Ptolemy's Theorem and Inequality:
from a Linear Algebra point of view

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Abstract

We cast Ptolemy's Theorem and inequality in terms of a quadratic form Pt . Consisting of the separation-lengths 6-vector twice contracted into a matrix $\boldsymbol{Pt}$. Which is an involution.

We also give the corresponding eigentheory; one of the semi-perimeter and the separimeter can be chosen among the eigenvectors. $\boldsymbol{Pt}$ is moreover indefinite, with Ptolemy's inequality corresponding to spacelike 6-vectors. And Ptolemy's Theorem to the bounding case of null 6-vectors. This description parallels how the Heron form distinguishes between zero area and positive area, while prohibiting negative-area separations magnitude data.

With Ptolemy's Theorem holding for cyclic quadrilaterals, we abstract a quantifier of acyclicness from Pt . We finally point to many extensions of the current Article's material.

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date stamps v1: 03-08-2024; v2: 05-08-2024

1 Introduction

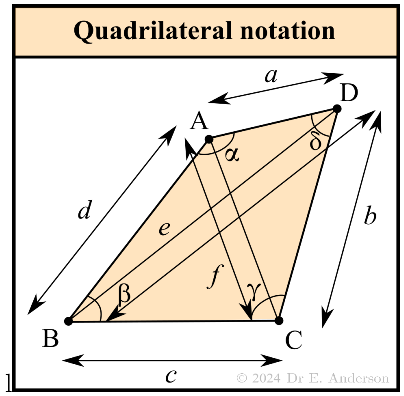

Figure 1:

Theorem 1 (Ptolemy) [2] Let ABCD be a cyclic quadrilateral. Then

$$|\mathrm{AB}|\,|\mathrm{BD}| \;=\; |\mathrm{AB}|\,|\mathrm{BD}| \;+\; |\mathrm{BC}|\,|\mathrm{AD}| \; . \tag{1}$$

Let us next bring in Fig 1's cyclic notation for the side lengths, as supplemented by the diagonal lengths. Then the above equation tidies up to the following expression.

$$e\,f \;=\; a\,c \;+\; b\,d \; . \tag{2}$$

Remark 1 Conceptually,

$$\prod(\,\text{diagonals}\,) \;=\; \sum\prod(\,\text{opposite sides}\,) \; . \tag{3}$$

For which

$$x \;=\; l$$

provides notation, the first piece nonstandard, referring to diagonals crossing, while the second piece is standard. Let us postpone conceptual discussion of l to [47].

Remark 2 This Theorem's converse is also true (if we choose for now to ignore degenerate quadrilaterals). I.e. if Ptolemy's condition (2) holds for a non-degenerate quadrilateral, then it is cyclic.

Remark 3 Styles of proof for [52] Ptolemy's Theorem include, firstly, by use of similar triangles [14, 15, 29, 23]. Secondly, by an auxiliary-point construction [19]

Thirdly, by use of trigonometry's two-angle formulae. To which Ptolemy's Theorem is in fact equivalent. Indeed, Ptolemy used this Theorem to build what we would now call Trigonometric tables for Astronomical use [2]. For Ptolemy was his epoch's leading Astronomer [9], in fact quite possibly the whole classical period's leading Astronomer.

Fourthly, by use of Inversive Geometry [15], which works since Ptolemy's Theorem indeed turns out to have inversive significance. Fifthly, by use of Projective Geometry techniques, such as a projection [35], cross-ratios [19] or Möbius transformations [25].

Exercise 1 a) Prove the converse of Ptolemy's Theorem using basic Euclidean Geometry.

b) Demonstrate equivalence between Ptolemy's Theorem and

$$\sin(\alpha \;+\; \beta) \;\;=\;\; \sin\,\alpha\,\cos\beta \;\;+\;\; \cos\alpha\,\sin\,\beta \; .$$

c) Deduce Stewart's Cevians-length Theorem [5, 16, 23, 33, 35, 44] as a Corollary of Ptolemy's Theorem.

2 The Ptolemy quadratic form and matrix

Theorem 1′ Ptolemy's Theorem can be recast as

$$Pt \ := \ \overline{\boldsymbol{r}} \cdot \underline{\underline{\boldsymbol{Pt}}} \cdot \overline{\boldsymbol{r}} \ = \ 0 \ . \tag{4}$$

For *separation lengths* 6-*vector* for the quadrilateral,

$$\boldsymbol{r} \ := \ \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \quad \text{in standard basis, or} \quad \begin{pmatrix} a \\ c \\ b \\ d \\ e \\ f \end{pmatrix}$$

in *paired-separations* basis.

Ptolemy matrix

$$\boldsymbol{Pt} \ := \ \begin{pmatrix} \mathbb{0} & \mathbb{I} & \mathbb{0} \\ \mathbb{I} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & -\mathcal{T} \end{pmatrix} \quad \text{in standard basis, or} \quad \begin{pmatrix} \mathcal{T} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathcal{T} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & -\mathcal{T} \end{pmatrix}$$

in paired-separations basis.

Where in turn $\mathbb{I}$ is the 2-d identity matrix. And

$$\mathcal{T} \ := \ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ :$$

the sole *transposition matrix* supported in 2-d .

Finally, Pt is the *Ptolemy quadratic form.*

Remark 1 Since we are equating to zero, we have absorbed a constant factor of $1/2$ in passing from traditional formulations to Linear Algebra ones.

Remark 2 Thus Ptolemy's Theorem – converse included – now reads that the Ptolemy quadratic form takes the *Ptolemy value* 0 iff the nondegenerate quadrilateral in question is cyclic. I.e.

$$Pt(\text{ABCD}) \ = \ Pt_0 \ := \ 0 \ \text{ iff } \ \text{ the nondegenerate ABCD is cyclic} \ .$$

3 Its eigentheory

Remark 1 The Ptolemy eigenequation is

$$0 \ = \ \det\left(\boldsymbol{Pt} \ - \ \lambda\mathbb{I} \right) \ = \ \left(\lambda^2 \ - \ 1 \right)^3 \ = \ (\lambda \ + \ 1)^3 \, (\lambda \ - \ 1)^3 \ .$$

$$\Rightarrow \lambda \ = \ \pm 1$$

are the Ptolemy eigenvalues. Each occurring with multiplicity 3 .

Remark 2 One can additionally read off from this that

$$\text{rank}(\boldsymbol{Pt}) \ = \ 6 \ .$$

Which, being the full rank supported, means that

$$\text{null}(\boldsymbol{Pt}) \ = \ 0 \ .$$

The Mathematicians' signature[1] is

$$s_{\text{Math}}(\boldsymbol{Pt}) \ = \ 3 \ .$$

[1]See [41, 52] for these definitions of signature, conventions in use included.

The Physicists' signature-in-summary is

$$s_{\text{Phys}}(\boldsymbol{Pt}) \ = \ 0 \ . \tag{5}$$

And the Physicists' signature-in-detail is

$$s_{\text{Phys-detail}}(\boldsymbol{Pt}) \ = \ + \ + \ + \ - \ - \ - \ .$$

Remark 3 Ptolemy's Theorem is thus a statement about the separation-lengths 6-vector $\boldsymbol{r}$ being null with respect to an indefinite 6×6 matrix. One ready consequence of this is that the separation-lengths of the cyclic quadrilaterals thus described are not free. For were they free, then Q would take all 3 possible signs.

Remark 4 In the standard basis, the Ptolemy matrix decomposes into 2 blocks: the 4×4 sides block and the 2×2 diagonals block. This corresponds to the 'sides' and 'diagonals' partition of the separations of a cyclic quadrilateral.[2] In the paired-separations basis, however, it decomposes into 3 2×2 blocks. I.e. one per pair. This is simpler to work with, as follows.

Here just 3 blockwise uses of a standard symmetric–antisymmetric combination provides the eigenvectors. I.e.

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} \ . \tag{6}$$

As then padded out, firstly with 4 postceding zeros. Then again with 2 anteceding and 2 postceding zeros. And then once more with 4 anteceding zeros.

Remark 5 In terms of the side-length variables, the above eigenbasis is, more succinctly,

$$\frac{a \ \pm \ c}{\sqrt{2}} \ , \quad \frac{b \ \pm \ d}{\sqrt{2}} \ , \quad \frac{e \ \pm \ f}{\sqrt{2}} \ .$$

I.e. Geometrically the diagonal-sum, diagonal difference and each opposite side pair's sum and difference.

Exercise 2 a) Show that the eigenexpansion version of Ptolemy's Theorem reads

$$(a + c)^2 \ - \ (a - c)^2 \ + \ (b + d)^2 \ - \ (b - d)^2 \ - \ (e + f)^2 \ + \ (e - f)^2 \ = \ 0 \ .$$

Provide an Algebraic interpretation for this equation.

b) Show that the semi-perimeter is also a normalized Ptolemy eigenvector. And orthogonal to the opposite side-pairs difference eigenvectors. Find and Geometrically interpret the eigenvector ϵ that completes this eigenbasis. Prove that

$$s^2 \ + \ \epsilon^2 \ - \ \frac{s_2}{2} \ = \ l \ . \tag{7}$$

Where

$$s_2 \ := \ \sum_{I \, = \, 1}^{4} s_I^2 \ .$$

c) Show that the *separimeter* [43]

$$\frac{1}{\sqrt{6}} \sum_{s \, = \, 1}^{6} s_s$$

is also a valid choice of eigenvector. Normalize it. Which eigenvectors considered so far are orthogonally compatible with this? Complete these to an orthonormal eigenbasis.

Remark 6 Our first eigenbasis above, and the semi-perimeter eigenbasis, respect the sides to diagonals block structure. The separimeter is however a separations-democratic [37] notion, by which eigenbases extending it do not respect this block split. This accounts for the separimeter not fitting in so well with the study of the Ptolemy matrix.

[2] Or more generally of a convex quadrilateral. Less experienced Readers might wish to check at this point that every cyclic quadrilateral must be convex...

4 The Ptolemy matrix is a fortiori an involution

Lemma 1

$$\boldsymbol{Pt}^2 \;=\; \mathbb{I}\;, \tag{8}$$

now standing for the 6-d identity matrix.

<u>Proof</u>

$$(-1)^2 \;=\; 1\;. \tag{9}$$

$$\mathcal{T}^2 \;=\; \mathbb{I}\;: \tag{10}$$

the 6-d identity matrix.

$$\boldsymbol{Pt}^2 \;=\; \begin{pmatrix} \mathcal{T} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathcal{T} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & \text{-}\mathcal{T} \end{pmatrix} \begin{pmatrix} \mathcal{T} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathcal{T} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & \text{-}\mathcal{T} \end{pmatrix} \;=\; \begin{pmatrix} \mathcal{T}^2 & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathcal{T}^2 & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & (-1)^2\,\mathcal{T}^2 \end{pmatrix} \;=\; \begin{pmatrix} \mathbb{I} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathbb{I} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & \mathbb{I} \end{pmatrix} \;=\; \mathbb{I}\;.$$

Where the first step evaluates $\boldsymbol{Pt}$ in the paired-separations basis. The second uses the matrix multiplication rule and refactorization. The third uses (9-10) And the fourth dissolves the blocks. □

Remark 1 This involution has a clear action on a single separation-magnitudes 6-vector, as follows.

$$\underline{\underline{\boldsymbol{Pt}}} \cdot \overline{\boldsymbol{r}} \;=\; \begin{pmatrix} \mathcal{T} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathcal{T} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & \text{-}\mathcal{T} \end{pmatrix} \begin{pmatrix} a \\ c \\ b \\ d \\ e \\ f \end{pmatrix} \;=\; \begin{pmatrix} c \\ a \\ d \\ b \\ \text{-}f \\ \text{-}e \end{pmatrix}\;. \tag{11}$$

Naming Remark 1 We henceforth call $\boldsymbol{Pt}$ the *Ptolemy involution*. A truer name for which is *separation-pairs flipping involution*. It being understood that, in the process, the diagonal lengths also each pick up a minus sign.

5 Ptolemy's inequality

Theorem 2 (Ptolemy's inequality) For ABCD a convex quadrilateral, Then

$$|\mathrm{AB}|\,|\mathrm{BD}| \quad \leq \quad |\mathrm{AB}|\,|\mathrm{BD}| \quad + \quad |\mathrm{BC}|\,|\mathrm{AD}| \ . \tag{12}$$

Or, in terms of Fig 1's cyclic notation for the side lengths, as supplemented by the diagonal lengths,

$$e\,f \quad \leq \quad a\,c \, + \, b\,d \ . \tag{13}$$

Conceptually

$$\prod (\,\mathrm{diagonals}\,) \quad \leq \quad \sum \prod (\,\mathrm{opposite\ sides}\,) \ , \quad \mathrm{i.e.}\ x \quad \leq \quad l \ . \tag{14}$$

Remark 1 The above is stated to cover the convex case. By using suitably directed quantities, however, Ptolemy's inequality generalizes to *arbitrary* quadrilaterals. In contrast, Ptolemy's Theorem clearly cannot, since it describes cyclic quadrilaterals (modulo degenerate cases).

Exercise 3 In fact, if degenerate quadrilaterals are entertained, then Ptolemy's inequality is not just saturated by Ptolemy's Theorem's cyclic quadrilaterals. For if both of the following hold then we also have equality.

a) ABCD is collinear.

b) Precisely 1 of B or D lies between A and C .

Prove this, and exhaust all other degenerate possibilities.

Remark 2 Thus for ABCD a non-collinear quadrilateral, the following is a *quantifier of departure from cyclicness* [39].

$$(\,\mathrm{Ptolemaic\ acyclicness}\,) \quad := \quad \sum \prod (\,\mathrm{opposite\ sides}\,) \; - \; \prod (\,\mathrm{diagonals}\,) \quad = \quad l \, - \, x \ . \tag{15}$$

By Ptolemy's inequality, this is a non-negative quantity. We qualify it as 'Ptolemaic', since we shall soon be entertaining further notions of acyclicness for quadrilaterals [47, 49, 52].

Remark 3 For instance the following styles of proof carry over from Ptolemy's Theorem to Ptolemy's inequality. Use of similar triangles, provided that we supplement it with the triangle inequality [52] . Use of Inversive Geometry [19]. Use of Projective Geometry.

Remark 4 A distinct useful proof involves using dot products of vectors [52]; the triangle inequality is guaranteed by Euclidean space here. This method is worth mentioning because it *obviously* generalizes to higher dimensions. In other words, *Ptolemy's inequality also holds for tetrahaedrons*, be these minimally realized in 3-d , or in some yet higher dimension.

Naming Remark 2 A truer name for Ptolemy's Theorem is *cyclic quadrilateral separations relation.* And one for Ptolemy's inequality is 4-*body problem separations inequality.* This includes building in that the inequality transcends to arbitrary spatial dimension. For 'N-body problem' carries this connotation, while 'quadrilateral' does not.

6 Linear Algebra of Ptolemy's inequality

Theorem 2′ Ptolemy's inequality can be recast as

$$Pt \; := \; \overline{\boldsymbol{r}} \cdot \underline{\underline{\boldsymbol{Pt}}} \cdot \overline{\boldsymbol{r}} \; \geq \; 0 \; . \tag{16}$$

Remark 1 Ptolemy's inequality in the unsaturated case can then be reinterpreted as follows. The separation-lengths 6-vector $\boldsymbol{r}$ must be spacelike with respect to our indefinite 6×6 involution $\boldsymbol{Pt}$. Which is, more specifically, of Physicists' signature $+ \; + \; + \; - \; - \; -$.

Remark 2 By way of comparison, the Minkowski spacetime of Special Relativity can be chosen to be $+ \; + \; + \; -$. While its also commonly used 2-d model is $+ \; -$. All three are *indefinite*. The second model has *null cones*, which collapse to *null wedges* in the third model. In each case separating between *spacelike* regions and *timelike* regions.

There being three independent directions for each sign in the first case alters the overall topology. Courant and Hilbert [21] fire a warning shot here: this is an *ultrahyperbolic space*, in which 'natural PDEs' would not be known to be well-posed.

And yet the distinction between a null vector and a spacelike vector remains. As does the difference between a null surface – swept out by null vectors – and a spacelike surface: swept out by spatial vectors.

Remark 3 *Thus in the space of all hexuples of separation-lengths, called separation magnitude space in [42, 37], contains a null space of cyclic quadrilaterals. Bounding a spacelike space of all permissible quadrilaterals. And not even for arbitrary quadrilaterals are the separation magnitudes free. For they cannot take every possible value. Indeed, they have no way of realizing any member of an a priori co-generic class of quadrilaterals separations data. Whose qualitative description is that these are the vectors that are timelike with respect to the Ptolemy involution* $\boldsymbol{Pt}$.

In this way, Ptolemy's inequality and Theorem play a role in the study of quadrilaterals' separation-magnitudes space. In the study of the subset of such that are cyclic quadrilaterals. And as regards how the separations-magnitude space of all quadrilaterals sits within the larger space of all hexuplets, whether or not these consistently specify quadrilaterals.

7 Outlook

Remark 1 Three further results in particular have been *nominatively* associated with Ptolemy's Theorem.

Pointer 1 Let

$$m \ := \ a\,d \ + \ b\,c \ , \quad n \ := \ a\,b \ + \ c\,d$$

be the 3-cycles as regards pairing up sides [47] of l. Then

$$\frac{e}{f} \ = \ \frac{m}{n}$$

has been called the '**second form of Ptolemy's Theorem**' [12, 31].

While the closely-related **cyclic quadrilateral diagonal-length formulae** [11, 12, 23, 31]

$$e \ = \ \sqrt{\frac{l\,m}{n}} \ , \quad f \ = \ \sqrt{\frac{l\,n}{m}}$$

were referred to as 'strong Ptolemy' in [36].

Pointer 2 A generalized Ptolemy Theorem is as follows. For say a convex quadrilateral,

$$e^2\,f^2 \ = \ a^2\,c^2 \ + \ b^2\,d^2 \ - \ 2\,a\,b\,c\,d\cos(\,\alpha \ + \ \gamma\,) \ .$$

This does include Ptolemy's Theorem as a subcase, as well as giving further insight into Ptolemy's inequality being a quantifier of acyclicness. It furthermore finds a role for $\boldsymbol{Pt}$ acting upon the vector $\boldsymbol{R}$ of $(\text{ separation-lengths })^2$.

Pointer 3 Casey's Theorem [8, 10, 11] is **a distinct generalized Ptolemy Theorem**. This considers 4 circles lying tangentially inside what had hitherto played the role of circumcircle of a cyclic quadrilateral. Then these circles' bitangents obey the same quadratic relation as Ptolemy's Theorem's separation lengths do. Thus the Linear Algebra of Casey's Theorem is identical to that of Ptolemy's. One needs however to re-interpret all the ensuing Linear Algebra objects in terms of these bitangents, so the above identification does not quite yet sort out this case [50].

Pointer 4 Further Linear Algebra ensues from combining Ptolemy's Theorem and what is usually referred to as **Euler's Quadrilateral Theorem** [6, 30, 32] and yet is more truly **Euler's 4-body Theorem** [26, 43].

Remark 2 We anticipate some further benefits from adopting the following batting order. Eigenclusterings [42, 37]. Ptolemy's Theorems and inequality and the cyclic-quadrilateral diagonal-length formulae (partly in the current Article and partly in [47]). Next, Euler's 4-body Theorem [43], followed by combining it and Ptolemy [46]. Then [47] Brahmagupta's area formula [3, 15, 19, 23, 40] and circumradius formulae (including Parameshvara's) [4, 31], both of which are for cyclic quadrilaterals. Then Bretschneider's area formulae [7, 19, 23, 40, 48]. And finally a generalized Ptolemy Theorem [48]. This batting order is Part II of [52]'s essay plan, and will be rehearsed to some extent in the first portion of [49].

Remark 3 One of the Authors' study [41] of the Heron matrix [22, 38] had found a similar role for indefinite quadratic forms realizing only the null and spacelike sign cases. There for zero area and positive area; negative area being inconsistent! In this study, furthermore, action on both $\boldsymbol{r}$ and $\boldsymbol{R}$ played a role. (Separations coinciding with sides for triangles, these are denoted by $\boldsymbol{s}$ and $\boldsymbol{S}$ in these Articles.) Unifying – at the level of Linear Algebra for flat space – Heron's formula, the cycle of cosine rules and the cycle of triangle inequalities. Interestingly, both of these features return upon considering Ptolemy's Theorem and inequality.

8 Epilogue

Remark 1 One of the Authors' study of Heron's formula led to [40, 45] a new: derivation of Hopf's little map [13, 27, 28, 51], proof of Smale's Little Theorem [17], and proof of Kendall's Little Theorem [18, 20, 24]. The last two of these results pertain to Shape Theory. All three of these results extend via Hopf's generalized map to complex-projective spaces that model the space of all N-a-gons in the flat plane.

Remark 2 Ptolemy's Theorem is not expected to play an initial role here at least, since it considers just cyclic quadrilatererals. So only further considerations of spaces of cyclic quadrilaterals might have some role for this. Be this intrinsic modelling of these spaces, or study of how they sit within their ambient $\mathbb{CP}^2$ of all quadrilaterals [34].

Remark 3 A suitably generally formulated version of Ptolemy's inequality is not however affected by this limitation.

Remark 4 For the sake of clarity, the separations magnitude space that we commented on above is not part of Shape Theory. Its consistent subset is, to some extent.

Examples 1-2 On the one hand, sides data does not suffice for quadrilaterals, as is clear from the infinite variety of rhombi. On the other hand, more obvious sides and relative angles prescriptions for this can be converted to separations data.

Example 3 This still leaves the following ambiguity. Given four points in general position, a quadrilateral can be drawn through them in 3 different orders of 'joining the dots'. In some positions, this gives 3 distinct re-entrant figures, while in others, 1 convex and 2 crossed figures. Shape Theory is however about the constellation of points itself, for which such an assignment is immaterial. Thus examples of this kind do not spoil separation-magnitude data's capacity to discern.

Example 4 Finally for triangles, for which there are no relative angles that are independent of side lengths, both sides legngth space and the triangleland shape sphere are well known [24, 52]. Already here, however, the separations-magnitude parametrization is opaque at the Metric Geometry level. It is rather [42, 37] 1 eigencluster magnitude ratio and the relative angle between the corresponding vectors that provides clarity. Which extends to $N - 2$ such pairs providing clarity for the N-a-gon [17, 24, 37].

Acknowledgments E.A. and K.E. thank S and A for previous discussions. And the other participants at the Institute of the Theory of STEM's "Linear Algebra of Quadrilaterals" Summer School 2024. E.A. also thanks C, Malcolm MacCallum, Reza Tavakol, Jeremy Butterfield and Enrique Alvarez for career support.

References

[1] Heron, alias Hero, of Alexandria, *Metrica* (60 C.E.). Historically documented in [9].

[2] C. Ptolemy of Alexandria, *Mathematike Syntaxis* though subsequent Arab preservation and promotion of this text gave it a name by which it has since become much more widely known: *Almagest.* (2nd Century C.E.).

We are referring to Book 1 thereof. For an English translation, see e.g. G.J. Toomer, *Ptolemy's Almagest* 2nd ed. (1998).

[3] Brahmagupta, (7th Century India).

[4] Parameshvara (15th Century India). For a review of this major result of his, see e.g. R.C. Gupta, "Parameshvara's Rule for the Circumradius of a Cyclic Quadrilateral", Historia Mathematica **4** 67 (1977).

[5] M. Stewart, *Some General Theorems of Considerable Use in the Higher Parts of Mathematics* (Sands, Edinburgh 1746).

[6] L. Euler worked on Geometry, among many other topics, in the 18th Century; this Theorem dates to 1748.

[7] C.A. Bretschneider, "Untersuchung der Trigonometrischen Relationen des Geradlinigen Viereckes" (Investigation of the Trigonometric Relations of Quadrilaterals) Archiv. der Math. **2** 225 (1842).

[8] J. Casey, "the Equations and Properties: (1) of the System of Circles Touching Three Circles in a Plane; (2) of the System of Spheres Touching Four Spheres in Space; (3) of the System of Circles Touching Three Circles on a Sphere; (4) of the System of Conics Inscribed to a Conic, and Touching Three Inscribed Conics in a Plane." Proc. Royal Irish Acad. **9** 396 (1866).

[9] T.A. Heath, *A History of Greek Mathematics. Volume 2, from Aristarchus to Diophantus* (C.U.P., Cambridge, 1921).

[10] C.V. Durell, *Modern Geometry: The Straight Line and Circle* (Macmillan, London 1928).

[11] R.A. Johnson, *Modern Geometry: an Elementary Treatise on the Geometry of the Triangle and the Circle* (Houghton Mifflin, Boston, MA 1929), republished as *Advanced Euclidean Geometry* (Dover, Mineola, N.Y. 1960).

[12] C.V. Durell and A. Robson, *Advanced Trigonometry* (Will and Son 1930; Courier and Dover, 2003).

[13] H. Hopf, "Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche", ("Concerning the Images of $\mathbb{S}^3$ on $\mathbb{S}^2$"), Math. Ann. (Berlin) Springer **104** 637 (1931). Reproduced in *Heinz Hopf, Collected Papers: Gesammelte Abhandlingen* ed. B. Eckmann (2001).

[14] Hazel Perfect, *Topics in Geometry* (Pergamon, London 1963).

[15] H.S.M. Coxeter and S.L. Greitzer, *Geometry Revisited* (M.A.A., Washington, D.C. 1967).

[16] A.S. Posamentier and C.T. Salkind, *Challenging Problems in Geometry* (MacMillan, New York 1970; Dover, Mineola, N.Y. 1996).

[17] S. Smale "Topology and Mechanics. II. The Planar N-Body Problem", Invent. Math. **11** 45 (1970).

[18] D.G. Kendall, "The Diffusion of Shape", Adv. Appl. Prob. **9** 428 (1977).

[19] Z.A. Melzak, *Invitation to Geometry* (Wiley, New York 1983).

[20] D.G. Kendall, "Shape Manifolds, Procrustean Metrics and Complex Projective Spaces", Bull. Lond. Math. Soc. **16** 81 (1984).

[21] R. Courant and D. Hilbert, *Methods of Mathematical Physics* Vol. 2 (Wiley, Chichester 1989).

[22] R.H. Buchholz, "Perfect Pyramids", Bull. Austral. Math. Soc. **45** 353 (1992).

[23] D.C. Kay, *College Geometry* (Harper Collins, 1994).

[24] D.G. Kendall, D. Barden, T.K. Carne and H. Le, *Shape and Shape Theory* (Wiley, Chichester 1999).

[25] J.R. Silvester, *Geometry Ancient and Modern* (O.U.P., New York 2001).

[26] G.A. Kandall, "Euler's Theorem for Generalized Quadrilaterals", College Math. J. **33** 403 (2002).

[27] D.W. Lyons, "An Elementary Introduction to the Hopf Fibration", Math. Magazine **76** 87 (2003), arXiv:2212.01642.

[28] H.K. Urbantke, "The Hopf Fibration Seven Times in Physics", J. Geom. Phys. **46** 125 (2003).

[29] A.D. Gardiner and C.J. Bradley, *Plane Euclidean Geometry : Theory and Problems* (U.K. Mathematics Trust, 2005).

[30] W. Dunham, “Quadrilaterally Speaking", in *The Edge of the Universe: Celebrating Ten Years of Math Horizons* ed. D. Haunsperger and S. Kennedy (M.A.A., Washington D.C. 2006)

[31] C. Alsina and R.B. Nelsen, “On the Diagonals of a Cyclic Quadrilateral", Forum Geometricorum **7** 147 (2007).

[32] C. Alsina and R.B. Nelsen, *Charming Proofs* (M.A.A., Washington D.C. 2010).

[33] I. Matić, “Olympiad Training Materials, Geometric Inequalities",
http://www.imomath.com/index.php?options=603&lmm=0 (2011).

[34] E. Anderson, “Relational Quadrilateralland. I. The Classical Theory", Int. J. Mod. Phys. **D23** 1450014 (2014), arXiv:1202.4186;
E. Anderson and S.A.R. Kneller, “Relational Quadrilateralland. II. The Quantum Theory", Int. J. Mod. Phys. **D23** 1450052 (2014), arXiv:1303.5645.

[35] I.E. Leonard, J.E. Lewis, A.C.F. Liu and G.W. Tokarsky, *Classical Geometry. Euclidean, Transformational, Inversive and Projective* (Wiley, Hoboken N.J. 2014).

[36] E. Chen, *Euclidean Geometry in Mathematical Olympiads* (M.A.A., 2016).

[37] E. Anderson, “The Smallest Shape Spaces. I. Shape Theory Posed, with Example of 3 Points on the Line", arXiv:1711.10054. For the updated version, see https://wordpress.com/page/conceptsofshape.space/1225 .

[38] “Two New Perspectives on Heron’s Formula", arXiv:1712.01441 For the updated version, see https://wordpress.com/page/conceptsofshape.space/1244 .

[39] Geometrical discussions between E. Anderson and S. Sánchez in 2018.

[40] E. Anderson, “Quadrilaterals in Shape Theory. II. Alternative Derivations of Shape Space: Successes and Limitations", arXiv:1810.05282. For the updated version, see https://wordpress.com/page/conceptsofshape.space/1237 .

[41] “The Fundamental Triangle Matrix" (2024), https://wordpress.com/page/conceptsofshape.space/1306 .

[42] “Lagrange Matrices: 3-Body Problem and General" (2024), https://wordpress.com/page/conceptsofshape.space/1436.

[43] “Euler’s Quadrilateral Theorem. I. A brief new Proof that is Physically Guaranteed to Generalize." (2024),
https://wordpress.com/page/conceptsofshape.space/1297 .

[44] “Generalizing Heron’s formula by use of Linear Algebra. I. Equi-Cevians and invertible Cevians", (2024), https://wordpress.com/page/conceptsofshape.space/1259 .

[45] “The Fundamental Triangle Matrix Commutes with the Lagrange and Apollonius Matrices. With implications for deriving Kendall and Hopf Theorems." (2024), https://wordpress.com/page/conceptsofshape.space/1308 .

[46] “Linear Algebra and Inequalities from combining Ptolemy’s Theorem and Inequality with Euler’s Quadrilateral Theorem", https://wordpress.com/page/conceptsofshape.space/1967 (2024).

[47] E. Anderson and K. Everard, “Linear Algebra of Cyclic Quadrilaterals: Ptolemy, Diagonal, Area, and Circumradius Formulae", https://wordpress.com/page/conceptsofshape.space/1971 (2024).

[48] E. Anderson, “Quantifiers of Acyclicness for Quadrilaterals", https://wordpress.com/page/conceptsofshape.space/1969 (2024).

[49] “Linear Algebra of Quadrilaterals. With particular emphasis on their Area Formulae." forthcoming 2024. https://wordpress.com/page/conceptsofshape.space/1303 .

[50] “Reinterpreting the Ptolemy Linear Algebra in the setting of Casey’s Theorem", forthcoming 2024.

[51] “Hopf’s Little Mathematics 32 times in Physics and Geometry", https://wordpress.com/page/conceptsofshape.space/1759 fothercoming 2024.

[52] *The Structure of Flat Geometry*, in some places called additionally *Widely-Applicable Mathematics. A. Improving understanding of everything by dual-wielding Combinatorics and Linear Algebra.* **3**, forthcoming 2025.