore upper-planar [28] posets.

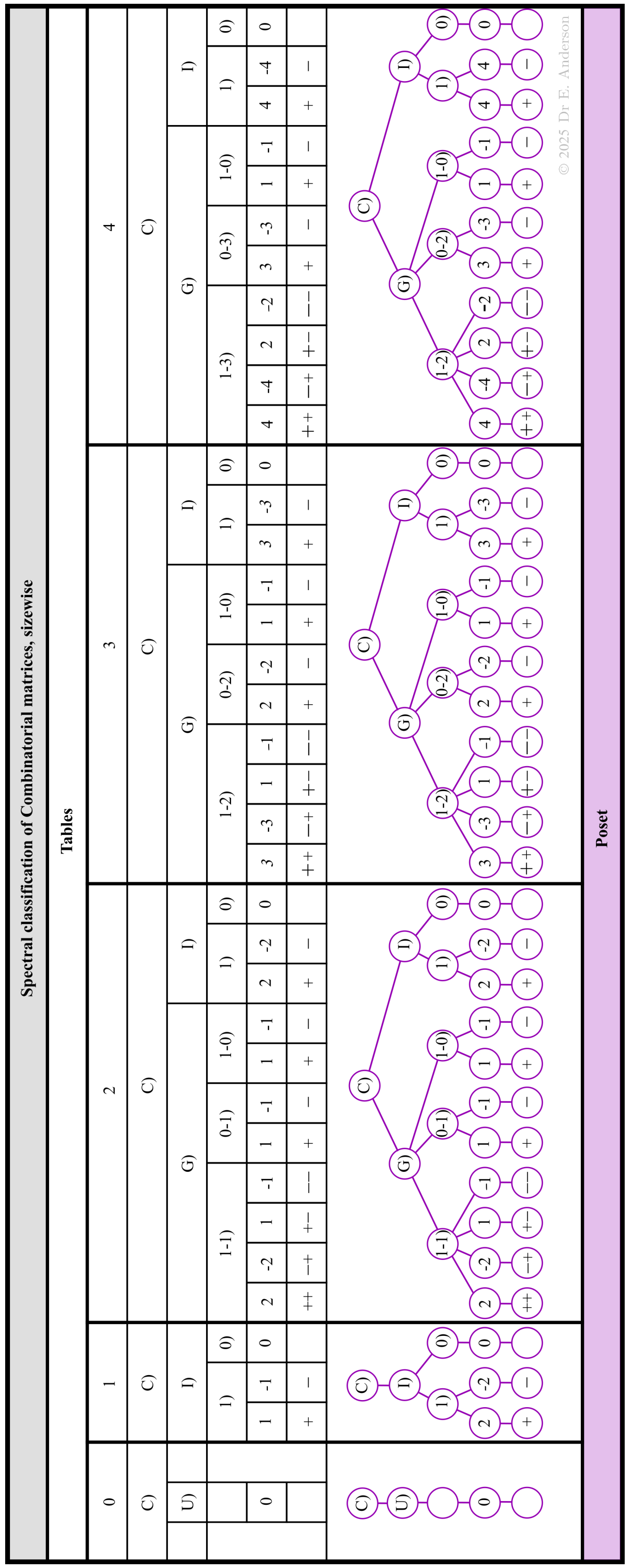
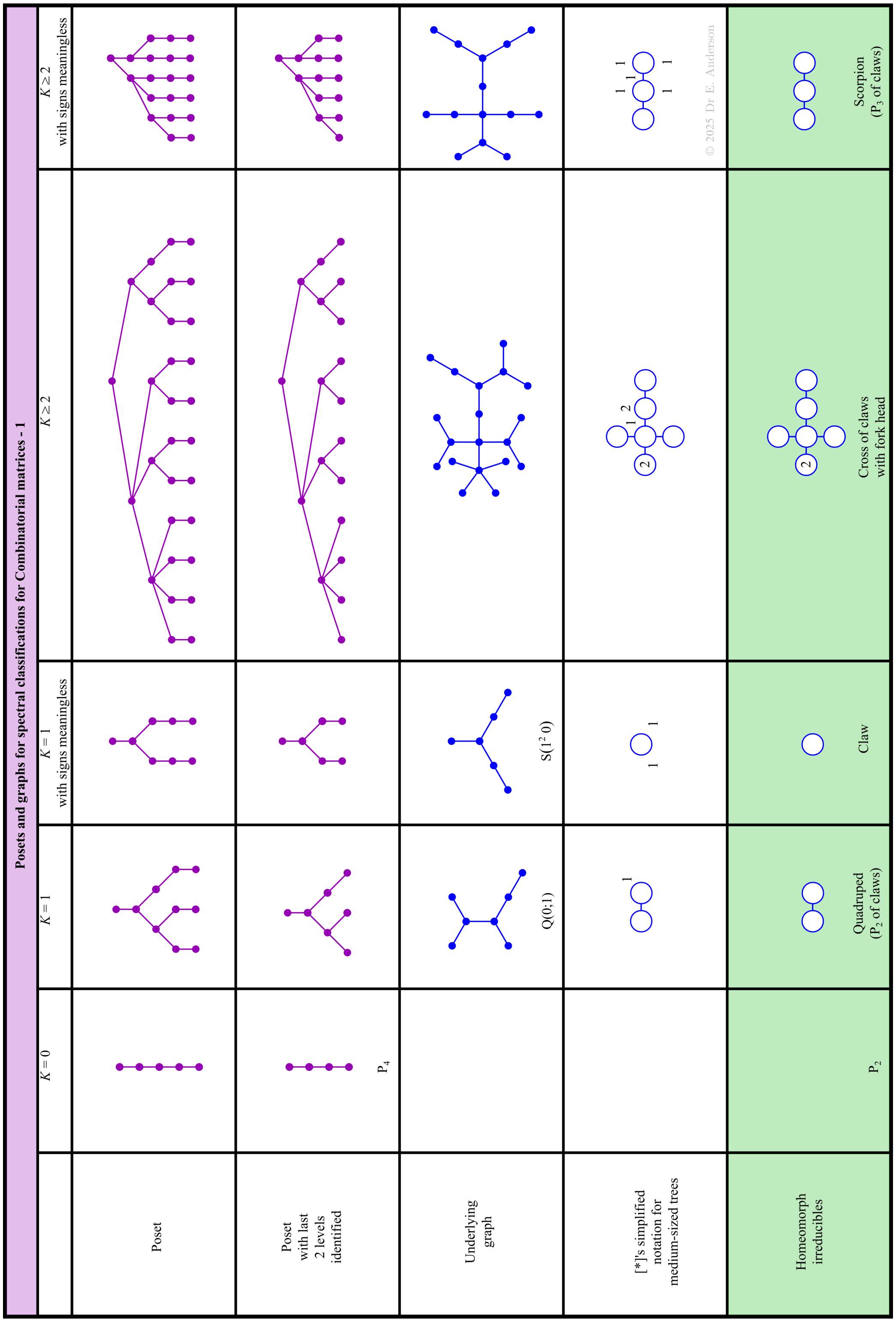

Spectral classification of Combinatorial matrices, sizewise
Tables
Poset
© 2025 Dr E. Anderson

Figure 6:



Posets and graphs for spectral classifications for Combinatorial matrices - 1
K = 0
K = 1
K = 1
with signs meaningless
K ≥ 2
K ≥ 2
with signs meaningless
Poset
Poset
with last
2 levels
identified
P4
Underlying
graph
Q(0;1)
S(1² 0)
[*]'s simplified
notation for
medium-sized trees
Homeomorph
irreducibles
P2
Quadruped
(P2 of claws)
Claw
Cross of claws
with fork head
Scorpion
(P3 of claws)
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Figure 7:

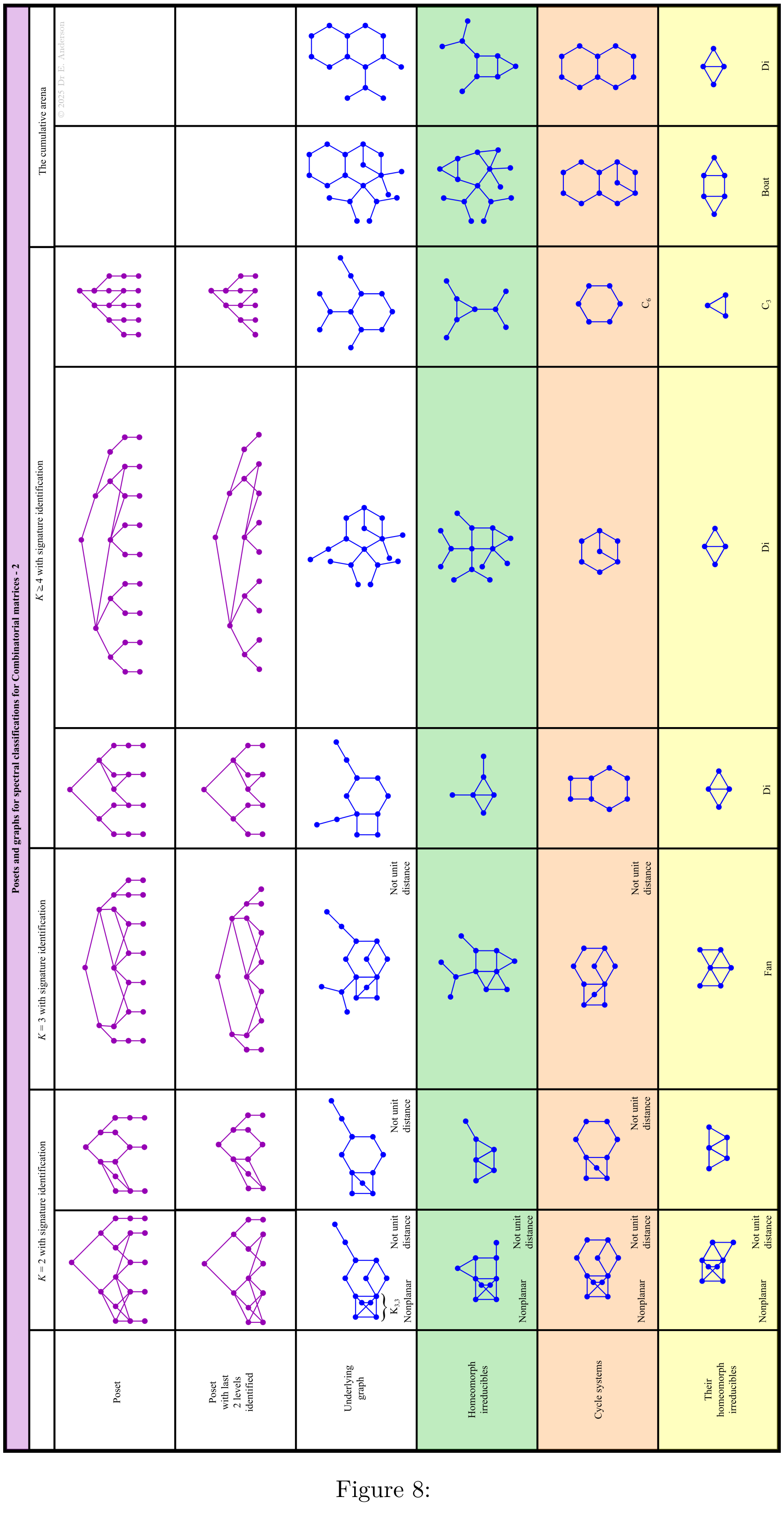
Posets and graphs for spectral classifications for Combinatorial matrices - 2
K = 2 with signature identification
K = 3 with signature identification
K ≥ 4 with signature identification
The cumulative arena
© 2025 Dr E. Anderson
Poset
Poset with last 2 levels identified
Underlying graph
Homeomorph irreducibles
Cycle systems
Their homeomorph irreducibles
Nonplanar
K3,3
Not unit distance
Fan
Di
C6
C3
Boat

Figure 8:

References

[1] Apollonius of Perga (3rd and 2nd Centuries B.C.E).

[2] Heron, alias Hero, of Alexandria, *Metrica* (60 C.E.)

[3] L. Euler's considered solving the associated linear system in integers in the 1760s; for an account of this, see in particular L.E. Dickson, *History of the Theory of Numbers. Vol. II: Diophantine Analysis* (Carnegie, Washington 1920).

[4] J.-L. Lagrange worked on Celestial Mechanics, among many other topics, in the late 18th Century.

[5] C.G.J Jacobi worked on Mechanics in the 1840's. See C.G.J. Jacobi's Collected Works (A.M.S., 1969).

[6] J.H.M. Wedderburn, "The Functional Equation $g(x^2) = 2\,a\,x + [g(x)]^2$ ", Ann. Math. **24** 121 (1922).

[7] R.A. Johnson, *Modern Geometry: an Elementary Treatise on the Geometry of the Triangle and the Circle* (Houghton Mifflin, Boston, MA 1929), republished as *Advanced Euclidean Geometry* (Dover, Mineola, N.Y. 1960).

[8] I.M.H. Etherington, "On Non-associative Combinations", Proc. Royal Soc. Edinburgh, **59** 153 (1939).

[9] H. Hopf, "Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche", ("Concerning the Images of $\mathbb{S}^3$ on $\mathbb{S}^2$ "), Math. Ann. (Berlin) **104** 637 (1931).

[10] P. Halmos, *Finite-Dimensional Vector Spaces*, (Springer-Verlag, N.Y. 1942).

[11] C. Lanczos, *The Variational Principles of Mechanics* (U. Toronto P., Toronto 1949).

[12] H.S.M. Coxeter, *Introduction to Geometry* (Wiley, New York 1961; 1989).

[13] R. Courant and D. Hilbert, *Methods of Mathematical Physics* Vol. 1 (Interscience, New York 1962; Wiley, Chichester 1989).

[14] Vol. 2.

[15] L.D. Landau and E.M. Lifshitz, *Quantum Mechanics* (Pergamon, New York 1965).

[16] A. Messiah, *Quantum Mechanics* (North Holland, Amsterdam 1967).

[17] D.E. Knuth, *The Art of Computer Programming* **1** *Fundamental Algorithms* (Addison-Wesley, 1968).

[18] E. Nering *Linear Algebra and Matrix Theory* (Wiley, New York 1970).

[19] See Part I of J-P. Serre, *Linear Representations of Finite Groups* (Springer–Verlag, New York, 1977).

[20] B.V. Landau, "An Asymptotic Expansion for the Wedderburn-Etherington Sequence", Mathematika, **24** 262 (1977).

[21] V.I. Arnol'd, *Mathematical Methods of Classical Mechanics* (Springer, New York 1978).

[22] N. Lloyd, *Degree Theory* (C.U.P., Cambridge 1978).

[23] M. Nakahara, *Geometry, Topology and Physics* (I.O.P., London 1990).

[24] R.H. Buchholz, "Perfect Pyramids", Bull. Austral. Math. Soc. **45** 353 (1992).

[25] F. Kirwan, *Complex Algebraic Curves* (C.U.P., Cambridge New York, 1992).

[26] D.C. Kay, *College Geometry* (Harper Collins, 1994).

[27] C.J. Isham, *Lectures on Quantum Theory* (Imperial College P., London 1995).

[28] A. Garg and R. Tamassia, "Upward Planarity Testing" Order **12** 109 (1995).

[29] R.G. Littlejohn and M. Reinsch, "Internal or Shape Coordinates in the N-body Problem", Phys. Rev. **A52** 2035 (1995).

[30] "Gauge Fields in the Separation of Rotations and Internal Motions in the N-Body Problem", Rev. Mod. Phys. **69** 213 (1997).

[31] R.P. Stanley, *Enumerative Combinatorics* Vol. 1 2nd ed. (C.U.P, Cambridge 2010). The 1st ed. dates back to 1997; use the 2nd ed. for technical reference.

[32] Lecture notes by W.T. Gowers, dating back at least as far as 1998.

[33] E. Merzbacher, *Quantum Mechanics* (Wiley, New York 1998).

[34] D.G. Kendall, D. Barden, T.K. Carne and H. Le, *Shape and Shape Theory* (Wiley, Chichester 1999).

[35] C.D. Meyer, *Matrix Analysis and Applied Linear Algebra* (S.I.A.M. 2000).

[36] G.H. Golub and H.A. van der Vorst, "Eigenvalue Computation in the 20th Century", J. Comput. App. Math. **123** 35 (2000).

[37] D.B. West, *Introduction to Graph Theory* (Prentice–Hall, Upper Saddle River N.J. 2001).

[38] D. Schmidt, "Central Configurations and Relative Equilibria for the N-Body Problem", in *Classical and Celestial Mechanics* ed. H Cabral and F. Diacu (P.U.P. , Princeton NJ 2002).

[39] B.A. Davey and H.A. Priestley, *Introduction to Lattices and Order* (C.U.P, Cambridge 2002).

[40] D.W. Lyons, "An Elementary Introduction to the Hopf Fibration", Math. Magazine **76** 87 (2003), arXiv:2212.01642.

[41] H.K. Urbantke, "The Hopf Fibration Seven Times in Physics", J. Geom. Phys. **46** 125 (2003).

[42] W. Fulton and J. Harris, *Representation Theory. A First Course* (Springer, New York 2004).

[43] H. Dym, *Linear Algebra in Action* (A.M.S, Providence RI 2006).

[44] G.F. Simmons, "The Formulas of Heron and Brahmagupta", in *Calculus Gems: Brief lives and memorable Mathematics* (M.A.A., Washington, D.C. 2007).

[45] S. Roman, *Advanced Linear Algebra* (Springer 2007).

[46] J.B. Conway, *A Course in Functional Analysis* (Springer, 2007).

[47] M. Bóna and P. Flajolet, "Isomorphism and Symmetries in Random Phylogenetic Trees", J. Applied Probability, **46** 1005 (2009), arXiv:0901.0696,

[48] G.H. Golub and C.F. Van Loan, *Matrix Computations* 4th ed. (Johns Hopkins, Baltimore MD, 2013).

[49] S. Weinberg, *Lectures on Quantum Mechanics* (C.U.P., Cambridge 2013).

[50] I.E. Leonard, J.E. Lewis, A.C.F. Liu and G.W. Tokarsky, *Classical Geometry. Euclidean, Transformational, Inversive and Projective* (Wiley, Hoboken N.J. 2014).

[51] R. Ghrist, *Elementary Applied Topology* (2014).

[52] L. Liberti and C. Lavor, "Six Mathematical gems from the history of Distance Geometry", International Transactions in Operational Research. **23** 897, arXiv:1502.02816.

[53] E. Anderson, "The Smallest Shape Spaces. I. Shape Theory Posed, with Example of 3 Points on the Line", arXiv:1711.10054. For the updated version, see https://wordpress.com/page/conceptsofshape.space/1225 . accepted by Period. Updating Rev. Applied Topology.

[54] E. Anderson, "Two New Perspectives on Heron's Formula", arXiv:1712.01441 Ann. Inst. Theory STEM **1** 1 (2025), https://institute-theory-stem.org/annals-of-the-institute-of-stem/ .

[55] Geometrical discussions between S. Sánchez and E.A. in 2017-2020.

[56] Combinatorial discussions between S. Sánchez, E.A. and A. Ford in 2017-2020.

[57] J.B.S. Temones, "Combinatorial Matrices" J. Physics Conference Series **1593** 10th International Conference on Applied Physics and Mathematics at: Tama Campus, Chuo University, Tokyo, Japan (2020).

[58] E. Anderson, *Applied Combinatorics*, Widely-Applicable Mathematics Series. A. Improving understanding of everything with a pinch of Combinatorics. **0**, (2022). Made freely available in response to the pandemic here: https://conceptsofshape.space/applied-combinatorics/ .

[59] G. Carlsson and M. Vejdemo-Johansson, *Topological Data Analysis with Applications* (C.U.P., Cambridge 2022).

[60] E. Anderson, "The Fundamental Triangle Matrix", Ann. Inst. Theory STEM **1** 2 (2025), https://institute-theory-stem.org/annals-of-the-institute-of-stem/ . Preprint: https://wordpress.com/page/conceptsofshape.space/1306 (2024).

[61] "The Fundamental Triangle Matrix Commutes with the Lagrange and Apollonius Matrices. With implications for deriving Kendall and Hopf Theorems." (2024) Ann. Inst. Theory STEM **1** 3 (2025), https://institute-theory-stem.org/annals-of-the-institute-of-stem/ . Preprint: https://wordpress.com/page/conceptsofshape.space/1308 (2024).

[62] "Only 2 of the Fundamental Triangle, Lagrange and Apollonius Matrices are Independent. With ensuing Algebras, Irreducibles and Splits" (2024 preprint) Ann. Inst. Theory STEM **1** 4 (2025), https://institute-theory-stem.org/annals-of-the-institute-of-stem/ . Preprint: https://wordpress.com/page/conceptsofshape.space/1310 (2024).

[63] E. Anderson and A. Ford (she/her), "Graph and Order Theory of N-Body Problem's Eigenclustering Networks", accepted by Period. Updating Rev. Applied Topology. Preprint: https://wordpress.com/page/conceptsofshape.space/2090 (2024),

[64] E. Anderson, "4-Body Problem: Ptolemy–Lagrange Algebra", Preprint: https://wordpress.com/page/conceptsofshape.space/2121 (2024).

[65] E. Anderson and K. Everard (she/her), "Linear Algebra of Cyclic Quadrilaterals: Ptolemy, Diagonal, Area, and Circumradius Formulae", Preprint: https://wordpress.com/page/conceptsofshape.space/1971 (2024).

[66] A. Ford, "Triangle, but not most Quadrilateral, Matrices are Combinatorial" (2024 preprint) Ann. Inst. Theory STEM **1** 5 (2025), https://institute-theory-stem.org/annals-of-the-institute-of-stem/ .

[67] E. Anderson, "Convex Quadrilaterals' Ptolemy–Lagrange–Bretschneider Algebras", forthcoming 2025, Preprint: https://wordpress.com/page/conceptsofshape.space/2278 (2024).

[68] R. Montgomery, *Four Open Questions for the N-Body Problem* (C.U.P., Cambridge 2025).

[69] The On-Line Encyclopedia of Integer Sequences, "Wedderburn-Etherington Numbers", https://oeis.org/A001190 .

[70] E. Anderson and A. Ford, "The 6-Graphs. Names and Properties", (2026).

[71] *The Structure of Flat Geometry*, in some places called additionally *Widely-Applicable Mathematics. A. Improving understanding of everything by dual-wielding Combinatorics and Linear Algebra.* **3**, forthcoming 2026.

[72] "Hopf's Little Mathematics 32 times in Physics and Geometry", https://wordpress.com/page/conceptsofshape.space/1759 , Invited Review for Period. Updating Rev. Applied Topology, extending a 2024 Summer School's material, forthcoming 2026.

[73] Online Encyclopaedia of Applied Graph and Order Theory, https://institute-theory-stem.org/online-encyclopaedia-of-graphs-and-orders/ .

[74] E. Anderson, *Linear Mathematics. Vector Spaces, Linear Methods and Tensors Throughout STEM*, *Widely-Applicable Mathematics. A. Improving understanding of everything with a pinch of Combinatorics* **1**, forthcoming 2027.

[75] "The Gauss–Bonnet Theorem and Generalizations", to be based on a 2026 Summer School's material, forthcoming 2027.