Eigenclustering Length-Exhange Theorems II.

Straight-P_3 alias K-counterpart of Euler-H

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Abstract

We recently gave new proofs of Apollonius' Theorem and Euler's Quadrilateral Theorem that generalize to all other eigenclustering networks for all N-body problems (in all dimensions). Eigenclustering vectors are elsewhere called (relative) Jacobi vectors. Eigenclustering networks are in $1:1$ correspondence with the unlabelled rooted binary tree graphs. And, with 1 exception, to the at-most-binary (AMB) trees, which are conveniently smaller. For such an *Eigenclustering Length-Exchange Theorem* to be nontrivial, one needs ≥ 1 eigenclustering vector that is not just a side to exchange! $N = 3$ is thus minimum, returning Apollonius' Median-Length Theorem as the AMB convention's smallest 2-path case of our family of Theorems. While Euler's Quadrilateral Theorem corresponds to the bent 3-path encoding the H-eigenclustering.

We now provide the straight 3-path counterpart, that corresponds to the K -eigenclustering network.

In the H case, separations are supplemented by the Newton line segment, alias crossbar of the H . Whose length is a measure of aparallelogramness. In contrast, for the K , they are supplemented by the spike and the handle of the K : its second and third strokes. These names arise from viewing the K as an axe, with the 3-body subsystem it picks out in the role of blade. So on the one hand, Euler's Theorem gives the crossbar length in terms of the separations. On the other hand, our new Theorem relates a sum of squares of the spike and the handle to the separations. Which is now a quantifier of departure from the central binary coincidence.

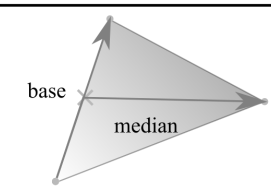
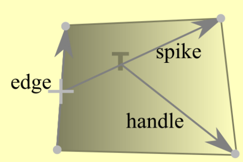
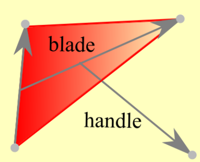
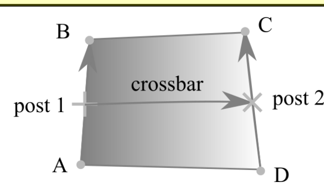
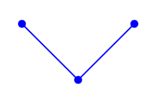
N	Eigen-clustering	Partition structure	a) Eigenclustering vectors	b) The axe conceptualization	c) Corresponding AMB tree graphs
3	(T)	(1 + 1) + 1			2-path P_2
4	K	((1 + 1) + 1) + 1			P_3-straight
	H	(1 + 1) + (1 + 1)			 P_3-bent

Figure 1:

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1 Introduction

In [24], we gave a new proof of Euler's Quadrilateral Theorem [2, 9, 10, 12, 13, 19] Which renders it clear that it is in fact a 4-body result (dimension independent). This proof is 'Physically guaranteed to generalize'. For it is built out of centre of mass (CoM) and moment of inertia (MoI) or radius-of-gyration (RoG) considerations: first and second moments.

[24] argues furthermore that my slightly earlier new proof of [22] Apollonius' Theorem [1, 7, 15, 29] is the 2-path P_2 version to [24]'s bent P_3 version of a working that holds for the following. *Any eigenclustering network for any point-or-particle number N in any dimension d* [27]. And for arbitrary masses. This is the full extent to which I am aware that the above 'Physically guaranteed to generalize' applies.

Eigenclustering vectors are elsewhere alias (relative) Jacobi vectors. Eigenclustering networks are in $1:1$ correspondence with [11, 27] the unlabelled rooted binary tree graphs [20]. And, with 1 exception, to the at-most-binary (AMB) trees [25, 27]. Which are conveniently smaller and thus what we choose to label cases by.

The only aspect of this that we use in the current Article is that the 4-body problem is minimum for eigenclustering network ambiguity. Where the straight versus bent P_3 ambiguity among AMB trees manifests. Corresponding to, in hitherto much more widely used words, the Jacobi-K versus -H ambiguity [3, 5, 8, 11, 16] at the level of the 4-Body Problem.

Euler's 4-body Theorem can then be viewed as an H-eigenclustering result. Its eigenclustering length-exchange is of the Newton line segment's length for side length data. Where the Newton line segment runs between the midpoints of the diagonals, by which its length is a quantifier of aparallelogramness (see Article I and Fig 2). While Apollonius' 3-body Theorem's eigenclustering length-exchange is already manifest in its truer name: Median-Length Theorem.

From this point on, the current Series shows what some further eigenclusterings yield instead. In the current Article, the case that we consider is the 4-bodies' K -eigenclustering network, which corresponds to P_3-straight . See Fig 1 for the 3 eigenclustering networks and trees mentioned in this Introduction. Sec 2 obtains what we need for the K -eigenclustering. Sec 3 then parallels [24]'s proof. Sec 4 summarizes this project so far at the level of a comparative table of namings. Appendix A provides some supporting Linear Algebra. [25, 26] subsequently provide further examples of such Theorems.

2 The K-eigenclustering

Structure 1 Consider the 4-body problem with equal masses. Here, the relative-separation-diagonalizing *relative Jacobi vectors* [3, 5, 8, 16] alias *eigenclustering vectors* [23, 24] can be chosen to form the following K-network [see also Fig 1.4.K)]. The $\overline{\boldsymbol{q}}$ are position coordinate vectors for our 4 points-or-particles.

$$\begin{array}{lcl} \text{edge} & : & \overline{\boldsymbol{R}}_g := \overline{\boldsymbol{q}}^B - \overline{\boldsymbol{q}}^A \\ \text{spike} & : & \overline{\boldsymbol{R}}_p := \overline{\boldsymbol{q}}^C - \frac{1}{2}\left(\overline{\boldsymbol{q}}^A + \overline{\boldsymbol{q}}^B\right) \\ \text{handle} & : & \overline{\boldsymbol{R}}_h := \overline{\boldsymbol{q}}^D - \frac{1}{3}\left(\overline{\boldsymbol{q}}^A + \overline{\boldsymbol{q}}^B + \overline{\boldsymbol{q}}^C\right) \end{array} \quad . \tag{1}$$

The corresponding eigenclustering masses (alias Jacobi masses and a subcase of reduced masses) are as follows.

$$\begin{array}{lcl} \text{edge mass} & : & \frac{1}{\mu_g} = \frac{1}{1} + \frac{1}{1} = 2 \Rightarrow \mu_g = \frac{1}{2} \\ \text{spike mass} & : & \frac{1}{\mu_p} = \frac{1}{2} + \frac{1}{1} = \frac{3}{2} \Rightarrow \mu_p = \frac{2}{3} \\ \text{handle mass} & : & \frac{1}{\mu_h} = \frac{1}{3} + \frac{1}{1} = \frac{4}{3} \Rightarrow \mu_h = \frac{3}{4} \end{array} \quad . \tag{2}$$

♠ ◇ ♣

Naming Remark 1 We name the K's 3 strokes as follows. This is from the axe conceptualization that Kneller and I exhibited in [14], as per Fig 1.b).[1] The *edge* is the leading face of the axe – one of the 6 separations supported by the 4-body's 3-simplex configuration. The *spike* is the thickness co-transversal [29] from the CoM of the edge to the third point-or-particle. Which 3-body subsystem constitutes the blade of the axe. Finally, the *handle* is the co-transversal from the blade's triple CoM T to the final point-or-particle.

Notational Remark 1 Let us denote *edge length*, *spike length* and *handle length* by

$$g \ , \quad p \ , \quad h \ .$$

This reflects that e is already booked as standard notation for one of the 3-simplex's 6 separations. While s is already in play for the totality of separations. So g stands for 'edge', being the only free letter therein. And p is the first free letter in 'spike'.

Remark 1 So to the 3-simplex's squared-length variables given in [24], we now add the following.

Definition 2 The $(\,\mbox{spike length}\,)^2$

$$P \ := \ p^2 \ .$$

The $(\,\mbox{handle length}\,)^2$

$$H \ := \ h^2 \ .$$

Of course, we call whichever $(\,\mbox{separation}\,)^2$ that we allot leading edge to the $(\,\mbox{edge length}\,)^2$

$$G \ := \ g^2 \ .$$

Definition 3 The corresponding *partial moments of inertia (MoI)* are, in order along the K's straight-$\mbox{P}_3$ edges,

$$\iota_G \ , \quad \iota_P \ , \quad \iota_H \ .$$

Notational Remark 2 The 4-body problem supports

$$\binom{4}{2} \ = \ 6$$

choices of leading edge. And then

$$\binom{2}{1} \ = \ 2$$

choices to complete the blade 3-subsystem. Then everything is fixed. So there are

$$6 \ \times \ 2 \ = \ 12$$

possible labellings of K-coordinates.

Which we index by $(\,S\,T\,)$. Or by $(\mbox{K}\,S\,T\,)$ if H's are also in play. For S an index running over separations. And T a 2-index.

[1] Though there we used 'face', 'thickness' and 'handle'.

3 K -Eigenclustering Length-Exchange Theorem

Theorem 1 [Anderson 2018] For p and h the spike and handle strokes of a K with the vertex separation $g = a$ as leading edge, the following hold.

a) **('Euler–Jacobi–Jacobi-K')**

$$2 (\iota_P + \iota_H) = - \iota_A + \iota_B + \iota_C + \iota_D + \iota_E + \iota_F \tag{3}$$

b) **('Euler–Jacobi-K')**

$$\frac{2}{3} P + \frac{3}{4} H = \frac{1}{4} (-A + B + C + D + E + F) . \tag{4}$$

<u>Proof</u> The below refers to the method in Fig 2 of Article I.

Our First Principle is the partial MoI expansion of the MoI in K-coordinates. I.e.

$$\iota = \iota_a + \iota_p + \iota_h . \tag{5}$$

Our Second Principle is the (separations! [23, 16]) democratic RoG formula (B) in Fig 2 of Article I.

a) Substitute the Greek (B) in (5), cancel terms and multiply by 2 .

b) Insert eq. (2)'s K-masses to return to the Latin world. □

Remark 1 This form of b) directly exhibits the K-masses involved ('manifest eigenclustering masses form'). Multiplying both sides by 2 casts the left-hand side in terms of the eigenclustering mass ratios.

Remark 2 b) can also be written more neatly as follows.

b′) (Rational form)

$$8 P + 9 H = 3 (-A + B + C + D + E + F) . \tag{6}$$

For all that this obscures that eigenclustering masses are in play.

Remark 3 In the original variables of the quadrilateral, we have the following.

c)

$$\frac{2}{3} p^2 + \frac{3}{4} h^2 = \frac{1}{4} \left(-a^2 + b^2 + c^2 + d^2 + e^2 + f^2 \right) . \tag{7}$$

While the rational counterpart is as follows.

c′)

$$8 p^2 + 9 h^2 = 3 \left(-a^2 + b^2 + c^2 + d^2 + e^2 + f^2 \right) . \tag{8}$$

Remark 4 In terms of the sum of separations,

a) reads

$$2 (\iota_p + \iota_h) = \iota_1 - \iota_g . \tag{9}$$

b) reads

$$\frac{2}{3} P + \frac{3}{4} H = \frac{1}{4} (S_1 - G) . \tag{10}$$

b′) reads

$$8 P + 9 H = 3 (S_1 - G) . \tag{11}$$

c) reads

$$\frac{2}{3} p^2 + \frac{3}{4} h^2 = \frac{1}{4} \left(s_2 - g^2 \right) . \tag{12}$$

Finally, c′) reads

$$8 p^2 + 9 h^2 = 3 \left(s_2 - g^2 \right) . \tag{13}$$

4 Sphynxnopsis

Can you think of a better name for a *Wheelerian comparison table* of *Namings*? (Fig 2)

Starting to name our family of Theorems			
Historical names albeit in tree lattice order!	Apollonius' Theorem [3rd and 2nd Centuries B.C.E.]	Theorem 1 [Anderson 2018]	Euler's Quadrilateral Theorem Euler's 4-Body Theorem [1748]
Jacobi alias eigenclustering network names	Jacobi-(T) 3-Body Theorem alias (T)-eigenclustering 3-Body Theorem alias 3-EMELET	Jacobi-K 4-Body Theorem alias K-eigenclustering 4-Body Theorem alias K-EMELET	Jacobi-H 4-Body Theorem alias H-eigenclustering 4-Body Theorem alias H-EMELET
Functional computation names	Median-Length Theorem	Spike-and-Handle Length Theorem	Crossbar-Length Theorem alias Newton-Length Theorem, Euler-Length Theorem and Newton-Gauss-Length Theorem
Names based on higher purpose as quantifiers of deviation	3-Body Anuniformity Theorem	4-Body Acentralbinarity Theorem	Aparallelogramness Theorem
Graph-Theoretic AMB-tree-indexed names, with advantage of extending to the whole truncated-AMB-tree-arena valued set of Theorems	3-Body 2-Path Theorem	Straight 3-Path 4-Body Theorem	Bent 3-Path 4-Body Theorem

Figure 2:

Remark 1 Names for the current Article's Theorem are highlighted in yellow. The penultimate one is explained in the Appendix.

Riddle 1 We leave what anuniformity may refer to as a riddle for the Reader (for Sphinxes are also Riddlers).

Remark 2 While [10] first envisaged the concept behind the name *Aparallelogramness Theorem*, this name itself arose in discussions between Sánchez and I [18]. This is highlighted in red to mark the truest name so far for the nicest theorem in the set. Some of its features are as follows. Since parallelograms have 4 vertices, in this case there is no need to say *4-body*. There are also many ways in which the parallelogram is more interesting than the other configurations which these Theorems quantify deviations from.

It should come as no surprise that the most interesting case, concurrently the one permitting the shortest name, comes with Euler's name attached. Elsewhere Euler founded Affine Geometry as an axiomatization of Parallelism. And also founded the very Graph Theory that indexes our whole family of Theorems! We bow to the master.

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A Linear Algebra of the K-Eigenclustering Length-Exchange Theorem

Remark 1 Theorem 1 – 4-Body K-Eigenclustering Length-Exchange Theorem alias everything in column 2 of Fig 2 – is a new result.

So we here re-run the Linear Algebra analysis that we had already conducted upon first mention for the 4-Body H-Eigenclustering Length-Exchange Theorem alias everything in Column 3 of Fig 2.

Structure 1 The *K-eigenclustering matrix* is

$$\boldsymbol{K} \ := \ \frac{1}{2} \begin{pmatrix} -1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -1 \end{pmatrix} \ . \tag{14}$$

With this scaling,

$$\underline{\boldsymbol{\mathcal{K}}} \ = \ \underline{\underline{\boldsymbol{K}}} \cdot \underline{\boldsymbol{\Sigma}} \ . \tag{15}$$

For *separation partial moments of inertia* 6-*vector* $\underline{\boldsymbol{\Sigma}}$. And *combined spike-and-handle inertia vector* $\boldsymbol{\mathcal{K}}$ with components

$$\mathcal{K}_S \ = \ \iota_{P\,S} \ + \ \iota_{H\,S} \ = \ {\rho_{p\,S}}^2 \ + \ {\rho_{h\,S}}^2 \quad (\text{ no sum }) \ . \tag{16}$$

Remark 2 Observe that this construct does not use all K-eigenclusters at once. Rather, so as to obtain a square and thus well-determined system, for each inter-vertex separation we pick precisely 1 K which has this as its leading edge.

Structure 2 The corresponding *spike-and-handle-squares* 6−*vector* is

$$\boldsymbol{K} \ = \ \frac{2}{3}\boldsymbol{P} \ + \ \frac{3}{4}\boldsymbol{H} \ .$$

For (spike length)^2 6-vector $\boldsymbol{P}$. And (handle length)^2 6-vector $\boldsymbol{H}$. Setting

$$\boldsymbol{K} \ = \ \boldsymbol{0}$$

then reveals our Theorem to be a quantifier of deviation from central binary collisions. Hence the penultimate name in column 2 of Fig 2.

Remark 3 Its eigenvalues are 1 with multiplicity 1 and -2 with multiplicity 5 . Its rank is 6 , which is the full rank supported, so the nullity is 0 . With reference to [21]'s conceptualization, its Mathematicians' signature is 5 and its Physicists' signature is 4 , and its Physicists' signature-in-detail is $- - - - - +$: hyperbolic.

The corresponding eigenvectors can be taken to be

$$\frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\1\\1\\1\\1\end{pmatrix} , \quad \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\\0\\0\\0\end{pmatrix} , \quad \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\\0\\0\\0\end{pmatrix} , \quad \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\0\\1\\0\\0\end{pmatrix} , \quad \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\0\\0\\1\\0\end{pmatrix} , \quad \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\0\\0\\0\\1\end{pmatrix} , \qquad (17)$$

The first is obligatory, while the remainder constitute a convenient pure-ellipticity [17] basis.

Remark 4 $\boldsymbol{K}$ is 6×6 , and invertible as implied by the full rank that can be read off its eigenspectrum.

Remark 5 We did not present any eigentheory in [24] for the following reason. The H-cycle matrix just returns the Heron matrix [6, 17] alias fundamental triangle matrix [21]. For which e.g. [21] already provided such an analysis (eigenvalues and eigenvectors first appeared in [17]). [21] furthermore gave 6 technically distinct routes to this matrix within the theory of triangles, 2 of which have 2 distinct conceptualizations; see [28] for yet more. By which the name 'Fundamental Triangle matrix' is well justified.

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