Euler's Bimedian-Length Theorem Generalized
to an Infinite Family of Eigenaxles between Equal-sized Wheels

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Abstract

We generalize Euler's bimedian-length theorem. Where we take 'bimedian' to mean a line segment between the midpoints of any opposite pair of separations of a 4-vertex configuration. Our program's generalization is from bimedians to whichever eigenstroke length for any N vertices in any dimension. Where an *eigenstroke length* is the magnitude of any eigenclustering vector that is not just a separation. And *eigenclustering vector* is a truer name for a relative Jacobi vector.

For equal masses on the vertices, the largest such generalization considered [45] generalizes Apollonius' median-length theorem as well. Giving generically a 3-coefficient formula. Euler's theorem is distinctive however in having just 1 coefficient. The current Article's narrower generalization is then to the other eigenstrokes that also have just 1 . These consist of the '*eigenaxles*: eigenstokes between equal-sized subsystems: the 'wheels'. Every even $N \geq 4$ contributes precisely 1 new eigenaxle, irrespectively of how the configuration is elsewhere eigenclustered.

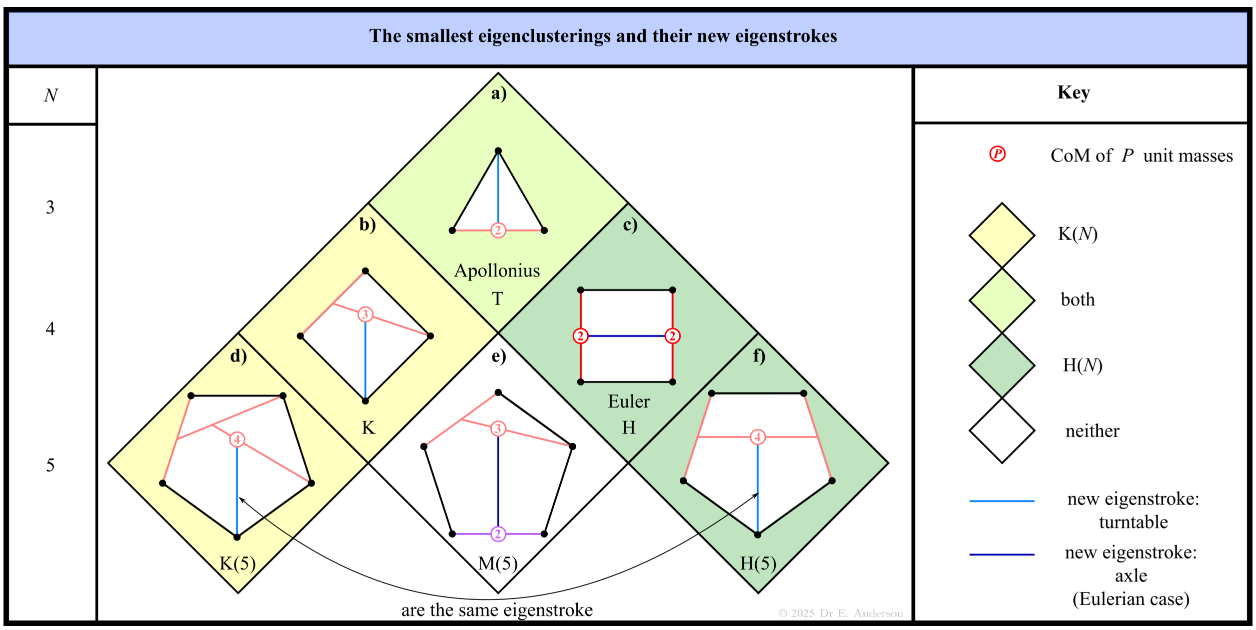

Figure 1:

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1 Introduction

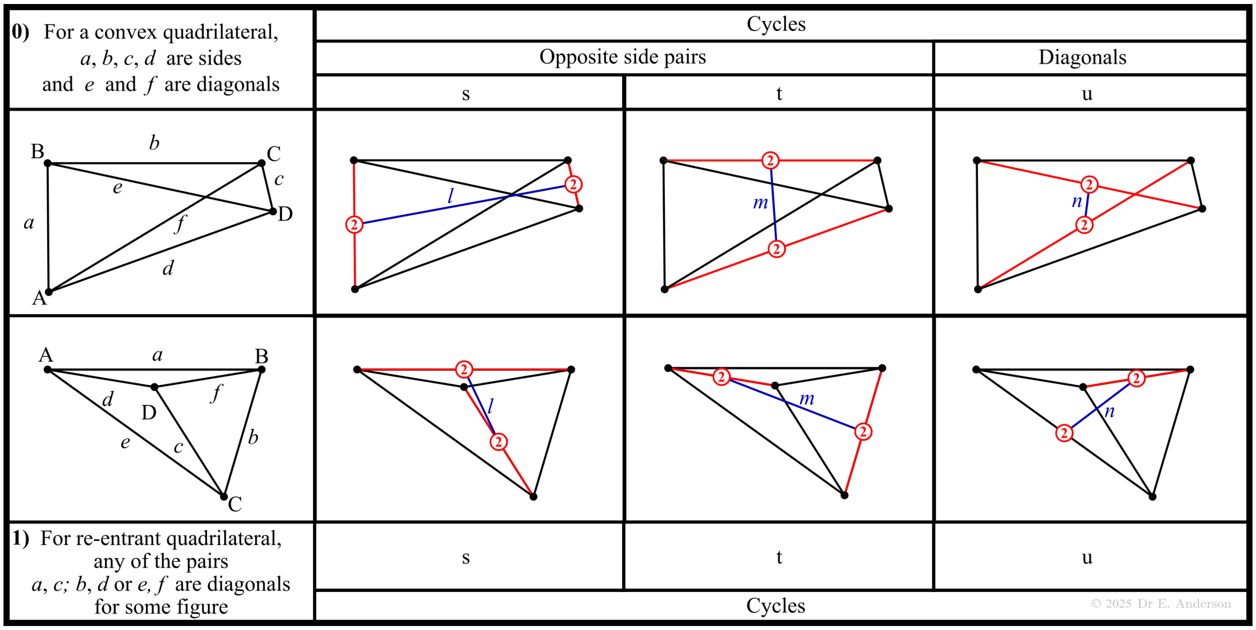

Figure 2:

1.1 The Theorem

For a convex quadrilateral, this Theorem of Euler's reads [3, 21, 23, 25, 33, 39, 52]

$$4\,n^2 \;=\; a^2 \;+\; b^2 \;+\; c^2 \;+\; d^2 \;-\; e^2 \;-\; f^2\;. \tag{1}$$

Consult Fig 2.0) notation for the sides and diagonals, which together comprise the separations. Also the object being solved for is the Newton(–Euler–Gauss) [2, 3, 5, 7, 9, 25] length n . Meaning the length between the midpoints of the 2 diagonals, as depicted in Subfig 0.u).

Furthermore, 3-cycles of this hold as well. For a convex quadrilateral, the other 2 cycles single out by sign reversal an opposite-sides pair rather than the diagonals. Now solving instead for the lengths of the line intervals l, m joining the midpoints of this pair; see Subfigs 0.s)-t).

For tetrahaedrons, the above 3 cycles continue to hold [3, 21, 39, 52]. Just now for 3 opposite-sides pairs without any carrying diagonal connotations. For re-entrant quadrilaterals, sides and diagonals are meaningful within each figure. And yet given just the vertex positions, any non-adjacent pair of separations can be interpreted as the diagonals for some 'joining of the dots' (Subfig 1). The Theorem also holds in 1-d as well, or in fact in any dimension. As established by reproving it in a dimension-irrelevant manner, such as by a moments method [39].

Naming Remark 1 This Theorem is often referred to as 'Euler's quadrilateral theorem'. And is often stated for convex quadrilaterals' case u only. These are however a modern-day misnomer and an unnecessary weakening. Indeed, Euler already knew that the result holds for arbitrary quadrilaterals and tetrahaedra. Incorporating the result's dimension-independence, *Euler's* 4-*vertex theorem* is thus a truer name (Geometry). As is *Euler's* 3-*simplex theorem* (Geometry, Combinatorics and Topology). Or *Euler's* 4-*body theorem* (Physics and Dynamics).

Notational Remark 1 The 3-cycle of lengths solved for as per above are known as *bimedians*. In particular, l, m, n are all standardly called bimedians for tetrahaedrons. Some sources ony use 'bimedian' [29] for a convex quadrilateral's l and m . However, given Euler's 3-cycle symmetry, and the consolidation of the usefulness of the whole-cycle notion by the above theorem, we take the following stance. The theorem enforces a whole 3-cycle notion of bimedian irrespective of dimension, being functionally a bimedian-length theorem in this sense. The s , t and u labels exhibited in the Figure as an alternative 3-cycle index values have the following origin. Particle Physics' account of 4-external point Feynman diagrams [17].

Notational Remark 2 Let us next adopt *weak Conway variables* and *notation*, using

$$\text{Length} := \text{length}^2 .$$

With capital letters for each Length corresponding to the lower-case letters used for each length. These render the above theorem linear:

$$4N = A + B + C + D - E - F . \qquad (2)$$

Variables of this kind continue to be useful in handling generalizations of this result [40, 41, 42, 44, 45, 46, 47] and more widely in Geometry [35, 52, 50].

1.2 The smallest eigenclusterings

Definition 1 An N-vertex configuration's *eigenclustering vectors* are LCs (linear combinations) of its relative separation vectors that diagonalize its inertia quadric, alias total moment of inertia. The current Article restricts itself entirely to the simplest and most Geometricaly natural case of equal-mass (EM) vertices. *Eigenclustering lengths* are then the corresponding magnitudes. A *nontrivial eigenclustering length*, alias *stroke*, is one that is not just a separation-length. 'Eigenclustering vector' is a truer name for what are more widely termed *relative Jacobi vectors* [6, 15, 19, 22]; the alias *relative Jacobi magnitude* is more occasionally used.

Definition 2 An *eigenclustering network* is a basis choice of eigenclustering vectors, with reference to how these fit together but without reference to whether or how the vertices are labelled.

Remark 1 Each of an N-vertex system's eigenclustering networks contains $n := N - 1$ LI (linearly independent) eigenclustering vectors, and thus introduces n eigenclustering lengths. This n is the dimension of relative space [37, 30, 52]. And LI refers to relative space as well, rather than to the Euclidean space that the vertices reside in.

Definition 3 An *eigenstroke vector* is our term for a nontrivial eigenclustering vector without reference to any particular eigenclustering network it belongs to. For it is between 2 subsystem CoMs (centres of mass) without any dependence on how these subsystems are eigenclustered. Or on how the rest of the full system is eigenclustered. This is with reference to our stroke, and the 2 partitioning subsystems that it runs between, in general forming but a larger subsystem of the full system at hand. Its corresponding magnitude is an *eigenstroke length*.

Remark 2 This distinction between strokes and eigenstrokes is useful because some eigenclustering vectors belong to multiple distinct eigenclustering networks. It also serves to pin the "eigen" descriptor on names referring to an eigenclustering vector-or-length free from reference to any eigenclustering network. For which using "eigenclustering" would either be ambiguous or require clunking up names with further words.

Example 3 $N = 3$ is minimum to have a nontrivial eigenclustering length. Its stroke is the triangle's median; see Fig 1.a).

Examples 4.K and .H $N = 4$ is minimum to have an eigenclustering network ambiguity: H versus K as per Subfigs b) and c). The H has a single stroke: the crossbar whose 3-cycle-invariant Geometrical name is, as abovementioned, the bimedian. While the K is the minimum eigenclustering network to contain 2 strokes: the slanty parts of the letter. Furthermore, the first of these is just the median of the 3-subsystem picked out, in which sense it is not a new stroke. This ambiguity, multiplicity of strokes in an eigenclustering, with however only 1 new stroke per N, are subsequently persistent.

Example 5 The minimum example in which the same stroke occurs in 2 different eigenclusterings is provided by $N = 5$. Here the last stroke of K(5) and of H(5) both run from the CoM of the same tetrahaedron subsystem to the remaining vertex. See Fig 2.d) and f) for this coincidence for the next-smallest handle eigenstroke. $N = 5$'s remaining 'middling' eigenclustering M(5) provides a different eigenstroke: the *turntable* in Subfig e).

1.3 The arena of generalization

Structure 1 The eigenclustering networks are in $1:1$ correspondence with [22, 43] the *unlabelled rooted binary trees* (*URBT*) [12, 20, 24, 34, 27, 28].

The number of eigenclustering networks [22, 43] (free from any vertex-labelling ambiguities) is the corresponding *Wedderburn–Etherington number* [8, 10, 14, 18, 49] $w(N)$. This is established by a straightforward isomorphism between eigenclustering networks on N vertices and the URBT on $2N - 1$. In which the end-nodes are the eigenclustering's vertices, and the internal nodes are all the CoMs that serve as junctures in the eigenclustering network. The total CoM included, which serves furthermore as the root. The URBT are in turn well-known to count out as the Wedderburn–Etherington numbers.

The URBT furthermore provide useful nomenclature for eigenclustering networks, in particular the left and right child about each internal node.

Structure 2 The *at-most binary* (*AMB*) presentation results from once defoliating the corresponding binary tree. In the eigenclustering applications, this involves removing all the primary vertices. while retaining all of the nontrivial CoMs used. These are rather smaller trees, now of order $2N - 1 - N = N - 1 = n$.

Naming Remark 2 AMB is most useful in naming eigenclusterings (and, intermediately, binary trees as well). For being rather smaller as per above, they have already been more widely and recognizably named. $N = 3$ is P_2 , K is P_3-straight and H is P_3-bent. Where straight and bent means rooted at an end-point, and the interior point respectively. P_2 needs no bending descriptor since its 2 vertices are equivalent. K(N) is P_n-straight, H(5) is Claw , and M(5) is P_4-bent.

1.4 The generalized result

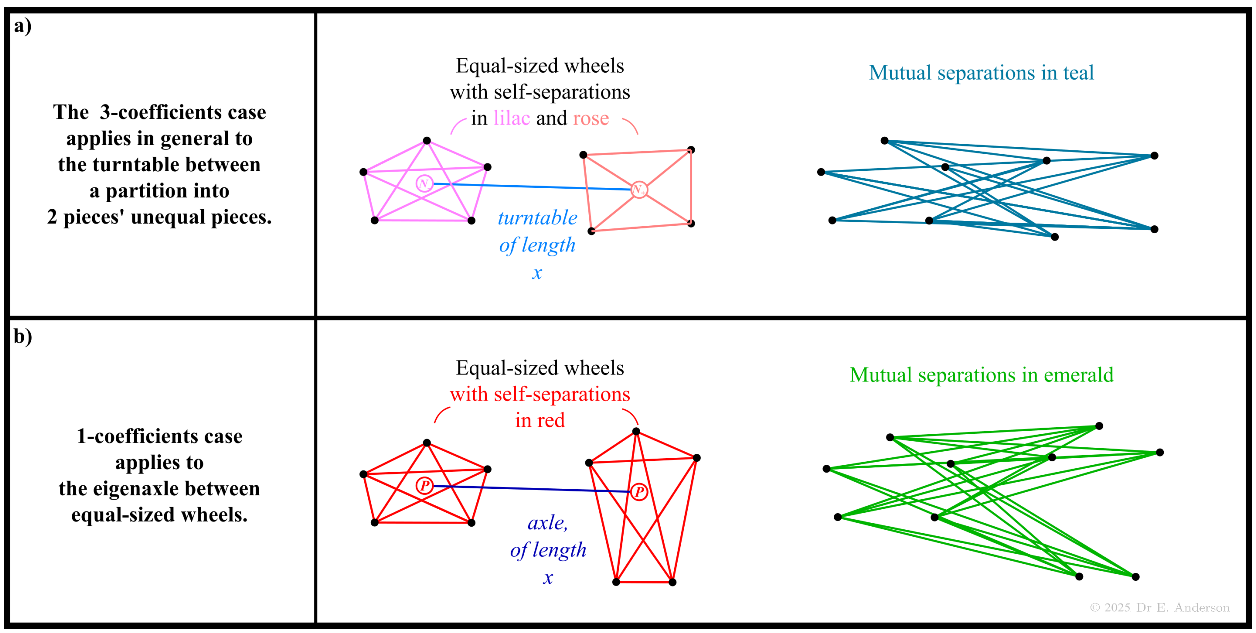

Figure 3:

A given eigenclustering network on N vertices contains r separations and $k = n - r$ strokes. The moments method of proof [31, 36, 39, 40, 41, 42, 45, 44, 46, 47] then returns the following. A – in weak Conway variable linear – equation for the reduced-mass-weighted LC of stroke lengths in terms of the separations. This is already a solution for $N = 3$ and the H, returning Apollonius' median-length theorem and Euler's bimedian-length theorem respectively. In all other nontrivial cases, we have an under-determined linear system: 1 equation in $k > 1$ unknowns. These are the corresponding *ELETs*: *eigenclustering Length-exchange theorems*.

However, we can keep on applying the moments method to the right and left children. Resulting in k linear equations in the k unknowns: well-determined. These are the corresponding *ELESs*: *eigenclustering Length-exchange systems*

The general solution for all ELESs was given in [47]. By using eigenstroke independence from how the rest of the configuration is eigenclustered. And how [46] had already solved the K(N) generalizations of K , whose

AMB aliases are the straight n-paths. This was given for arbitrary-mass (AM) vertices as the *Solved-for ELTs* (*eigenstroke-length theorems*). With the EM subcase, that the current Article requires, picked out as a Corollary. This takes the following form.

Corollary 1: Solved form for EMELT [32, 47] For EM, the schematic form simplifies to the ' 3-coefficient formula':

$$T = \frac{1}{N_- N_+} \sum_{\mathrm{I}_- = 1}^{N_-} \sum_{\mathrm{I}_+ = 1}^{N_+} R^{\mathrm{I}_- \mathrm{I}_+} - \sum_{\mathrm{H} = \mp} \frac{1}{N_\mathrm{H}{}^2} \sum_{\substack{\mathrm{I}_\mathrm{H}, \mathrm{J}_\mathrm{H} = 1 \\ \mathrm{I}_\mathrm{H} < \mathrm{J}_\mathrm{H}}}^{N_\mathrm{H}} R^{\mathrm{I}_\mathrm{H} \mathrm{I}_\mathrm{H}} . \tag{3}$$

Remark 3 The above Corollary generalizes Apollonius's median-length theorem [1, 16, 26] to an explicit formula for the following. The Length T of the last eigenstroke[1] for any EM N-vertex configuration. Between the left child and the right child, which are the 2 self subsystems. The mutual separations are then between the left and right child subsystems. Also, the R are the separation lengths. N_- is the order of the left child and N_+ is the order of the right child. The 3 coefficients are

$$N_\mp{}^{-1} \quad \text{and} \quad (N_- N_+)^{-1} . \tag{4}$$

Remark 4 This is of the conceptual form

$$T = \frac{1}{N_- N_+} \sum_{\text{mutual}} R^m - \frac{1}{N_-{}^2} \sum_{\text{self}_-} R^{s-} - \frac{1}{N_+{}^2} \sum_{\text{self}_+} R^{s+} . \tag{5}$$

Remark 5 The corresponding integer-coefficients form is

$$N_-{}^2 N_+{}^2 T = N_- N_+ \sum_{\mathrm{I}_- = 1}^{N_-} \sum_{\mathrm{I}_+ = 1}^{N_+} R^{\mathrm{I}_- \mathrm{I}_+} - \sum_{\mathrm{H} = \mp} N_{\bar{\mathrm{H}}}{}^2 \sum_{\substack{\mathrm{I}_\mathrm{H}, \mathrm{J}_\mathrm{H} = 1 \\ \mathrm{I}_\mathrm{H} < \mathrm{J}_\mathrm{H}}}^{N_\mathrm{H}} R^{\mathrm{I}_\mathrm{H} \mathrm{I}_\mathrm{H}} . \tag{6}$$

Where $\bar{H}$ takes the value that H does not.

Remark 6 For a size P subsystem, just work within this subsystem with P in place of N and obtain its last eigenstroke' length.

Remark 7 Euler's bimedian-length theorem arises in the H eigenclustering, which extends to form a chain family with 1 H(N) per N . Our generalization, however, involves instead

$$N_- = N_+ : \tag{7}$$

dividing the vertices into 2 equal-order parts. Denoting their common value as P , all the coefficients conflate:

$$N_\mp{}^2 = 4 P^2 = N_- N_+ . \tag{8}$$

So (7) is the 1-coefficient condition. One ready consequence is that

$$N = N_- + N_+ = 2 P , \tag{9}$$

so N is forced to be even.

In our generalization, the turntable between two partitioning subsystems is the eigenaxle between same-sized wheels. In the sense of equal numbers of vertices, of total mass, or of self edges.

Naming Remark 3 *Equal-order parts* is an input name for the current Article's subcase, while *1-coefficient* ELT is an output name. A further name is *Eulerian*, due to its generalizing a property of Euler's case. Rendering all the excluded cases *non-Eulerian.*

1.5 Outline of the rest of the Article

Sec 2 reformulates the above Corollary in terms of separation indices. Sec 3 recasts Euler's theorem in terms of our conceptualization, as the simplest $p = 1$ example. Gives the minimum counterexample to the H(N) giving just 1-coefficient formulae. Provides the minimum new generalization of Euler's theorem. Includes the $p = 3$, i.e. $N = 8$, case. As well as the $N = 2^q$ subfamily, alongside brief theoretical justification of this subfamily and of its minimum member beyond Euler: the aforementioned $N = 8$ case.

[1] A truer name for stroke is *eigen*[*transversal*]. Where [transversal] [38, 52] is a projectively-dual notion comprising co-transversals such as medians and transversals [4, 9] such as bimedians.

2 Separation-index notation

2.1 General case

Notational Remark 1 Denote the vertices by

$$\mathrm{A}_{\mathrm{I}} \ , \quad \mathrm{I} \ = \ 1 \ \ \text{to} \ \ N \ .$$

Split these into

$$\mathrm{A}^{-}_{\mathrm{I}_-} \ , \quad \mathrm{I}_- \ = \ 1 \ \ \text{to} \ \ N_- \quad \text{and} \quad \mathrm{A}^{+}_{\mathrm{I}_+} \ , \quad \mathrm{I}_- \ = \ 1 \ \ \text{to} \ \ N_+ \ .$$

With

$$N \ = \ N_- \ + \ N_+ \ . \tag{10}$$

Also denote the separations by

$$S_s \ , \quad s \ = \ 1 \ \ \text{to} \ \ \binom{N}{2} \ .$$

The above vertex split partitions these into the following 3 pieces. The self parts

$$S^{-}_{s_-} \ , \quad s_- \ = \ 1 \ \ \text{to} \ \ \binom{N_-}{2} \quad \text{and} \quad S^{+}_{s_+} \ , \quad s_+ \ = \ 1 \ \ \text{to} \ \ \binom{N_+}{2} \ .$$

And the mutual part

$$M_m \ , \quad m \ = \ 1 \ \ \text{to} \ \ N_- \, N_+ \ .$$

Remark 1 Let us check that the split of the separations counts out right:

$$\#(\,\text{mutual separations}\,) \ + \ \#(\,\text{self separations}\,) \ =$$

$$N_- \, N_+ \ + \ \sum_{H \ = \ \mp} \binom{N_H}{2} \ = \ N_- \, N_+ \ + \ \sum_{H \ = \ \mp} \frac{N_H \ (\, N_H \ - \ 1\,)}{2}$$

$$= \ \frac{2\, N_- \, N_+ \ + \ {N_-}^2 \ + \ {N_+}^2 \ - \ (\, N_- \ + \ N_+\,)}{2} \ = \ \frac{(\, N_- \ + \ N_+\,)^2 \ - \ (\, N_- \ + \ N_+\,)}{2} \ =$$

$$\frac{N^2 \ - \ N}{2} \ = \ \frac{N \ (\, N \ - \ 1\,)}{2} \ = \ \binom{N}{2} \ = \ \#(\,\text{separations}\,) \ .$$

Structure 1 Interpreting the set of vertices paired with the set of separations as the vertex- and edge-sets of a graph,

$$\{\, \mathrm{A}^{\mp}; \mathrm{S}^{\mp} \,\} \ = \ \mathrm{K}_{N_\mp} \ : \ \ \text{complete graphs} \ . \tag{11}$$

$$\{\, \mathrm{A}; \mathrm{M} \,\} \ = \ \mathrm{K}_{N_-\, , \, N_+} \ : \ \ \text{complete bipartite graph} \ . \tag{12}$$

Remark 2 In separation-indexed variables, the current Article's theorem takes the following solved form.

$$T \ = \ \frac{1}{N_- \, N_+} \sum_{m \ = \ 1}^{N_- \, N_+} M_m \ - \ \sum_{H \ = \ \mp} \frac{1}{{N_H}^2} \sum_{s_H \ = \ 1}^{\frac{N_H \ (\, N_H \ - \ 1\,)}{2}} {S^H}_{s_H} \ . \tag{13}$$

Or equivalently the following integer-coefficients form.

$${N_-}^2 \, {N_+}^2 \, T \ = \ N_- \, N_+ \sum_{m \ = \ 1}^{N_- \, N_+} M_m \ - \ \sum_{H \ = \ \mp} {N_{\bar{H}}}^2 \sum_{s_H \ = \ 1}^{\frac{N_H \ (\, N_H \ - \ 1\,)}{2}} {S^H}_{s_H} \ . \tag{14}$$

2.2 Equal-parts = 1-coefficient = Eulerian subcase

Notational Remark 2 Now

$$N_- \;=\; P \;=\; N_+ \;. \tag{15}$$

So

$$N \;=\; 2\,P \;. \tag{16}$$

$$I_{\mp} \;=\; 1 \;\text{ to }\; P \;.$$

$$s_{\mp} \;=\; 1 \;\text{ to }\; \binom{P}{2} \;.$$

$$m \;=\; 1 \;\text{ to }\; P^2 \;.$$

Remark 3 The count simplifies as follows.

$$\#(\,\text{mutual separations}\,) \;+\; \#(\,\text{self separations}\,) \;=$$

$$P^2 \;+\; 2\binom{P}{2} \;=\; P^2 \;+\; 2\,\frac{P\,(\,P\;-\;1\,)}{2} \;=\; \frac{2\,P\,(\,2\,P\;-\;1\,)}{2} \;=$$

$$=\; \frac{N\,(\,N\;-\;1\,)}{2} \;=\; \binom{N}{2} \;=\; \#(\,\text{separations}\,)\;.$$

Structure 2 Restricting Structure 1 to our special case,

$$\{\,\mathrm{A}^{\mp};\mathrm{S}^{\mp}\,\} \;=\; \mathrm{K}_P \;:\; \text{ equal complete graphs }. \tag{17}$$

$$\{\,\mathrm{A};\,\mathrm{M}\,\} \;=\; \mathrm{K}_{P\,P} \;:\; \text{ complete bipartite graph with equal parts }. \tag{18}$$

Remark 4 By (7), the current Article's Corollary then takes the following solving form.

$$X \;=\; \frac{1}{P^2}\left(\;\sum_{m\,=\,1}^{P^2} M_m \;-\; \sum_{H\,=\,\mp}\;\sum_{s_H\,=\,1}^{\frac{P\,(\,P\,-\,1\,)}{2}} {S^H}_{s_H}\right)\;. \tag{19}$$

Or equivalently, the following integer-coefficients form.

$$P^2\,X \;=\; \sum_{m\,=\,1}^{P^2} M_m \;-\; \sum_{H\,=\,\mp}\;\sum_{s_H\,=\,1}^{\frac{P\,(\,P\,-\,1\,)}{2}} {S^H}_{s_H}\;. \tag{20}$$

3 The smallest examples with discussion and pointers

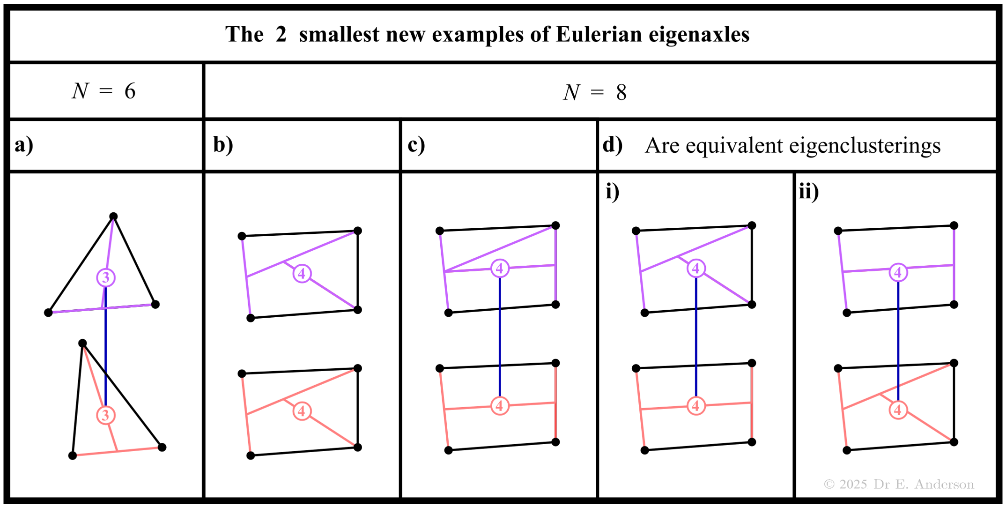

Figure 4:

Example A $p = 1$ and thus $N = 4$ returns Euler's bimedian-length theorem, now formulated as follows.

$$4\,X \;=\; \sum_{m\,=\,1}^{4} M_m \;-\; \sum_{H\,=\,\mp}\;\sum_{s_H\,=\,1}^{1} S^H{}_{s_H} \;. \tag{21}$$

Conceptually revealing intermediaries in passing between this and (2) are

$$4\,X \;=\; M_1 \;+\; M_2 \;+\; M_3 \;+\; M_4 \;-\; S^- \;-\; S^+ \;. \tag{22}$$

And in 2-d ,

$$4\,X \;=\; A_1 \;+\; A_2 \;+\; A_3 \;+\; A_4 \;-\; D_1 \;-\; D_2 \;. \tag{23}$$

I.e. the 2 parts are the 2 diagonals, which also comprise the 2 self contributions supported. While the mutual contributions' bipartite graph is

$$K_{2,\,2} \;=\; C_4 \;, \tag{24}$$

which 4-cycle forms the quadrilateral's sides.

Example B $p = 2$ and thus $N = 6$ provides the smallest example of a Eulerian eigenstroke length. Here,

$$9\,X \;=\; \sum_{m\,=\,1}^{9} M_m \;-\; \sum_{H\,=\,\mp}\;\sum_{s_H\,=\,1}^{3} S^H{}_{s_H} \;. \tag{25}$$

So there are $2 \times 3 = 6$ self terms.

Aside 1 [44] obtained a 2-coefficient formula for the $\mathrm{K}(N)$ including Apollonius' median-length theorem. This arises from the $n\,|\,1$ partition returning

$$n^{-2}\;,\quad n^{-1} \times 1^{-1} \;=\; n^{-1}\;,\quad 1^{-1} \times 1^{-1} \;=\; 1\;.$$

But also the single-vertex supports no separations, by which the 1 coefficient is wiped out as well. So in effect

$$\text{w.l.o.g. } N_- \;=\; n \text{ is the 2-coefficient condition,} \tag{26}$$

with coefficients

$$n^{-2} \text{ and } n^{-1}\;. \tag{27}$$

Yielding the solved form

$$O \;=\; \frac{1}{n}\sum_{m\,=\,1}^{n} M_m \;-\; \frac{1}{n^2}\sum_{i\,=\,1}^{\frac{n\,(\,n\,-\,1\,)}{2}} S_j \;. \tag{28}$$

Or equivalently the integer-coefficients form

$$n^2 O = n \sum_{m=1}^{n} M_m - \sum_{i=1}^{\frac{n(n-1)}{2}} S_j \, . \tag{29}$$

Where we use O rather than T to mark that all of these eigenstrokes are specifically cotransversals.

2-coefficients is the 'Apollonian' case: most closely matching Apollonius' theorem. Which follows from the regularity of $\mathrm{K}(N)$ that is aptly summarized by its P_n-straight AMB name. 'Apollonian' and 'Eulerian' exhaust the ways in which an eigenstroke length formula can have < 3 coefficients. There is just 1 'Apollonian' and 0 or 1 'Eulerian' case per $N \geq 3$. While the 3-coefficient cases grow unboundedly with N . Thus we term this last case of the tripartition by number of coefficients *generic*; also see the end summary in Fig 5.

Counterexample B H(5)'s last stroke has not 1 but 3 coefficients. Using Fig 1's notation,

$$T = \frac{E + F + G + H + I + J}{6} - \frac{A + B + C}{9} - \frac{D}{4} \, . \tag{30}$$

Or in integer-coefficient polynomial form,

$$36T = 6(E + F + G + H + I + J) - 4(A + B + C) - 9D \, . \tag{31}$$

This is the minimum generic example.

Example C $p = 3$ and thus $N = 8$ yields

$$16X = \sum_{m=1}^{16} M_m - \sum_{H = \mp} \sum_{s_H = 1}^{6} S^H{}_{s_H} \, . \tag{32}$$

So there are now $2 \times 6 = 12$ self terms.

This case is theoretically interesting since it has 3 eigenclustering wirings (Subfigs 4.b-d). And not 4 , due to eigenclustering networks being left- and right-child ($\mp$) inistiguishability. By which for us Subfigs 4.d).i)-ii) are of the same eigenclustering. Equivalently, $\mp$ are distinct but meaningless labels for eigenclustering, just as spin up and spin down are in Quantum Theory. Due to this subtlety [43], eigenclustering network counts turn out to be Wedderburn–Etherington numbers rather than half-Catalan numbers [48]. With $N = 8$ being the minimum counterexample to these being the same. By a count discrepancy of precisely 1 , as covered by the above discussion of Subfig d).

Example D and Pointer 1 The $N = 2^Q$ series, corresponding to $P = 2^q$ for $q := Q - 1$, is of further theoretical interest due to supporting special symmetry properties and minimum counterexamples. See [51] for more about this.

Euler's bimedian-length theorem and Example 3 are the first 2 members of this subseries. For the arbitrary member of this subseries, the current Article's Corollary's solved form becomes

$$X = 2^{-2q} \left(\sum_{m=1}^{2^{2q}} M_m - \sum_{H = \mp} \sum_{s_H = 1}^{2^{q-1}(2^q - 1)} S^H{}_{s_H} \right) \, . \tag{33}$$

Equivalently its integer-coefficients form is

$$2^{2q} X = \sum_{m=1}^{2^{2q}} M_m - \sum_{H = \mp} \sum_{s_H = 1}^{2^{q-1}(2^q - 1)} S^H{}_{s_H} \, . \tag{34}$$

So in this case there are $2^{2q} - 2^q$ self terms.

Pointer 2 A further application of the current Article shall be announced within a few days, to considerable fanfare.

Case	Number of coefficients	Condition	Count
Eulerian	1	$N_- = N_+$, only attainable for N even	1 case per even N
Apollonius	2	w.l.o.g. $N_- = n$	1 case per N
Generic	3	otherwise	all other cases

Figure 5:

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