A New ‘Physical’ Proof of Apollonius’ Theorem

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Abstract

A new ‘Physical’ proof of Apollonius’ Theorem is offered. It generalizes to arbitrary dimension. Thus including the collinear triangles needed for Kendall’s Shape Theory that Pythagorean proofs however leave out. It furthermore generalizes to Euler’s Quadrilateral Theorem. And beyond: uncovering a binary tree-graph indexed family of Theorems for all eigenclustering networks supported by whichever N-body problem.

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Apollonius’s Theorem The length of a triangle’s median is determined by its sides data as follows.

$$m_a{}^2 \;=\; \frac{2\left(b^2 + c^2\right) - a^2}{4} \; . \tag{1}$$

This is for the median m_a that bisects the side a .

Remark 1 Cycles of this equation clearly hold as well.

Remark 2 The standard proof is as a simple Corollary of Pythagoras’ Theorem [11].

Definition 1 $A := a^2$ and cycles are the $(\,\text{sides}\,)^2$ variables. And $M_A := m_a{}^2$ and cycles are the $(\,\text{medians}\,)^2$ variables.

Remark 3 In terms of these, Apollonius’ Theorem reads

$$M_{\mathrm{A}} \;=\; \frac{2\,(B + C) - A}{4} \; .$$

Remark 4 In the current Note, I give a distinct ‘Physical’ proof of Apollonius’ Theorem.

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Motivation 1 This handles a small blip in the proof based on Pythagoras’ Theorem. Namely that Pythagoras’ Theorem cannot be applied to the collinear triangles. Which Kendall’s Shape Theory [3, 5, 7, 12, 13, 14] requires inclusion of.

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Remark 5 I preliminarily require the democratic alias cluster-independent alias cycle-independent [1] formula for the moment of inertia (MoI) of a triangle. Or, for unit masses, equivalently for its radius of gyration.

Lemma 1 The centre-of-mass (CoM) MoI of a triangle with unit masses at each vertex is

$$I \;=\; \frac{1}{3}\sum_{\text{cycles}} A \; . \tag{2}$$

<u>Proof</u> This must be a symmetric homogeneous-linear function in A, B, C . Thus

$$I \;=\; k\sum_{\text{cycles}} A \; . \tag{3}$$

[1]The nomenclature and underlying literature is further commented on in a longer version [18] of the current Note.

We can then determine what k is by examining the equilateral triangle. A very basic Physical argument establishes that the medians are concurrent, meeting at the CoM. Which cuts the medians in a $2 : 1$ ratio. Thus the radial distance from the CoM to each vertex is

$$r_a = \frac{2}{3} m_a \ . \tag{4}$$

Also the equilateral triangle's medians are perpendicular to their corresponding bases and thus serve as this triangle's height h :

$$m_a = h \ . \tag{5}$$

Claim:

$$h = \frac{\sqrt{3}}{2} a \ . \tag{6}$$

Thus concatenating the previous three equations,

$$r_a = \frac{2}{3} h = \frac{2}{3} \frac{\sqrt{3}}{2} a = \frac{a}{\sqrt{3}} \ . \tag{7}$$

Thus from the definition of MoI, about the CoM,

$$I := \sum (\,\text{mass}\,)(\,\text{distance to CoM}\,)^2 = 3 {r_a}^2 = 3 \frac{a^2}{3} = A = \langle A \rangle := \frac{1}{3} \sum_{\text{cycles}} A \ . \tag{8}$$

Where the penultimate step is by isotropy and the last step is by the definition of average.

It remains to establish the claim. Three distinct conceptualizations for this are as follows.

Route 1 Using Pythagoras' Theorem *just for the equilateral triangle*,

$$h = \sqrt{a^2 - \left(\frac{a}{2}\right)^2} = \sqrt{\frac{3\,a^2}{4}} = \frac{\sqrt{3}}{2} a \ . \tag{9}$$

Route 2 Use

$$\frac{a\,h}{2} = \frac{(\,\text{base}\,) \times (\,\text{height}\,)}{2} = \mathit{Area} = \sqrt{s \prod_{\text{cycles}} (s - a)} = \sqrt{\frac{3\,a}{2} \left(\frac{a}{2}\right)^3} = \frac{\sqrt{3}\,a^2}{4} \ . \tag{10}$$

Where the third step is Heron's formula for semi-perimeter s . And the fourth evaluates s for the equilateral triangle.

Finally, since $a \neq 0$ for equilateral triangles E (being instead the *maximum coincidence-or-collision* O),

$$h = \frac{\sqrt{3}}{2} a$$

is arrived at.

Route 3

$$\frac{a\,h}{2} = \frac{(\,\text{base}\,) \times (\,\text{height}\,)}{2} = \mathit{Area} = = \frac{1}{2} \underline{\boldsymbol{a}} \times \underline{\boldsymbol{b}} = \frac{1}{2} a\,b \sin \gamma = \frac{a^2}{2} \sin \frac{\pi}{3} = \frac{a^2}{2} \frac{\sqrt{3}}{2} \ .$$

Where $\boldsymbol{a}$ and $\boldsymbol{b}$ are for side vectors. Then apply Route 2's last paragraph's argument. □

Remark 6 Virtues and peccadilloes of each of these routes are discussed in [18]. An extremization working valid for general vertex number N can be found in [15], under the name of 'democratic radius of gyration (RoG) Lemma'. Interconversion between RoG and MoI is of course elementary.

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Proof of Apollonius' Theorem

$$I \ = \ I_{\text{base}} \ + \ I_{\text{median}} \ = \ {\rho_{\text{base}}}^2 \ + \ {\rho_{\text{median}}}^2 \ = \ \frac{1}{2}\,A \ + \ \frac{2}{3}\,M_A \ . \tag{11}$$

Where the first step splits the total moment of inertia into base and median partial moments of inertia. The second step expresses these in terms of the corresponding mass-weighted relative Jacobi coordinates [4, 6, 13, 14, 19]. And the third step expresses these in terms of the Jacobi reduced masses and the squares of Geometry's own variables.

Equating this with Lemma 1's expression, we arrive at the following.

$$\frac{2}{3}\,M_A \ + \ \frac{A}{2} \ = \ \frac{A \ + \ B \ + \ C}{3} \ \Rightarrow \tag{12}$$

$$\frac{2}{3}\,M_A \ = \ \frac{2\,(\,B \ + \ C\,) \ - \ A}{3} \ \Rightarrow \tag{13}$$

$$M_A \ = \ \frac{2\,(\,B \ + \ C\,) \ - \ A}{4} \ , \tag{14}$$

as required. □

End Remark 1 See [18] for how Apollonius' Theorem conversely implies Lemma 1's democratic MoI formula. For how this proof manages to include the collinear case. And for a comparison of the 1- and ($\geq$ 2)-d theory of medians.

Motivation 2 This proof generalizes to [16] Euler's Quadrilateral Theorem [2, 8, 9]. And beyond, with great fanfare, giving one Theorem per N-body problem eigencluster shape (= type of Jacobi coordinates). Which in turn are indexed by the unlabelled binary tree graphs. The N-body stipulation includes that all of this holding in arbitrary dimension.

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