An infinite Series of Generalizations of
Apollonius' Theorem

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Abstract

Apollonius' Theorem gives the length of a triangle's median in terms of the lengths of its sides. We generalize this to an infinite series of formulae for the lengths of the equal-masses N-simplex's $K(N)$-eigenclustering's strokes. These simple formulae closely resemble Apollonius' Theorem in form. Which Theorem is indeed their first nontrivial member, corresponding to viewing the median as the first non-side stroke of a $K(N)$ eigenclustering. Eigenclustering vectors are alias relative Jacobi vectors. While $K(N)$ is the obvious N-body problem generalization of the 4-body problem's K-eigenclustering, alias Jacobi-K .

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1 Eigenclusterings

Remark 1 The N-simplex in some $\mathbb{R}^d$ supports

$$\binom{N}{2}$$

separations between its verttex points. Of which only

$$n := N - 1$$

are linearly independent. The N-simplex can also be viewed as a Physical N-body problem's N-particle configuration in $\mathbb{R}^d$. For $N \geq 3$, a choice of n relative separations does not diagonalize the inertia quadric. But linear bases of *eigenclustering vectors* do. Eigenclustering vectors [32, 36] are alias *relative Jacobi vectors* [9, 17, 20, 25]. While at the present level of reduction, the inertia quadric is equivalent to the Euclidean metric on relative space [32, 44] in Physics-free terms.

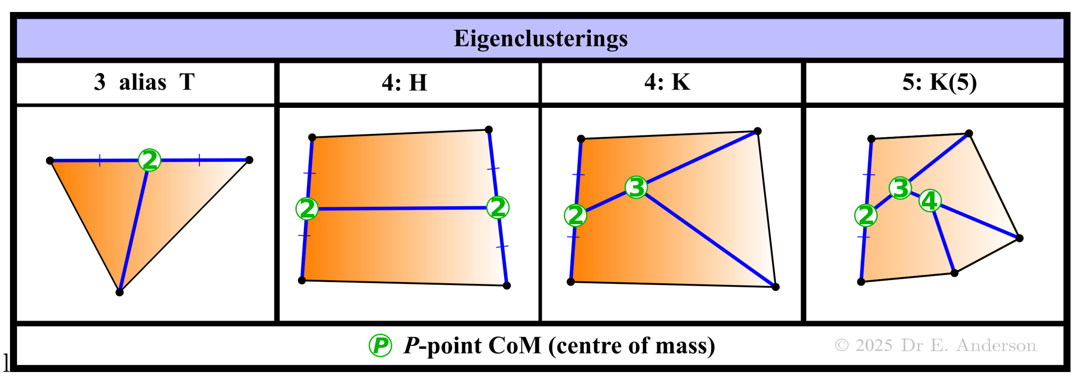

Figure 1:

Example 1 $N = 3$ supports a single eigenclustering network: the T-shape of Fig 1.3.

Example 2 $N = 4$ exhibits a first eigenclustering network ambiguity. Namely, H- versus K-eigenclustering, alias the Jacobi-H and -K ; see Fig 1.4.

Remark 2 Such eigenclustering network ambiguities growingly persist for all subsequent N . More specifically, the N-body problem supports $w(N)$ eigenclustering networks, where w denotes the *Wedderburn–Etherington numbers* [11, 13, 16, 19, 41]. This is since the eigenclustering networks are in $1:1$ correspondence with [25, 40] the unlabelled rooted binary trees [22, 27, 33, 40].

Example 3 Every N supports a generalized-K eigenclustering $\mathrm{K}(N)$, formed by keeping on adding 1 point at a time to the growing network. This recursive construction corresponds to forming the straight-P_n path in the AMB tree representation. Where AMB stands for *at-most binary*, which is a more efficient representation than the binary version for our domain of interest, $N \geq 3$. Whereupon it is related to [40] the somewhat larger binary tree representation by defoliation [33]. See Fig 1.5 for $\mathrm{K}(5)$.

2 ELETs

Remark 1 For each $N \geq 3$, every eigenclustering network supports a nontrivial *ELET*. I.e. an *Eigenclustering Length-Exchange Theorem* [36, 37, 38, 39, 40] . In the current Letter, we restrict ourselves to the EMELET subcase, meaning with equal (unit) masses in the Physical setting. While 'mass-weighting' vertex points remains meaningful in the Geometrical setting, equal 'masses' is also a natural simplicity condition here.

Example 1 The first nontrivial EMELET, for $N = 3$, is *Apollonius' Theorem* [1, 18, 30, 34, 44]

$$M_A = \frac{2\,(\,B + C\,) - A}{4} . \tag{1}$$

Notational Remark 1 We use caps for $(\,\text{length}\,)^2$ variables. A and cycles are the $(\,\text{side-lengths}\,)^2$, a and cycles, of a triangle. While M_A is the $(\,\text{length}\,)^2$ of the median m_a corresponding to side a (see Fig 2.a). Following e.g. Conway, squared variables are well-known to be useful in studying triangles [42]. Much of this usefulness is turning out to transcend to N-body problems [44].

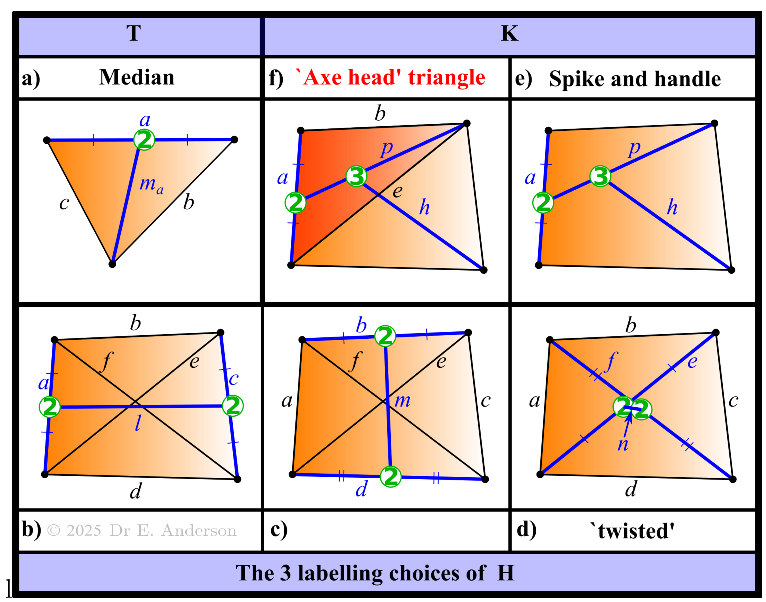

Figure 2:

Example 2 The $N = 4$ H-EMELET also turns out to be a familiar result [36], for it returns Euler's 4-Body Theorem [5, 26, 23, 24, 28, 36, 44],

$$4\,N \;=\; A \;+\; C \;+\; B \;+\; D \;-\; E \;-\; F\;. \tag{2}$$

Where N is $(\,\mbox{Newton length}\,)^2$. And Newton length [4, 7, 10, 12, 28, 31] is the distance between the midpoints of the 2 diagonals of a quadrilateral with sides a, b, c, d . This corresponds to the 'twisted' H in Fig 2.b).

Though Euler's 3-cycles corresponding to the H's in Figs 2.c-d) hold as well. Returning

$$4\,L \;=\; B \;+\; D \;+\; E \;+\; F \;-\; A \;-\; C\;, \tag{3}$$

$$4\,M \;=\; E \;+\; F \;+\; A \;+\; C \;-\; B \;-\; D\;. \tag{4}$$

Which opposite pair of separations plays the role of diagonals is thereby moot. Which is essential for non-convex quadrilaterals, and tetrahaedrons, to be included [23, 36].

In this way, Euler's 4-Body Theorem is indeed a truer name. For all that it has been more widely referred to as Euler's Quadrilateral Theorem... Apollonius' Theorem also happens to matchingly hold irrespective of dimension [34, 44].

Example 3 The $N = 4$ K-EMELET [37] was recently made public by the Author [37]. The 'rational' form for this [paralleling (2)] is

$$8\,P \;+\; 9\,H \;=\; 3\,(\;B \;+\; C \;+\; D \;+\; E \;+\; F \;-\; A\;)\;. \tag{5}$$

Though the following form is more convenient for the current Letter's considerations.

$$\frac{A}{2} \;+\; \frac{2}{3}\,P \;+\; \frac{3}{4}\,H \;\;=\;\; \sum_{s\,=\,1}^{6} A_s \;\;=\;\; \sum_{s\,\in\,\mathfrak{s}\mbox{ep}(4)} A_s\;. \tag{6}$$

Where $\mathfrak{s}\mbox{ep}(N)$ denotes the set of separations contained in an N-body problem configuration. P is the $(\,\mbox{length}\,)^2$ of the 'spike' p and H that of the 'handle' h . These names and notation follow from viewing the K as an axe [29, 37]; see Fig 2.f).

3 Discussion

Remark 1 Prima facie, the K-EMELET computes a linear combination of 2 eigenclustering $(\,\mbox{lengths}\,)^2$. While Apollonius and Euler each isolate a single eigenclustering length, thus solving for it. So is the K-EMELET less interesting due to this feature?

Remark 2 In [43], however, it was shown that adjoining Apollonius' Theorem for the 'axe-head' triangle (Fig 2.f) to (6) gives a linear system for P and H . Which is furthermore already-decoupled. In the sense that we can separately solve Apollonius for P first. And then interpret the K-EMELET as an equation for H alone.

This yields

$$H \;=\; \frac{3\,(\,C \;+\; D \;+\; F\,) \;-\; (\,A \;+\; B \;+\; E\,)}{9}\;. \tag{7}$$

Which looks strikingly like a generalization of Apollonius' Theorem, in particular given the conceptual analysis in Fig 3.

The purpose of this Letter is then to prove Fig 3.c)'s hypothesis.

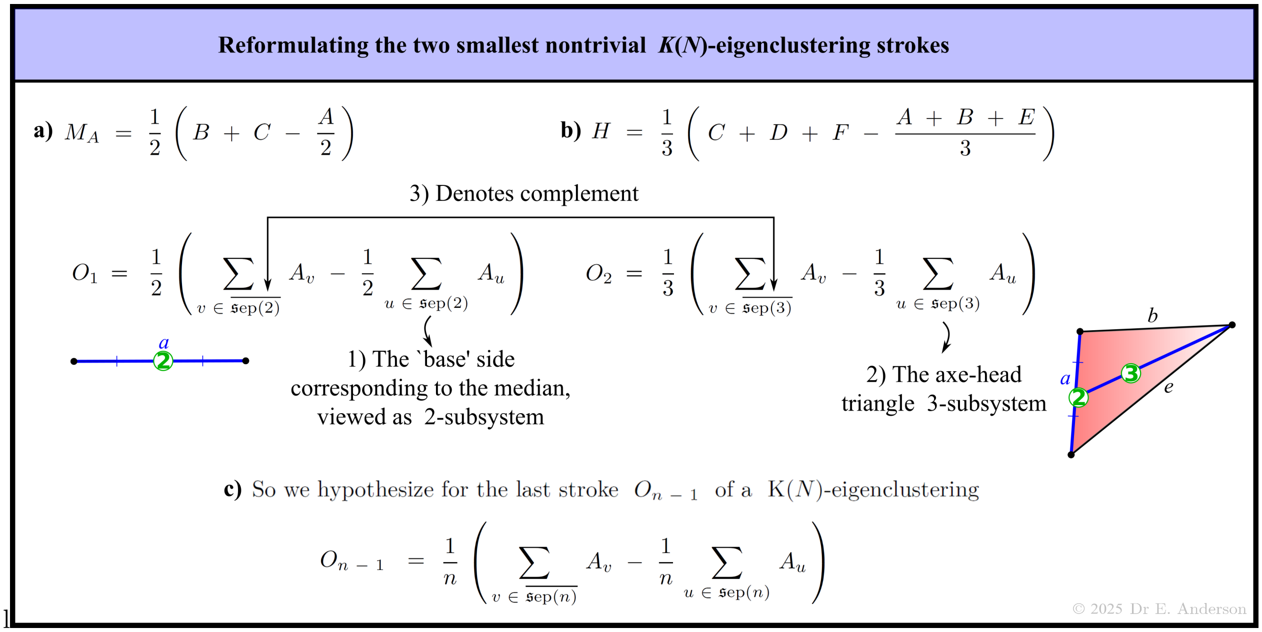

Figure 3:

Remark 3 Before we start, let us recollect the following result that was also recently made public by the Author [38].

Theorem 0 (K(N)-EMELET)

$$\sum_{j=1}^{n} \frac{j}{j+1} O_{j-1} = \frac{1}{N} \sum_{s=1}^{\binom{N}{2}} A_s = \frac{1}{N} \sum_{s \in \mathfrak{sep}(N)} A_s \, . \tag{8}$$

Naming Remark 1 O_0 is just the $(\text{side-length})^2$ A, while none of the other O_k correspond to sides. Let us call an eigenclustering stroke that is not just a side *nontrivial*. Our aim is then to exchange all the nontrivial eigenclustering strokes for sides data. This aim is what the 'ELE' in 'ELET' refers to.

Remark 4 In general, an ELET will not achieve this aim by itself, due to under-determination. But forming a linear system out of a given eigenclustering basis' picked out subsystems' ELETs can overcome this. $\mathrm{K}(N)$ benefits from being an already-decoupled linear system. So one can start at the side and then just move along the K solving for one stroke at a time. Where moving along the K corresponds to moving along the straight-P_n path in the AMB representation. Leading to the current Letter's simple closed formula for the $\mathrm{K}(N)$-EMELET's nontrivial eigenclustering lengths.

Remark 5 For an arbitary eigenclustering network, the O_k are [transversals] [35]. I.e. a projectively-dual portmanteau name for transversals [6, 8, 12] – through a vertex – and cotransversals – instead cutting a side. Which are then collectively denoted by T_k [38].

For triangles, Cevians [3, 14, 15, 21, 30] are cotransversals while Menelians [2, 14, 15, 21, 30] are transversals. The quadrilateral's Newton line interval is also a transversal. But for $K(N)$, all the [transversals] are cotransversals. For which we use the notation O_k . (C is already booked as part of the notation for sides.)

4 Proof

Take Fig 3.c)'s hypothesis to be an induction hypothesis for $n =$ some $p - 1$.

For $p = 1$ (and thus $N = 2$, $n = 1$, $n - 1 = 0$), this collapses to the identity

$$A = O_0 = A \, . \tag{9}$$

Which is certainly true.

For $N = p$, (10) gives

$$\sum_{j=1}^{p} \frac{j}{j+1} O_{j-1} = \frac{1}{P} \sum_{s \in \mathfrak{s}\mathrm{ep}(P)} A_s \, . \tag{10}$$

Where

$$P := p + 1 \, . \tag{11}$$

But also for $n = p - 1$, (8) gives

$$\sum_{j=1}^{p-1} \frac{j}{j+1} O_{j-1} = \frac{1}{p} \sum_{u \in \mathfrak{s}\mathrm{ep}(p)} A_u \, . \tag{12}$$

Take $(10) - (12)$, while performing the indexing-set split into

$$\mathfrak{s}\mathrm{ep}(P) = \mathfrak{s}\mathrm{ep}(p) \amalg \overline{\mathfrak{s}\mathrm{ep}(p)} \, .$$

Thus

$$\frac{p}{P} O_{p-1} = \frac{1}{P} \sum_{v \in \overline{\mathfrak{s}\mathrm{ep}(p)}} A_v + \left(\frac{1}{P} - \frac{1}{p} \right) \sum_{u \in \mathfrak{s}\mathrm{ep}(p)} A_u \, . \tag{13}$$

But

$$\frac{1}{P} - \frac{1}{p} = \frac{p - P}{p\,P} \stackrel{(11)}{=} - \frac{1}{p\,P} \, . \tag{14}$$

Thus (13) becomes

$$O_{p-1} = \frac{P}{p} \frac{1}{P} \left(\sum_{v \in \overline{\mathfrak{s}\mathrm{ep}(p)}} A_v - \frac{1}{p} \sum_{u \in \mathfrak{s}\mathrm{ep}(p)} A_u \right) \, . \tag{15}$$

Which cancels down to our desired result. □

5 Naming our result

Remark 1 Thus we have proven the following.

Theorem 1 The $K(N)$ eigenclustering's $(q - 1)$th stroke's $(\,\mathrm{length}\,)^2$ is given by the following.

$$O_{q-1} = \frac{1}{q} \left(\sum_{v \in \overline{\mathfrak{s}\mathrm{ep}(q)}} A_v - \frac{1}{q} \sum_{u \in \mathfrak{s}\mathrm{ep}(q)} A_u \right) \, . \tag{16}$$

For $\mathfrak{s}\mathrm{ep}(q)$ the set of separations of the q-subsystem which the $K(N)$ eigenclustering in question is adapted to.

Remark 2 The first nontrivial such is Apollonius' Theorem. While the first new such is the outcome of solving the K-EMELET coupled to Apollonius' Theorem for the corresponding 'axe-head' 3-subsystem.

Naming Remark 2 *Apollonius–Jacobi–*$\mathrm{K}(N)$ *Theorem* is thus a reasonable name. The $\mathrm{K}(N)$'s own truer name is straight-P_n, while 'eigenclustering' is a truer name for 'relative Jacobi'. So *straight*-P_n-*Eigenclustering Length Theorem* is a truer name. This confers a new name – P_2-*Eigenclustering Length Theorem* – to Apollonius' Theorem. [This path is short enough to not need to specify a rooting, which is the function that 'straight' performs.]

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