4-Body Problem: Ptolemy–Lagrange Algebra

E. Anderson*

Abstract

We consider algebras generated by the following 4 4-body 4×4 matrices. The equal-masses Lagrange matrix $\boldsymbol{L}$, which can furthermore be viewed as a projector $\boldsymbol{P}$. And the 3-cycle of Ptolemy matrices $\boldsymbol{Pt}_o$, which are furthermore involutions, $\boldsymbol{J}_o$.

We observe that $\boldsymbol{P}$, $\boldsymbol{J}_o$ and the identity matrix are linearly dependent. That all of these commute. And that the plain matrix product algebra is a commutative monoid. Which is the direct product of the Ptolemy realization of the Klein 4-group and the single projector's minimum nontrivial commutative monoid. And that while a common eigenbasis can be found for $\boldsymbol{P}$ and any one of the $\boldsymbol{J}_o$, the eigenspaces do not coincide.

Our common eigenbasis requires specifically an H-eigenclustering network. Motivating us to collect further pros and cons for H versus K : the sole alternative supported by $N = 4$. We finally compare our results with those for the previously considered combination of the Lagrange projector and the Apollonius sides-to-medians-involution for 3 bodies.

* Dr.E.Anderson.Maths.Physics *at* protonmail.com . Institute for the Theory of STEM

1 Introduction

1.1 The Lagrange matrix

The 4-body *equal-masses Lagrange matrix* [7, 24, 42] is [36]

$$\boldsymbol{L} = \frac{1}{4}\begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix} . \tag{1}$$

This can also be viewed as [42, 36] a *projection* alias *projection operator* shortening to *projector* $\boldsymbol{P}$. Corresponding to the condition

$$\boldsymbol{P}^2 = \boldsymbol{P} \tag{2}$$

holding. Specifically, it is the projector from the *constellation space*

$$\mathfrak{q}(d, 4) = \left(\mathbb{R}^2\right)^d = \mathbb{R}^{2d} . \tag{3}$$

To the *relative space*

$$\mathfrak{R}\text{elative}(d, 4) = \left(\mathbb{R}^2\right)^d = \mathbb{R}^{2d} . \tag{4}$$

For d the spatial dimension.

With action sending 4 position d-vectors to 3 linearly-independent relative-separation d-vectors. By which *position-separations matrix* is another, in some sense truer, name for it.

A much wider audience will have come across projections of this kind in the context of projecting out the centre of mass position from the N-point-or-particle configurations.

$\boldsymbol{P}$ furthermore admits an action on the quadrilateral's side-length vector

$$\boldsymbol{s} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} . \tag{5}$$

1.2 The Ptolemy sides matrices

The *first Ptolemy sides matrix* is [52]

$$\boldsymbol{Pt}_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \left(\begin{array}{c|c} \mathbb{0} & \mathbb{I} \\ \hline \mathbb{I} & \mathbb{0} \end{array} \right) . \tag{6}$$

In the sides-diagonals split formulation [53] of Ptolemy's First Theorem and inequality [2, 17, 18, 29, 20, 23, 25], this matrix encodes the sides piece. It is presented in the same *sides basis* as (5) for $\boldsymbol{P}$. With use of 2×2 blocks in its last expression, where $\mathbb{I}$ denotes identity matrix.

The following *opposite side-pairs basis* is more convenient for it.

$$\boldsymbol{s} = \begin{pmatrix} a \\ c \\ b \\ d \end{pmatrix} . \tag{7}$$

For here it takes rather the following form [53].

$$\boldsymbol{Pt}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \left(\begin{array}{c|c} \mathcal{T} & \mathbb{0} \\ \hline \mathbb{0} & \mathcal{T} \end{array} \right) . \tag{8}$$

Where

$$\mathcal{T} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : \tag{9}$$

the sole *transposition matrix* supported by the 2-object set. Which is more convenient since it is now blockwise-diagonal with all blocks of size 2 , trivializing the elucidation of its eigentheory.

The *second and third Ptolemy sides matrices* are the following 3-cycles [53] of (6).

$$\boldsymbol{Pt}_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \left(\begin{array}{c|c} \mathbb{0} & \mathcal{T} \\ \hline \mathcal{T} & \mathbb{0} \end{array} \right) . \tag{10}$$

$$\boldsymbol{Pt}_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \left(\begin{array}{c|c} \mathcal{T} & \mathbb{0} \\ \hline \mathbb{0} & \mathcal{T} \end{array} \right) . \tag{11}$$

These encode [53] the numerator and denominator of the sides piece in Ptolemy's second Theorem [14, 15, 32]. Observe that the above basis change switches $\boldsymbol{Pt}_1$ and $\boldsymbol{Pt}_3$'s explicit forms while leaving $\boldsymbol{Pt}_2$ invariant. Let us jointly denote these 3 matrices by $\boldsymbol{Pt}_o$.

All Ptolemy matrices are *involutions*. Meaning that they obey

$$\boldsymbol{J}^2 \;=\; \mathbb{I} \,. \tag{12}$$

While not being the identity matrix $\mathbb{I}$ itself. We thus reissue $\boldsymbol{J}_o$ as notation for the $\boldsymbol{Pt}_o$.

We also include the identity among our matrices for the following reasons.

Firstly, it features in the definition of involution.

Secondly, it has the property of commuting with all compatible-sized matrices.

For some purposes, we view this as $\boldsymbol{Pt}_0$, indexing all 4 together by $\boldsymbol{Pt}_\omega$. We also permit the notations $\boldsymbol{J}_0$ and $\boldsymbol{J}_\omega$ by not strictly enforcing the second line in the above standard definition of involution.

A truer name for Ptolemy-sides matrices is *sides-pair-transposition* matrices. Meaning selecting a pair thus fixing the other, and then transposing each.

1.3 Outline

In Sec 2, we show that $\boldsymbol{P}$ and $\boldsymbol{J}_\omega$ are actually an LD (linearly dependent) set. That all of these commute. That while a common eigenbasis can be found for $\boldsymbol{P}$ and any one of the $\boldsymbol{J}_o$, the eigenspaces do not coincide. And that the plain matrix product algebra is a commutative monoid [28, 31]. Which is the direct product of the Ptolemy realization of the Klein 4-group and the single projector's minimum nontrivial commutative monoid.

In Sec 3.1 these results are compared with those for the 2 3-body 3×3 matrices $\boldsymbol{P}$ and $\boldsymbol{J}$. For $\boldsymbol{P}$ the corresponding Lagrange projector. And $\boldsymbol{J}$ the Apollonius involution [35, 43, 44] between sides and medians. Encoding the cycle of [23, 1] Apollonius Median-Length Theorems [1, 34, 12, 41]. Which matrices, and relations, are provided for ease of comparison in Appendix A.

Our common eigenbasis result requires specifically an H-*eigenclustering network*. Motivating us to collect in Sec 3.2 pros and cons for this and over the sole alternative supported by $N = 4$: the K-*eigenclustering network*. Eigenclusterings have hitherto often been referred to as *(relative) Jacobi coordinates* [10, 21, 42, 36, 45, 60].

2 Ptolemy–Lagrange Algebra

2.1 LD relation

Remark 1 A priori, our matrices are all 4×4 and symmetric. And thus belong to an arena $\mathfrak{S}\mbox{ym}_4$ with 10 degrees of freedom (d.o.f.).

Remark 2 The Lagrange matrix can however be reformulated as follows.

$$\boldsymbol{L} \;=\; \left(\frac{3\,\mathbb{I} \;-\; \mathcal{T} \;|\; \text{-}\mathbb{I} \;-\; \mathcal{T}}{\text{-}\mathbb{I} \;-\; \mathcal{T} \;|\; 3\,\mathbb{I} \;-\; \mathcal{T}}\right) \; . \tag{13}$$

So more specifically our matrices are all of the form

$$\boldsymbol{M} \;=\; \left(\frac{\alpha\,\mathbb{I} \;+\; \beta\,\mathcal{T} \;|\; \gamma\,\mathbb{I} \;+\; \delta\,\mathcal{T}}{\bullet \qquad |\; \epsilon\,\mathbb{I} \;+\; \phi\,\mathcal{T}}\right) \; . \tag{14}$$

The arena $\mathfrak{m}$ of which has just 6 d.o.f.

Remark 3 5 matrices belonging to an arena with 6 d.o.f. are generically linearly independent (LI). Our particular 5 are however linearly-dependent (LD). This is since they obey the following linear relation.

Lemma 1

$$\boldsymbol{L} \;=\; \mathbb{I} \;-\; \langle \boldsymbol{Pt} \rangle \; . \tag{15}$$

In words,

(4-body Lagrange matrix) = (identity) − (the average Ptolemy matrix) .

With specific reference to the 4-average.

<u>Proof</u> By another rearrangement,

$$4\,\boldsymbol{L} \;=\; 4\,\mathbb{I} \;-\; \mathbb{1} \; . \tag{16}$$

Where $\mathbb{1}$ is the matrix of 1 's.

But also adding up,

$$\sum_{\omega\,=\,0}^{3} \boldsymbol{Pt}_{\omega} \;=\; \mathbb{1} \; . \tag{17}$$

Eliminate $\mathbb{1}$ between these:

$$4\,(\,\mathbb{I} \;-\; \boldsymbol{L}\,) \;=\; \sum_{\omega\,=\,0}^{3} \boldsymbol{Pt}_{\omega} \; . \tag{18}$$

Divide by 4 and finally evoke the definition of average. □

2.2 Zero-commutator algebras formed

Proposition 1 i) [53]

$$[\,\boldsymbol{Pt}_{o}\,,\, \boldsymbol{Pt}_{o'}\,] \;=\; 0 \; . \tag{19}$$

ii)

$$[\,\boldsymbol{Pt}_{o}\,,\, \boldsymbol{L}\,] \;=\; 0 \; . \tag{20}$$

Exercise 1$^-$ Prove this directly by evaluating the matrix commutators.

Proof 2 of ii). Sub Lemma 1 in to obtain

$$[\,\boldsymbol{Pt}_o\,,\,\mathbb{I}\,-\,\langle\boldsymbol{Pt}\rangle\,] \;=\; [\,\boldsymbol{Pt}_o\,,\,\mathbb{I}\,] \,-\, \frac{1}{4}\sum_{o\,=\,1}^{3}[\,\boldsymbol{Pt}_o\,,\,\boldsymbol{Pt}_{o'}\,] \;=\; 0\,-\,0\,=\,0\,. \tag{21}$$

Where the second step is by the definition of average and use of linearity. While the third step uses Proposition 1 alongside that everything commutes with the identity. □

Remark 1 ii) is thus not an independent commutation relation. Thus i)'s discrete zero-commutator algebra of 3 Ptolemy matrices suffices for the joint study of the Lagrange and Ptolemy matrices. Though $\boldsymbol{L}$ and 1 or 2 of the $\boldsymbol{Pt}_o$ are more suitable for some applications.

Keeping 2 maintains the maximal algebra.

While dropping $\boldsymbol{Pt}_2$ and $\boldsymbol{Pt}_3$ corresponds to combining working on relative space with considering a single Ptolemy inequality.

In these ways, ii) does see some use.

Remark 2 So overall this Subsec yields 2- or 3-generator zero-commutator algebras. These can be viewed as very basic Lie algebras [19]: no nonzero structure constants. Of linear combinations (LCs) of our matrices. Subsec 2.7 provides a distinct interpretation.

2.3 Projection and involutions

Proposition 2 [53] All Ptolemy matrices are involutions.

Remark 1 [53] consider a separations and a diagonals Ptolemy matrix as well, so Proposition 2 covers 5 matrices.

Remark 2 Due to this, and to equal-masses Lagrange matrices being projectors, let us rewrite the previous Subsecs' equations in Algebraic rather than Geometric form.

$$[\,\boldsymbol{J}_o\,,\,\boldsymbol{J}_{o'}\,] \;=\; 0\,. \tag{22}$$

$$[\,\boldsymbol{J}_o\,,\,\boldsymbol{P}\,] \;=\; 0\,. \tag{23}$$

Which is not independent since

$$\boldsymbol{P} \;=\; \mathbb{I}\,-\,\langle\boldsymbol{J}\rangle\,. \tag{24}$$

Also

$$4\,(\,\mathbb{I}\,-\,\boldsymbol{P}\,) \;=\; \mathbb{1}\,. \tag{25}$$

And

$$\mathbb{1} \;=\; \sum_{\omega\,=\,0}^{3}\boldsymbol{J}_\omega \;=\; \mathbb{I}\,+\,\sum_{o\,=\,0}^{3}\boldsymbol{J}_o\,. \tag{26}$$

2.4 The Ptolemy–Lagrange Eigentheory

Remark 1 See Figs 1 and 3 for this. And e.g. [53, 61, 60] for the notions and conventions used.

Remark 2 Observe in particular that while shared eigenvectors can be found (for instance as exhibited), the eigenspace structure is not shared. For $\boldsymbol{P}$ has a $3 + 1$ split eigenspace. While the $\boldsymbol{J}_o$ have $2 + 2$ split eigenspaces. This involves taking the indicated LC to pass from row 4's simple sparse block-aligned eigenvectors to row 3's shared ones.

Algebraic properties of current Article's 4-body matrices

Matrix	Eigenvalues			Rank	Nullity	s_{Math}	s_{Phys}	$s_{Phys\text{-}detail}$	Notes			
Lagrange projection $\boldsymbol{L} = \boldsymbol{P}$	0	1		3	1	3	2	+ + +0	Diagonalizable	Degenerate, with elliptic nondegenerate sector	Projection	Commute with each other
	1	3										
	1	3										
	1	1										
Ptolemy sides involutions $\boldsymbol{Pt}_o = \boldsymbol{J}_o$	1	-1		4	0	2	0	+ + - -		Ultra-hyperbolic	Involutions	
	2	2										
	2	2										
	1	1										

Key			
	λ_e	Eigenvalue	
	α_e	Algebraic	multiplicities
	γ_e	Geometric	
	μ_e	Minimal	

Figure 1:

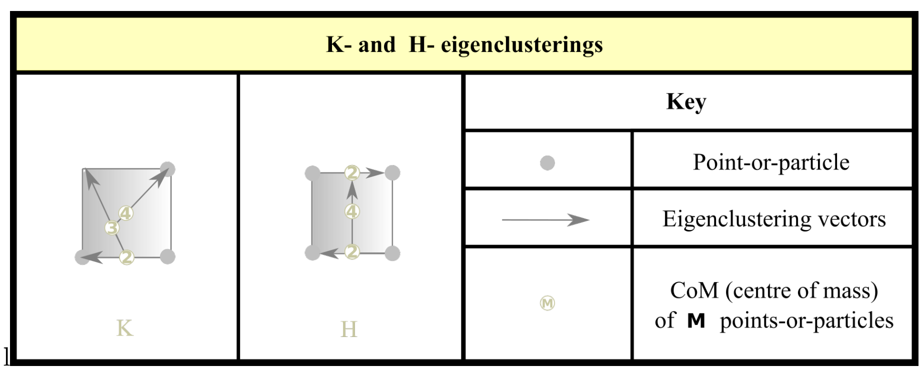

Figure 2:

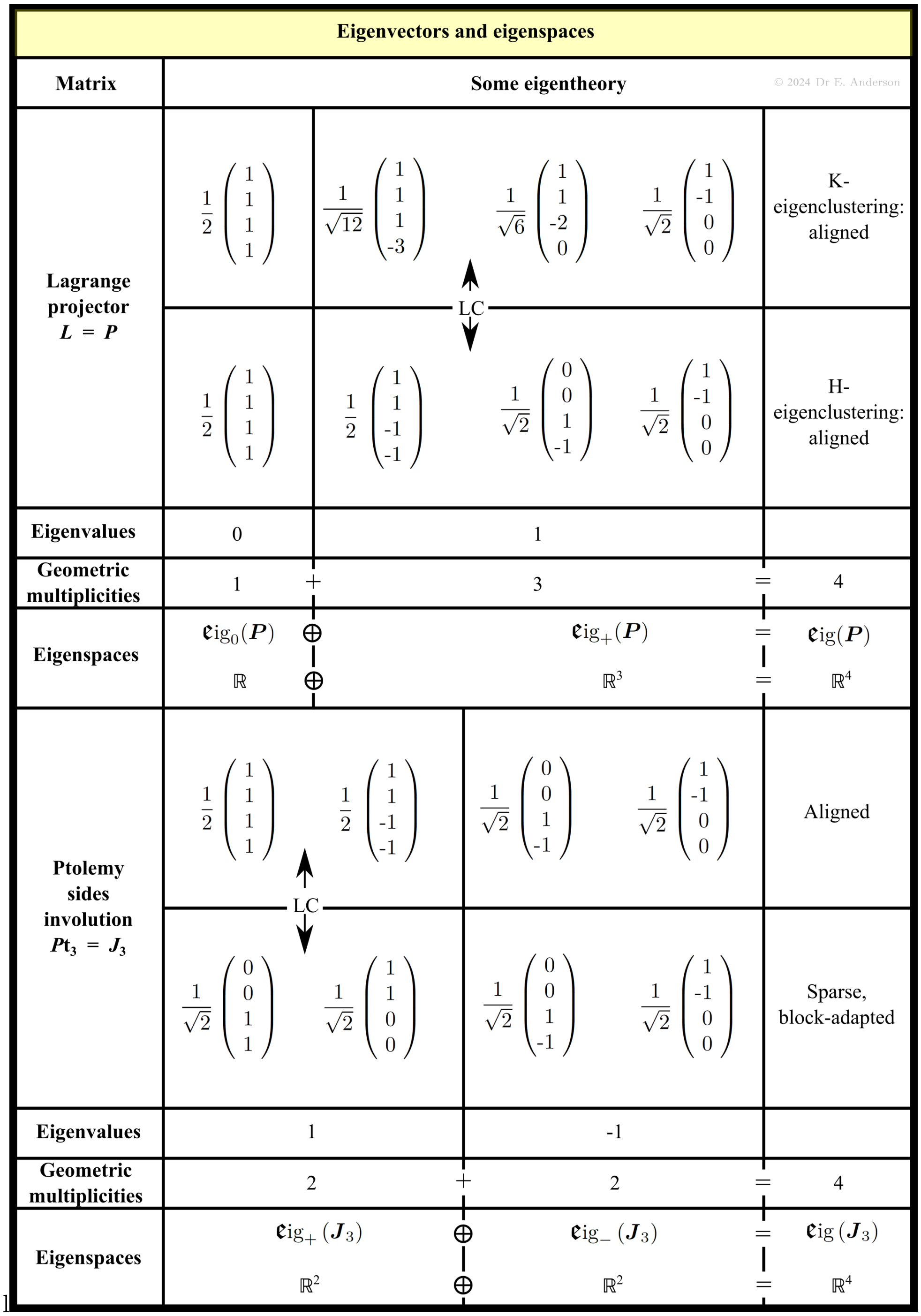

Figure 3:

Remark 3 Also observe that we have attained compatibility within the Lagrange matrix's H-eigenclustering network (Fig 2.b). $N = 4$'s sole alternative to this network – the K-eigenclustering of Figs 2.a) and the top row of Fig 3 – cannot accommodate such a compatibility. This is because a 2-d eigensubspace is involved in the transformation moving between H and K . While for the Ptolemy matrices, one of the 2 eigenvectors required for this is partitioned off from the other.

Exercise 2^- Work out both 2-way LCs in Fig 3.

2.5 Trace–tracefree decomposition

Structure 1 $\mathbb{1}$ admits the following irreducible split.

$$\mathbb{1} \;=\; \mathbb{I} \;+\; \mathbb{T} \;. \tag{27}$$

For *trace part* $\mathbb{I}$ and *tracefree part*

$$\mathbb{T} \;=\; \mathbb{1} \;-\; \mathbb{I} \;. \tag{28}$$

Proposition 3 i) Making the matrix of 1 's the subject,

$$\mathbb{1} \;=\; \frac{4}{3}\left(\boldsymbol{P} \;+\; \sum_{o\,=\,1}^{3} \boldsymbol{J}_o\right) \;. \tag{29}$$

ii) Making the trace irreducible the subject,

$$\mathbb{I} \;=\; \frac{1}{3}\,(\,4\,\boldsymbol{P} \;-\; \langle \boldsymbol{J} \rangle\,)\;\;. \tag{30}$$

iii) Concurrently making the tracefree irreducible and the Ptolemy-involutions-sum the subject,

$$\mathbb{T} \;=\; \sum_{o\,=\,1}^{3} \boldsymbol{J} \;. \tag{31}$$

iv) Making the Lagrange projector the subject,

$$\boldsymbol{P} \;=\; \mathbb{I} \;-\; \frac{1}{4}\mathbb{1} \;\;=\;\; \frac{1}{4}\,(\,3\,\mathbb{I} \;-\; \mathbb{T}\,)\;, \tag{32}$$

now in the sense of 3-average.

Remark 1 iii) is particularly nice as the isolation of an irreducible also isolating one of our Geometrical objects.

Exercise 3$^-$ Prove Proposition 3.

2.6 The Ptolemy–Lagrange multiplicative algebra

Remark 1 In considering plain matrix multiplication rather than commutator product, incorporation of $+$ is not obligatory. We can thus ask what algebra is generated by joint consideration of the Lagrange and Ptolemy-sides matrices.

Preliminary Structure 1 We already know how the $\boldsymbol{Pt}_o$ behave in isolation. Namely, Everard and I [53] showed that they form the Klein-4-group $V_4 \;=\; C_2 \;\times\; C_2$ (Fig 1.a). Incidentally, this means that only 2 of them are independent multiplicative generators. According to

$$\boldsymbol{J}_o \;\cdot\; \boldsymbol{J}_{o'} \;\;=\;\; \boldsymbol{J}_{o''} \;. \tag{33}$$

Where $o \;\neq\; o'$ and o'' is the remaining index value. It is furthermore entirely arbitrary which 2 are taken to be the generators.

Preliminary Structure 2 And how the Lagrange matrix generator behaves by itself [44]. The times table is now as per Fig 1.b). Groups have the property that each element features precisely once in any given row or column. Thus this times table immediately diagnoses that our algebra is not a group. All commutative group axioms hold bar the inverse property, signifying that we have a commutative monoid [28, 31]. The inverse property fails since nontrivial projections have nonzero kernel, and are thus not invertible. A consequence of this is that cancellability fails:

$$\boldsymbol{P}^2 \;=\; \boldsymbol{P} \quad\not\Rightarrow\quad \boldsymbol{P} \;=\; \mathbb{I} \;.$$

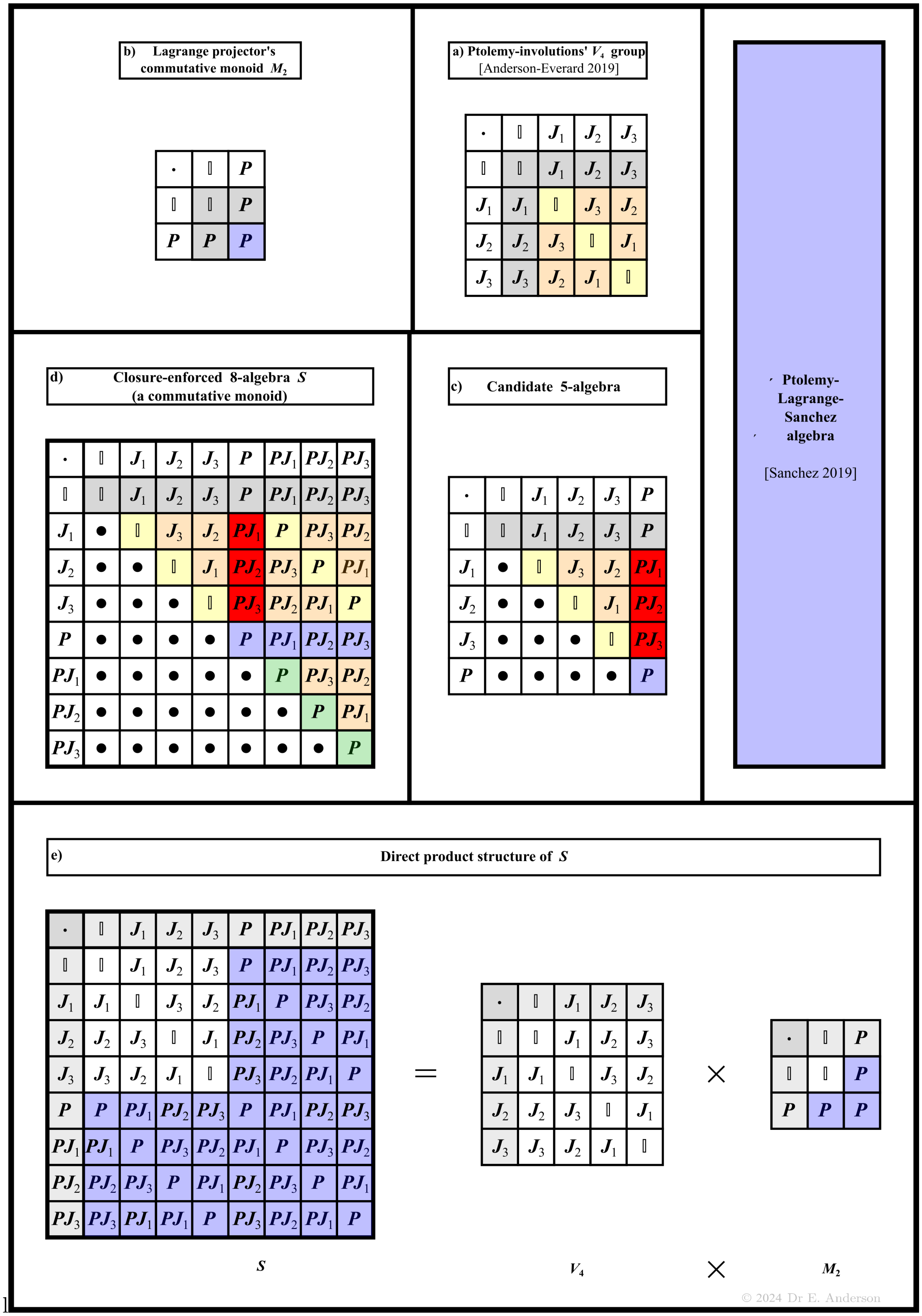

Figure 4:

More specifically, the identity and a single nontrivial projection form the smallest commutative monoid, M_2 .

Proposition 4 [Anderson–Sánchez 2019] i) $\boldsymbol{P}$ and $\boldsymbol{J}_o$ generate the 8-element commutative monoid S whose times table is exhibited in Fig 1.c).

ii) Furthermore,

$$S = V_4 \times M_2 = C_2 \times C_2 \times M_2 . \tag{34}$$

Proof 0) Commutativity $\Rightarrow$ the times table is symmetric about its lead diagonal.

1) The first row is always taken care of by the identity property.

2) The yellow and blue portions of the incipient algebra's diagonal (Fig 1.d) follow by the involution and projection properties respectively.

3) The orange wedge follows from (33).

4) The last column discovers 3 perms of $\boldsymbol{P}$ (red). This gives 3 new columns to work out.

5) Involution, (33) and projection deal respectively with the new yellow, orange and blue entries.

6)

$$\boldsymbol{P} \cdot \boldsymbol{J}_o \cdot \boldsymbol{P} \cdot \boldsymbol{J}_o = \boldsymbol{P}^2 \cdot {\boldsymbol{J}_o}^2 .$$

So finally projection and involution finish off the diagonal's extension (green).

This obeys all commutative group axioms bar the inverse property, and is thus a commutative monoid.[1]

ii) Reconceptualize our emergent consistent closure-enforced 8-algebra independently of how we obtained it. Ditch the bullet shorthand enabled by 0) since it is not useful *in seeking for product structures*. With the generators ordered as shown in Fig 4.d), quarter the table into squares (LHS of Subfig e). The blue square's entries are each $\boldsymbol{P} \cdot$ the white square's corresponding entries. Which is itself the above Ptolemy sides involution realization of $V_4 = C_2 \times C_2$. But this meets the definition of direct product algebra, via the shading in the RHS of Subfig e). □

Remark 2 Blue now indicates projector action. The fourth square's blue is then really double-blue $=$ blue by the projection property.

2.7 Ptolemy–Lagrange zero-commutator algebras revisited

Now that we are wearing the Abstract Algebra hat, another interpretation for these is as follows. They close because all products return the zero matrix $\mathbb{0}$. This is automatically included, since we are working with a $\mathbb{R}^2$ or $\mathbb{R}^3$ vector space of LCs of our matrices. For which $\mathbb{0}$ serves as zero.

They are commutative and associative. But do not enjoy the identity property, since commutation with zero returns zero rather than the original element. Nor are they cancellable. Since for

$$A \neq B , \ [A, C] = [B, C] \ \text{ fails to cancel down to } \ A = B . \tag{35}$$

Thus they constitute *commutative semigroups* [31] (**Proposition 5**).

[1] Given that $\boldsymbol{P}$ is among the incipient generators, this is as much as can be hoped for.

3 Scholium

3.1 Comparison with Apollonius–Lagrange algebras

Remark 1 The above generalize the commutator and plain-multiplication algebras [44] formed by the following. The 3-body 3×3 Lagrange projector $\boldsymbol{P}$ [42, 36] and a single Apollonius sides-medians involution $\boldsymbol{J}$ [35, 43, 44]. Which we outline in Appendix A for ease of comparison.

This is a more minimal example of a zero-commutator algebra, commutative semigroup, and commutative monoid. This exhibits no LD, so the zero-commutator algebra possesses 2 generators. Though the single projector is an even more minimal example of commutative monoid. Our example is however the minimal example of a nontrivial direct-product commutative monoid. In the sense of being the smallest direct product of two nontrivial factors. That carry between them the minimum amount of nontrivial commutative-monoid departure from a commutative group.

Remark 2 But the direct analogue of Apollonius's Theorem for 4 bodies is either Euler's Quadrilateral Theorem [5, 26, 27, 30, 32, 46] for the Newton line [4, 9] interval's length. Alias the crossbar length of the H-eigenclustering. Or its K- counterpart [47]. For $N = 4$ exhibits this eigenclustering network ambiguity.[2] And yet these give 3- and 6-matrices respectively [46, 47], which are not compatible with the Lagrange 4-matrix.

Remark 3 Instead, *Ptolemy's results surprisingly turn out to be able to take over the role of partnering the Lagrange projector with involution matrices.* These results are of a conceptual type for which $N = 4$ is minimum. For they concern diagonal information being controlled by side information. And $N = 4$ is clearly minimum for notions of diagonal to be supported. Or equivalently for separations and sides to be non-coincident notions, since some separations are now diagonals.

Pointer 1 For triangles, the *fundamental triangle matrix* $\boldsymbol{F}$ enters as a third 3×3 matrix. E.g. from Heron's formula [5, 22, 35], the cycle of cosine rules or the cycle of triangle inequalities [40]. *Now* the triangle's matrix theory picks up its own LD [44]:

$$\boldsymbol{F} = \boldsymbol{A} - \boldsymbol{L} = \boldsymbol{J} - \boldsymbol{P} . \tag{36}$$

With plenty of consequences (Appendix A.3, [44, 56]).

Analogously extending the 4-body Lagrange and Ptolemy-sides matrix algebra via the content of quadrilateral area formulae picks up some multiplicity [13, 38, 39, 59, 54].

Example 1 For Brahmagupta's cyclic-quadrilateral area formula [3, 18, 20, 23], a competing poset of zero-commutator algebras arises [53]. In the sense that the following 2 patches are incompatible.

Firstly, the Ptolemy vector of matrices (whether or not with 1 component exchanged for the Lagrange matrix).

Secondly, one Brahmagupta factor matrix and the unique Ptolemy matrix $\boldsymbol{Pt}_1$ that is commutator-compatible with it.

Example 2 For Bretschneider's second convex-quadrilateral area formula [11, 13, 16, 20, 23], full compatibility ensues [55].

Remark 2 The Geometrically-meaningful sum (or average) of the 3 Ptolemy sides matrices has Representation-Theoretic dual nationality as an irreducible Proposition 3.iii). $N = 3$ does not possess a directly analogous result.

When viewed as a difference, $\boldsymbol{F}$ does however enjoy Geometric, Algebraic and Representation-Theoretic triple nationality [44]. Namely, it is all of the difference of Geometry's Apollonius and Lagrange: the first

[2] See [8, 27, 33, 46, 47, 51, 48, 49, 50, 45, 57, 58] for development of truer names for these lengths, networks and Theorems.

form of (36). Of Algebra's involution and projector: the second form of (36). And of the 2 irreducibles by the last form in (48). See Sec 3.1, [44] and [56] for yet further routes to $\boldsymbol{F}$.

So on the one hand, $N = 3$ supports an exhalted relation, but only once its area formula has entered the fray.

On the other hand, $N = 4$'s involutions support one by themselves.

Remark 4 For the 3-body Apollonius–Lagrange matrix algebra, both eigenvectors and eigenspaces are shared (Fig 6). Thus the current Article's shared eigenvectors and yet not shared eigenspaces result is minimum at $N = 4$ for this effect to occur in the N-body and Flat Geometry contexts.

Pointer 2 There are further eigenvector alignment and eigenspace non-alignment effects among the above extensions by quadrilateral area formulae's matrices.

3.2 H- versus K-eigenclustering

Remark 1 It is well known that each of H- and K-eigenclustering models a different Physics, Geometry, Statistics... situation.

Namely, the H is adapted to double-binary configurations. Such as 2 planet-moon subsystems. Or 2 hydrogen atoms.

While the K is adapted to the nest: a binary within a triple subsystem. Such as a tight binary pair of stars with a more loosely-bound third star and finally the system's planet. Or the nuclei and then alongside the bound electron of the molecular hydrogen ion H_2^+ , and finally a loose electron.

Remark 2 Some further technical reasons to pick H or K arising from the current program are as follows.

Motivation 1 for H In studying the 4-body problem, the Ptolemy inequality plays a major role and the Ptolemy Theorem an occasional role. Then picking the H- rather than K-eigenclustering is preferable due to permitting a fully shared basis of eigenvectors. For K , at most the centre of mass – here alias perimeter – and one side-pair difference eigenvectors are shared: just a 50% success rate (Fig 3).

Motivation 2 for H Euler's Quadrilateral Theorem for the H-eigenclustering is nicer than its K-counterpart. For the H case exchanges crossbar length for separation-length information. While the K case only exchanges an aggregate of 2 stroke lengths – spike and handle [33, 47], for separation-length information.

Motivation 1 for K Every N has a K corresponding to the simplest tree graph available: the path P_{N-1} . Corresponding to adding points-or-particles one at a time. The H instead corresponds to starting by bringing in as many pairs as possible.

Motivation 2 for K One can work systematically with each N 's K for many purposes [48, 57]. But for odd N , a point-or-particle is left over after bringing in as many pairs as possible. Leading to even and odd cases of H often needing to be treated separately [45, 50].

Motivation 1 against H Forr some purposes, the H is not extremal; in particular it does not always pick out maximal symmetry [58].

Remark 3 In view of the previous three items, arbitrary-N H is thereby a less systematically tractable, less unique and less mathematically distinguished at the level of eigenclustering.

A Apollonius–Lagrange algebra

A.1 The matrices

Structure 1 The equal-masses 3-body 3×3 *Lagrange matrix* ([42, 36, 43, 44])

$$\boldsymbol{L} := \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} . \tag{37}$$

Which corresponds to placing an equal mass at each vertex of the triangle. This is again a projector of the type detailed in the Introduction, so we also denote it by $\boldsymbol{P}$.

Structure 2 The *Apollonius matrix* [35, 41, 43] is

$$\boldsymbol{O} := \frac{1}{4}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} . \tag{38}$$

This arises from passing from sides to medians by the cycle of [23, 1] Apollonius Median-Length Theorems [1, 34, 12, 41]. Rescaling to

$$\boldsymbol{A} := \frac{4}{3}\boldsymbol{O} , \tag{39}$$

we obtain an involution, which we thus additionally Algebraically denote by $\boldsymbol{J}$. Specifically

$$\boldsymbol{A} = \boldsymbol{J} = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} . \tag{40}$$

This is the *Apollonius* or *sides–medians involution.*

Proposition A [43]

$$[\,\boldsymbol{P}\,,\,\boldsymbol{J}\,] = 0 . \tag{41}$$

Which can be viewed as a trivial (commutative) Lie algebra on 2 generators.

A.2 Their eigentheory

This is provided in Figs 5 and 6 .

Algebraic properties of current Article's cyclic quadrilateral matrices

Matrix	Eigenvalues			Rank	Nullity	s_{Math}	s_{Phys}	$s_{Phys\text{-}detail}$	Notes			
Lagrange projector $\boldsymbol{L} = \boldsymbol{P}$	0	1		2	1	2	2	+ + 0	Diagonalizable	Degenerate with elliptic nondegenerate sector	Projection	Commute with each other
	1	2										
	1	2										
	1	1										
Apollonius sides-medians involution $\boldsymbol{J}$	1	-1		3	0	2	1	+ - -		Hyperbolic	Involution	
	1	2										
	1	2										
	1	1										

Figure 5:

Eigenvectors and eigenspaces			
Matrix	**Some eigentheory**		
Lagrange projector $\boldsymbol{L} = \boldsymbol{P}$	$\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}$ $\frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\-2\end{pmatrix}$	
Eigenvalues	0	1	
Geometric multiplicities	1	+ 2	= 3
Eigenspaces	$\mathfrak{E}\mathrm{ig}_0(\boldsymbol{P})$ $\mathbb{R}$	$\oplus$ $\mathfrak{E}\mathrm{ig}_+(\boldsymbol{P})$ $\oplus$ $\mathbb{R}^2$	$=$ $\mathfrak{E}\mathrm{ig}(\boldsymbol{P})$ $=$ $\mathbb{R}^3$
Apollonius sides-medians involution $\boldsymbol{J}$	$\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}$ $\frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\-2\end{pmatrix}$	
Eigenvalues	1	-1	
Geometric multiplicities	1	+ 2	= 3
Eigenspaces	$\mathfrak{E}\mathrm{ig}_+(\boldsymbol{J})$ $\mathbb{R}$	$\oplus$ $\mathfrak{E}\mathrm{ig}_-(\boldsymbol{J})$ $\oplus$ $\mathbb{R}^2$	$=$ $\mathfrak{E}\mathrm{ig}(\boldsymbol{J})$ $=$ $\mathbb{R}^3$

Figure 6:

A.3 Triangle matrices' block formulation

Proposition B i) Making the matrix of 1 's the subject,

$$\mathbb{1} = 3(\boldsymbol{J} + \boldsymbol{P}) . \tag{42}$$

ii) Making the trace irreducible the subject,

$$\mathbb{I} = \boldsymbol{J} + 2\boldsymbol{P} . \tag{43}$$

iii) Making the tracefree irreducible the subject,

$$\mathbb{T} = 2\boldsymbol{J} + \boldsymbol{P} . \tag{44}$$

iv) Making $\boldsymbol{J}$ the subject,

$$\boldsymbol{J} = \frac{2}{3}\mathbb{1} - \mathbb{I} = \frac{1}{3}(2\mathbb{T} - \mathbb{I}) . \tag{45}$$

v) Making $\boldsymbol{P}$ the subject,

$$\boldsymbol{P} = \mathbb{I} - \frac{1}{3}\mathbb{1} = \frac{1}{3}(2\mathbb{I} - \mathbb{T}) . \tag{46}$$

vi)

$$\boldsymbol{J} + \boldsymbol{P} = \frac{1}{3}\mathbb{1} = \frac{1}{3}(\mathbb{I} + \mathbb{T}) . \tag{47}$$

vii)

$$\boldsymbol{J} - \boldsymbol{P} = \mathbb{1} - 2\mathbb{I} = \mathbb{T} - \mathbb{I} . \tag{48}$$

Remark 1 These last two serve as adapted variables [44]. With the second of these moreover constituting one of the many routes to the fundamental triangle matrix $\boldsymbol{F}$; see Sec 3.1.

A.4 Their multiplicative algebra

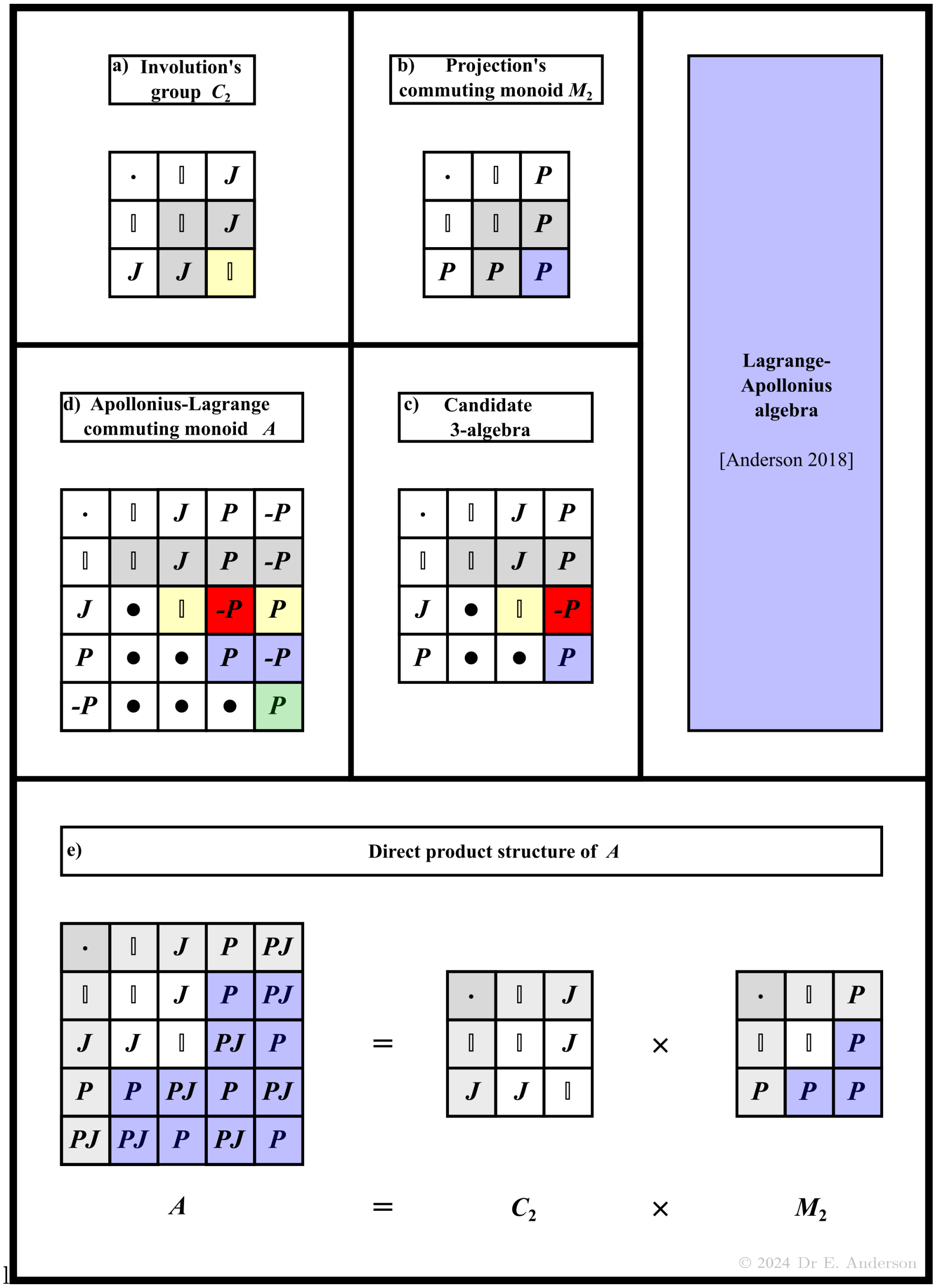

Figure 7:

Lemma A [43, 44] Any involution and the identity form the commutative group C_2 .

Lemma B [44] Any projector $\boldsymbol{P}$ and the identity form the commutative monoid

$$M_2$$

whose times table is in Fig 7.b).

Proposition C [44] i) Together, $\boldsymbol{P}$ and $\boldsymbol{J}$ generate 4-element monoid, whose times table is in Fig 7.d).

ii) Furthermore, this is the product commutative monoid (Fig 7.e)

$$A = C_2 \times M_2 . \tag{49}$$

iii) The two forms displayed are equivalent by the relation

$$\boldsymbol{P} \cdot \boldsymbol{J} = -\boldsymbol{P} . \tag{50}$$

Proposition D The Apollonius–Lagrange zero-commutator algebra can also be viewed as a 2-generator commutative semigroup.

Exercise 4 i) Derive (50).

ii) Express (33) in terms of the epsilon tensor, paying careful attention to what spaces all indices involved reside in.

iii) Prove Proposition B.

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References

[1] Apollonius of Perga (3rd and 2nd Centuries B.C.E.).

[2] C. Ptolemy of Alexandria, *Mathematike Syntaxis* though subsequent Arab preservation and promotion of this text gave it a name by which it has since become much more widely known: *Almagest.* (2nd Century C.E.).

We are referring to Book 1 thereof. For an English translation, see e.g. G.J. Toomer, *Ptolemy's Almagest* 2nd ed. (1998).

[3] Brahmagupta, (7th Century India).

[4] I. Newton evoked the Newton line in proving a Theorem about Quadrilaterals. For an English translation, see e.g. *The Mathematical Papers of Isaac Newton* ed. D. Whiteside (C.U.P., Cambridge 1967–1981).

[5] L. Euler worked on Geometry, among many other topics, in the 18th Century; this Theorem dates to 1748.

[6] For an account of Euler's contribution in this regard, see e.g. L.E. Dickson, *History of the Theory of Numbers* (Carnegie, Washington 1920).

[7] J.-L. Lagrange worked on Celestial Mechanics, among many other topics, in the late 18th Century.

[8] L. Carnot, *Essai sur la Théorie des Transversales* (Courcier, Paris, 1806).

[9] C.F. Gauss established further properties of the Newton line in the 19th Century, in the context of the complete quadrilateral. By which the term *Newton–Gauss line* is also used. For a popular account, of this work of Gauss see e.g. D. Wells, *The Penguin Dictionary of Curious and Interesting Geometry* (1991).

[10] C.G.J. Jacobi worked on Mechanics, among many other topics, in the 1840s.

[11] C.A. Bretschneider, "Untersuchung der Trigonometrischen Relationen des Geradlinigen Viereckes" (Investigation of the Trigonometric Relations of Quadrilaterals) Archiv. der Math. **2** 225 (1842).

[12] C. Godfrey and A.W. Siddons, *Modern Geometry* (University Press, 1908).

[13] R.C. Archibald, "The Area of a Quadrilateral", Amer. Math Monthly **29** 29 (1922).

[14] R.A. Johnson, *Modern Geometry: an Elementary Treatise on the Geometry of the Triangle and the Circle* (Houghton Mifflin, Boston, MA 1929), republished as *Advanced Euclidean Geometry* (Dover, Mineola, N.Y. 1960).

[15] C.V. Durell and A. Robson, *Advanced Trigonometry* (Will and Son 1930; Courier and Dover, 2003).

[16] J.L. Coolidge, "A Historically Interesting Formula for the Area of a Quadrilateral", Amer. Math. Monthly **46** 345 (1939).

[17] Hazel Perfect, *Topics in Geometry* (Pergamon, London 1963).

[18] H.S.M. Coxeter and S.L. Greitzer, *Geometry Revisited* (M.A.A., Washington, D.C. 1967).

[19] R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications* (Wiley, New York 1974; Dover, New York 2006).

[20] Z.A. Melzak, *Invitation to Geometry* (Wiley, New York 1983; Dover, Mineola N.Y. 2008).

[21] C. Marchal, *Celestial Mechanics* (Elsevier, Tokyo 1990).

[22] R.H. Buchholz, "Perfect Pyramids", Bull. Austral. Math. Soc. **45** 353 (1992).

[23] D.C. Kay, *College Geometry* (Harper Collins, 1994).

[24] D. Lynden-Bell, "A Relative Newtonian Mechanics", in *Mach's principle: From Newton's Bucket to Quantum Gravity* ed. J.B. Barbour and H. Pfister (Birkhäuser, Boston 1995).

[25] J.R. Silvester, *Geometry Ancient and Modern* (O.U.P., New York 2001).

[26] G.A. Kandall, "Euler's Theorem for Generalized Quadrilaterals", College Math. J. **33** 403 (2002).

[27] S.-H. Chou and S. He, "On the Regularity and Uniformness Conditions on Quadrilateral Grids", Computer Methods App. Math. Eng. **191** 5149 (2002).

[28] P.M. Cohn, *Basic Algebra. Groups, Rings and Fields* (Springer, London 2003).

[29] A.D. Gardiner and C.J. Bradley, *Plane Euclidean Geometry : Theory and Problems* (U.K. Mathematics Trust, 2005).

[30] W. Dunham, "Quadrilaterally Speaking", in *The Edge of the Universe: Celebrating Ten Years of Math Horizons* ed. D. Haunsperger and S. Kennedy (M.A.A., Washington D.C. 2006)

[31] N. Jacobson, *Basic Algebra* **1** (Dover, Mineola N.Y. 2009).

[32] C. Alsina and R.B. Nelsen, *Charming Proofs* (M.A.A., Washington D.C. 2010).

[33] E. Anderson, "Relational Quadrilateralland. I. The Classical Theory", Int. J. Mod. Phys. **D23** 1450014 (2014), arXiv:1202.4186;

E. Anderson and S.A.R. Kneller, "Relational Quadrilateralland. II. The Quantum Theory", Int. J. Mod. Phys. **D23** 1450052 (2014), arXiv:1303.5645.

[34] I.E. Leonard, J.E. Lewis, A.C.F. Liu and G.W. Tokarsky, *Classical Geometry. Euclidean, Transformational, Inversive and Projective* (Wiley, Hoboken N.J. 2014).

[35] E. Anderson, "Two New Perspectives on Heron's Formula", arXiv:1712.01441 For the updated version, see https://wordpress.com/page/conceptsofshape.space/1244 .

[36] "The Smallest Shape Spaces. I. Shape Theory Posed, with Example of 3 Points on the Line", arXiv:1711.10054. For the updated version, see https://wordpress.com/page/conceptsofshape.space/1225 .

[37] Geometrical discussions between E. Anderson and S. Sánchez in 2018.

[38] E. Anderson, "Quadrilaterals in Shape Theory. II. Alternative Derivations of Shape Space: Successes and Limitations", arXiv:1810.05282. For the updated version, see https://wordpress.com/page/conceptsofshape.space/1237 .

[39] M. Josefsson, *Area Formulas for certain Convex Quadrilaterals*, Math. Gazette **103** 20 (2019).

[40] "The Fundamental Triangle Matrix" (2024), https://wordpress.com/page/conceptsofshape.space/1306 .

[41] "A New 'Physical' Proof of Apollonius' Theorem" (2024), https://wordpress.com/page/conceptsofshape.space/1353 .

[42] "Lagrange Matrices: 3-Body Problem and General" (2024), https://wordpress.com/page/conceptsofshape.space/1436.

[43] "The Fundamental Triangle Matrix Commutes with the Lagrange and Apollonius Matrices. With implications for deriving Kendall and Hopf Theorems." (2024), https://wordpress.com/page/conceptsofshape.space/1308 .

[44] "Only 2 of the Fundamental Triangle, Lagrange and Apollonius Matrices are Independent. With ensuing Algebras, Irreducibles and Splits." (2024), https://wordpress.com/page/conceptsofshape.space/1310 .

[45] E. Anderson and A. Ford, "Graph and Order Theory of N-Body Problem's Eigenclustering Networks", https://wordpress.com/page/conceptsofshape.space/2090 (2024).

[46] E. Anderson, "Euler's Quadrilateral Theorem. I. A brief new Proof that is Physically Guaranteed to Generalize." alias "Eigenclustering-Length Exchange Theorems. I. P_2 and bent-P_3 . (2024), https://wordpress.com/page/conceptsofshape.space/1297 .

[47] "Eigenclustering-Length Exchange Theorems. II. Straight-P_3 alias K-counterpart of Euler-H . (2024), https://wordpress.com/page/conceptsofshape.space/1299 .

[48] "III. The Bottom Series: Straight-P_n . (2024), https://wordpress.com/page/conceptsofshape.space/1301 .

[49] "IV. 5-Body Problem" . (2024), https://wordpress.com/page/conceptsofshape.space/1512 ;
"V. 6-Body Problem" . (2024), https://wordpress.com/page/conceptsofshape.space/1514 .

[50] "VI. The Top Series: Chain of Claws (with Leaf)" . (2024), https://wordpress.com/page/conceptsofshape.space/1516.

[51] "Generalizing Heron's formula by use of Linear Algebra. II. Möbius, Jacobi and Routh for Concurrent Cevians" (2024), https://wordpress.com/page/conceptsofshape.space/1261 .

[52] E. Anderson and K. Everard, "Ptolemy's Theorem and Inequality: from a Linear Algebra point of view" https://wordpress.com/page/conceptsofshape.space/1938 .

[53] "Linear Algebra of Cyclic Quadrilaterals: Ptolemy, Diagonal, Area, and Circumradius Formulae", https://wordpress.com/page/conceptsofshape.space/1971 (2024).

[54] E. Anderson, "Quadrilaterals' Hidden Casimir", forthcoming 2024.

[55] "Convex Quadrilaterals' Ptolemy–Lagrange–Bretschneider Algebras", forthcoming (2024).

[56] "Hopf's Little Mathematics 32 times in Physics and Geometry", https://wordpress.com/page/conceptsofshape.space/1759 fothercoming 2024.

[57] "N-body Problems' Eigenclusterings: Linear Algebra and Order Theory" https://wordpress.com/page/conceptsofshape.space/1558 (2024).

[58] E. Anderson and A. Ford, "Combinatorial Features of Eigenclustering for $N = 11$- to 33-Body Problems", forthcoming 2024.

[59] E. Anderson, "Linear Algebra of Quadrilaterals. With particular emphasis on their Area Formulae." forthcoming 2024. https://wordpress.com/page/conceptsofshape.space/1303 .

[60] *The Structure of Flat Geometry*, in some places called additionally *Widely-Applicable Mathematics. A. Improving understanding of everything by dual-wielding Combinatorics and Linear Algebra.* **3**, forthcoming 2025.

[61] *Linear Mathematics. Vector Spaces, Linear Methods and Tensors Throughout STEM*, *Widely-Applicable Mathematics. A. Improving understanding of everything with a pinch of Combinatorics* **1**, forthcoming 2025.