Eigentheory of Combinatorial Matrices: a More General Setting for the CoM-Relative Split

E. Anderson*

Abstract

Following Ford, I use algebraic properties of general sets of Combinatorial matrices so as to account for properties previously observed for 3 matrices that encode many properties of triangles. In the generic case, there is a 1-d eigenspace and its complement. A split eigenbasis for which is given by the unit vector, one interpretation of which is centre of mass (CoM) position, and some basis of relative differences. The equal-mass generalized Jacobi K-vectors are then picked out as eigenbases for the latter by the induced representation method. More generally, given any set of compatible Combinatorial matrices, their eigenbases can be taken to be shared. And, to the exclusion of the zero matrix, their eigenspaces are shared as well.

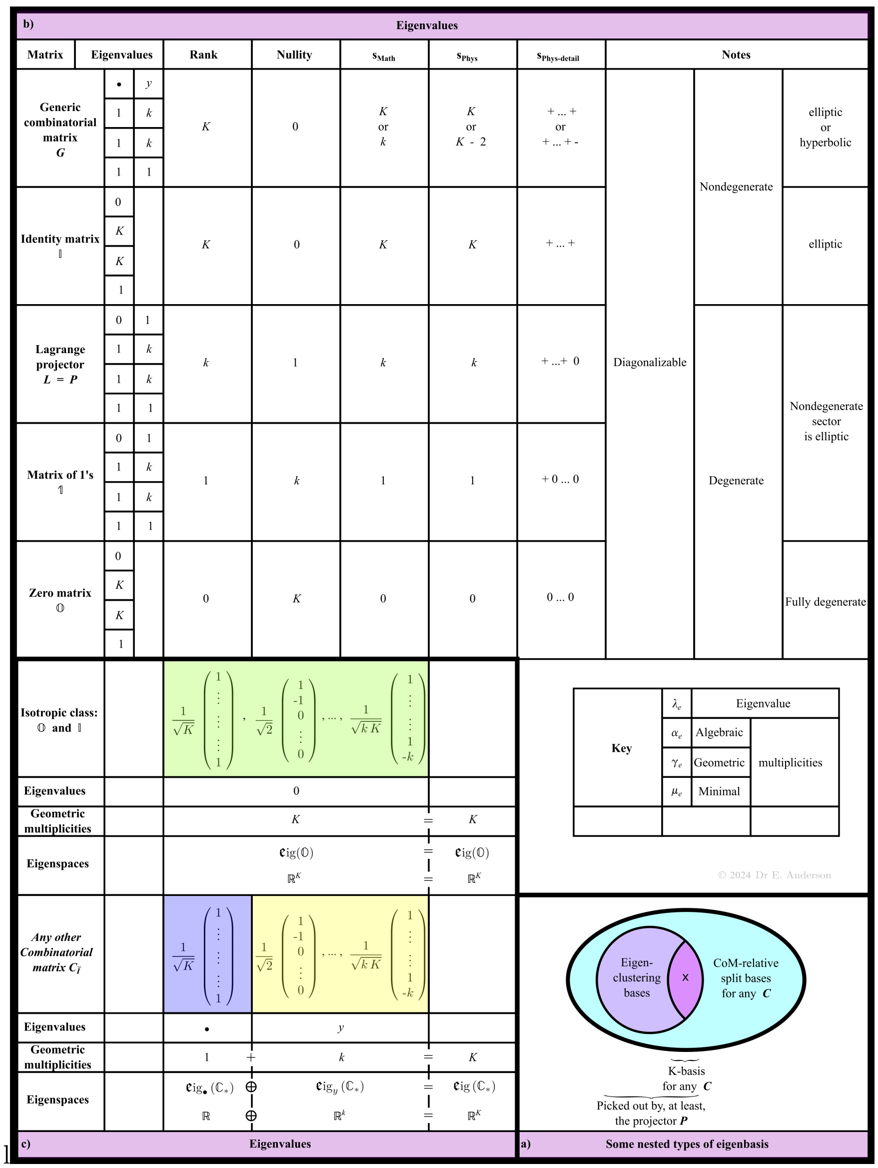

Figure 1:

* Dr.E.Anderson.Maths.Physics *at* protonmail.com . Institute for the Theory of STEM.

Date stamp: v1 25-10-2024.

1 Introduction

Definition 1 A *Combinatorial matrix* [1] is a square matrix of the following form.

$$\begin{pmatrix} x + y & x & \dots & x \\ x & & & \vdots \\ \vdots & & & x \\ x & \dots & x & x + y \end{pmatrix} = y \mathbb{I} + x \mathbb{1} . \quad (1)$$

For our purposes, considering $x, y \in \mathbb{R}$ suffices.

Remark 1 A. Ford [14] has very recently demonstrated that various properties that I attributed to [5, 7, 9, 10] the 3 triangle matrices follow purely because these all happen to be Combinatorial matrices.

Naming and Notation Remark 1 With reference to [14]'s definitions, I term the notation

$$(x + y , \ x)_K \quad (2)$$

for 1 *Ford's symbol of the zeroth kind.* This is a Wheelerian use of zeroth for 'most fundamental'. For this symbol corresponds to the Representation-Theoretically privileged use of irreducibles.

I also introduce the name *Ford's symbol of the first kind* for the incipient-block square-brackets version. And *Ford's symbol of the minus-oneth kind* for the trace-reversed block's angled-bracket version. Since this reversal is a reflection of the first about the zeroth. I have also chosen to reverse Ford's allocation of round and square brackets. So that the block is square (and angular like its reflection).

Irreducible, *block* and *trace-reversed* are respectively truer names.

Structure 1 The 3 triangle matrices are then as follows (dropping the K subscripts since they are all 3 's).

1) The (equal-masses) *Lagrange projector* alias *positions-to-relative separations matrix* [8],

$$\boldsymbol{P} := \frac{1}{3} (2, \text{-}1) . \quad (3)$$

2) The *Apollonius involutor* alias *sides-medians length-exchange involutor* [5].

$$\boldsymbol{J} := \frac{1}{3} (\text{-}1, 2) . \quad (4)$$

3) The *Heron matrix* [3], alias *sides-data triangle area formula matrix.* And subsequently alias *fundamental triangle matrix* $\boldsymbol{F}$ out of its also occurring in many other places in the theory of the triangle. Which is

$$\boldsymbol{F} := (\text{-}1, 1) . \quad (5)$$

Remark 2 In particular, [14] showed that many Commutator- and Abstract-Algebra of results for triangle matrices just follow from their being a set of Combinatorial matrices. The current note complements this by revisiting their basic Linear Algebra to similar effect.

2 Combinatorial matrices' eigenspectrum

Proposition 1 G) A Combinatorial matrix's eigenspectrum consists of the following.

$$\bullet \; := \; K \, x \; + \; y \;\; \text{with algebraic multiplicity} \;\; 1 \; , \tag{6}$$

$$y \;\; \text{with algebraic multiplicity} \;\; k \; := \; K \; - \; 1 \; , \tag{7}$$

Unless the Combinatorial matrix is of one of the 3 following exceptional kinds.

0)

$$x \; = \; 0 \; = \; y \; , \tag{8}$$

for which the sole eigenvalue is

$$0 \;\; \text{with algebraic multiplicity} \;\; K \; . \tag{9}$$

1)

$$y \; = \; 0 \; \neq \; x \; , \tag{10}$$

for which the sole eigenvalue is

$$y \;\; \text{with algebraic multiplicity} \;\; K \; . \tag{11}$$

U)

$$K \; = \; 0 \; , \tag{12}$$

for which there are no eigenvalues at all.

Remark 0 The generic case G) is proven in e.g. [6]. Without mention however of the 3 exceptional cases. Whose conceptual names are the *unmatrix* U , the *zero matrix* $\mathbb{O}$ and the *matrices proportional to the identity*. Which, collectively constitute the *isotropic matrices*. The cases 0) and 1) can for some purposes be jointly summarized by I)

$$k \, \mathbb{I} \; . \tag{13}$$

Though not as regards eigenspectra, since the $k \; = \; 0$ and $k \; \neq \; 0$ cases then differ as stated above. See also Fig 1.b) for further details.

The more often omitted (or even unimagined) case U) is isotropic since *all* directions being the same is satisfied when there are *no* directions. The unmatrix then realizes the uneigenspectrum: an incarnation of the empty set consisting of no eigenvalues.

Remark G.1 Also for $K \; = \; 1$, $k \; = \; 0$. One would then omit saying that y occurs with algebraic multiplicity 0 .

Remark G.2 For $K \; \geq \; 2$, we have a $k \, | \, 1$ partition of the underlying vector space by eigenvalue into the corresponding eigenspaces.

Remark G.3 $K \; = \; 3$ is minimum for this partition to be into a larger piece and a smaller piece (Fig 1.c). And thus to have a dimensionally-nontrivial eigenspace. These features clearly persist for all subsequent N .

Remark 2 The previous three remarks are all for case G). For case I), the eigenspace is unsplit (a 1-piece partition). While for case U), the eigenspace is the unpartition of the empty set.

3 Combinatorial matrices' eigenvectors

Proposition 2 With reference to a cover by some of the above-defined cases, a Combinatorial matrix's normalized eigenvectors take the following corresponding forms.

G) The normalized unit vector. Alongside any normalized basis choice of relative-difference vectors.

I) Any normalized basis for $\mathbb{R}^K$ will do.

U) The *uneigenbasis* consisting of an empty set's worth of eigenvectors.

Proof G). For the distinct ('principal') eigenvalue, the eigenvector equation is

$$(-k\,x\,,x\,)_K \cdot \boldsymbol{x} \;=\; \mathbf{0}\;. \tag{14}$$

Which is solved by

$$x_1 \;=\; ... \;=\; x_K \;=\; 1\;. \tag{15}$$

Finally divide by the corresponding normalization factor

$$\sqrt{\sum_{i\,=\,1}^{K} 1^2} \;=\; \sqrt{K}\;. \tag{16}$$

For the other eigenspace's eigenvalue,

$$x\,\mathbb{1} \cdot \boldsymbol{x} \;=\; \mathbf{0}\;. \tag{17}$$

Which is solved as claimed.

I) The eigenvector equation now reads

$$(\,0\,,0\,)_K \cdot \boldsymbol{x} \;=\; \mathbf{0}\;. \tag{18}$$

Which places no restrictions on what $\boldsymbol{x}$ can serve as an eigenvector.

U) Now there is no eigenvector equation, but no vectors to restrict either. The restriction of the empty set by the (empty set of equations) is of course just the empty set again. □

4 Some particular eigenbases

Structure 1 A particular set of eigenbases of relative-difference vectors is provided by the *relative Jacobi vectors* [2], subsequently alias *eigenclustering vectors* [11].

For $K \;=\; 1$, there is just an empty set of such.

While for $K \;\geq\; 2$, these are always nontrivially realized, and always carries a coordinate-label-choice multiplicity.

For $K \;\geq\; 4$, there is a network ambiguity as well, starting with the well-known 'Jacobi-K versus -H ' dichotomy [11].

Together with the normalized unit vector, each of these sets is enlarged to the corresponding (absolute) Jacobi vectors.

Remark 1 Hitherto, in the Mechanics literature, the Jacobi vectors have been taken to be eigenbases for the Lagrange matrix. Which exists for any N-body problem in any $\mathbb{R}^d$. In this setting, the normalized unit vector is the CoM position vector. And the relative differences are linear combinations of relative separations, whose arena is relative space [8].

Remark 2 In the case of the triangle, these eigenvectors were subsequently found to be shared by $\boldsymbol{J}$ and $\boldsymbol{F}$. My initial interpretation for this was that $\boldsymbol{P}$ brings in the Jacobi vectors. Which subsequently happen to be shared by $\boldsymbol{J}$ and $\boldsymbol{F}$. But it has now become clear that some aspects of $\boldsymbol{P}$-primality are a misconception, along the following lines.

Remark 3 Rather, *any Combinatorial matrix can be taken to have a CoM-relative split eigenbasis.*

Remark 4 Eigenclustering eigenbases are only a small subset of these (Fig 1.a).

Remark 5 There is furthermore a well-defined way of picking out specifically the generalized K-Jacobi vectors alias straight-path eigenclusters [11]. Namely that Ford's padding operation supports an induced representation construction. The padding operation extends size-K Combinatorial matrices to size-$(K + 1)$ such. While construction supported in turn pads the unnormalized vector of 1's with another 1. And the existing relative Jacobi vectors with one 0 each. The final eigenvector is then uniquely fixed up to sign by orthonormality.

This rests on the permutation group S_K acting on K objects for each K. And the subgroup inclusion

$$S_k < S_{k+1} = S_K \, . \tag{19}$$

The Lagrange projector version of the above argument recently featured in [9]. We thus arrive at the following.

5 Sharing eigenbases and eigenspaces

Proposition 3 Any Combinatorial matrix can be naturally equipped with a generalized-K eigenclustering eigenbasis.

Naming Remark 2 By which *combinatorial-matrix eigenvectors*, or for short *eigencombinatorial vectors*, is in turn a truer name than generalized-K eigenclustering vectors.

Proposition 4 Any generic set of compatible Combinatorial matrices can be taken to share eigenbasis. If none of them are isotropic, then they additionally share the underlying $k \,|\, 1$ partition into eigenspaces as labelled by eigenvalue (Fig 1.c).

Proposition 5 All combinatorial matrices are minimally-minimal. Meaning that their minimal polynomials are of the lowest-possible order.

Exercise 1 Prove this.

Corollary 1 Since they are a subcase of a set of compatible Combinatorial properties, the 3 triangle matrices enjoy Proposition 3 to 5's properties.

Open Question 1 For $\boldsymbol{P}$, the eigenclustering vectors are picked out by networks of subsystem CoMs. What does this interpretation generalize to if considered instead for an arbitrary generic Combinatorial matrix?

[This provides a sense in which $\boldsymbol{P}$-primality has not yet been demonstrated to be a misconcepetion!]

End Note *The following remain of merit* [12, 13, 15, 17]. *Which sets of compatible quadrilateral matrices commute, form multiplicative commutative monoids, share eigenspaces and share eigenbases.* For here not all of the matrices in any of these sets considered are Combinatorial.

Acknowledgments I thank A. Ford, K. Everard and S. Sánchez for discussions. And other participants at the "Linear Algebra of Quadrilaterals" Summer School 2024 at the Institute for the Theory of STEM. Also in respectful memory of Niall ó Murchadha and Jimmy York Jr. Discussions long past with whom about trace-reversed matrices subsequently translated into some of my discussions with A. Ford, by which these entered her work as well.

References

[1] D.E. Knuth, *The Art of Computer Programming* **1** *Fundamental Algorithms* (Addison-Wesley, 1968).

[2] C. Marchal, *Celestial Mechanics* (Elsevier, Tokyo 1990).

[3] R.H. Buchholz, "Perfect Pyramids", Bull. Austral. Math. Soc. **45** 353 (1992).

[4] R.G. Littlejohn and M. Reinsch, "Gauge Fields in the Separation of Rotations and Internal Motions in the N-Body Problem", Rev. Mod. Phys. **69** 213 (1997).

[5] E. Anderson, "Two New Perspectives on Heron's Formula", arXiv:1712.01441 For the updated version, see https://wordpress.com/page/conceptsofshape.space/1244 .

[6] J.B.S. Temones, "Combinatorial Matrices" J. Physics Conference Series **1593** 10th International Conference on Applied Physics and MathematicsAt: Tama Campus, Chuo University, Tokyo, Japan (2020).

[7] E. Anderson, "The Fundamental Triangle Matrix" (2024), https://wordpress.com/page/conceptsofshape.space/1306.

[8] "Lagrange Matrices: 3-Body Problem and General" (2024), https://wordpress.com/page/conceptsofshape.space/1436 .

[9] "The Fundamental Triangle Matrix Commutes with the Lagrange and Apollonius Matrices. With implications for deriving Kendall and Hopf Theorems." (2024), https://wordpress.com/page/conceptsofshape.space/1308 .

[10] "Only 2 of the Fundamental Triangle, Lagrange and Apollonius Matrices are Independent. With ensuing Algebras, Irreducibles and Splits" (2024), https://wordpress.com/page/conceptsofshape.space/1310 .

[11] E. Anderson and A. Ford, "Graph and Order Theory of N-Body Problem's Eigenclustering Networks", https://wordpress.com/page/conceptsofshape.space/2090 (2024).

[12] E. Anderson, "4-Body Problem: Ptolemy–Lagrange Algebra", (2024) https://wordpress.com/page/conceptsofshape.space/2121 .

[13] E. Anderson and K. Everard, "Linear Algebra of Cyclic Quadrilaterals: Ptolemy, Diagonal, Area, and Circumradius Formulae", https://wordpress.com/page/conceptsofshape.space/1971 (2024).

[14] A. Ford, "Triangle, but not most Quadrilateral, Matrices are Combinatorial" Preprint 2 on https://institute-theory-stem.org/geometry/

[15] E. Anderson, "Convex Quadrilaterals' Ptolemy–Lagrange–Bretschneider Algebras", forthcoming 2024, https://wordpress.com/page/conceptsofshape.space/2278 .

[16] "Hopf's Little Mathematics 32 times in Physics and Geometry", https://wordpress.com/page/conceptsofshape.space/1759 forthcoming 2024.

[17] "Linear-Algebraic reappraisal of modern Area Formulae for Quadrilaterals", forthcoming 2025.