Combinatorial Matrices' Kallista symbol:

Projective, Spectral, isotropic and multiplicative

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Abstract

Combinatorial matrices have recently been studied by Ford and the Author. We here add a fourth kind of Ford symbol, which carries both Projective and Spectral connotations. And unlike the previous three, manages to have the very simplest multiplicative property. Revealing that Combinatorial matrices' eigenvalues combine in the simplest possible way under products and sums: an uncommon good fortune among square matrices. $K \times K$ Combinatorial matrices support not only K-square eigenexpansions but also 2-square eigenexpansions. This is underpinned by Combinatorial matrices mostly consisting of an isotropic block. A new style of proof for each K's Combinatorial matrices commuting with each other is also included.

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1 Introduction

Definition 1 A *Combinatorial matrix* [1] is a $K \times K$ matrix of the following form.

$$\begin{pmatrix} x+y & x & \dots & x \\ x & & & \vdots \\ \vdots & & & x \\ x & \dots & x & x+y \end{pmatrix} = y\mathbb{I} + x\mathbb{1} =: (x+y, y)_K . \tag{1}$$

Remark 1 While Combinatorics often involves $x, y \in \mathbb{Z}$ or $\mathbb{N}$, Flat Geometry applications [8, 9, 10] extend interest to $\mathbb{R}$. Here $\mathbb{I}$ is the identity matrix and $\mathbb{1}$ is the matrix of 1s. While $(\ ,\ )_K$ is the Ford symbol of the zeroth kind [12, 13], truer name *irreducible symbol*. With reference to the trace–tracefree basis, with basis matrices $\mathbb{I}$ and $\mathbb{T}$: 1's off-diagonal and 0's on-diagonal. Whenever we work with a fixed K, we simplify this notation to $(\ ,\ )$.

Remark 2 For a fixed K, the totality of these constitute the arena $\mathfrak{C}M_K(2)$. Which is a 2-d vector space.

Remark 3 Combinatorial matrices were expanded relative to various other bases in [10, 12]. We now consider a further such, which accumulates various interesting properties.

Some previous bases involved $\mathbb{1}$. We consider this again, now however viewing

$$K^{-1}\underline{\overline{\mathbb{1}}} = (1, 1)$$

in the following further way. As the dyad of normalized equal-entries vectors

$$\overline{\boldsymbol{n}}\,\underline{\boldsymbol{n}} .$$

Which is furthermore an orthogonal projector

$$\underline{\overline{\boldsymbol{P}}}_{\perp} \tag{2}$$

onto the 1-d total-sum direction space [13]

$$\mathfrak{D}\mathrm{r}(1) . \tag{3}$$

Whose complement is k-dimensional difference space [13]

$$\mathfrak{D}\mathrm{f}(k) . \tag{4}$$

With corresponding orthogonal projector

$$\underline{\overline{\boldsymbol{P}}}_{\perp\perp} = \underline{\overline{\boldsymbol{P}}} = \underline{\overline{\mathbb{I}}} - \overline{\boldsymbol{n}}\,\underline{\boldsymbol{n}} . \tag{5}$$

For Combinatorial matrices are blessed with [13] at least a $k := K - 1$ fold degenerate eigenvalue. Resting upon at least an $O(k)$-symmetry isotropy being forced [13]. In the generic case, the corresponding eigensubspace is difference space. The only alternative to this is conflation with the remaining eigenvalue. Yielding a single unsplit K-fold degenerate eigenspace. Now resting upon the whole matrix being $O(K)$-symmetry isotropic [13].

In the N-body problem context, (3) is the centre of mass (CoM) position label. While (5) is the CoM-removing projector And (4) takes the form of relative label space $\mathfrak{Rel}(k)$: the space of linearly-independent (LI) separation vector labels. Labels first enter $1, 2, ...$ to the points. Leading to the separations being labelled by their pair of boundary points, e.g. $1\,2$. CoM position carries both a spatial vector index and a point label index. So does the relative space of LI separations [18]. But which are only the label part plays an active role: the full version of everything just tensors with the spatial identity matrix [18].

In this context, $\boldsymbol{P}$ is furthermore numerically (if not Physically-dimensionally [18]) the Lagrange matrix [16]. The overarching theme is that upon translating to the CoM frame, all remaining degrees of freedom are relative separations. With the Lagrange matrix arising by [18] extremizing the arbitrary-translation correction to the inertia quadric with respect to its translation variable. This is how translations, CoMs and separations are inter-related. For the general Combinatorial matrix, upon translating to the *total sum*, all remaining freedom lies correspondingly in *difference variables*.

Remark 4 The current Note's incipient idea is to take $\boldsymbol{P}_\perp$ and $\boldsymbol{P} = K^{-1}\,(\,k, -1\,)$ as our basis. Which is a priori motivated by its being privileged by both of its basis matrices being projectors. While the below results build up additional a posteriori motivations.

2 Eigentheory results

Lemma 1

$$(\,x + y, x\,)\,\boldsymbol{P} = y\,\boldsymbol{P} = \lambda\,\boldsymbol{P} =: \lambda_k\,\boldsymbol{P}\ . \tag{6}$$

$$(\,x + y, x\,)\,\boldsymbol{P}_\perp = (\,K\,x + y\,)\,\boldsymbol{P}_\perp = \lambda_\perp\,\boldsymbol{P}_\perp =: \lambda_1\,\boldsymbol{P}_\perp\ . \tag{7}$$

Where λ is the k-fold degenerate eigenvalue λ_k : with algebraic multiplicity k [13]. While $\lambda_\perp$ is the complementary eigenspace's lone eigenvalue λ_1 : with algebraic multiplicity 1 .

Corollary 1

$$(\,x + y, x\,) = \lambda\,\boldsymbol{P} + \lambda_\perp\,\boldsymbol{P}_\perp\ . \tag{8}$$

Naming Remark 1 This prompts introducing a fourth 'Ford symbol': the *Projective symbol*

$$\{\,\lambda, \lambda_\perp\,\}_K\ . \tag{9}$$

Where the first entry is the $\boldsymbol{P}$ component and the second the $\boldsymbol{P}_\perp$ component. But these entries are furthermore the spectrum of eigenvalues (without multiplicities), so another truer name for it is *Spectral symbol*. That a pair of eigenvalues covers the whole spectrum in this sense is underpinned by Combinatorial matrices having an at least k-fold degenerate eigenvalue [13]. Resting in turn on an at least $O(k)$-symmetry isotropy. Whence a third name: *isotropy symbol* with reference to an at-most codimension-1 isotropy.

Lemma 2

$$\{\,p\,\lambda_1, p\,\lambda_{2\,\perp}\,\} = p\,\{\,\lambda_1, \lambda_{2\,\perp}\,\}\ . \tag{10}$$

$$\{\,\lambda_1, \lambda_{1\,\perp}\,\} + \{\,\lambda_2, \lambda_{2\,\perp}\,\} = \{\,\lambda_1 + \lambda_2, \lambda_{1\,\perp} + \lambda_{2\,\perp}\,\}\ . \tag{11}$$

$$\{\,\lambda_1, \lambda_{1\,\perp}\,\}\{\,\lambda_2, \lambda_{2\,\perp}\,\} = \{\,\lambda_1\,\lambda_2, \lambda_{1\,\perp}\,\lambda_{2\,\perp}\,\}\ . \tag{12}$$

Remark 1 So passing to using our symbol preserves the simplest linearity properties, while attaining the simplest product formula. Nor is it at all usual for the eigenvalues of the product of 2 matrices to be the product of the eigenvalues of the 2 matrices. And yet we have hereby established that this holds for any 2 Combinatorial matrices of the same size... And similarly with 'sum' in place of 'product'!

Naming Remark 2 Let us celebrate by, firstly, placing a fourth name on this Combinatorial matrix symbol: *multiplicative symbol*. Secondly, by pointing out that at present this symbol 'conceptually outnumbers' our irreducible symbol by 4 to 1. I.e. it is known to have multiplicative, Projective, Spectral and symmetry-protected significance. To its rival carrying Representation-Theoretic significance. By the above confluence of properties from across Mathematics, we call $\{ \, , \, \}_K$ the Combinatorial matrix *Kallista symbol*.

Corollary 2 And thirdly by extending to the following.

$$\sum_{i=1}^{R} \{ \lambda_i, \lambda_{i\perp} \} = \left\{ \sum_{i=1}^{R} \lambda_i, \sum_{i=1}^{R} \lambda_{i\perp} \right\} , \quad \prod_{i=1}^{R} \{ \lambda_i, \lambda_{i\perp} \} = \left\{ \prod_{i=1}^{R} \lambda_i, \prod_{i=1}^{R} \lambda_{i\perp} \right\} . \quad (13)$$

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Remark 2 Next working with the quadratic form version – acting on a Combinatorial vector $\boldsymbol{C}$, we readily obtain the following results. In the N-body problem case, it turns out to be quite useful [9, 18] to take $\boldsymbol{C} = \boldsymbol{S}$: the vector of side-lengths squared. These are 1 dependency away from being LI. But there are also K of them, while we are only expecting k differences. So everything checks out at the level of well-determinedness.

Proposition 1 2-squares expansion.

$$||\boldsymbol{C}||_{\{ \lambda, \lambda_\perp \}}{}^2 = \lambda_\perp ||\boldsymbol{C}||_{\boldsymbol{P}_\perp}{}^2 + \lambda ||\boldsymbol{C}||_{\boldsymbol{P}}{}^2 = \lambda_\perp K^{-1} U^2 + \lambda ||\boldsymbol{C}||_{\boldsymbol{P}}{}^2 . \quad (14)$$

Proposition 2 K-squares expansion in $\mathrm{K}(K)$ network choice of basis.

$$||\boldsymbol{C}||\{ \lambda, \lambda_\perp \}^2 = \lambda_\perp K^{-1} U^2 + \lambda \sum_{q=1}^{k} q^{-1} Q^{-1} \left(\sum_{p=q}^{k} C_p - q \, C_Q \right)^2 . \quad (15)$$

Remark 3 In the N-body problem context, firstly the total sum of the objects U is realized as R: the square of the *radius of gyration*. Secondly, the first distinct K network is called the Jacobi-K [3, 16, 18] for the 4-body problem. This corresponds to [6, 11] the straight-3-path unlabelled rooted binary tree (URBT) [2]. The subsequent K networks are the corresponding straight-path URBTs for each larger path. [13] showed that these bases remain meaningful for arbitrary Combinatorial matrices. Thirdly, the above codimension-1 isotropy corresponds in the N-body setting to the *democracy transformations* [4].

Corollary 3

$$||\boldsymbol{c}||_{\{ \lambda, \lambda_\perp \}}{}^2 = \lambda_\perp K^{-1} + \lambda ||\boldsymbol{c}||_{\boldsymbol{P}}{}^2 = \lambda_\perp K^{-1} + \lambda \sum_{q=1}^{k} q^{-1} Q^{-1} \left|\left| \sum_{p=q}^{k} \boldsymbol{c}_p - q \, \boldsymbol{c}_Q \right|\right|^2 . \quad (16)$$

Remark 4 The K-squares expansion uses the whole eigenspectrum – with algebraic multiplicity – based on

$$\mathfrak{M}_K = Im_{\boldsymbol{P}_\perp}(\mathfrak{M}_K) \oplus Im_{\boldsymbol{P}}(\mathfrak{V}_K) = \mathfrak{D}\mathrm{r}(1) \oplus \mathfrak{D}\mathrm{f}(k) .$$

For $\mathfrak{M}_K$ the space of $K \times K$ matrices. In contrast, 2-squares expansion uses that,

$$\mathfrak{C}\mathrm{M}(2) = \mathfrak{D}\mathrm{r}(1) \oplus \mathfrak{D}\mathrm{f}(1) .$$

Remark 5 Where we have introduced the ratio variables

$$\boldsymbol{c} := U^{-1} \boldsymbol{C} . \quad (17)$$

And which is enabled by the above at most codimension-1 isotropy.

3 Examples

Examples 1-7: in our brackets notation. The CoM-removing projector itself is

$$\boldsymbol{P} \ = \ \{\,1,\, 0\,\} \ .$$

Its complement is

$$\boldsymbol{P}_\perp \ = \ \{\,0,\, 1\,\} \ .$$

The zero matrix is

$$\mathbb{0} \ = \ \{\,0,\, 0\,\} \ .$$

And the identity matrix is

$$\mathbb{1} \ = \ \{\,1,\, 1\,\} \ .$$

The previous two are both examples of isotropic matrices, the general case of which is

$$r\,\{\,1,\, 1\,\} \ , \quad r \ \in \ \mathbb{R} \ .$$

The fundamental 2-simplex matrix, shared by the triangle inequality, the cosine rule, and Heron's formula, [7, 8, 9, 10] is

$$\boldsymbol{F} \ = \ \{\text{-}2,\, 1\} \ .$$

The Apollonius involutor [7, 9, 10] is

$$\boldsymbol{J} \ = \ \frac{1}{3}\,\{\text{-}1,\, 1\} \ .$$

Example 8 Of sums of K squares. In the $K \ = \ N \ = \ 3$-body problem context, Proposition 2 returns $\mathit{Aniso}^2 \ + \ \mathit{Anelp}^2$ for its last 2 squares [7, 8, 9, 18]. Standing for *anisoscelesness*: departure from isoscelesness. And departure from being in equilateral proportion, with reference to the base to median ratio. In terms of side-lengths[2] $A,\, B, C$,

$$\mathit{Aniso} \ = \ \frac{A \ - \ B}{\sqrt{2}} \ , \quad \mathit{Anelp} \ = \ \frac{A \ + \ B \ - \ 2\,C}{\sqrt{6}} \ .$$

Thus generalizing from sides in the 3-body problem to differences of primary objects in Combinatorics, Proposition 2 recovers as a subcase

$$||\boldsymbol{C}||_{(\,x \ + \ y, x\,)}{}^2 \ = \ \frac{3\,x \ + \ y}{3}\,U^2 \ + \ \frac{y}{2}\left[\,(\,A \ - \ B\,)^2 \ + \ \frac{(\,A \ + \ B \ - \ 2\,C\,)^2}{3}\,\right] \ . \tag{18}$$

Which can also be written in the following generalized Euler 3-cycle form.[1]

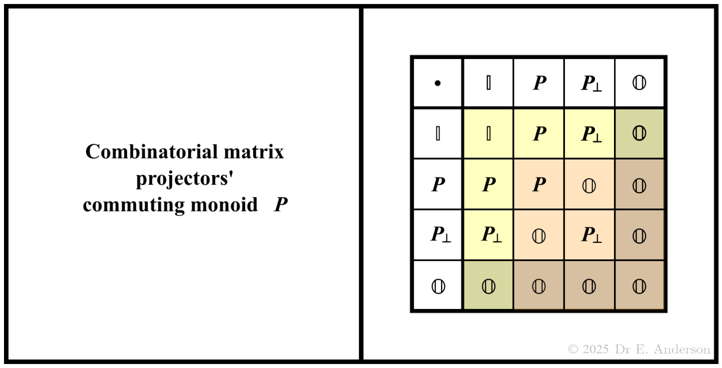
$$||\boldsymbol{C}||_{(\,x \ + \ y,\, x\,)}{}^2 \ = \ \frac{3\,x \ + \ y}{3}\sum_{\text{3-cycles}} A\,(\,A \ + \ 2\,B\,) \ + \ \frac{2\,y}{3}\sum_{\text{3-cycles}} A\,(\,A \ - \ B\,)$$

$$= \ \sum_{\text{3-cycles}} A\,[\,(\,x \ + \ y\,)\,A \ + \ 2\,x\,B\,] \ . \tag{19}$$

Setting x and y to match $\boldsymbol{F}, \boldsymbol{J}$ and $\boldsymbol{P}$ in turn recovers some previous results from [7, 10]. In particular, since $\boldsymbol{P}$ is itself the projector, the eigenvalue $\lambda_\perp$ corresponding to $\boldsymbol{P}_\perp$ is 0 . And so the $U \ = \ R$ contribution to $\boldsymbol{P}_\perp$ drops out entirely.

[1] This is usually for triangles, or 2-simplices. But we find that the arbitrary Combinatorial matrix has a counterpart! See [18] for triangle-free names for *Aniso* and *Anelp* .

4 Algebras



$\bullet$	$\mathbb{I}$	$\boldsymbol{P}$	$\boldsymbol{P}_\perp$	$\mathbb{0}$
$\mathbb{I}$	$\mathbb{I}$	$\boldsymbol{P}$	$\boldsymbol{P}_\perp$	$\mathbb{0}$
$\boldsymbol{P}$	$\boldsymbol{P}$	$\boldsymbol{P}$	$\mathbb{0}$	$\mathbb{0}$
$\boldsymbol{P}_\perp$	$\boldsymbol{P}_\perp$	$\mathbb{0}$	$\boldsymbol{P}_\perp$	$\mathbb{0}$
$\mathbb{0}$	$\mathbb{0}$	$\mathbb{0}$	$\mathbb{0}$	$\mathbb{0}$

Figure 1:

Proposition 3 Under multiplication, $\boldsymbol{P}$ and $\boldsymbol{P}_\perp$ alongside the identity $\mathbb{I}$ and zero $\mathbb{0}$ form the commutative monoid [5] whose times table is in Fig 1.

Proposition 4 The above four inputs also form a zero-commutator algebra.

Remark 1 Proposition 3's times table follows from just the identity property, the projector property and projector complementarity. Proposition 3's inputs reflect that, on the one hand, we can freely append an identity element. On the other hand, we are forced to include the zero since we discover it as the product of our two projectors.

Remark 2 For Proposition 4, the only nontrivial bracket to check is

$$[\,\overline{\boldsymbol{P}}\,,\,\overline{\boldsymbol{P}}_\perp\,] \;=\; \overline{\boldsymbol{P}}\cdot\overline{\boldsymbol{P}}_\perp \;-\; \overline{\boldsymbol{P}}_\perp\cdot\overline{\boldsymbol{P}} \;=\; \mathbb{0} \;-\; \mathbb{0} \;=\; \mathbb{0}\;.$$

Where step 1 is by the definition of commutator. And step 2 makes 2 uses of complementarity.

Remark 3 The current Section is the generalized Combinatorial matrices' counterpart of the triangle (or more generally 2-simplex) algebras presented in [10]. And of various quadrilateral algebras presented in [14, 15, 17].

5 Conclusion

Remark 1 A first motivation for our Kallista symbol for Combinatorial matrices is that it corresponds to the Projective choice of basis. In the current Note, this appeared as our a priori reason to entertain this symbol. A posteriori, we found 3 further motivations to strengthen this.

Remark 2 Using our new basis, the product rule (12) and the commutative monoid concurrently manage to take particularly simple forms. In contrast to the various basis choices in [10], the current Article's commutative monoid's is distinctive by its times table necessarily involving the zero. That compatible Combinatorial matrices always commute was recently pointed out by Ford [12]. Proposition 4 however provides an alternative style of proof for this! I.e. every Combinatorial matrix can be written as the weighted sum of a complementary pair of projectors. Which commute with each other, and so all compatible sized Combinatorial matrices commute with each other.

Remark 3 A third motivation is Spectral: our basis displays the eigenvalues as its components. And clarifies that these combine particularly simply for Combinatorial matrices under both addition and multiplication. General square matrices' eigenvalues do not have these properties! By which our basis displays a 2-term eigenexpansion. This is rendered possible by all Combinatorial matrices having an eigenvalue that is at least k-fold degenerate by symmetry. Due to the underlying at-least $O(k)$-symmetry – and thus at most codimension-1 – isotropy.

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