Fundamental Triangle Matrix:
Linear Algebra and Commutator interplay with Lagrange and Apollonius matrices. With implications for deriving Hopf and Kendall's Little Results.

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Abstract

All of the triangle inequality, the cosine rule and Heron's formula can be formulated in terms of the same fundamental triangle matrix, $\boldsymbol{F}$. We now consider its interplay with two further 3-body matrices. Firstly, the *Lagrange matrix* $\boldsymbol{L}$, alias $\boldsymbol{P}$ since it is a projector. $\boldsymbol{F}$ and $\boldsymbol{P}$ commute, share eigenspaces, and can be taken to share eigenbasis of eigenvectors. The 3-body *eigenclustering vectors*, alias *relative Jacobi vectors*, that are well-known from $\boldsymbol{L}$, are thereby also natural in considering $\boldsymbol{F}$.

In the Geometrically simplest unit vertex masses conceptualization, these eigenvectors are Geometrically a side vector and the corresponding median vector. By which how sides and medians transform into each other becomes of interest. And so our last matrix the *sides–medians* or *Apollonius involution* $\boldsymbol{J}$ is encountered. All three of $\boldsymbol{P}$, $\boldsymbol{J}$, $\boldsymbol{F}$ commute, . share all eigenspaces, and can be taken to share eigenbasis of eigenvectors.

These considerations give a new proof of the medians-data counterpart of Heron's formula. And, for a particular choice of eigenbasis, principles for which we cover here, a new derivation of Hopf's Little Mathematics from just Heron's formula. Leading into a new derivation from just Heron's formula of Kendall's Little Theorem. I.e. that the shape space of triangles – modulo similarities – is metric-geometrically a sphere.

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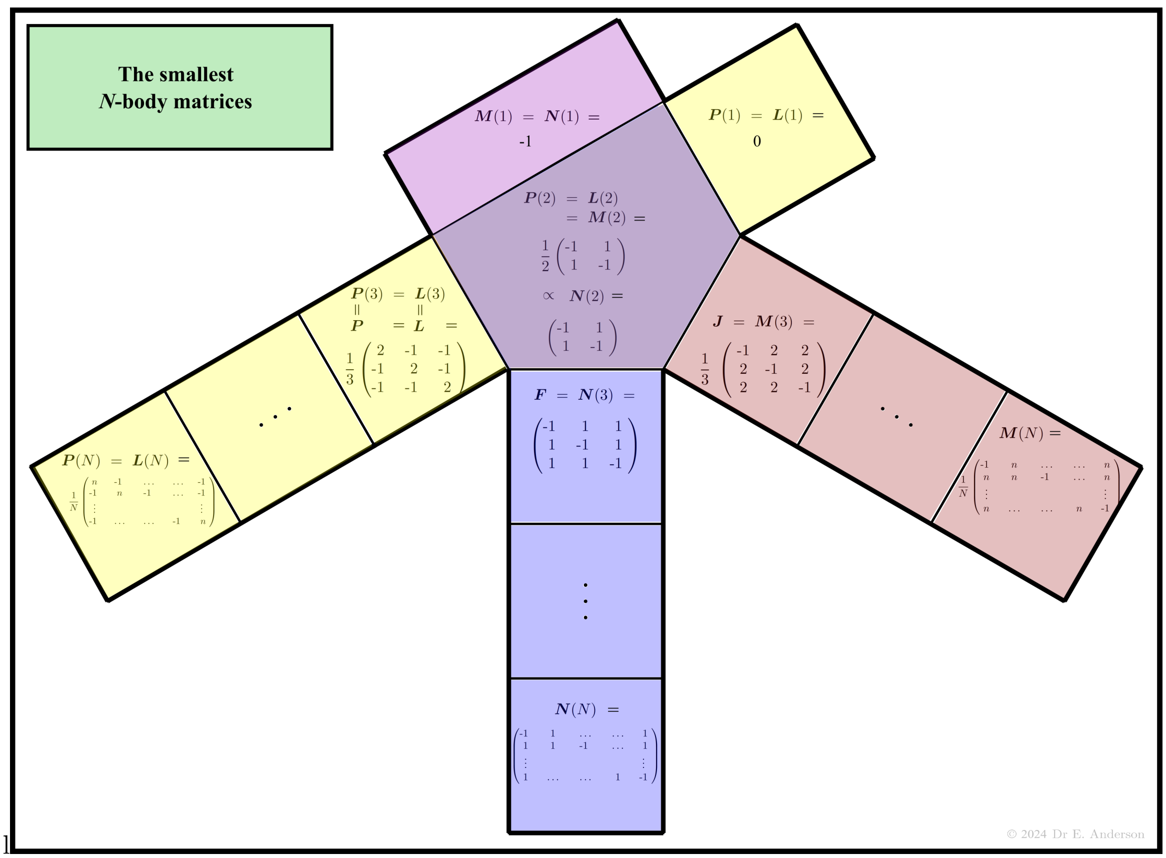

Figure 1:

1 Introduction

1.1 The Lagrange projector

Remark 1 The current Article concerns the interplay between 3 3-body 3×3 matrices. The first of these is of a conceptual type that occurs for all material N (meaning ≥ 1) . One way of viewing these is as Lagrange matrices, which we set up as follows.

Structure 0 Consider a Mechanical system of N point-particle masses in $\mathbb{R}^d$ [22, 30]. Its *inertia quadric* is

$$I = \sum_{I=1}^{N} m_I \, \boldsymbol{q}^{I\,2} = \sum_{I \neq J = 1}^{N} \frac{m_I \, m_J}{M} \boldsymbol{r}^{IJ\,2} .$$

Where, in the first – *absolute* – form, the m_I are masses and the $\boldsymbol{q}^I$ are position vectors. Which, as well as carrying the internal particle-label index I running from 1 to N , are spatial vectors. In the second – *relative* form [4, 36],

$$\boldsymbol{r}^{IJ} := \boldsymbol{q}^J - \boldsymbol{q}^I$$

are the *relative separation vectors of* each particle pair. And M is the total mass of the system.

Structure 1 The interconversion between the $\boldsymbol{q}^I$ and $\boldsymbol{r}^{IJ}$ is encoded by the *Lagrange matrix* $\boldsymbol{L}$ as follows.

$$||\boldsymbol{r}||^2 = ||\boldsymbol{q}||_{\boldsymbol{L}}{}^2 = \overline{\boldsymbol{q}} \cdot \underline{\underline{\boldsymbol{L}}} \cdot \overline{\boldsymbol{q}} \, . \quad (1)$$

So each Lagrange matrix is the corresponding N 's *relative-separations to position-coordinates matrix.*

Structure 1′ Another way of viewing these matrices is as *projectors* $\boldsymbol{P}$ obeying

$$\boldsymbol{P}^2 = \boldsymbol{P} \, . \quad (2)$$

It is a projection from the constellation space $\mathfrak{q}(d, N)$ of N-point-or-particle configurations in $\mathbb{R}^d$. To the relative space $\mathfrak{R}\text{elative}(d, N)$ of linearly independent (LI) relative separations [56, 49]. These spaces are furthermore $\mathbb{R}^{dN}$ and $\mathbb{R}^{dn}$ respectively, where

$$n := N - 1. \quad (3)$$

It is much more widely known as taking out the centre of mass (CoM). Which can furthermore be viewed as quotienting down from constellation space to relative space. Or indeed as projecting out the CoM position.

Remark 2 When equal masses are accorded to the N -body's points-or-particles, the above two notions coincide . In this case, we refer to the matrix as the *Lagrange projector*. See [56, 49] for recent reviews.

♠ ◇ ♣

Remark 3 $N = 1$ return just

$$L = 0 \, : \quad (4)$$

the zero scalar. See Fig 1 for the two next smallest cases, which play pivotal roles in the current Article.

1.2 The Apollonius involution

Remark 1 The current Article's second matrix arises as follows [60].

Apollonius' sides-to-medians Theorem [1, 45, 33, 48]. The square of the length of the median m_a emanating from vertex A of a triangle is given by the following.

$$ {m_a}^2 \ = \ \frac{1}{4} \left(2\,b^2 \ + \ 2\,c^2 \ - \ a^2 \right) \ . \tag{5} $$

Remark 2 Considering all cycles of of (5) is quite common in the literature; see e.g. [33, 45].

Remark 3 In squared variables,

$$ M_A \ = \ \frac{1}{4} \left(2\,B \ + \ 2\,C \ - \ A \right) \quad \text{and cycles} \ . \tag{6} $$

Structure 2 This cycle can furthermore be packaged into [51] the *sides-to-medians* alias *Apollonius matrix* $\boldsymbol{O}$. I.e.

$$ \boldsymbol{O} \ := \ \frac{1}{4} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \ . \tag{7} $$

Such that

$$ \overline{\boldsymbol{M}} \ = \ \underline{\overline{\boldsymbol{O}}} \cdot \overline{\boldsymbol{S}} \ , \tag{8} $$

for $\boldsymbol{M}$ the $(\,medians\,)^2$ *vector*.

Remark 4 $\boldsymbol{O}$ furthermore turns out to be proportional to the following involution [51].

$$ \boldsymbol{J} \ := \ \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \ . \tag{9} $$

Clearly,

$$ \boldsymbol{J} \ = \ \frac{4}{3}\,\boldsymbol{O} \ . \tag{10} $$

Involution means that

$$ \boldsymbol{J}^2 \ = \ \mathbb{I} \ . \tag{11} $$

With 2 being the smallest power for which the identity matrix arises. I.e. this definition is to the exclusion of the identity matrix itself.

1.3 The fundamental triangle matrix $\boldsymbol{F}$

Structure 3 We already introduced $\boldsymbol{F}$ in Article 1, though the Reader can also conveniently pick out its form at the centre of Fig 1. $\boldsymbol{F}$ arises as the core of the quadratic form formulation [3, 32, 51] of Heron's formula [2, 10]. And in very many further a priori conceptually distinct ways [55, 57, 71]. As befits not only a citizen of Kallista but also one which turns out to be twinned with the illustrious *Hopf's Little Map* [7, 20, 28, 41, 42, 71].

Aside 1 The $\boldsymbol{M}(N)$ and $\boldsymbol{N}(N)$ matrices in Fig 1 are provided to give context to the following. That $\boldsymbol{L}(2) = \boldsymbol{P}(2)$ – a key object in the current Article [52] – is a conflation of all 3 of our 3-body matrices. Though for $N = 2$, the projector property overwrites the others. Nor are $\boldsymbol{M}(N)$ and $\boldsymbol{N}(N)$ realized beyond $N = 3$. Do the following Exercise and see End-Note 2 in Sec 5 for further pointers in this regard.

Exercise 1 Show that no projector can also be an involution. That $\boldsymbol{L}(N)$ is a projector for every $N \in \mathbb{N}$. And that $\boldsymbol{M}(N)$ is only an involution for $N = 1$ and 3.

1.4 Outline of the rest of this Article

In Sec 2, we jointly tabulate some basic eigentheory for each of our 3 matrices. Observing that all of them share eigenspaces, and thus can be made to share a basis of eigenvectors. We also cover the $N = 2$ joint precursor of these 3 matrices. For which the Lagrange projector overwrites subsequent N's other two matrices' properties. We use this precursor to fix a basis for the nontrivial – 2-d – eigenspace. Such fixing has been called 'hierarchical', though we argue for the technically sharper term 'induced representation'. This, and Mechanics and Geometry principles contrasted more generally, is the subject of Sec 3.

In Sec 4, we show that all 3 of our matrices commute with each other. While some commuting sets of matrices have common eigenbases, others do not.

Exercise 2 Find two commuting matrices for which no common eigenbasis can be found.

We end with some further interplays in Sec 5. In particular, the 'hierarchical' or 'induced' eigenbasis leads from Heron's formula to Hopf's Little map. This motivates the current Article presenting specific arguments for fixing this eigenbasis. And can be built up to a simple new proof of Kendall's Little Theorem from just Heron's formula. This Theorem structurally extends Smale's Little Theorem [17, 55]. In the sense of the shape space of triangles – modulo similarities – being not only Topologically but also Metric-Geometrically a sphere. We also show that involution combines with one of the zero commutators to return a new Linear Algebra proof [51, 59] of the medians-data Heron formula [6, 24, 33, 40].

2 Eigentheory

2.1 Eigenvalue level analysis

<table>
<tr><th colspan="12">Algebraic properties of current Article's cyclic quadrilateral matrices</th></tr>
<tr><th>N</th><th>Matrix</th><th>Eigenvalues</th><th>Rank</th><th>Nullity</th><th>s_{Math}</th><th>s_{Phys}</th><th>$s_{Phys-detail}$</th><th colspan="4">Notes</th></tr>
<tr><td>1</td><td>Lagrange identity scalar
$L = 0$</td><td>1
1
1
1</td><td>01</td><td>0</td><td>0</td><td>0</td><td>0</td><td>Scalar: already diagonal</td><td>Null</td><td>Projection with null image</td><td rowspan="2">Commutation is trivial</td></tr>
<tr><td>2</td><td>Lagrange projector
$\boldsymbol{L} = \boldsymbol{P}$</td><td>0 1
1 1
1 1
1 1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>+ 0</td><td rowspan="4">Diagonalizable</td><td rowspan="2">Degenerate with elliptic nondegenerate sector</td><td rowspan="2">Projection</td></tr>
<tr><td rowspan="3">3</td><td>Lagrange projector
$\boldsymbol{L} = \boldsymbol{P}$</td><td>0 1
1 2
1 2
1 1</td><td>2</td><td>1</td><td>2</td><td>2</td><td>+ + 0</td><td rowspan="3">Commute with each other</td></tr>
<tr><td>Apollonius sides-medians involution
$\boldsymbol{O} = \boldsymbol{J}$</td><td>1 -1
1 2
1 2
1 1</td><td>3</td><td>0</td><td>2</td><td>1</td><td>+ - -</td><td>Hyperbolic</td><td>Involution</td></tr>
<tr><td>Fundamental triangle matrix
$\boldsymbol{F}$</td><td>1 -2
1 2
1 2
1 1</td><td>3</td><td>0</td><td>2</td><td>1</td><td>+ - -</td><td>Hyperbolic</td><td></td></tr>
</table>

<table>
<tr><td rowspan="4">Key</td><td>λ_e</td><td colspan="2">Eigenvalue</td></tr>
<tr><td>α_e</td><td>Algebraic</td><td rowspan="3">multiplicities</td></tr>
<tr><td>γ_e</td><td>Geometric</td></tr>
<tr><td>μ_e</td><td>Minimal</td></tr>
</table>

Figure 2:

Remark 1 See Fig 2. With reference to Article 1's notions, conventions and notation.

Example $N = 1$ This is already diagonal, so there is no eigentheory to work out.

Example $N = 2$ The symmetry–antisymmetry outcome here is very familiar; c.f. e.g. vibration modes or spin.

Example $N = 3$ Here some of the eigentheory of $\boldsymbol{F}$ and $\boldsymbol{J}$ was previously worked out in [51]. While that of $\boldsymbol{P}$ is longstanding.

Exercise 3 More experienced Physicists among the Readers might wish to compare the above analysis for $\boldsymbol{L}$ with the following in dimensions 4 , 2 and especially 3 . Carroll spacetime: the zero speed-of-light limit [12, 14] of Minkowski spacetime. So that here "*it takes all the running you can do, to keep in the same place*" [5]. Which also pertains to the 'strong gravity' limit of General Relativity [19, 21]. Compare additionally with the metric induced by a 3-metric on a 2-surface [25].

2.2 Eigenvectors and eigenspaces

Eigenvectors and eigenspaces						
N	**Matrix**	**Some eigentheory** © 2024 Dr E. Anderson				
1	**Lagrange projector $\boldsymbol{L} = \boldsymbol{P}$**			0		
	Eigenvalues			0		
	Geometric multiplicities			1	=	1
	Eigenspaces			$\mathfrak{e}\mathrm{ig}_+(\boldsymbol{P})$ $\mathbb{R}$	= =	$\mathfrak{e}\mathrm{ig}(\boldsymbol{P})$ $\mathbb{R}$
2	**Lagrange projector $\boldsymbol{L} = \boldsymbol{P}$**	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$		$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$		
	Eigenvalues	0		1		
	Geometric multiplicities	1	+	1	=	2
	Eigenspaces	$\mathfrak{e}\mathrm{ig}_0(\boldsymbol{P})$ $\mathbb{R}$	$\oplus$ $\oplus$	$\mathfrak{e}\mathrm{ig}_+(\boldsymbol{P})$ $\mathbb{R}^2$	= =	$\mathfrak{e}\mathrm{ig}(\boldsymbol{P})$ $\mathbb{R}^3$
3	**Lagrange projector $\boldsymbol{L} = \boldsymbol{P}$**	$\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$		$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}$ $\frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\-2\end{pmatrix}$		
	Eigenvalues	0		1		
	Geometric multiplicities	1	+	2	=	3
	Eigenspaces	$\mathfrak{e}\mathrm{ig}_0(\boldsymbol{P})$ $\mathbb{R}$	$\oplus$ $\oplus$	$\mathfrak{e}\mathrm{ig}_+(\boldsymbol{P})$ $\mathbb{R}^2$	= =	$\mathfrak{e}\mathrm{ig}(\boldsymbol{P})$ $\mathbb{R}^3$
	Apollonius sides-medians involution $\boldsymbol{J}$	$\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$		$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}$ $\frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\-2\end{pmatrix}$		
	Eigenvalues	1		-1		
	Geometric multiplicities	1	+	2	=	3
	Eigenspaces	$\mathfrak{e}\mathrm{ig}_+(\boldsymbol{J})$ $\mathbb{R}$	$\oplus$ $\oplus$	$\mathfrak{e}\mathrm{ig}_-(\boldsymbol{J})$ $\mathbb{R}^2$	= =	$\mathfrak{e}\mathrm{ig}(\boldsymbol{J})$ $\mathbb{R}^3$
	Fundamental triangle matrix $\boldsymbol{F}$	$\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$		$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}$ $\frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\-2\end{pmatrix}$		
	Eigenvalues	1		-2		
	Geometric multiplicities	1	+	2	=	3
	Eigenspaces	$\mathfrak{e}\mathrm{ig}_{-2}(\boldsymbol{F})$ $\mathbb{R}$	$\oplus$ $\oplus$	$\mathfrak{e}\mathrm{ig}_+(\boldsymbol{F})$ $\mathbb{R}^2$	= =	$\mathfrak{e}\mathrm{ig}(\boldsymbol{F})$ $\mathbb{R}^3$

Figure 3:

Remark 1 The 1-d eigenspace corresponds to centre of mass (CoM), semi-perimeter, radius of gyration (RoG), or total moment of inertia (MoI), depending on the meaning of the underlying vector space. This eigenspace is realized for every $N \geq 1$.

Example $N = 1$ Here this is the sole eigenspace (cyan in Fig 3).

Example $N = 2$ Here our 2×2 matrix's eigenvectors simply return the symmetric–antisymmetric eigenvector pair (sky and ivory in Fig 3). This is unambiguous up to signs.

Example $N = 3$ Here all 3 3×3 3-body matrices share eigenspaces. As per the blue and yellow highlights in Fig 3.

3 Arguing for the particular eigenbasis given

Example $N = 2$ Here the eigenbasis is fixed up to the sign of the antisymmetric vector.

Example $N = 3$ But now a 2-d eigenspace is realized. Thus there is now a 1-degree of freedom family of eigenbases. The particular one exhibited is one of the 3 label permutations of the eigenclustering (alias relative Jacobi) vectors. This 'is standard' for $\boldsymbol{L}$. And can then be viewed as inherited by the other two matrices. Implementing how our study of medians, and of triangle areas, can be taken to less redundantly reside within the relative space that the Lagrange projector projects constellation space down to.

Remark 2 But given that this specific basis is needed for some interesting results, we should examine what selection principles underlie this 'standard choice' of eigenbasis for $\boldsymbol{L}$ itself.

3.1 Mechanics Selection Principle

Remark 1 In Mechanics, each choice of eigenclustering coordinates corresponds to modelling a *clustering hierarchy*. For $N = 3$, this takes the following form.

One considers the relative separation vector between some pair of particles, picking out a particular binary subsystem that our eigenbasis is adapted to the study of. One then discovers that the following orthonormally completes this eigenbasis. The *inter-clustering separation vector* between the binary subsystem's CoM and the third particle.

Remark 2 $N = 3$ supports 3 choices for this procedure if the 3 particles are distinguishable. For there are then 3 distinguishable relative separation vectors. Corresponding to the 3 ways in which 1 particle can be left out in assigning a binary subsystem.

Remark 3 Molecular Physicists call quantities that are independent of such labelling choices *democracy invariants*; see e.g. [13, 15, 16, 27, 29, 37, 38, 49] for details.

3.2 Geometry Selection Principle

Remark 1 We pick one eigenvector to correspond to a side vector of the (possibly degenerate) triangle. And thus to a side of the triangle. Under the most Geometrically-natural assumption of equal point masses, orthonormality then forces the last eigenvector to be the corresponding median vector. Since sides and medians are significant notions in Geometry, this choice of basis is privileged.

Remark 2 There is furthermore a residual basis-choice freedom as regards which of the 3 sides to use. Geometrically corresponds to the threefold cycling manifested by assigning distinguishable labels to the triangle's vertices. Indistinguishability of such then realizes an $A_3 = C_3$ permutation group symmetry.

3.3 Attaining Mechanics–Geometry compatibility

Remark 1 The previous two Subsections can be rendered equivalent.

Remark 2 On the one hand, our Mechanics account makes no reference to particle masses. On the other hand, our Geometry account amounts to specializing to equal point masses.

Lesser unification The Mechanics account can be restricted to the equal masses.

Greater unification The Geometry account's median can be generalized to a Cevian. Which has the effect of incorporating arbitrary point masses [60].

Remark 3 The minimal model for $N = 3$'s democracy transformations underlying the democracy invariants is again C_3 . So Mechanics' 'democracy' is Geometry's 'cycle'.

Remark 4 Finally, if mirror images are inequivalent, a further group generator is picked up. It is in this way that the full permutation group of the 3 labels, S_3 , comes to be realized. Whether or not mirror images are inequivalent is a modelling ambiguity.

3.4 Group- and Representation-Theoretic principles

Remark 1 CoMs enter via $\boldsymbol{L}$ admitting a natural action on the position coordinates. The resulting eigenbasis is then shared by $\boldsymbol{F}$. For all that the most natural action of this is on $(\text{sides})^2$ vectors. This is not however a discrepancy, given that in Group Theory, natural actions on *whichever aspects of* the objects under study is a source of insight.[1] It is also straightforward to see that the position coordinates can be taken to be part of the theory of the triangle.

Remark 2 Furthermore, the position coordinates admit natural 2-body problem subsystems. The above Physical and Geometrical arguments can be contemplated for one of these as well. Now a single separation vector suffices. And the label permutation symmetry is $S_2 = C_2$.[2]

Remark 3 The $N = 3$ CoM position vector can be envisaged as arising by padding its $N = 2$ counterpart with a 1 and then renormalizing. This corresponds to passing from a 2-variable symmetric homogeneous-linear function to a 3-variable one. And furthermore, via this group action on functions, to *inducing* a matrix member of a S_3 representation from one of S_2 .

Remark 4 The $N = 3$ side vector can similarly be envisaged as arising by padding its *fixed* $N = 2$ counterpart with a 0 . This corresponds to passing from a 2-variable pairwise-antisymmetric homogeneous-linear function to a subcase of its 3-variable counterpart. And so to *inducing* another matrix member of a representation.

Remark 5 Along these lines, picking an eigenbasis adapted to a 'hierachy of subsystems' comes to be recognized as a subcase of one of Representation Theory's most powerful constructs. Namely, the *Method of Induced Representations* [43].

Remark 6 We have now done enough to justify the current Article's shared eigenbasis. By inducing from $N = 2$'s fixed eigenbasis and then applying orthonormal completion. See [73, 72] for the next smallest explicit examples ($N = 4, 5, 6$) of the current Subsection's method.

Remark 7 Why are CoM notions applicable to Geometry, Statistics and yet further settings? It is but a first-moment concept. And is realized whenever one's model has Affine structure [23, 10, 34].

[1] When Dr Jan Saxl taught me, he often emphasized this point, for which I am very grateful.
[2] There is now no democracy, nor sides-cycle, though there is still a vertices 2-cycle.

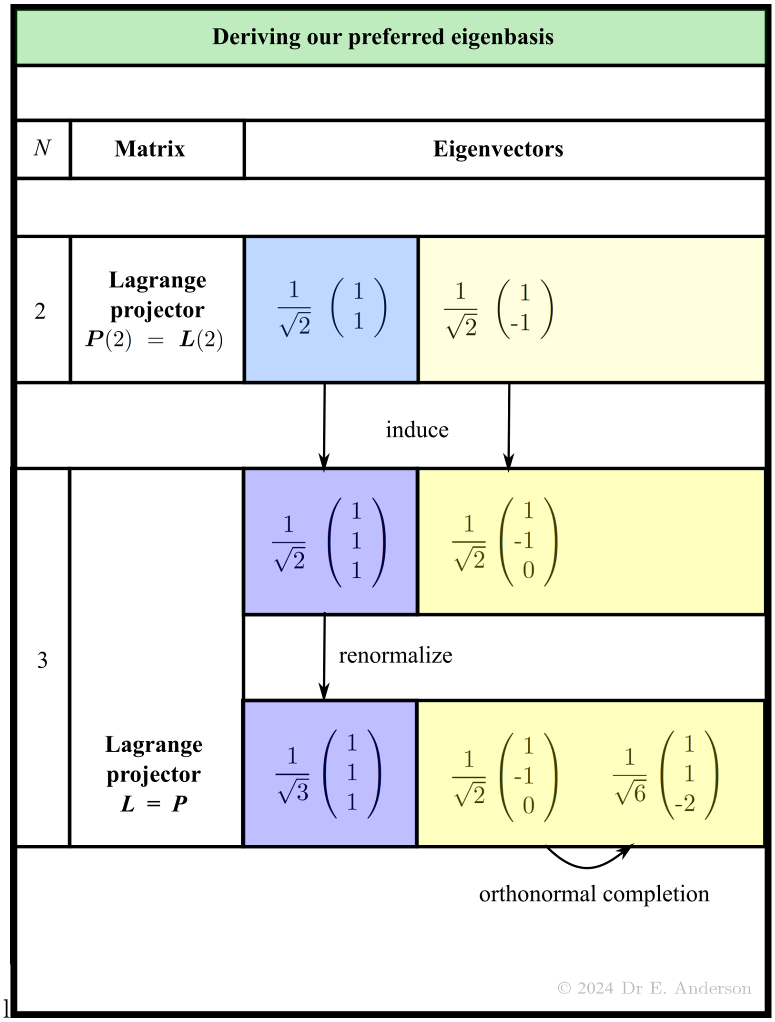
Deriving our preferred eigenbasis
N
Matrix
Eigenvectors
2
Lagrange projector
$\boldsymbol{P}(2) = \boldsymbol{L}(2)$
$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$
induce
3
$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$
renormalize
Lagrange projector
$\boldsymbol{L} = \boldsymbol{P}$
$\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$
$\frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$
orthonormal completion
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Figure 4:

4 Commutation relations

Proposition 1 a).i)

$$[\,\boldsymbol{O}\,,\,\boldsymbol{L}\,] \;=\; \boldsymbol{0}\;. \tag{12}$$

ii)

$$[\,\boldsymbol{L}\,,\,\boldsymbol{F}\,] \;=\; \boldsymbol{0}\;. \tag{13}$$

iii) [51]

$$[\,\boldsymbol{O}\,,\,\boldsymbol{F}\,] \;=\; \boldsymbol{0}\;. \tag{14}$$

b) More specifically,

$$\underline{\overline{\boldsymbol{L}}} \cdot \underline{\overline{\boldsymbol{O}}} \;=\; -\underline{\overline{\boldsymbol{L}}} \;=\; \underline{\overline{\boldsymbol{O}}} \cdot \underline{\overline{\boldsymbol{L}}}\;. \tag{15}$$

$$\underline{\overline{\boldsymbol{L}}} \cdot \underline{\overline{\boldsymbol{F}}} \;=\; -2\,\underline{\overline{\boldsymbol{L}}} \;=\; \underline{\overline{\boldsymbol{F}}} \cdot \underline{\overline{\boldsymbol{L}}}\;. \tag{16}$$

$$\underline{\overline{\boldsymbol{O}}} \cdot \underline{\overline{\boldsymbol{F}}} \;=\; \underline{\overline{\boldsymbol{Q}}} \;=\; \underline{\overline{\boldsymbol{F}}} \cdot \underline{\overline{\boldsymbol{O}}}\;. \tag{17}$$

Where

$$\boldsymbol{Q} \;:=\; \frac{1}{3}\begin{pmatrix} 5 & -1 & -1 \\ -1 & 5 & -1 \\ -1 & -1 & 5 \end{pmatrix}\;. \tag{18}$$

<u>Proof</u> See Fig 5. □

Establishing triangle matrices' zero commutators

$$\underline{\overline{\boldsymbol{L}}} \cdot \underline{\overline{\boldsymbol{O}}} = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} = \underline{\overline{\boldsymbol{O}}} \cdot \underline{\overline{\boldsymbol{L}}}$$

$$= -\underline{\overline{\boldsymbol{L}}}$$

$$\underline{\overline{\boldsymbol{L}}} \cdot \underline{\overline{\boldsymbol{F}}} = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} = \underline{\overline{\boldsymbol{F}}} \cdot \underline{\overline{\boldsymbol{L}}}$$

$$= -2\,\underline{\overline{\boldsymbol{L}}}$$

$$\underline{\overline{\boldsymbol{O}}} \cdot \underline{\overline{\boldsymbol{F}}} = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 5 & -1 & -1 \\ -1 & 5 & -1 \\ -1 & -1 & 5 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} = \underline{\overline{\boldsymbol{F}}} \cdot \underline{\overline{\boldsymbol{O}}}$$

$$= \underline{\overline{\boldsymbol{Q}}}$$

Figure 5:

5 Further interplays

5.1 Emergent medians and their own Heron-type formula

Interplay 1 The Lagrange matrix returns eigenclustering vectors alias relative Jacobi vectors $\boldsymbol{\rho}_i$, $i = 1$ to n . For $N = 3$ with equal masses, medians drop out of the calculation (inducing and then orthonormally completing). As $\boldsymbol{\rho}_2$ while the corresponding side vector arises as $\boldsymbol{\rho}_1$. This suggests treating sides and medians co-primarily: an eigenclustering perspective. And leads us to investigate the transformations taking sides to medians and vice versa. We then find there is an involution $\boldsymbol{J}$ between median- and side-lengths.

Interplay 2 The involution property (11) and the commutation relation (14) – now in $\boldsymbol{J}$ notation – combine to give a new proof [51, 59] of the following. The medians-data counterpart [6, 24, 33, 40, 51, 74] of Heron's formula.

For

$$T^2 = \underline{\boldsymbol{S}}^{\mathrm{T}} \cdot \underline{\overline{\boldsymbol{F}}} \cdot \overline{\boldsymbol{S}} = \left(\frac{4}{3} \underline{\overline{\boldsymbol{J}}} \cdot \overline{\boldsymbol{M}} \right)^{\mathrm{T}} \cdot \underline{\overline{\boldsymbol{F}}} \cdot \left(\frac{4}{3} \underline{\overline{\boldsymbol{J}}} \cdot \overline{\boldsymbol{M}} \right)$$

$$= \left(\frac{4}{3}\right)^2 \underline{\boldsymbol{M}}^{\mathrm{T}} \cdot \underline{\overline{\boldsymbol{J}}}^{\mathrm{T}} \cdot \underline{\overline{\boldsymbol{F}}} \cdot \underline{\overline{\boldsymbol{J}}} \cdot \overline{\boldsymbol{M}} = \left(\frac{4}{3}\right)^2 \underline{\boldsymbol{M}}^{\mathrm{T}} \cdot \underline{\overline{\boldsymbol{J}}} \cdot \underline{\overline{\boldsymbol{F}}} \cdot \underline{\overline{\boldsymbol{J}}} \cdot \overline{\boldsymbol{M}} = \left(\frac{4}{3}\right)^2 \underline{\boldsymbol{M}}^{\mathrm{T}} \cdot \underline{\overline{\boldsymbol{J}}} \cdot \underline{\overline{\boldsymbol{J}}} \cdot \underline{\overline{\boldsymbol{F}}} \cdot \overline{\boldsymbol{M}}$$

$$= \left(\frac{4}{3}\right)^2 \underline{\boldsymbol{M}}^{\mathrm{T}} \cdot \underline{\overline{\mathbb{I}}} \cdot \underline{\overline{\boldsymbol{F}}} \cdot \overline{\boldsymbol{M}} = \left(\frac{4}{3}\right)^2 \underline{\boldsymbol{M}}^{\mathrm{T}} \cdot \underline{\overline{\boldsymbol{F}}} \cdot \overline{\boldsymbol{M}} .$$

Finally work backwards parallelling the calculation obtaining the expanded Heron quadratic form from the square root formulation to obtain the following.

$$Area = \frac{4}{3} \sqrt{ m \prod_{\text{cycles}} (m - m_a) } = \frac{4}{3} \sqrt{ m \, (m - m_a) \, (m - m_b) \, (m - m_c) } . \quad (19)$$

5.2 Rediscovering Hopf's Little Mathematics

Structure 1
We normalize by division by the MoI I . Which, for equal masses, is numerically equivalent to division by the square of the RoG. A democratic formula for this for a triangle is [56, 49]

$$R = \frac{a^2 + b^2 + c^2}{3} \quad (20)$$

Article 1's J is also proportional to this.
The MoI I is also particularly cleanly expressible in terms of the ρ^i ,

$$I = \rho^2 = {\rho_1}^2 + {\rho_2}^2 . \quad (21)$$

Then also

$$\boldsymbol{\nu}_i := \frac{\boldsymbol{\rho}_i}{\rho} := \frac{\boldsymbol{\rho}_i}{\sqrt{I}} . \quad (22)$$

Interplay 3 Using our shared basis, Heron's quadratic form for the tetra-area T and its 3 eigenvectors can be mass-weighted and normalized to the following. 1 and the 3 *Hopf coordinates*

$$\alpha\text{niso} = 2 \underline{\boldsymbol{\nu}}_1 \cdot \overline{\boldsymbol{\nu}}_2 =: \text{hopf}_1 . \quad (23)$$

$$t = 2 (\underline{\boldsymbol{\nu}}_1 \times \overline{\boldsymbol{\nu}}_2)_3 =: \text{hopf}_2 . \quad (24)$$

$$\epsilon\text{llip} = ||\underline{\boldsymbol{\nu}}_1||^2 - \left|\left|\underline{\boldsymbol{\nu}_2}\right|\right|^2 =: \text{hopf}_3 . \quad (25)$$

In this way, the eigentheory of the Heron matrix (at least in context of being nested within relative space) suffices to derive the following [51, 53]. *Hopf's little map* [7]

$$H : \mathbb{S}^3 \longrightarrow \mathbb{S}^2 . \quad (26)$$

Exercise 4$^-$ Show that the above 3 Hopf quantities indeed square to give 1 . By which they implement the usual on-$\mathbb{S}^2$ condition.

Remark 1 Though we already obtained this on-sphere condition without evoking eigenclustering, in Article 1. This holding amounts to Smale's Little Theorem [17] that the space of triangles modulo similarities is topologically $\mathbb{S}^2$.

Remark 2 For some versions of Hopf's little map [71], we need to adjoin [51, 49] unit-sphere and coning maps. Hopf's Little Map [7, 28, 31, 41, 47] is rather ubiquitous in Geometry and Physics [20, 8, 9, 11, 18, 29, 38, 42, 71].

Remark 3 The above derivation of Hopf's Little Map follows from evoking the a single common eigenbasis of eigenvectors that is adapted to both $\boldsymbol{P}$ and $\boldsymbol{F}$ at once. Namely, the eigenbasis argued for in Sec 3. In this sense, the working is a double diagonalization. The above 'nested in relative space' comment is implemented by the other matrix involved – $\boldsymbol{L}$ – being the projector onto this from constellation space.

Remark 4 In the process, the following further Geometrical interpretations for the eigenvectors drops out [51, 50]. Anisosceleness is equivalently a measure of the extent to which the triangle's median is left- or right-leaning. While ellipticity has as a truer name '*departure from equilateral proportion*'. With reference to its base-to-median length ratio.

5.3 Rediscovering Kendall's Little Theorem

Interplay 4 Taking the associated Hopf little fibration [31, 35] and *projecting down the fibres* [28] returns Kendall's Little Theorem [26, 39]. That the space of triangles modulo similarities is additionally a sphere at the level of Metric Geometry. We cover this in detail in [71, 74]. Alternative derivations involving the chordal metric, or either Geometric or Mechanical reduction, can be found in e.g. [39, 44, 46, 53].

5.4 And back to medians

Interplay 5 The sides-medians involution's symmetry explains why medians' eigenvectors include their own version of an anisoscelesness and of an ellipticity [58].

Interplay 6 And why medians-as-data exhibit a close analogue of Heron's formula. This and Heron's formula differ by an overall numerical factor of $4/3$. Which [51] identified as the ratio of eigenclustering (i.e. relative Jacobi) masses. By which the mass-weighted relative Jacobi coordinates version of Heron and medians-Heron end up identical.

No deeper explanation was previously offered as to why to combine Jacobi coordinates with the Heron matrix and working in terms of medians. But we have now shown that a common eigenbasis can be used for all of $\boldsymbol{F}$, $\boldsymbol{L}$ and $\boldsymbol{J}$. Eigenclustering coordinates then both constitute a natural choice, and point us toward treating medians and sides as co-primary. Now not only because sides and medians enter the equal-masses Jacobi coordinates on the same footing. But also because the ensuing standard sides- and medians-Heron's formulae take identical mathematical form in Jacobi coordinates.

Interplay 7 We can also obtain the Hopf map from the medians Heron formula.

End Note 1 This article has covered a number of ways in which the three matrices $\boldsymbol{F}$, $\boldsymbol{L}$ and $\boldsymbol{J}$ interact to form a partial theory of triangles. This is continued in Article 3 at the level of Algebra.

Endnote 2 How special the sides–medians transformation and consequent medians–Heron formula are is being considered elsewhere [67, 59, 60, 61]. In the process, $\boldsymbol{M}(N)$ is found to be a wrong guess beyond $N = 3$, becoming a poset rather than a chain [62, 63, 73]. Nor is $\boldsymbol{N}(N)$ realized beyond $N = 3$. Now with $\boldsymbol{F} = \boldsymbol{N}(3)$'s citizen of Kallista accolades [55, 57, 71] getting split up between a number of different quadrilateral and tetrahaedron objects [53, 64, 66, 65, 68, 69, 70]. In contrast, $\boldsymbol{L}(N) = \boldsymbol{P}(N)$ is a 'stable anchor rope for the material N-body problems' [52, 49]. In that it realizes a chain of projectors $\forall \, \boldsymbol{N} \in \mathbb{N}$.

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