Stewart's Theorem generalized to
1 Theorem per Eigenclustering Network per N-Simplex.
II. K(N) alias straight-P_n .

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Abstract

In the current Series, we use a 'physical' moments method to derive a family of N-simplex generalizations of Stewart's Cevian-Length Theorem for triangles. Namely, the Arbitrary-Mass Eigenclustering Length-Exchange Theorems (AMELETs).

In the current Article, we contemplate the K(N)-eigenclusterings. Which generalize the 4-Body Problem's Jacobi-K eigenclustering. And correspond to the straight n-path graphs in the at-most binary tree representation. This case's nice series regularity readily permits a general solution for the individual eigenclustering lengths in terms of the sides. And a homogeneous-coordinate unification of Article I's split multi-linear formulation.

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1 Solving for each stroke in each K(N)

Remark 1 As explained in Article I [17], our underlying principle [10, 13] is to equate 2 different expressions for the inertia quadric,

$$\iota_{\mathrm{Sep}} \;=\; \iota_{\mathrm{Eig}} \;. \tag{1}$$

Where the second expression is for a specific choice of eigenclustering network [4, 6, 8, 12, 9, 19]. The current Article contemplates the K(N)-eigenclusterings; see [16] for this notion and corresponding notation. Which form the infinite series of paths P_n-straight for $n \;:=\; N \;-\; 1$ in the at-most binary (AMB) tree representation [15, 17]. In this setting, our principle yields the following for the $(\,length\,)^2$ of a linear combination of K(N) 's strokes O_k . Thus generalizing [14]'s result from equal to arbitrary masses.

Theorem 0 (K(N)-AMELET)

$$\sum_{j\,=\,1}^{n} \frac{\mathrm{M}_j}{\mathrm{M}_{j\,+\,1}}\, O_{j\,-\,1} \;\;=\;\; \frac{1}{\mathrm{M}} \sum_{\substack{\mathrm{I,\,J}\;=\;1\\ \mathrm{I}\;<\;\mathrm{J}}}^{N} \mathrm{I\,J}\, r^{\mathrm{IJ}\,2} \;. \tag{2}$$

Where M_k is the mass of the k-vertex subsystem picked out by the K(N) eigenclustering.

Remark 2 We next observe that K($P \;+\; 1$)-AMELET $-$ K(P)-AMELET isolates the $(\,\mathrm{length}\,)^2$ of the $(\,p \;-\; 1\,)$th stroke.

Via

$$\frac{\mathrm{M}_p}{\mathrm{M}_P}\,\mathrm{A}_P\, O_{p\,-\,1} \;\;=\;\; \sum_{\mathrm{V}\,\in\,\overline{\mathfrak{S}\mathrm{ys}(p)}} \mathrm{A_V}\, A_{\mathrm{V}} \;+\; \left(\frac{1}{\mathrm{M}_P} \;-\; \frac{1}{\mathrm{M}_p}\right) \sum_{\mathrm{U,\,U'}\,\in\,\mathfrak{S}\mathrm{ys}(p)} \mathrm{A_U A_{U'}}\, A_{\mathrm{U}}$$

$$=\;\; \sum_{\mathrm{V}\,\in\,\overline{\mathfrak{S}\mathrm{ys}(p)}} \mathrm{A_V}\, A_{\mathrm{V}} \;-\; \left(\frac{\mathrm{A}_P}{\mathrm{M}_P\mathrm{M}_p}\right) \sum_{\mathrm{U}\,\in\,\mathfrak{S}\mathrm{ys}(p)} \mathrm{A_U A_{U'}}\, A_{\mathrm{U}} \;.$$

So cancelling off the arbitrary A_P and the consecutive-subsystem mass ratio $\dfrac{\mathrm{M}_p}{\mathrm{M}_P}$,

$$O_{p\,-\,1} \;=\; \frac{1}{\mathrm{M}_p} \sum_{\mathrm{V}\,\in\,\overline{\mathfrak{S}\mathrm{ys}(p)}} \mathrm{A_V}\, A_{\mathrm{VP}} \;-\; \frac{1}{\mathrm{M}_p{}^2} \sum_{\mathrm{U,\,U'}\,\in\,\mathfrak{S}\mathrm{ys}(p)} \mathrm{A_U A_{U'}}\, A_{\mathrm{U}} \;.$$

Thus passing to mass-fraction variables

$$\xi_{\mathrm{I}} \;:=\; \frac{\mathrm{A_I}}{\mathrm{M}_p} \;,$$

we have arrived at the following generalization of [16]'s equal-masses result.

Theorem 1 (Explicit solution for each stroke-length in K(N)). d) Redundant mass-ratio form.

$$O_{p\,-\,1} \;=\; \sum_{\mathrm{V}\,\in\,\overline{\mathfrak{S}\mathrm{ys}(p)}} \xi_{\mathrm{V}}\, A_{\mathrm{VP}} \;-\; \sum_{\mathrm{U},\,\mathrm{U}'\,\in\,\mathfrak{S}\mathrm{ys}(p)} \xi_{\mathrm{U}}\xi_{\mathrm{U}'}\, A_{\mathrm{UU}'} \;. \tag{3}$$

Remark 3 The K(P)-AMELET is introduced by hand above. However, Article I's splitting phenomenon argument transcends from K to K(N) , $N \geq 4$. By which the K(P)-AMELET also arises intrinsically to the K($P + 1$)-AMELET itself. From the presence of an irrelevant ratio that the answer cannot possibly depend on, splitting this equation into 2 pieces.

Remark 4 Also we only need 2 equations to solve for each stroke. This is as opposed to having a system of $n - 1$ equations in $n - 1$ unknowns. And amounts to a 'Markovian' decoupling: for each new stroke length, the only prior information we need to remember is the previous stroke-length.

2 Multi-linear formulations

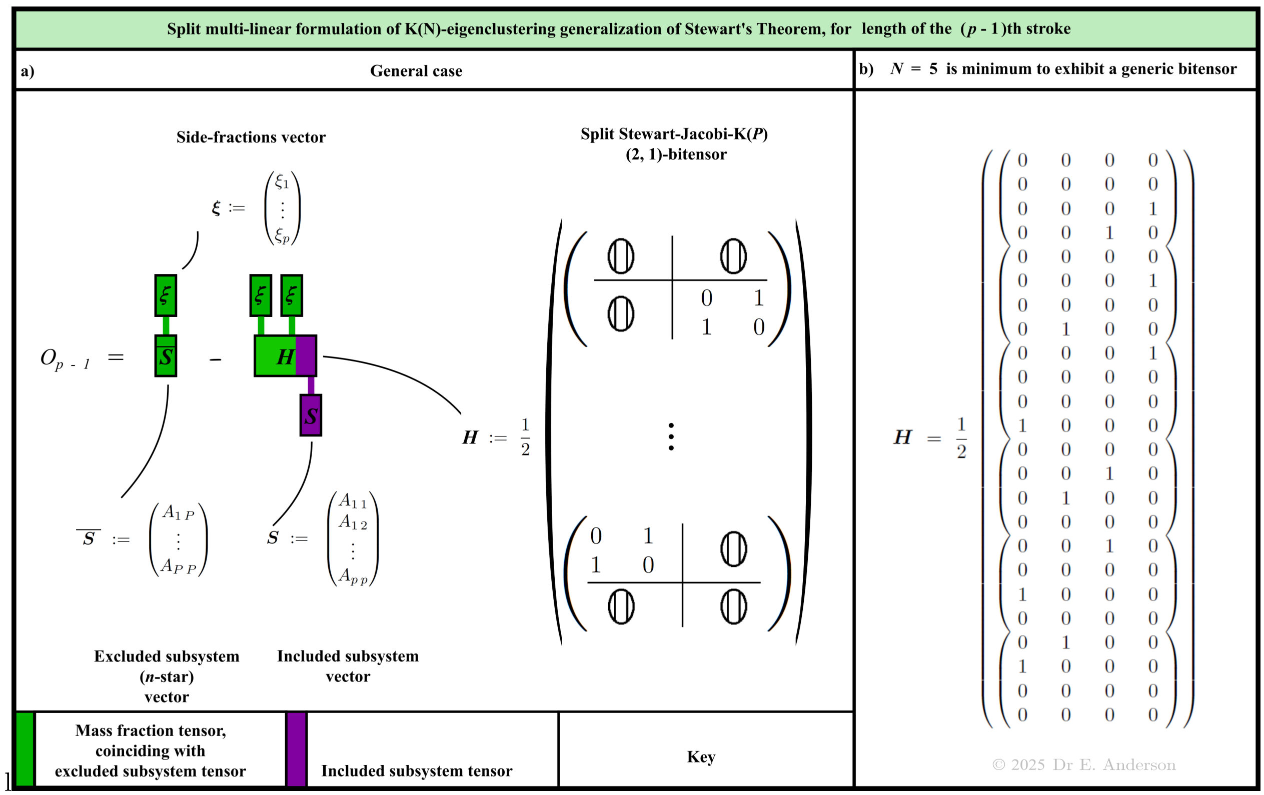

Figure 1:

Remark 1 We first continue Article 1's approach with the split multi-linear algebra form of Fig 1. $N = 5$ is minimum for a $(2, 1)$-bitensor to manifest in the role of $\boldsymbol{C}$. Corresponding to all subsequent members of the series starting with Article 1's half-transpositor and half-permanentor being pencil-shaped. Meaning with 2 equal shorter directions and 1 longer one. We depict the shorter one in emerald for 'excluded subsystem' and the longer one in indigo for included subsystem.

There are

$$\binom{P}{2} \quad \text{indigos but only} \quad P \quad \text{emeralds} \quad .$$

So equating these, we obtain

$$P\,(\,P - 3\,) \;=\; 0 \;.$$

Thus

$$P \;=\; 0 \;\text{ or }\; 3 \;.$$

So $P \geq 4$ supports no further such coincidences. This translates to the pencil-shaped $(2, 1)$-bitensors being persistently realized $\forall N \geq 5$.

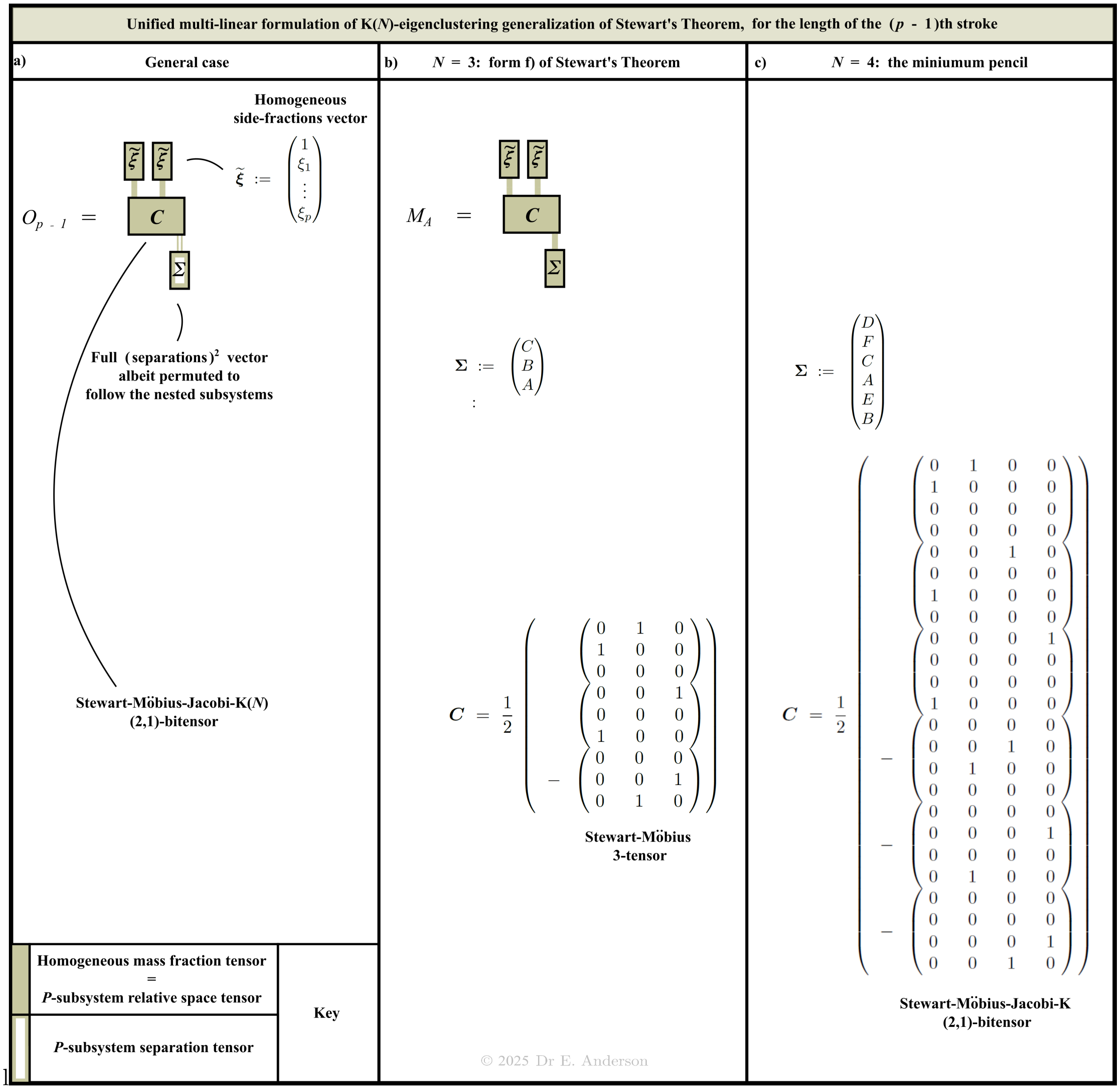

Figure 2:

Remark 2 The ξ_i are ratio variables. Thus a homogeneous-coordinate representation formed by adjoining 1 to the vector of these is also very natural (and Projectively-significant). This technique dates back to Möbius [3], and is discussed in e.g. [7]. It unifies the multi-linear algebra of the K(N)-AMELET generalization of Stewart's Theorem in the sense of Fig 2. I.e. there is no longer any distinction between terms linear and quadratic in the ξ_i . Albeit included-or-excluded retains an imprint at the level of the signs entering the corresponding $(2, 1)$-bitensor.

Viewed in this way, the nth case is closely akin to the quadratic part of the split presentation of the Nth case. So for instance, it is now $N = 3$'s Stewart's Theorem that involves a 3-tensor 'permanentor'. Albeit modulo 2 signs (Subfig b). Yielding a further form f) for Stewart's Theorem [see Article I for forms 0) to e)]. And it is now $N = 4$ is now minimum to exhibit pencil-shaped $(2, 1)$-bitensors (Subfig c).

Pointer 1 Let us leave the following to Article III. How this unified multi-linearization relates to Article I's H-AMELET that generalizes Euler's 4-Body Theorem [2, 11, 19]. And conceptual analysis of, and thus a name for, $\boldsymbol{C}$.

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