N-Body Problem: Eigenclustering Network Counts
return the Wedderburn–Etherington Numbers

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Abstract

For N-body configurations, eigenclustering vectors are alias (relative) Jacobi vectors. $N = 4$ supports 2 distinct eigenclustering networks: H and K . Which source of ambiguity growingly persists for subsequent N .

We now establish that the counts of eigenclustering networks are given by the Wedderburn–Etherington numbers. While providing one algebraic and two tree representations for this that are natural to the eigenclustering context. The smaller tree representation – rooted at-most binary – provides a good candidate for a systematic notation for eigenclusterings. K is here the straight 3-path while H is the bent 3-path.

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1 Introduction

Let us consider N-body configurations. These play major roles in Dynamics [4, 5, 6, 9, 11, 12, 13, 28], Molecular Physics [14, 18] and Shape Statistics [10, 17, 19, 25]. Their natural home is however Flat Geometry [39]. To cover all of these applications at once, let us conceptualize in terms of points-or-particles [26]. Where the latter are classical nonrelativistic point particles.

$N = 0$ does not even have any points-or-particles.

$N = 1$ has 1 , but supports no separations.

$N = 2$ possesses a single separation.

$N = 3$ has 3 separations, but now [26] only 2 are linearly independent (LI). Furthermore, the inertia quadric is no longer diagonal in terms of these [26]. This is resolved by passing to eigenclusterings. Separation vectors are thereby converted to *eigenclustering vectors* [26]: now in general separation vectors between subsystem centres of mass (CoM). Which are elsewhere alias *(relative) Jacobi vectors* [5, 11, 14, 18, 28]. For $N = 3$, there are 3 labelling choices for these if the points-or-particles are themselves labelled. For equal point-or-particle masses, these consist of each of the 3 side vectors paired with its corresponding median vector. These collapse to a single 'T' shape (column 1 of Fig 1) if the points-or-particles are furthermore indistinguishable.

$N = 4$ is minimum for 2 distinct eigenclustering network to be realized: the first H and K in column 1 of Fig 1. Such eigenclustering network ambiguities are independent of labelling ambiguities.

So how many eigenclustering network $e(N)$ do N-body configurations support, under the assumption of indistinguishable point-or-particles?

2 The count using Graph Theory

Remark 1 Picking an eigenclustering amounts to forming a network of CoMs for the arbitrary N-body configuration, by coarse-graining two subsystems into one at each step.

These eigenclustering networks are rooted binary trees with N leaves (column 3 of Fig 1). Where the role of leaves is played by the constituent points-or-particles, and that of root by the system's total mass.

Lemma 1 These are equivalently rooted binary trees with $2N - 1$ vertices.

<u>Proof</u> Binary trees are tri-regular [29]. With specifically

$$N \text{ leaves of degree } 1 . \tag{1}$$

$$1 \text{ root of degree } 2 . \tag{2}$$

$$I \text{ internal vertices of degree } 3 . \tag{3}$$

Introduce also the Graph-Theoretically standard notation V and E for the total numbers of vertices and edges respectively.

Binary trees obey the following simultaneous linear equations.

$$V - E = 1 , \tag{4}$$

$$V - I = N + 1 , \tag{5}$$

$$2E - 3I = N + 2 . \tag{6}$$

We are here treating N as an input datum and the other 3 quantities as unknowns, so this is a well-determined system.

For by tri-regularity, (1-3) exhaust all possible vertices. So

$$V = N + 1 + I ,$$

which rearranges to (5).

Also, for any graph, the number of edges E obeys Euler's Degree-sum Theorem. Whose degreewise-expanded form is

$$2E = \sum_i i \, d_i . \tag{7}$$

By the above exhaustion again, this returns

$$2E = N + 2 + 3I ,$$

which rearranges to (6).

But also binary trees are trees. For which Euler's even more famous relation

$$V - E + F = 2 \tag{8}$$

– for F the number of faces – reduces to the following.

$$V - E + 1 = 2 ,$$

which cancels down to (4). This is since trees are acyclic, and so only have the one – exterior – face.

Finally, to solve our system, (5) - (4) yields

$$E = I + N .$$

Substitute in (6) to obtain

$$2N + 2I - 3I = N + 2 .$$

Which cancels down to

$$I = N - 2 .$$

Substituting this in (5),

$$V = N - 2 + N + 1 .$$

Which simplifies to our desired answer. □

Remark 2 In symbols,

$$e(N) = t^*_2(2N - 1) . \qquad (9)$$

Where the t stands for tree, the 2 for binary and the $*$ for rooted. At the level of arenas [29],

$$\mathfrak{EN}(N) \cong \mathfrak{T}\mbox{ree}^*_2(2N - 1) . \qquad (10)$$

Where $\mathfrak{EN}(N)$ stands for arena of eigenclustering networks supported by the N-body problem.

3 The count using Algebra

Remark 1 The ways of performing Remark 1 of Sec 2's procedure are in $1 : 1$ correspondence with the sequential binary bracketings of the sum of N 1's [27]. Where the 1's encode the indistinguishable points-or-particles. And the binary bracketing encodes the two subsystems into one operation.

The total number of 1's within a bracket counts a subsystem's number of points-or-particles. Which then indexes the strength of the CoM produced by the bracket. The resulting strings of 1's, sums and brackets are displayed in column 2 of Fig 1. For $N = 0$ to 5. [40] shall cover up to $N = 8$ in more detail than the current Article.

Remark 2 These strings are in $1 : 1$ correspondence with the number of ways for bracketing x^N for a multiplication that is commutative but non-associative. In symbols,

$$e(N) = c\overline{a}(N) . \qquad (11)$$

Where the c stands for commutative, the a for associative and the bar for 'non'.

4 Our result

Theorem 1 Eigenclustering network count returns the Wedderburn–Etherington numbers.

<u>Proof</u> Each of the following standardly return these numbers. The count of Lemma 1's objects provides a Graph Theory proof. While the count of Remark 2 of Sec 3's objects – Etherington's algebraic representation [8] – provides an Algebra proof. □

Remark 1 The *Wedderburn–Etherington numbers* $w(N)$ [7, 8, 15, 24] are standard and tabulated. See Fig 2.b) for the first 20, and [36] for more and links to many more.

Remark 2 See Fig 2.a) for a symbolic rendition of Theorem 1. See [40] for Etherington's algebraic representation for $N = 0$ to 8.

5 Discussion

Remark 1 Up to $N = 8$,

$$e(N) = w(N) = t(N) : \tag{12}$$

the number of unlabelled trees on N vertices [15, 24, 37]. But

$$e(9) = w(9) = 46 \neq 47 = t(9) . \tag{13}$$

Remark 2 Also, up to $N = 7$,

$$e(N) = w(N) = h(N) : \tag{14}$$

the *half-Catalan numbers* [15, 38]. But

$$e(8) = w(8) = 23 \neq 24 = h(8) . \tag{15}$$

In this case, the divergence follows from the breakdown of the independence embodied by the half-Catalan numbers' quadratic recurrence relation of convolution type. I.e.

$$e(N) = \sum_{K = 1}^{\lfloor \frac{N}{2} \rfloor} e(K)\, e(N - K) . \tag{16}$$

The $N = 8$ case is minimum for such a breakdown. For the $4\,|\,4$ eigenclustering partition does not distinguish between the following. Partitioning the first 4 as $2\,|\,2$ and the second as $3\,|\,1$ and vice versa.

Remark 3 We are interested in finding practical nomenclature for small-N eigenclustering networks. In this regard, the above rooted-binary-tree and algebraic representations are rather long.

One suggestion is to defoliate the trees down to smaller trees. This sends $2N - 1$ down to just $N - 1$. What we then arrive at are *rooted at-most binary trees* (last column of Fig 1). Which now model the network of *nontrivial CoMs*: to the exclusion of the individual points-or-particles. This remains well-adapted to eigenclustering's Physical content.

These objects are in $1 : 1$ correspondence with the previous since 'defoliate all leaves' is an isomorphism. Whose inverse is well-defined on binary tees: 'binarily (re)foliate'. See Fig 2.c) for a final symbolic rendition, where ≤ 2 denotes 'at-most binary'. While, at the level of arenas,

$$\mathfrak{T}\mathrm{ree}^*_2(2N - 1) \cong \mathfrak{T}\mathrm{ree}^*_{\leq 2}(N - 1) . \tag{17}$$

Pointer 1 A companion Article [40] provides Order-Theoretic rather than just counting considerations.

Pointer 2 Our present reason for considering eigenclustering networks is that we have recently established the following. That Apollonius' Median-Length Theorem [1, 23, 39] and Euler's Quadrilateral Theorem [3, 20, 21, 22, 31, 39] extend to [32, 33, 34, 35] one Length-Exchange Theorem per eigenclustering network. With a nontrivial such Theorem being realized for each N supporting nontrivial eigenclusterings: not just separations. So that they have other eigenclustering lengths to exchange for separations. I.e. for every $N \geq 3$, there is one such Theorem per eigenclustering network. In Euler's case, it is the Newton line interval's [2] length which is being exchanged [31].

We have thus now established that for a given N, the number of such Theorems is the corresponding Wedderburn–Etherington number $w(N)$. Which can also be described as a $\mathfrak{T}\mathrm{ree}^*_2(2N - 1)$-valued family of Theorems. Or, removing trivial content, to a $\mathfrak{T}\mathrm{ree}^*_{\leq 2}(N - 1)$-valued family of Theorems.

$N = 4$'s K and the H each generalize to give the bottom and top elements of $\mathfrak{T}\mathrm{ree}^*_{\leq 2}(N - 1)$ for each N. The corresponding $N - 1$-vertex tree graphs for these are as follows. The straight-P_n. And the path of claws (with leaf). Comprising the path formed by $\frac{N - 3}{2}$ claws for N odd. And the preceding odd case with an extra upturned leaf from its root for N even. Aside from being Order-Theoretically privileged (see [40] for further details), these [35] join the straight paths [33] in having a specific series form for their Length-Exchange Theorems.

Eigenclusterings and some representations for them

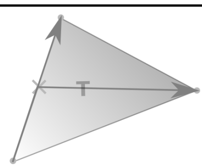
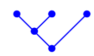
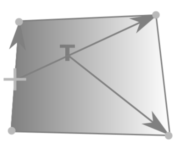
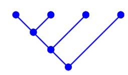
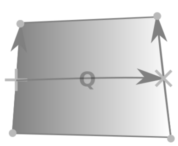
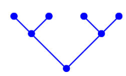
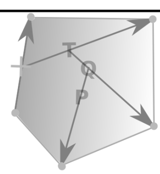
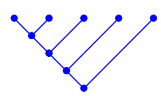
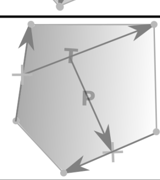
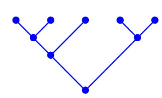
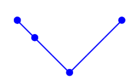
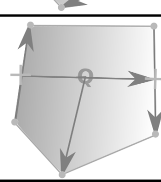
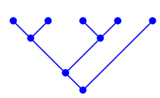
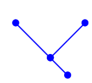
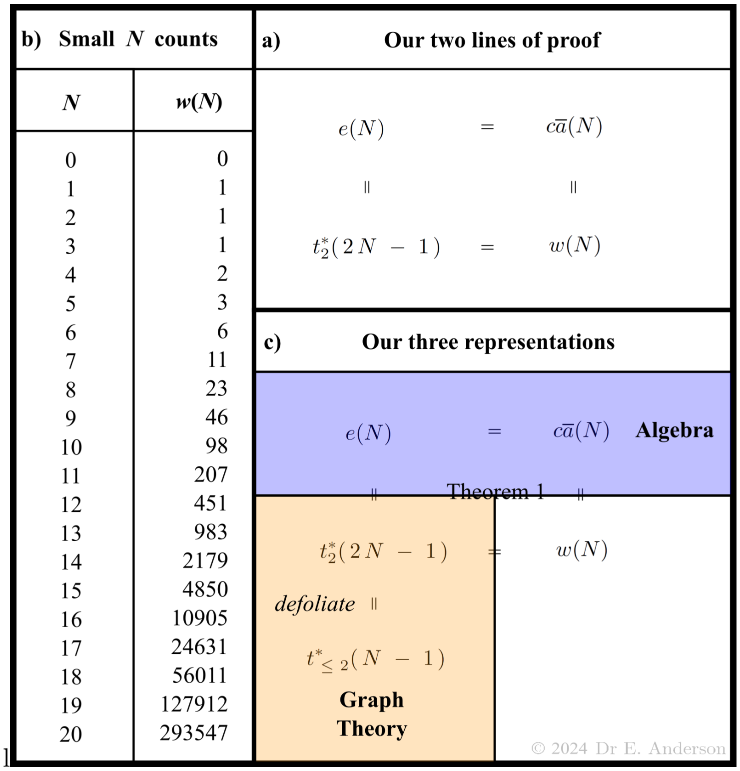
N		Eigenclustering	Algebraic rep	$t_2^*(2N-1)$ rep	$t^*_{\leq 2}(N-1)$ rep
0		ϕ	ϕ	ϕ	ϕ
1			1		ϕ
2			1 + 1		
3			(1 + 1) + 1		
4	K		((1 + 1) + 1) + 1		
	H		(1 + 1) + (1 + 1)		
5	K		(((1 + 1) + 1) + 1) + 1		
	M		((1 + 1) + 1) + (1 + 1)		
	H		((1 + 1) + (1 + 1)) + 1		

Figure 1:



N	$w(N)$
0	0
1	1
2	1
3	1
4	2
5	3
6	6
7	11
8	23
9	46
10	98
11	207
12	451
13	983
14	2179
15	4850
16	10905
17	24631
18	56011
19	127912
20	293547

Figure 2:

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