Eigenclustering-Length Exchange Theorems.

III. The Bottom Series: Straight-P_n

Edward Anderson*

Abstract

We have found a set of N-body problem Theorems, or alternatively for N-simplex configurations. Each is for a weighted sum of squares of those eigenclustering (= relative Jacobi) magnitudes which are not already (Lagrange) relative separations in terms of those which are. By which these are Eigenclustering Length-Exchange Theorems (ELETs). In the current Series, we restrict ourselves to equal-mass such, with reference to placing masses at the vertices; this is the most Geometrically-natural case. So we are more specifically considering EMELETs.

Eigenclustering networks become ambiguous for $N \geq 4$. In fact, these are unlabelled rooted binary tree (URBT)-valued. Though for the purpose of systematically naming our Theorems, it is more convenient to defoliate these once. Thus passing to the unlabelled rooted AMB trees: at-most binary.

Then the minimum nontrivial $N = 3$ EMELET returns Apollonius' Theorem for the median length in terms of sides data as the 2-path P_2 case. While the P_3-bent- (alias 4-body H-)EMELET returns Euler's 4-Body Theorem for the Newton length in terms of separations data. In the current article, we show that trees smaller than P_2 do not contribute any ELETs. And, more substantially, we provide the explicit form of the EMELET for all straight P_n , $n \geq 2$. Where $n := N - 1$.

* Dr.E.Anderson.Maths.Physics *at* protonmail.com . Institute for the Theory of STEM.

Date stamp v1: 29-03-2024 v2: 23-01-2025.

1 Introduction

We recently gave new proofs [26, 29] of Apollonius' Theorem [1, 9, 18, 33] and Euler's 4-Body Theorem [3, 12, 13, 16, 17, 23] These use just centre of mass (CoM) and moment of inertia (MoI) concepts: first and second moments. By which they readily generalize to give corresponding Theorems for each eigenclustering network supported by each N-body problem in arbitrary dimension. For a weighted sum of squares of those eigenclustering magnitudes which are not already relative separations in terms of those which are.

Eigenclustering networks [32, 33] are more widely known as types of *Jacobi coordinate system* [8, 10, 11, 14]. In Sec 2, we set up separation coordinates and then the particular choice of eigenclustering coordinates that we require. Apollonius' Theorem in its original context is for a triangle: how to compute the length of a median from sides data. Via e.g. our proof, this generalizes to being a 3-body problem result, thus in particular holding in 1-d as well. The 3-body problem supports just the one eigenclustering network. While the most well-known subcase of Euler's 4-Body Theorem – Euler's Quadrilateral Theorem – is for the Newton length [2, 5, 6, 7, 17, 19] between the midpoints of the diagonals, in terms of separations data. Via e.g. our proof,[1] this generalizes to being a 4-body problem result, thus holding in 1- and 3-d as well. It corresponds to the H-eigenclustering network for the 4-body problem, wherein one alternative is supported: the K-eigenclustering network.

In this manner, our Theorems are Eigenclustering Length-Exchange Theorems (ELETs). More specifically, the current Series considers EMELETs. With reference to placing equal masses at the vertices, which is the most Geometrically-natural modelling situation.

Eigenclustering networks moreover correspond to [14, 32] the unlabelled rooted binary trees (*URBT*) [24]. We subsequently index eigenclustering networks, and thus our Theorems, by the corresponding trees. We

[1] This is not the only way of obtaining this generalization, however; see e.g. [12].

explained this correspondence in Article I. As well as how it is more convenient to use the corresponding AMB trees – at most binary – that result from defoliating the URBT. See [32] for many further details about this.

From this point of view, the 3-body problem's eigenclustering network corresponds to the 2-path tree P_2. And the 4-body problem's H versus K eigenclustering-network ambiguity corresponds to the bent versus straight P_3 rooting ambiguity. See Fig 1. These respectively support Apollonius' Theorem as the 3-EMELET. Euler's 4-Body Theorem as the H-EMELET. And the corresponding K-EMELET unveiled in [30].

We explain in Sec 3 how the space of nontrivial ELETs is a slight truncation of the URBT, with the Apollonius P_2 as its bottom element. We then present our new straight n-path Theorem in Sec 4. For $N \geq 5$, this corresponds to the $\mathrm{K}(N)$-eigenclustering generalization of $N = 4$'s K-eigenclustering. In Sec 5 we check that this recovers Apollonius' Theorem (P_2 case) and my Jacobi-K Theorem (P_3-straight case). And then finally give the explicit form for the next smallest case: the P_4-straight eigenclustering for the 5-body problem. This eigenclustering is displayed in the last row of Fig 1.

2 Eigenclustering coordinates and their masses

2.1 Position and separation levels of structure

Structure 0 We denote the position vectors of our model's points-or-particles by $\boldsymbol{q}^I$, $I = 1$ to N.

Structure 1 We denote the *(Lagrange) (relative) separation vectors* between them by $\boldsymbol{r}^{IJ}$, $J \geq I$. We also re-index these 2-index combinations by the following single index.

$$S = 1 \text{ to } \binom{N}{2} = \frac{N(N-1)}{2} = \text{dim}(\text{ separation space [27] }) . \tag{1}$$

We also use s^S to denote the corresponding magnitudes: the *separations* themselves. For a particular planar convex Geometrical figure's realization, separations can be subdivided as follows. Into *sides* $a^I = s^I$, $I = 1$ to N. And *diagonals* d^D. With index

$$D = 1 \text{ to } \frac{N(N-1)}{2} - N = \frac{N(N-3)}{2} ; \tag{2}$$

see Fig 1.a) for the first few instances. Thereby, the first N S^S have 'dual nationality' as sides.

Structure 1′ With the $\boldsymbol{r}$ not all being LI for $N \geq 3$, we introduce some relative space basis for them that we denote by $\widetilde{\boldsymbol{r}}^i$. With

$$i = 1 \text{ to } n := N - 1 = \text{dim}(\text{ relative space [27] }) .$$

Remark 1 For $N \geq 3$, the inertia quadric is not however diagonal with respect to such a basis [27]. To acquire this property, we pass to the following.

2.2 Eigenclustering vectors, alias relative Jacobi vectors

Structure 1″ *Eigenclustering vectors* [27], alias *relative Jacobi vectors* [4, 8, 11, 20, 33] are an alternative basis for the relative space information that is diagonalizing. We present this here in the case in which the underlying point-or-particle masses are equal.

Examples 0 to 2 For $N \leq 2$, the eigenclustering concept is unnecessary.

Example 3 For $N = 3$, they are uniquely specified up to 3 relabellings. Fixed by which side we choose to be the base of the triangle.

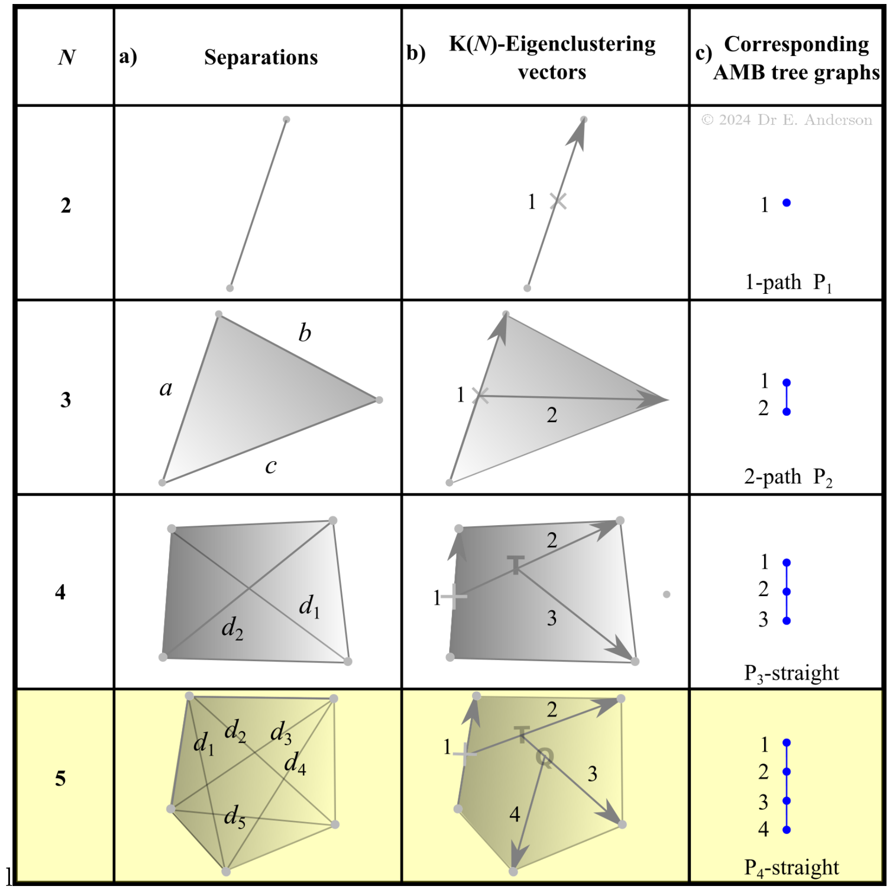

Figure 1:

Example 4 For $N \geq 4$, there are further eigenclustering network ambiguities. Starting with the H versus K ambiguity for the 4-body problem. Which corresponds to the bent versus straight P_3 rooting ambiguity. Indeed, the possible eigenclustering networks for the N-body problem correspond to the unlabelled trees on N vertices. In the current Article, we consider the P_n-straight alias $\mathrm{K}(N)$ the 4-body problem's P_3 alias K. Fig 1 illustrates this and the previous Subsection's main notions up to $N = 5$.

2.3 Straight-path eigenclusterings

Remark 1 For our path choice, each N extends the preceding N's eigenbasis by one eigenvector. From the total CoM so far (X , T , Q , P... in Fig 1) to the extra point-or-particle. Thereby, one can immediately write down the general Nth case (Fig 2.a).

Structure 2 Every eigenclustering network contains ≥ 1 relative separation. For path eigenclusterings, it is precisely 1. Eigenclustering line segments which are not separations are some kind of [transversal]. This is a term introduced in [28] to cover both co-transversals such as medians and transversals such as the Newton line in the H case. See Fig 2.a) for explicit formula for the P_n-straight cases' eigenclustering vectors.

Naming Remark 1 Instead of naming individual transversals as we did in [29, 30] for H and K respectively, we now just label them by their edges along P_n-straight. Edge 1 is the 2-body separation. Edge 2 extends this to a 3-body eigenclustering basis, and so on. See Fig 1.b).

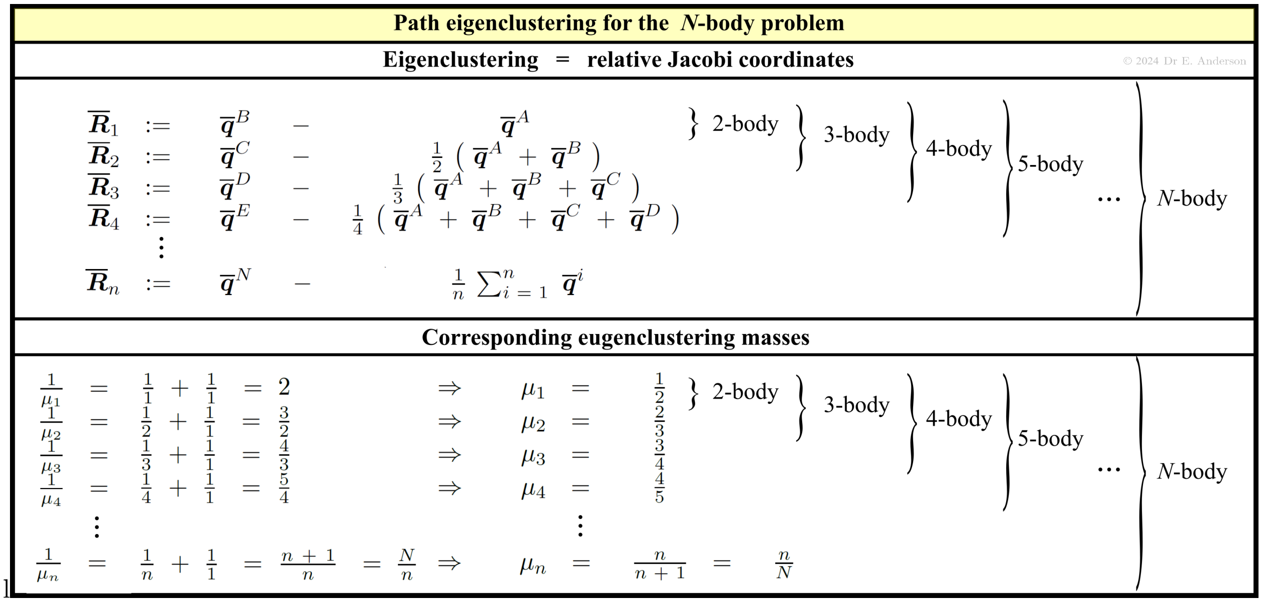

Figure 2:

Notational Remark 1 For path eigenclusterings, the corresponding edge-labelled [transversal] length variables are

$$t_2 \ , \quad t_3 \ , \quad \dots \ , \quad t_n \ . \tag{3}$$

Thus the *path-eigenclustering* [*transversal*] *subspace* of relative space has the following dimension.

$$n - 1 = N - 2 \ .$$

Let us also use

$$T_2 \ , \quad \dots \ , \quad T_n \tag{4}$$

to denote their squares.

2.4 Eigenclustering masses and the 'Greek world'

Structure 1 Eigenclustering masses are a subcase of reduced masses. We compute these in Fig 2.b) for $N = 2$ to 5 . The general case for this is obvious as well. Observe that equal particle masses *does not* guarantee equal reduced masses, eigenclustering masses included.

Notational Remark 2 Eigenclustring masses lead to corresponding mass-scaled length variables [27]. Let us use Greek letter versions of symbols to denote mass-weighted counterparts. Chief among these in the current Article is ι , which, unadorned, stands for total moment of inertia. With indices, this stands for the corresponding partial moments. Though we use furthermore α corresponding to side a and $\tau_{\bar{t}}$ for those corresponding to [transversals] $t_{\bar{t}}$.

Notational Remark 3 Finally let us use $\mathcal{O}_p$ to denote the sum of the pth powers of the objects $\mathcal{O}$. [21, 26, 25, 29, 30] benefited from many simplifications from sequential use of squared variables and power sums. This usefulness largely recurs in the current Article.

3 How our result trivializes for $N \leq 2$: the 3 smallest binary trees

Remark 1 The system of equations that our Theorem follows from consists of the following.

1) The MoI ι's partial moments expansion in our eigenclustering basis,

$$\iota \ = \ \sum_{i \, = \, 1}^{n} \iota_i \ =: \ \iota_1 \ . \tag{5}$$

2) And the separations-democratic radius of gyration (RoG) formula [27]: an equable expansion in the separations. Which, in the equal-masses case used in the current Series, is

$$(\,\mathrm{RoG}\,)^2 \ := \ R \ = \ \frac{1}{N^2} \sum_{\substack{I, \, J \, = \, 1 \\ I \, < \, J}}^{N} \ ||\overline{\boldsymbol{r}}^{I\,J}||^2 \ . \tag{6}$$

By Sec 2, these expansions contain

$$n \ := \ N \ - \ 1 \tag{7}$$

and

$$\frac{N\,(\,N \ - \ 1\,)}{2} \tag{8}$$

terms respectively. We are to solve our system for the [transversal] lengths. For our Theorem is a device from computing some (perhaps weighted) sum of such lengths squared in terms of purely separation-length data. This fits the bill: firstly, a truer name is Apollonius' *Median-Length* Theorem [26]. Secondly, Euler's 4-Body Theorem can be taken to compute the length of the Newton line segment [29].

Question 1 So what happens for the cases even smaller than Apollonius' Theorem?

Example 2 For the 2-body problem, there is just the 1 separation and no [transversals]. At the level of counting,

$$\frac{2\,(\,2 \ - \ 1\,)}{2} \ = \ 1 \ = \ 2 \ - \ 1 \ .$$

So both expansions become trivial, in the sense that there is only 1 piece in each. And our system of equations degenerates to 2 copies of the same identity equation,

$$\iota \ = \ \alpha \ . \tag{9}$$

And this is fine, since there are also no [transversals] to solve for in this case!

The 2-body problem is thus too simple to have an Eigenclustering Length-Exchange Theorem. This case is indexed by the

$$\mathrm{pt} \ = \ \mathrm{D}_1 \ = \ \mathrm{P}_1 \ \ \text{AMB tree} \ .$$

Which are respectively, its single-point point-cloud notation, its totally-disconnected graph notation and its 1-path reconceptualiation.

Example 1 For the 1-body problem, there are no separations or [transversals]. Indeed now both counts return 0 . This means that both of our expansions contain no terms. Our system has thus again degenerated to 2 copies of the same equation, which is now furthermore a zero equation:

$$\iota \ = \ 0 \ . \tag{10}$$

And this is fine, since there are no [transversals] at all to solve for in this case! So now the partial moments expansion is even more trivial: it contains *no* partial moments.

This case is indexed by the same AMB tree as above. This corresponds to the only place where the binary-to-AMB defoliation fails to give an isomorphism [32]. For, on the one hand, defoliating P_3 -bent returns the point. But, on the other hand, defoliating the point also returns the point, because now the input tree already has no leaves at all! Thus if discussing $N = 1, 2$ we need AMB labelling to distinguish between the two. I.e. the P_3-straight AMB tree versus the pt AMB tree.

Example 0 For the 0-body problem, there are no points-or-particles, let alone separations or [transversals]. Now neither a MoI nor a RoG exist. So there are no equations at all! And this is fine, since there are no [transversals] to solve for in this case either... This case is indexed by

$$\text{U} \ : \ \text{whichever of the unpoint, untree or unpath incarnations of } \emptyset \ . \tag{11}$$

Structure 1 Our Theorems are thus indexed in more detail by

$$\mathfrak{T}\text{ree}_{2*} \ - \ \{\,\text{U, D, P}\,\} \ . \tag{12}$$

For $\mathfrak{T}\text{ree}_{2*}$ the arena of unlabelled rooted binary trees. On which restriction it is isomorphic to

$$\mathfrak{T}\text{ree}_{\,\leq\, 2*} \ - \ \{\,\text{U, D, P}\,\} \ . \tag{13}$$

For $\mathfrak{T}\text{ree}_{\,\leq\, 2*}$ the arena of unlabelled rooted AMB trees.

While the current article involves just

$$\mathfrak{p}\text{ath} \ - \ \{\,\text{U, D, P}\,\} \ \cong \ \mathbb{N}_0 \ - \ \{\,0,\,1,\,2\,\} \ \cong \ \mathbb{N} \ . \tag{14}$$

For $\mathfrak{p}$ath the arena of unlabelled paths. Thereby, Apollonius's Theorem – as the P_2 case – is, from an Order-Theoretic [15, 24] point of view, the *bottom element* of our slightly truncated arenas of path and tree graphs.

4 The P_n-straight Eigencluster [Transversal]-lengths^2 Theorem

Theorem 1 [Anderson 2018]

a) (Mass-weighted rational form)

$$N \sum_{\bar{t}\,=\,2}^{n} \mathcal{T}_{\bar{t}} \ = \ 2\,\Sigma_1 \ - \ N\,\alpha \ . \tag{15}$$

b) (Geometrical [transversal]-lengths subject form)

$$2 \sum_{\bar{t}\,=\,2}^{n} \frac{\bar{t}}{\bar{t} \, + \, 1}\, T_{\bar{t}} \ = \ \frac{2}{N}\, S_1 \ - \ A \ . \tag{16}$$

<u>Proof</u> Our First Principle (A) is the partial MoI expansion of in H-coordinates.

Our Second Principle (B) is the separations-democratic RoG formula.

Squared variables and sum variables sequentially save us symbols.

Latin $\longleftrightarrow$ Greek 'translations' (inter-conversions) must always remember to (un)deploy eigenclustering mw factors.

Everything else is in Fig 3.

Remark 1 In the Geometrical variables, the eigenclustering masses render placing the general case in rational form unwieldy. This can however be done for each small example of interest ([30] and Sec 5).

Corollary 1 a)

$$N \sum_{\bar{t}\,=\,2}^{n} \mathcal{T}_{\bar{t}} \ = \ (\,2 \ - \ N\,)\,\alpha \ + \ 2 \sum_{\bar{S}\,=\,2}^{\frac{N\,(\,N\,-\,1\,)}{2}} \Sigma_{\bar{S}} \ . \tag{17}$$

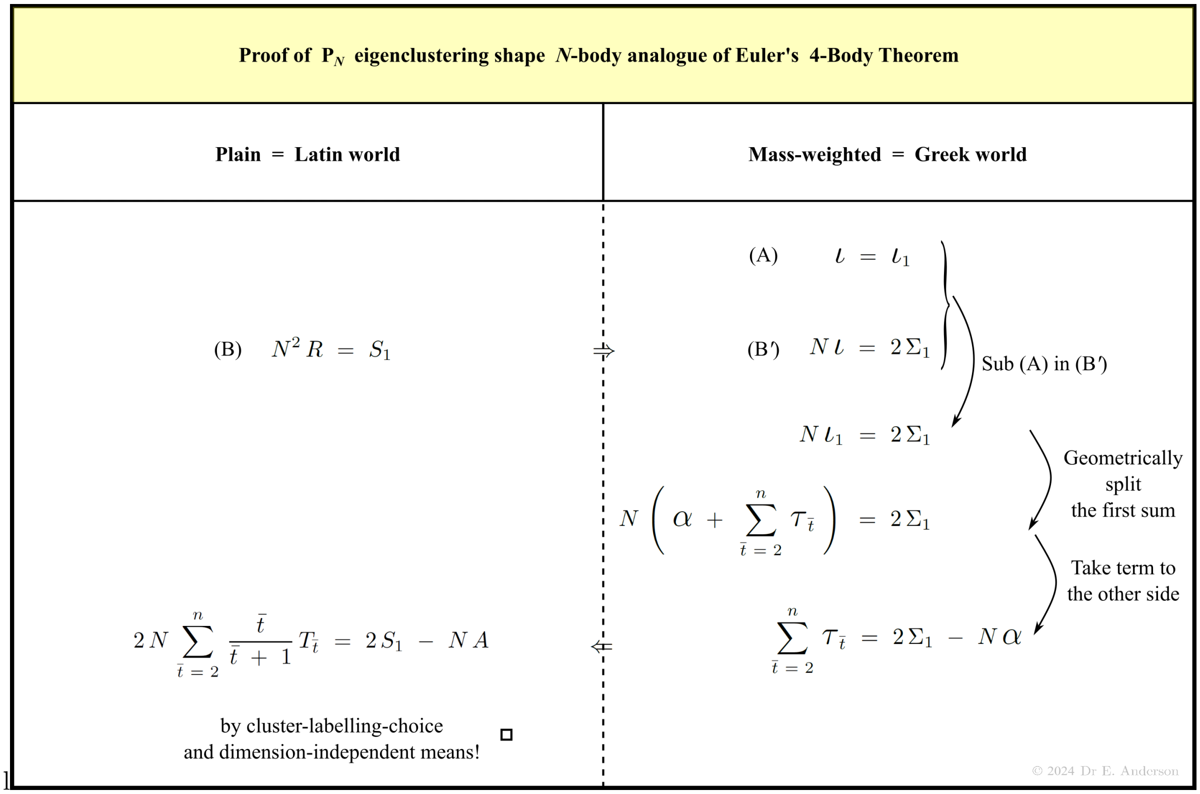

Figure 3:

b)

$$\sum_{\bar{t} = 2}^{n} \frac{\bar{t}}{\bar{t} + 1} T_{\bar{t}} = \frac{2 - N}{2 N} A + \frac{1}{N} \sum_{\bar{S} = 2}^{\frac{N(N-1)}{2}} S_{\bar{S}} \tag{18}$$

Proof This follows from also splitting the second sum in Fig 2's second column. And then cancelling contributions from our two bases' 'dual nationality' element, α □ .

Remark 2 The Theorem's statements benefit from brevity. But it is the Corollary's forms that most directly translate to a Linear-Algebraic formulation of the cycle of such results over all possible separations.

5 The first few nontrivial examples

Example 3

a) reads

$$3 \mu_{\alpha} = \alpha + 2 \left(\beta + \gamma \right) . \tag{19}$$

b) gives

$$M_A = \frac{B + C - A}{4} \tag{20}$$

I.e.

$$m_a{}^2 = \frac{b^2 + c^2 - a^2}{4} :$$

a standard form for Apollonius' Theorem recovered. For m_a the median corresponding to side a .

Example 4 [30] recovered.

a) reads

$$2\,(\,\mathcal{T}_1\;+\;\mathcal{T}_2\,)\;=\;\iota_1\;-\;\alpha\;. \tag{21}$$

b) reads

$$\frac{2}{3}T_1\;+\;\frac{3}{4}T_2\;=\;\frac{1}{4}\,(\,S_1\;-\;A\,)\;. \tag{22}$$

For which a rational form is

$$8\,T_1\;+\;9\,T_2\;=\;3\,(\,S_1\;-\;A\,)\;. \tag{23}$$

Or, in terms of the original Geometrical variables,

$$\frac{2}{3}{t_1}^2\;+\;\frac{3}{4}{t_2}^2\;=\;\frac{1}{4}\left(\,s_2\;-\;a^2\,\right)\;. \tag{24}$$

With rational form

$$8\,{t_1}^2\;+\;9\,{t_2}^2\;=\;3\left(\,s_2\;-\;a^2\,\right)\;. \tag{25}$$

Example 5 For P_5 ,

a) reads

$$5\,(\,\mathcal{T}_1\;+\;\mathcal{T}_2\;+\;\mathcal{T}_3\,)\;=\;2\,\iota_1\;-\;3\,\alpha\;. \tag{26}$$

b) reads

$$\frac{2}{3}T_1\;+\;\frac{3}{4}T_2\;+\;\frac{4}{5}T_3\;=\;\frac{1}{10}\,(\,2\,S_1\;-\;3\,A\,)\;. \tag{27}$$

For which a rational form is

$$40\,T_1\;+\;45\,T_2\;+\;48\,T_3\;=\;12\,S_1\;-\;18\,A\;. \tag{28}$$

Or, in terms of the original Geometrical variables,

$$\frac{2}{3}{t_1}^2\;+\;\frac{3}{4}{t_2}^2\;+\;\frac{4}{5}{t_3}^2\;=\;\frac{1}{10}\left(\,2\,s_2\;-\;3\,a^2\,\right)\;. \tag{29}$$

With rational form

$$40\,{t_1}^2\;+\;45\,{t_2}^2\;+\;48\,{t_3}^2\;=\;12\,S_1\;-\;18\,a^2\;. \tag{30}$$

End Remark [31] go on to consider further eigenclusterings that are not straight paths.

Acknowledgments I thank A for covering my back by proofreading and trouble-shooting this Series. A and E for ongoing related collaborations. And S for incipient discussions about the Lagrange matrix, the projector superceding this, and the ensuing relative space and eigenclustering network notions. As well as setting up the moments proof, understanding its scope, and inter-relating numerous representations for eigenclustering networks. Which A and I formalized and further developed in [32]. And C, Malcolm MacCallum, Reza Tavakol, Jeremy Butterfield and Enrique Alvarez for support with my career.

References

[1] Apollonius of Perga (3rd and 2nd Centuries B.C.E.).

[2] I. Newton evoked the Newton line in proving a Theorem about Quadrilaterals. For an English translation, see e.g. *The Mathematical Papers of Isaac Newton* ed. D. Whiteside (C.U.P., Cambridge 1967–1981).

[3] L. Euler worked on Geometry, among many other topics, in the 18th Century; this Theorem dates to 1748.

[4] C.G.J. Jacobi worked on Mechanics, among many other topics, in the 1840s.

[5] C.F. Gauss established further properties of the Newton line in the 19th Century. For a popular account, of this work of Gauss see e.g. D. Wells, *The Penguin Dictionary of Curious and Interesting Geometry* (1991).

[6] C.V. Durell, *Modern Geometry: The Straight Line and Circle* (University of California Libraries, 1920).

[7] R.A. Johnson, *Modern Geometry* alias *Advanced Euclidean Geometry* (Houghton, Boston, 1929; reprinted by Dover, Mineola N.Y. 1960).

[8] C. Marchal, *Celestial Mechanics* (Elsevier, Tokyo 1990).

[9] D.C. Kay, *College Geometry* (Harper Collins, 1994).

[10] R.G. Littlejohn and M. Reinsch, "Internal or Shape Coordinates in the N-body Problem", Phys. Rev. **A52** 2035 (1995).

[11] R.G. Littlejohn and M. Reinsch, "Gauge Fields in the Separation of Rotations and Internal Motions in the N-Body Problem", Rev. Mod. Phys. **69** 213 (1997).

[12] G.A. Kandall, "Euler's Theorem for Generalized Quadrilaterals", College Math. J. **33** 403 (2002).

[13] S.-H. Chou and S. He, "On the Regularity and Uniformness Conditions on Quadrilateral Grids", Computer Methods App. Math. Eng. **191** 5149 (2002).

[14] H. Cabral and F. Diacu, *Classical and Celestial Mechanics* (P.U.P. , Princeton NJ 2002).

[15] B.A. Davey and H.A. Priestley, *Introduction to Lattices and Order* (C.U.P, Cambridge 2002).

[16] W. Dunham, "Quadrilaterally Speaking", in *The Edge of the Universe: Celebrating Ten Years of Math Horizons* ed. D. Haunsperger and S. Kennedy (M.A.A., Washington D.C. 2006)

[17] C. Alsina and R. B. Nelsen, *Charming Proofs* (M.A.A., Washington D.C. 2010).

[18] I.E. Leonard, J.E. Lewis, A.C.F. Liu and G.W. Tokarsky, *Classical Geometry. Euclidean, Transformational, Inversive and Projective* (Wiley, Hoboken N.J. 2014).

[19] E.A. Weinstein, " Inequalities in Quadrilateral involving Newton Line", Int. J. Geom. **5** 54 (2016).

[20] E. Anderson, "The Smallest Shape Spaces. I. Shape Theory Posed, with Example of 3 Points on the Line", arXiv:1711.10054. For the updated version, see https://wordpress.com/page/conceptsofshape.space/1225 .

[21] "Two New Perspectives on Heron's Formula", arXiv:1712.01441 For the updated version, see https://wordpress.com/page/conceptsofshape.space/1244 .

[22] " N-Body Problem: Smallest N's for Qualitative Nontrivialities. I.", arXiv:1807.08391; For the updated version, see https://wordpress.com/page/conceptsofshape.space/1235 .

[23] C. Alsina and R.B. Nelsen, *A Cornucopia of Quadrilaterals* (MAA Press, 2020) .

[24] E. Anderson, *Applied Combinatorics*, Widely-Applicable Mathematics Series. A. Improving understanding of everything with a pinch of Combinatorics. **0**, (2022). Made freely available in response to the pandemic here: https://conceptsofshape.space/applied-combinatorics/ .

[25] "The Fundamental Triangle Matrix" (2024), https://wordpress.com/page/conceptsofshape.space/1306 ;

"Only 2 of the Fundamental Triangle, Lagrange and Apollonius Matrices are Independent. With ensuing Algebras, Irreducibles and Splits" (2024), https://wordpress.com/page/conceptsofshape.space/1310 .

[26] "A New 'Physical' Proof of Apollonius' Theorem" (2024), https://wordpress.com/page/conceptsofshape.space/1353 .

[27] "Lagrange Matrices: 3-Body Problem and General" (2024), https://wordpress.com/page/conceptsofshape.space/1436 .

[28] II. Möbius, Jacobi and Routh for Concurrent Cevians" (2024), https://wordpress.com/page/conceptsofshape.space/1261.

[29] "Euler's Quadrilateral Theorem. I. A brief new Proof that is Physically Guaranteed to Generalize." alias "Eigenclustering-Length Exchange Theorems. I. P_2 and bent-P_3 . (2024), https://wordpress.com/page/conceptsofshape.space/1297 .

[30] "Eigenclustering-Length Exchange Theorems. II. Straight-P_3 alias K-counterpart of Euler-H . (2024), https://wordpress.com/page/conceptsofshape.space/1299 .

[31] "IV. 5-Body Problem" . (2024), https://wordpress.com/page/conceptsofshape.space/1512 ;

"V. 6-Body Problem" . (2025), https://wordpress.com/page/conceptsofshape.space/1514 ;

"VI. The Top Series: Chain of Claws (with Leaf)" . (2025), https://wordpress.com/page/conceptsofshape.space/1516.

[32] E. Anderson and A. Ford, "Graph and Order Theory of N-Body Problem's Eigenclustering Networks", https://wordpress.com/page/conceptsofshape.space/2090 (2024).

[33] E. Anderson, *The Structure of Flat Geometry*, *Widely-Applicable Mathematics. A. Improving understanding of everything by dual-wielding Combinatorics and Linear Algebra.* **3**, forthcoming 2024.