# Generalizing Heron's formula via Multi-Linear Algebra

## I. Equi-Cevians and Invertible Cevians


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**Abstract**

It has long been known that medians admit their own Heron's formula. How special are medians in this regard?

We recently re-proved the medians-data Heron formula using Linear Algebra. This working's sides-to-medians Apollonius matrix $\boldsymbol{O}$ generalizes to the sides-to-arbitrary-Cevians Stewart matrix $\boldsymbol{T}$. This provides us with the arena within which we assess the medians case's robustness: the space of triples of Cevians. The key properties of $\boldsymbol{O}$ for the medians proof are involutivity up to proportion and commutativity with the fundamental triangle matrix $\boldsymbol{F}$ at the core of Heron's formula.

We first show that $\boldsymbol{T}$ is only symmetric for the medians' $\boldsymbol{O}$. A knock-on effect of which is needing to weaken our involutivity up to proportion and commutativity requirements. We nonetheless next find a 1-parameter family – the equi-Cevians – for which our weakened conditions hold. By which our Linear Algebra proof extends and we obtain a 1-parameter family of Heron's formulae directly built out of $\boldsymbol{F}$. Which translates to these having a square-root of 4 factors formulation in which the 'semi-equi-Cevimeter' plays the role of the semi-perimeter or 'semi-medimeter'. Where *equi-Cevians* are triples of Cevians that cut their corresponding sides in equal proportion to each other.

We finally show that Heron's formula extends to invertible $\boldsymbol{T}$ but elsewise arbitrary Cevian data. Here $\boldsymbol{F}$ is not directly manifested; we rather have a 2-tensor transformation of $\boldsymbol{F}$ under $\boldsymbol{T}^{-1}$.


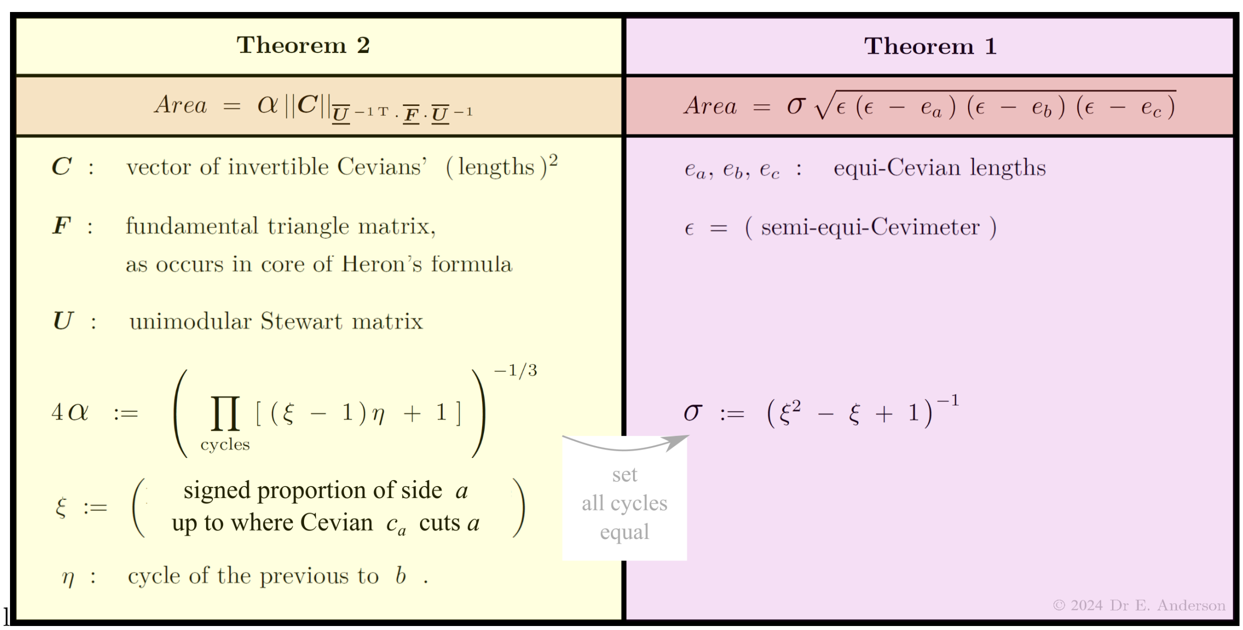


Figure 1:


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# 1 Introduction

## 1.1 Heron's formula

**Notation 1** We use the standard cyclic notation for the triangle (Fig 2).

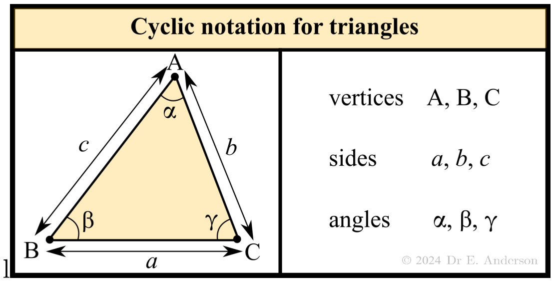


Figure 2:

**Definition 1** The *semi-perimeter* of a triangle is

$$s \ := \ \frac{1}{2} \sum_{\text{cycles}} a_i \ = \ \frac{1}{2} \, ( \, a \ + \ b \ + \ c \, ) \ . \tag{1}$$

**Theorem A (Heron's formula)** [2, 16] The area of a triangle is given by

$$Area \ = \ \sqrt{ s \prod_{\text{cycles}} ( \, s \ - \ a \, ) } \ = \ \sqrt{ s \, ( \, s \ - \ a \, ) ( \, s \ - \ b \, ) ( \, s \ - \ c \, ) } \ . \tag{2}$$

**Naming Remark 1** *Sides-data triangle area formula* is a truer name.

**Remark 1** Squaring and multiplying by $16$, we obtain the following expanded version of Heron's formula.

$$T^2 \ = \ \sum_{\text{cycles}} S_i \, ( \, 2 \, S_j \ - \ S_i \, ) \ = \ 2 \, ( \, A \, B \ + \ B \, C \ + \ C \, A \, ) \ - \ \left( \, A^2 \ + \ B^2 \ + \ C^2 \, \right) \ . \tag{3}$$

Where the $S_i$ are the *squared-sides variables*

$$A \ := \ a^2 \ \text{ and cycles } \ . \tag{4}$$

$j \neq i$ . And

$$T \ = \ 4 \ \times \ Area \tag{5}$$

is the *tetra-area*. Which is a useful variable in Shape Theory [24, 33, 51], as part of [33, 34, 37] the Hopf map [10, 18, 19, 20, 30, 49], but also even just in Flat Geometry [51].

**Remark 2** In these sides-squared variables, (3) can furthermore be expressed as the following *fundamental triangle quadratic form*.[1]

$$T^2 \ = \ ||\boldsymbol{S}||_{\boldsymbol{F}}{}^2 \ := \ \underline{\boldsymbol{S}}^{\mathrm{T}} \cdot \underline{\overline{\boldsymbol{F}}} \cdot \overline{\boldsymbol{S}} \ . \tag{6}$$

For $( \, sides \, )^2$ *vector* $\boldsymbol{S}$ . And *fundamental triangle matrix*

$$\boldsymbol{F} \ := \ \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \ . \tag{7}$$

[1] The 'fundamental triangle' name is argued for in [39, 42, 44]. Previously used names are various subsets of *Heron–Euler–Buchholz quadratic form* and *matrix* [21, 6, 37].

## 1.2 The sides-medians involution

**Definition 1** A *median* in a triangle is a line segment running from one of its vertices to the midpoint of the opposite side. A cyclic notation for these is provided in Subfig 3.b).

**Theorem B (Apollonius' sides-to-medians Theorem)** [1, 29, 22, 31]. The square of the length of the median $m_a$ emanating from vertex A of a triangle is given by the following.

$$m_a{}^2 \;=\; \frac{1}{4}\left(2\,b^2 \;+\; 2\,c^2 \;-\; a^2\right) \;. \tag{8}$$

**Remark 1** Considering all cycles of of (8) is quite common in the literature; see e.g. [22, 29].

**Remark 2** In squared variables,

$$M_A \;=\; \frac{1}{4}\,(\,2\,B \;+\; 2\,C \;-\; A\,) \quad \text{and cycles} \;\;. \tag{9}$$

**Remark 3** This cycle can furthermore be packaged into [34] the *sides-to-medians* alias *Apollonius matrix* $\boldsymbol{O}$ . I.e.

$$\boldsymbol{O} \;:=\; \frac{1}{4}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \;. \tag{10}$$

Such that

$$\overline{\boldsymbol{M}} \;=\; \underline{\overline{\boldsymbol{O}}} \cdot \overline{\boldsymbol{S}} \;, \tag{11}$$

for $\boldsymbol{M}$ the $(\,medians\,)^2$ *vector*.

♠ ◇ ♣

**Remark 4** Interestingly, $\boldsymbol{O}$ is furthermore proportional to the following involution [34].

$$\boldsymbol{J} \;:=\; \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \;. \tag{12}$$

In terms of this,

$$\overline{\boldsymbol{M}} \;=\; \frac{3}{4}\,\underline{\boldsymbol{J}} \cdot \overline{\boldsymbol{S}} \;. \tag{13}$$

*Involution* means that

$$\boldsymbol{J}^2 \;=\; \mathbb{I} \;. \tag{14}$$

A consequence of which is

$$\boldsymbol{J}^{-1} \;=\; \boldsymbol{J} \;. \tag{15}$$

So furthermore

$$\underline{\boldsymbol{J}} \cdot \overline{\boldsymbol{M}} \;\overset{(15)}{=}\; \underline{\boldsymbol{J}}^{-1} \cdot \overline{\boldsymbol{M}} \;\overset{(13)}{=}\; \underline{\boldsymbol{J}}^{-1} \cdot \frac{3}{4}\underline{\boldsymbol{J}} \cdot \overline{\boldsymbol{S}} \;=\; \frac{3}{4}\underline{\boldsymbol{J}}^{-1} \cdot \underline{\boldsymbol{J}} \cdot \overline{\boldsymbol{S}} \;=\; \frac{3}{4}\underline{\mathbb{I}} \cdot \overline{\boldsymbol{S}} \;=\; \frac{3}{4}\underline{\mathbb{I}} \cdot \overline{\boldsymbol{S}} \;.$$

Where steps 4 to 6 just pull out a common factor, and use the inverse and identity properties. And thus

$$\overline{\boldsymbol{S}} \;=\; \frac{4}{3}\,\underline{\boldsymbol{J}} \cdot \overline{\boldsymbol{M}} \;. \tag{16}$$

**Remark 5** Interestingly, $\boldsymbol{J}$ is furthermore a commutant of $\boldsymbol{F}$ [34]:

$$\underline{\overline{\mathbb{0}}} \;=\; \underline{\boldsymbol{J}} \cdot \underline{\boldsymbol{F}} \;-\; \underline{\boldsymbol{F}} \cdot \underline{\boldsymbol{J}} \;=\; [\,\underline{\boldsymbol{J}}\,,\underline{\boldsymbol{F}}\,] \;. \tag{17}$$

## 1.3 The medians-Heron formula

**Theorem C** (Medians-Heron formula).[2]
a)

$$T^2 \;=\; \left(\frac{4}{3}\right)^2 ||\boldsymbol{M}||_{\boldsymbol{F}}{}^2 \;. \tag{18}$$

b)

$$Area \;=\; \frac{4}{3}\sqrt{m \prod_{\text{cycles}} (\, m \,-\, m_a \,)} \;=\; \frac{4}{3}\sqrt{m\,(\,m \,-\, m_a\,)\,(\,m \,-\, m_b\,)\,(\,m \,-\, m_c\,)} \;. \tag{19}$$

Where the '*semi-medimeter*' [34, 35] of a triangle is given by

$$m \;:=\; \frac{1}{2}\sum_{\text{cycles}} m_i \;=\; \frac{1}{2}\,(\,m_a \,+\, m_b \,+\, m_c\,) \;. \tag{20}$$

<u>Proof</u>

$$T^2 \;=\; \underline{\boldsymbol{S}}^{\mathrm{T}} \cdot \underline{\overline{\boldsymbol{F}}} \cdot \overline{\boldsymbol{S}} \;=\; \left(\frac{4}{3}\,\underline{\overline{\boldsymbol{J}}} \cdot \overline{\boldsymbol{M}}\right)^{\mathrm{T}} \cdot \underline{\overline{\boldsymbol{F}}} \cdot \left(\frac{4}{3}\,\underline{\overline{\boldsymbol{J}}} \cdot \overline{\boldsymbol{M}}\right)$$

$$= \left(\frac{4}{3}\right)^2 \underline{\boldsymbol{M}}^{\mathrm{T}} \cdot \underline{\overline{\boldsymbol{J}}}^{\mathrm{T}} \cdot \underline{\overline{\boldsymbol{F}}} \cdot \underline{\overline{\boldsymbol{J}}} \cdot \overline{\boldsymbol{M}} \;=\; \left(\frac{4}{3}\right)^2 \underline{\boldsymbol{M}}^{\mathrm{T}} \cdot \underline{\overline{\boldsymbol{J}}} \cdot \underline{\overline{\boldsymbol{F}}} \cdot \underline{\overline{\boldsymbol{J}}} \cdot \overline{\boldsymbol{M}} \;=\; \left(\frac{4}{3}\right)^2 \underline{\boldsymbol{M}}^{\mathrm{T}} \cdot \underline{\overline{\boldsymbol{J}}} \cdot \underline{\overline{\boldsymbol{J}}} \cdot \underline{\overline{\boldsymbol{F}}} \cdot \overline{\boldsymbol{M}}$$

$$= \left(\frac{4}{3}\right)^2 \underline{\boldsymbol{M}}^{\mathrm{T}} \cdot \underline{\overline{\mathbb{I}}} \cdot \underline{\overline{\boldsymbol{F}}} \cdot \overline{\boldsymbol{M}} \;=\; \left(\frac{4}{3}\right)^2 \underline{\boldsymbol{M}}^{\mathrm{T}} \cdot \underline{\overline{\boldsymbol{F}}} \cdot \overline{\boldsymbol{M}} \;.$$

Where the first step is (6). The second is (16). The third is

$$(\,\overline{\boldsymbol{v}} \cdot \underline{\overline{\boldsymbol{A}}}\,)^{\mathrm{T}} \;=\; \underline{\boldsymbol{v}}^{\mathrm{T}} \cdot \underline{\overline{\boldsymbol{A}}}^{\mathrm{T}} \;. \tag{21}$$

The fourth is that $\boldsymbol{J}$ is symmetric, from its explicit form (12). The fifth is the commutativity (17). The sixth is the involution property. And the seventh is the identity property.

b) Work backwards through Subsec 1.1's progression. □

**Naming Remark 2** A truer name is *medians-data triangle area formula.*

**Remark 2** The factor of

$$\frac{4}{3}$$

in (19) can be interpreted as *Jacobi mass ratio* $\mu$ for the triangle with equal masses at its vertices [34]. Consequently the mass-weighted Jacobi coordinates [17, 23, 32, 43, 50] version of the medians–Heron formula is identical in form to [34] the usual (sides-)Heron formula.

[2] The first place some version of the medians-Heron formula is the second form of b) as an Exercise in Hobson [8], with a later edition [9] carrying a trigonometric proof. Among modern textbooks, e.g. [15, 22] cover the medians-Heron formula. E.g. [26] provides a geometric proof and an algebraic proof. a) and the Linear-Algebraic proof presented here are from [34].

## 1.4 Cevians and Stewart's Theorem

**Remark 1** Another way of arriving at Apollonius' Theorem (8) is as a medians Corollary to Stewart's Theorem. The general case of which's natural setting is for Cevians.

**Definition 1** A *Cevian* [3, 4, 11, 12, 25, 31] is any line from a vertex of a triangle to its opposite side (if needs be extended).

**Notational Remark 1** See Fig 3.a) for notation for the general case. And Subfigs b)-d) for specific examples, with notation for their line-interval lengths and concurrent points. In particular, medians are among the Cevians, being those special Cevians that each bisect their corresponding side.

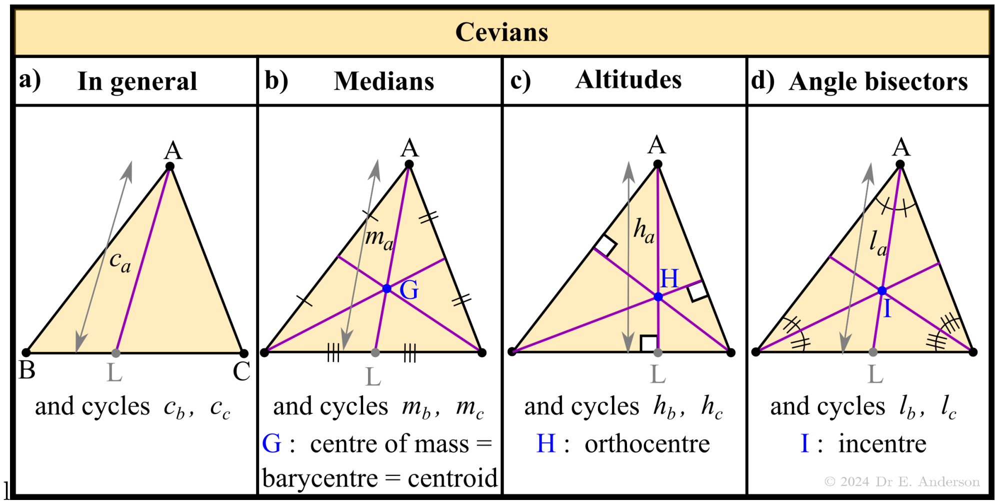


Figure 3:

**Naming Remark 3** Indeed, the name 'Cevian' is a portmanteau of 'Ceva' and 'median'.

**Definition 1** The *directed length* between points P, Q is given by the following.

$$\overline{\mathrm{PQ}} := \begin{cases} \mathrm{PQ} & \text{if running from P to Q} \\ \text{-PQ} & \text{if running from Q to P} \end{cases} . \tag{22}$$

**Theorem D (Stewart's Cevian-length Theorem)** [5, 13, 22, 29, 31] a) Let A be a Cevian's vertex and L be the point at which the opposite side (extended) is cut. Then

$$\mathrm{AL}^2 = \frac{\overline{\mathrm{LC}}}{\overline{\mathrm{BC}}}\mathrm{AB}^2 + \frac{\overline{\mathrm{BL}}}{\overline{\mathrm{BC}}}\mathrm{AC}^2 - \overline{\mathrm{BL}}\,\overline{\mathrm{LC}} . \tag{23}$$

b) Using the corresponding likewise-directed length variables, the squared length of our Cevian is given by

$$l^2 = \frac{y}{a}c^2 + \frac{x}{a}b^2 - x\,y . \tag{24}$$

c) Or equivalently,

$$a\left(l^2 + x\,y\right) = x\,b^2 + y\,c^2 . \tag{25}$$

**Exercise 1** Check that a) $\Rightarrow$ b) $\Rightarrow$ c). And that Stewart's Theorem indeed returns Apollonius' Theorem in the case of medians.

### 1.5 Outline of the rest of this Article

**Remark 1** The purpose of the current Series [45, 47, 46] – also referred to as Articles II, III and IV – is to conduct the following robustness test on Subsec 1.3's proof. To what extent does this working generalize upon passing from medians to Cevians?

**Remark 2** In the current Article, we first form and study the Stewart matrix that encodes all cycles of Stewart's Theorem. We next find a 1-parameter family of equi-Cevians (Sec 3) each of which possesses a Heron's formula to the same extent that medians do: with $\boldsymbol{F}$ manifestly realized. We also observe that (Sec 4) Heron's formula extends to invertible $\boldsymbol{T}$ but elsewise arbitrary Cevian data. Now as a 2-tensor transformation of the fundamental triangle matrix $\boldsymbol{F}$ under the inverse Stewart matrix $\boldsymbol{T}^{-1}$ . The rest of the current Series is outlined in Sec 5.

## 2 The 'Stewart' sides-to-Cevians transformation matrix

### 2.1 Dimensionless formulation

**Remark 1** The Stewart matrix's natural arena is for arbitrary triples of Cevians: one per vertex-opposite-side pair. Let us denote this arena by

$$\mathfrak{Ceva}(3) \ ;$$

$\mathfrak{Ceva}$ will be used elsewhere [47] to denote the arena of individual Cevians. The (Apollonius) sides–medians involution $\boldsymbol{J}$ [34, 40, 42] thereby generalizes to arbitrary (Stewart) sides–Cevians transformations. However, as we shall document below, none of the three properties of $\boldsymbol{J}$ used in Sec 1.3's proof – symmetry involutivity and commutativity with $\boldsymbol{F}$ is generic.

**Notational Remark 1** Let $\boldsymbol{C}$ be the vectors whose components are the lengths squared of our arbitrary triple of Cevians,

$$C_A \ = \ {c_a}^2 \ \text{ and cycles .} \tag{26}$$

**Notational Remark 2** Also let

$$\xi \ := \ \frac{x}{a} \ \text{ and cycles } \ \eta \ , \ \ \zeta \ : \tag{27}$$

the *fraction-of-a-side* signed-ratio variables forming the vector $\boldsymbol{\xi}$ .

**Proposition 1** The dimensionless formulation of Stewart's Theorem is as follows.

a)

$${c_a}^2 \ = \ \xi\,(\,\xi \ - \ 1\,)\,a^2 \ + \ \xi\,b^2 \ + \ (\,1 \ - \ \xi\,)\,c^2 \ . \tag{28}$$

b) Or, in squared variables, the linear equation

$$C_A \ = \ \xi\,(\,\xi \ - \ 1\,)\,A \ + \ \xi\,B \ + \ (\,1 \ - \ \xi\,)\,C \ . \tag{29}$$

**Exercise 2** Prove this.

**Proposition 2** The *sides-to-Cevians transformation* is

$$\overline{\boldsymbol{C}} \ = \ \underline{\overline{\boldsymbol{T}}} \cdot \overline{\boldsymbol{S}} \ . \tag{30}$$

For *Stewart matrix*

$$\boldsymbol{T} \ := \ \boldsymbol{T}(\boldsymbol{\xi}) \ := \ \begin{pmatrix} \xi\,(\,\xi \ - \ 1\,) & \xi & 1 \ - \ \xi \\ 1 \ - \ \eta & \eta\,(\,\eta \ - \ 1\,) & \eta \\ \zeta & 1 \ - \ \zeta & \zeta\,(\,\zeta \ - \ 1\,) \end{pmatrix} \ . \tag{31}$$

<u>Proof</u> Package the 3 cycles of Proposition 1.b)'s linear equations into a matrix equation. □

**Remark 2** So far as we are aware, this is a new dimensionless formulation of the Stewart matrix. Corresponding furthermore to a 3-parameter parametrization of $\mathfrak{C}\text{eva}(3)$ .

**Remark 3**

$$\det \boldsymbol{T} \;=\; \prod_{\text{cycles}} \left[\, (\, \xi \,-\, 1\,)\, \eta \,+\, 1 \,\right] . \tag{32}$$

**Exercise 1** Prove this using basic algebra only.

## 2.2 A unique property of the medians

**Proposition 3** The Stewart matrix $\boldsymbol{T}$ is symmetric iff the Cevians in question are all medians.

Proof Set

$$\boldsymbol{T} \;=\; \boldsymbol{T}^{\mathrm{T}} \tag{33}$$

– where the T-superscript denotes transpose – and equate components. The diagonal components return mere identities. While the off-diagonal components yield 2 copies of the following system.

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} \;=\; \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} . \tag{34}$$

The matrix in question being invertible, there is a unique solution. Which is

$$\xi \;=\; \frac{1}{2} \quad \text{and cycles} \quad : \quad \text{medians} . \;\square \tag{35}$$

## 2.3 A first two consequences

**Remark 1** For other than the medians case, the Stewart matrix's asymmetry renders transposition necessary in formulating Heron's formula for Cevian data.

**Remark 2** It also turns out that a suitable generalization of the involution property is the following Proposition's condition.

**Proposition 4**

$$\underline{\overline{\boldsymbol{T}}}^{\mathrm{T}} \cdot \underline{\overline{\boldsymbol{T}}} \;=\; k^2 \,\underline{\overline{\mathbb{I}}} \tag{36}$$

gives the following system of equations.

a)

$$\xi^3 \,-\, \xi^2 \,-\, \eta^3 \,+\, 2\,\eta^2 \,-\, \eta \,+\, \zeta \,-\, \zeta^2 \;=\; 0 \quad \text{and cycles} . \tag{37}$$

$$\left(\, \xi^4 \,-\, 2\,\xi^3 \,+\, \xi^2 \,+\, \zeta^2 \,+\, \eta^2 \,-\, 2\,\eta \,+\, 1 \,\right)^3 \;=\; k^2 \quad \text{and cycles} . \tag{38}$$

b) Equivalently,

$$\xi^2\,(\,1 \,-\, \xi\,) \,+\, \eta\,(\,\eta \,-\, 1\,)^2 \;=\; \zeta(\,1 \,-\, \zeta\,) \quad \text{and cycles} . \tag{39}$$

$$\xi^2\,(\,\xi \,-\, 1\,)^2 \,+\, (\,\eta \,-\, 1\,)^2 \,+\, \zeta^2 \;=\; k^{2/3} \quad \text{and cycles} . \tag{40}$$

Proof a) Multiplying out the matrices, the off-diagonal components yield two copies of (37). Whereas the diagonal components return (38).

b) Factorize. And, for the second equation, then take the cube root of both sides. $\square$

## 2.4 (Non)singular triples of Cevians

**Definition 1** The *singular* $\boldsymbol{T}$ are precisely those $\boldsymbol{N}$ for which

$$\det \boldsymbol{T} \;=\; \det \boldsymbol{N} \;=\; 0 \;. \tag{41}$$

The corresponding singular triples of Cevians form the *arena of singular Cevian triples*,

$$\mathfrak{Sing}(3) \;\subset\; \mathfrak{Ceva}(3) \;.$$

Whereas the *nonsingular* $\boldsymbol{T}$ are precisely those $\boldsymbol{G}$ for which

$$\det \boldsymbol{T} \;=\; \det \boldsymbol{G} \;\neq\; 0 \;. \tag{42}$$

The corresponding invertible triples of Cevians form the *arena of nonsingular* alias *invertible Cevian triples*,

$$\mathfrak{Inv}(3) \;\subset\; \mathfrak{Ceva}(3) \;, \;\; \mathfrak{Ceva}(3) \;=\; \mathfrak{Inv}(3) \;\amalg\; \mathfrak{Sing}(3) \;.$$

**Remark 1** We show in Article III that $\mathfrak{Inv}(3)$ is 3-$d$ whereas $\mathfrak{Sing}(3)$ is just 2-$d$ . This corresponds to nonsingular triples of Cevians being generic; below we work solely within this generic case.

**Remark 2** $k$ is not a free parameter, however, by the *dilation-unimodular split.* This is a type of irreducible tensor split, and thus well-grounded in Representation Theory [14, 27]. Where *unimodular* means of unit determinant. Within our 3-$d$ arena $\mathfrak{Inv}(3) \subset \mathfrak{Ceva}(3)$ , this split is as follows.

$$\boldsymbol{G} \;=\; (\det \boldsymbol{G})^{1/3}\, \boldsymbol{U} \;. \tag{43}$$

Where the *unimodular Stewart matrix* $\boldsymbol{U}$ is just defined by rearranging:

$$\boldsymbol{U} \;:=\; (\det \boldsymbol{G})^{-1/3}\, \boldsymbol{G} \;. \tag{44}$$

**Proposition 5** a) Invertible matrices admit a dilation–unimodular split

$$\underline{\overline{\boldsymbol{G}}} \;=\; \underline{\overline{\boldsymbol{D}}} \cdot \underline{\overline{\boldsymbol{U}}} \;. \tag{45}$$

b) For invertible Stewart matrices, (36) takes the simplified form

$$\underline{\overline{\boldsymbol{U}}}^{\mathrm{T}} \cdot \underline{\overline{\boldsymbol{U}}} \;=\; \underline{\overline{\mathbb{I}}} \;. \tag{46}$$

c) (45) is a fortiori a scale-rotation split.

d) The second equation (38) in (36)'s system takes the following more specific form.

$$\big(\, \xi^2\, (\xi \,-\, 1)^2 \;+\; (\eta \,-\, 1)^2 \;+\; \zeta^2 \,\big)^3 \;=\; (\det \boldsymbol{G})^2 \;. \tag{47}$$

Proof a)

$$\underline{\overline{\boldsymbol{G}}} \;=\; (\det \boldsymbol{G})^{1/3}\, \underline{\overline{\boldsymbol{U}}} \;=\; (\det \boldsymbol{G})^{1/3} \left( \underline{\overline{\mathbb{I}}} \cdot \underline{\overline{\boldsymbol{U}}} \right) \;=\; \Big( (\det \boldsymbol{G})^{1/3}\, \underline{\overline{\mathbb{I}}} \Big) \cdot \underline{\overline{\boldsymbol{U}}} \;.$$

Step 1 is by (43). Step 2 is an insertion of the identity. Step 3 is by associativity. Whose final term in large brackets can be interpreted as a dilation $\boldsymbol{D}$ .

b) Divide through by $\tau^2$ and then make 2 uses of the definition of unimodular matrix.

c) By the uniqueness of inverse, b) is the orthogonality condition.

d) Equate coefficients to fix that $k \;=\; (\det \boldsymbol{G})^{1/3}$ . Finally use Proposition 4.b)'s form for the left-hand-side. □

**Remark 3** By substituting (32) in the second of system's equations (47), we arrive at a cyclic triple of 12th-order trinomials. This is coupled to the first equation's cyclic triple of cubic trinomials. Generically, this system is over-determined by a factor of 2 : 6 equations in 3 unknowns.

# 3 The equi-Cevian case

## 3.1 The equi-Cevian solution

**Proposition 6**

$$\boldsymbol{\xi} \;=\; \xi\,\mathbf{1}\;, \quad \text{i.e.}\;\; \xi \;=\; \eta \;=\; \zeta\;, \tag{48}$$

solves the above system.

<u>Proof</u> In this case, the first equation readily reduces to the trivial identity $0 \;=\; 0$ .

Substitute (48) in (31) to collapse the Stewart matrix to the following.

$$\boldsymbol{T}(\xi) \;:=\; \boldsymbol{T}(\,\xi\,\mathbf{1}\,) \;=\; \begin{pmatrix} \xi\,(\,\xi\;-\;1\,) & \xi & 1\;-\;\xi \\ 1\;-\;\xi & \xi\,(\,\xi\;-\;1\,) & \xi \\ \xi & 1\;-\;\xi & \xi\,(\,\xi\;-\;1\,) \end{pmatrix}\;. \tag{49}$$

With determinant

$$\det\boldsymbol{T}(\xi) \;=\; \mathcal{T}^3\;. \tag{50}$$

Where

$$\mathcal{T} \;:=\; \xi^2 \;-\; \xi \;+\; 1\;. \tag{51}$$

Also the LHS of (47) collapses to

$$\mathcal{T}^2\;. \tag{52}$$

So our second equation becomes just

$$\left(\,\mathcal{T}^3\,\right)^2 \;=\; \left(\,\mathcal{T}^2\,\right)^3\;.$$

Which also readily holds identically. □

**Naming Remark 4** Condition (48) turns out to be significant. Let us thus give this case a name: the *equi-Cevians*. Meaning that they each cut their corresponding side with the same side-fraction ratio as each other.

**Notational Remark 1** We subsequently write $\boldsymbol{E}$ for $\boldsymbol{C}_{\text{equi}}$ and $\boldsymbol{Q}$ for $\boldsymbol{T}_{\text{equi}} \;=\; \boldsymbol{T}(\,\xi\,\mathbf{1}\,) \;=\; \boldsymbol{T}(\xi)$ .

**Remark 1** Equi-Cevians are uniquely parametrized by their common ratio's value, thus forming a $\mathbb{R}$ of distinct cases. Thereby, we are only considering a non-generic subset

$$\mathfrak{E}\text{qui}(3) \;\subset\; \mathfrak{C}\text{eva}(3)\;. \tag{53}$$

In Article III, this will be upgraded to a subspace condition.

**Remark 2** The triple of medians, and the triple of sides are themselves examples of equi-Cevians. Medians correspond to

$$\xi \;=\; \frac{1}{2}\;.$$

Sides get represented twice,

$$\xi \;=\; 0\;,\;\; 1\;:$$

the ambiguity of choosing the clockwise or anticlockwise sides to serve as Cevians.

### 3.2 Equi-Cevian data's 1-parameter generalization of Heron's formula

**Proposition 7** In the equi-Cevian case, we have the following 1-parameter family of Heron's formulae.

$$T^2 \;=\; (\det \boldsymbol{Q})^{-4/3}\, \underline{\boldsymbol{E}}^{\mathrm{T}} \cdot \underline{\overline{\boldsymbol{Q}}} \cdot \underline{\overline{\boldsymbol{F}}} \cdot \underline{\overline{\boldsymbol{Q}}}^{\mathrm{T}} \cdot \overline{\boldsymbol{E}} \; . \tag{54}$$

<u>Proof</u> Step -1 . Claim: $\det \boldsymbol{Q} \;\neq\; 0$ .

For $\det\boldsymbol{Q} \;=\; \mathcal{T}^3$ . And $\mathcal{T} \;=\; 0$ possesses just a complex pair of roots. Which are thus not realized by any Cevians in the *real* Euclidean plane...

Step 0: Thus we can always meaningfully formulate the *Cevians-to-sides transformation*,

$$\overline{\boldsymbol{S}} \;=\; \underline{\overline{\boldsymbol{Q}}}^{-1} \cdot \overline{\boldsymbol{E}} \;: \tag{55}$$

Next substitute this into Heron's formula (6) to obtain the first step of the following.

$$T^2 \;=\; \left(\underline{\overline{\boldsymbol{Q}}}^{-1} \cdot \overline{\boldsymbol{E}}\right)^{\mathrm{T}} \cdot \underline{\overline{\boldsymbol{F}}} \cdot \left(\underline{\overline{\boldsymbol{Q}}}^{-1} \cdot \overline{\boldsymbol{E}}\right) \;=\; \underline{\boldsymbol{E}}^{\mathrm{T}} \cdot \underline{\overline{\boldsymbol{Q}}}^{-1\,\mathrm{T}} \cdot \underline{\overline{\boldsymbol{F}}} \cdot \underline{\overline{\boldsymbol{Q}}}^{-1} \cdot \overline{\boldsymbol{E}} \;=$$

$$\underline{\boldsymbol{E}}^{\mathrm{T}} \cdot \frac{\underline{\overline{\boldsymbol{Q}}}}{k^2} \cdot \underline{\overline{\boldsymbol{F}}} \cdot \frac{\underline{\overline{\boldsymbol{Q}}}^{\mathrm{T}}}{k^2} \cdot \overline{\boldsymbol{E}} \;=\; (\det \boldsymbol{Q})^{-4/3}\, \underline{\boldsymbol{E}}^{\mathrm{T}} \cdot \underline{\overline{\boldsymbol{Q}}} \cdot \underline{\overline{\boldsymbol{F}}} \cdot \underline{\overline{\boldsymbol{Q}}}^{\mathrm{T}} \cdot \overline{\boldsymbol{E}} \; . \tag{56}$$

Where the second step uses (21), and the third (36)'s proportionality. Finally, the fourth step evaluates the coefficient of proportionality. □

**Proposition 8** In the equi-Cevian case,
a)

$$\underline{\overline{\boldsymbol{Q}}} \cdot \underline{\overline{\boldsymbol{F}}} \cdot \underline{\overline{\boldsymbol{Q}}}^{\mathrm{T}} \;=\; \mathcal{T}^2\, \underline{\overline{\boldsymbol{F}}} \; . \tag{57}$$

b) Equivalently,

$$\underline{\overline{\boldsymbol{W}}} \cdot \underline{\overline{\boldsymbol{F}}} \cdot \underline{\overline{\boldsymbol{W}}}^{\mathrm{T}} \;=\; \underline{\overline{\boldsymbol{F}}} \; . \tag{58}$$

Where $\boldsymbol{W}$ is a unimodular equi-Cevian matrix.

<u>Proof</u> a) Substitute in (49) and (7). Then apply matrix multiplication, factorization and (51).

b) Divide through by $\mathcal{T}^2$ and make 2 uses of the definition of unimodular matrix. □

**Remark 1** This result is a weaker substitute for the median case's commutativity condition (17), which however turns out to satisfy our purposes below. As necessitated by $\boldsymbol{Q}$ no longer being symmetric (a third consequence!)

♠ ◇ ♣

**Theorem 1 (Equi-Cevian Heron's Formulae) [Anderson 2018]** Consider equi-Cevians $e_i$ and cycles corresponding to the side-split ratios having equal values (48). Then we have the following 1-parameter family of Heron's formulae for the area of the triangle based on $e_i$ as data.
a)

$$T^2 \;=\; \sigma^2\, ||\boldsymbol{E}||_{\boldsymbol{F}}{}^2 \; . \tag{59}$$

For

$$\sigma \;:=\; \mathcal{T}^{-1} \;=\; \left(\xi^2 \;-\; \xi \;+\; 1\right)^{-1} \; . \tag{60}$$

b)

$$T^2 \;=\; \sigma^2 \sum_{\text{cycles}} e_i \;(\, 2\, e_j \;-\; e_i \,) \; . \tag{61}$$

c) Or, in the most traditional format,

$$\mathit{Area} \;=\; \left(\, \xi^2 \;-\; \xi \;+\; 1 \,\right)^{-1} \sqrt{\epsilon\, (\epsilon \;-\; e_a\,)\, (\epsilon \;-\; e_b\,)\, (\epsilon \;-\; e_c\,)} \; . \tag{62}$$

Where

$$\epsilon \;:=\; \frac{1}{2} \sum_{\text{cycles}} e_i \;=\; \frac{1}{2}\, (\, e_a \;+\; e_b \;+\; e_c\,) \; . \tag{63}$$

Proof a) Use (50) and Proposition 8 in Proposition 7 to obtain

$$T^2 \;=\; \mathcal{T}^{-4}\,\underline{\boldsymbol{E}}^{\mathrm{T}} \cdot \mathcal{T}^2\,\underline{\overline{\boldsymbol{F}}} \cdot \overline{\boldsymbol{E}} \;=\; \mathcal{T}^{-2}\,\underline{\boldsymbol{E}}^{\mathrm{T}} \cdot \underline{\overline{\boldsymbol{F}}} \cdot \overline{\boldsymbol{E}} \;=\; \sigma^2\,||\boldsymbol{E}||_{\boldsymbol{F}}{}^2 \;.$$

Where the second step gathers terms. And the third uses (60) and the definition of $||\ ||$ .

b)-c) Run backwards along the Introduction's set of formulations of the standard (sides-data) Heron's formula. □

**Example 1** $\xi \;=\; 0$ and $1$ are the two available ways of encoding sides. In the first case,

$$\sigma \;=\; \left(0^2 \;-\; 0 \;+\; 1\right)^{-1} \;=\; (\,0 \;-\; 0 \;+\; 1\,)^{-1} \;=\; 1^{-1} \;=\; 1\;.$$

In the second case,

$$\sigma \;=\; \left(1^2 \;-\; 1 \;+\; 1\right)^{-1} \;=\; (\,1 \;-\; 1 \;+\; 1\,)^{-1} \;=\; (\,0 \;+\; 1\,)^{-1} \;=\; 1^{-1} \;=\; 1\;.$$

Thereby the standard Heron's formula is recovered.

**Example 2** While for medians, $\xi \;=\; 1/2$ , so

$$\sigma \;=\; \left[\left(\frac{1}{2}\right)^2 \;-\; \frac{1}{2} \;+\; 1\right]^{-1} \;=\; \left(\frac{1}{4} \;-\; \frac{1}{2} \;+\; 1\right)^{-1} \;=\; \left(\frac{3}{4}\right)^{-1} \;=\; \frac{4}{3}\;.$$

By which the medians-data Heron's formula is recovered.

**Naming Remark 5** $\epsilon$ is the equi-Cevians' subcase of the *semi-Cevimeter*. I.e. the half-sum of our triple of Cevian intervals' lengths. In immediate analogy to the semi-perimeter for sides and the semi-medimeter for medians.

**Naming Remark 6** A truer name for Theorem 1 is *Equi-Cevians data Triangle-Area Formulae.*

**Remark 2** Let us finally check how the medians case of Proposition 8's condition indeed recovers Sec 1.3's commutativity condition.

$$\left(\frac{3}{4}\right)^2 \underline{\overline{\boldsymbol{F}}} \;=\; \mathcal{T}^2\,\underline{\overline{\boldsymbol{F}}} \;=\; \underline{\overline{\boldsymbol{O}}} \cdot \underline{\overline{\boldsymbol{F}}} \cdot \underline{\overline{\boldsymbol{O}}}^{\mathrm{T}} \;=\; \underline{\overline{\boldsymbol{O}}} \cdot \underline{\overline{\boldsymbol{F}}} \cdot \underline{\overline{\boldsymbol{O}}}$$

$$\Rightarrow\;\; \left(\frac{3}{4}\right)^2 \underline{\overline{\boldsymbol{O}}}^{-1} \cdot \underline{\overline{\boldsymbol{F}}} \;=\; \underline{\overline{\boldsymbol{O}}}^{-1} \cdot \underline{\overline{\boldsymbol{O}}} \cdot \underline{\overline{\boldsymbol{F}}} \cdot \underline{\overline{\boldsymbol{O}}} = \underline{\overline{\mathbb{I}}} \cdot \underline{\overline{\boldsymbol{F}}} \cdot \underline{\overline{\boldsymbol{O}}} \;=\; \underline{\overline{\boldsymbol{F}}} \cdot \underline{\overline{\boldsymbol{O}}}$$

$$\Rightarrow\;\; \left(\frac{3}{4}\underline{\overline{\boldsymbol{O}}}\right) \cdot \underline{\overline{\boldsymbol{F}}} \;=\; \underline{\overline{\boldsymbol{F}}} \cdot \left(\frac{4}{3}\underline{\overline{\boldsymbol{O}}}\right) \;\;\Rightarrow\;\; \underline{\overline{\boldsymbol{J}}} \cdot \underline{\overline{\boldsymbol{F}}} = \underline{\overline{\boldsymbol{F}}} \cdot \underline{\overline{\boldsymbol{J}}} \;\;\Rightarrow\;\; [\,\underline{\overline{\boldsymbol{J}}}\,,\,\underline{\overline{\boldsymbol{F}}}\,] \;=\; \underline{\overline{\mathbb{0}}}\;.$$

# 4 Invertible but elsewise general Cevian data's 3-parameter generalization of Heron's formula

**Remark 1** We can also obtain a more general type of Heron's formula as follows. Alongside asymmetry rendering transposes necessary, non-involutivity renders inverses necessary. And 'general' is in the sense of belonging to $\mathfrak{Inv}(3)$ rather than $\mathfrak{Sing}(3)$ .

**Theorem 2 (Invertible-Cevians Heron's Formulae) [Anderson 2019]** For $\boldsymbol{T}$ nonsingular, Heron's formula in terms of elsewise arbitrary Cevian length data is given by the following.

a)

$$T^2 = (\det \boldsymbol{G})^{-4/3} \underline{\boldsymbol{C}}^{\mathrm{T}} \cdot \underline{\overline{\boldsymbol{G}}}^{-1\,\mathrm{T}} \cdot \underline{\overline{\boldsymbol{F}}} \cdot \underline{\overline{\boldsymbol{G}}}^{-1} \cdot \overline{\boldsymbol{C}} = (\det \boldsymbol{G})^{-4/3} ||\boldsymbol{C}||_{\underline{\overline{\boldsymbol{G}}}^{-1\,\mathrm{T}} \cdot \underline{\overline{\boldsymbol{F}}} \cdot \underline{\overline{\boldsymbol{G}}}^{-1}}{}^2 \, . \tag{64}$$

b)

$$\begin{aligned} T &= (\det \boldsymbol{G})^{-1/3} \sqrt{\underline{\boldsymbol{C}}^{\mathrm{T}} \cdot \underline{\overline{\boldsymbol{U}}}^{-1\,\mathrm{T}} \cdot \underline{\overline{\boldsymbol{F}}} \cdot \underline{\overline{\boldsymbol{U}}}^{-1} \cdot \overline{\boldsymbol{C}}} \\ &= \left( \prod_{\text{cycles}} [\,(\xi - 1)\,\eta + 1\,] \right)^{-1/3} ||\boldsymbol{C}||_{\underline{\overline{\boldsymbol{U}}}^{-1\,\mathrm{T}} \cdot \underline{\overline{\boldsymbol{F}}} \cdot \underline{\overline{\boldsymbol{U}}}^{-1}} \, . \end{aligned} \tag{65}$$

<u>Proof</u> a) The zeroth to second steps of Proposition 7's proof extend from $\boldsymbol{Q}$ to all invertible $\boldsymbol{G}$ . Finish off with the definition of $||\ ||$ .

b) The unimodality split still holds, so we can insert this into a). Next apply expression (32) for the determinant. Finish off with the definition of $||\ ||$ . $\Box$

**Naming Remark 7** A truer name for Theorem 2 is *invertible-Cevians data triangle-area formulae.*

**Impasse 1** We can however no longer interpret the unimodal matrix as an involution or the weakening (36). The third step of Proposition 7 is thereby blocked.

**Remark 2** Let us finally parallel how [34] denoted the medians-Heron Formula's numerical prefactor by $\kappa$ until its nature was better understood. By rewriting Theorem 3 as

$$T = \mathcal{K} \, ||\boldsymbol{E}||_{\underline{\overline{\boldsymbol{U}}}^{-1\,\mathrm{T}} \cdot \underline{\overline{\boldsymbol{F}}} \cdot \underline{\overline{\boldsymbol{U}}}^{-1}} \, . \tag{66}$$

**Remark 3** Finally, to relate to the 'coverpage' Fig 1,

$$4\,\mathcal{\alpha} := \mathcal{K} = (\det \boldsymbol{G})^{-1/3} = \left( \prod_{\text{cycles}} [\,(\xi - 1)\,\eta + 1\,] \right)^{-1/3} . \tag{67}$$

**Exercise 3**$^{+}$ We suggest that the Reader pauses awhile here, to see if you can find any reasons why $\mathcal{\sigma}$ or $\mathcal{K}$ should or should not be a (function of) some Jacobi mass ratio.

# 5 Conclusion

## 5.1 Sphynxnopsis

The 'Apollonius' sides-to-medians transformation matrix $\underline{\underline{\boldsymbol{O}}}$ has $3$ special properties. Namely, symmetry, proportionality to an involution $\underline{\underline{\boldsymbol{J}}}$, and commutativity with the fundamental triangle matrix $\underline{\underline{\boldsymbol{F}}}$. These combine to guarantee that the standard (sides-data) Heron's formula is partnered by the medians-data Heron's formula. With both built directly out of $\underline{\underline{\boldsymbol{F}}}$, by which each has the most traditional and recognizable square root formulation as well.

Medians generalize to Cevians. The analogue of $\underline{\underline{\boldsymbol{T}}}$ is now the 'Stewart' sides-to-Cevians transformation matrix, $\underline{\underline{\boldsymbol{T}}}$. For which we have provided a $3$-parameter dimensionless-ratio formulation. Forming the arena (space of mathematical objects) $\mathfrak{Ceva}(3)$.

In answer to the Abstract's opening question, medians are special in the following ways. Symmetry holds for no other triples of Cevians. While the other two properties weakly extend to the $1$-parameter family $\mathfrak{Equi}(3)$ of equi-Cevians: those Cevian triples that each cut their corresponding side with the same side-fraction ratio as each other. Equi-Cevian data consequently supports a new $1$-parameter family of Heron's formulae also built directly out of $\underline{\underline{\boldsymbol{F}}}$, as per Theorem 1. I.e. directly manifesting the Heron quadratic form.

Also invertible Cevian data, forming the $3$-$d$ subset $\mathfrak{Inv}(3)$, supports a new $3$-parameter family of Heron's formulae as per Theorem 2. Now in general not built directly out of $\underline{\underline{\boldsymbol{F}}}$, but rather its $2$-tensor transformation under the inverse of the invertible Stewart matrix, $\underline{\underline{\boldsymbol{G}}}^{-1}$. Or, in an alternative formulation, under the inverse of the unimodular Stewart matrix, $\underline{\underline{\boldsymbol{U}}}^{-1}$.

**Naming Endnote 1** What do we mean by 'weakened'? On the one hand, we have replaced medians' involutivity by equi-Cevians' relation (36). Which is *homogeneous similarity* in the Geometrical sense of orthogonality up to proportion. On the other hand, we have replaced medians' $\boldsymbol{F}$-commutativity by equi-Cevians' (57). Which is a weighted version of $2-$tensor invariance for $\underline{\underline{\boldsymbol{F}}}$ under the sides-to-equi-Cevians matrices $\underline{\underline{\boldsymbol{Q}}}$ [36]. Corresponding to the *homogeneous-Euclidean* alias *Cartesian* notion of tensors.

If (36) holds, then (57) can be recast as

$$\underline{\underline{\boldsymbol{Q}}} \cdot \underline{\underline{\boldsymbol{F}}} \cdot \underline{\underline{\boldsymbol{Q}}}^{-1} \;=\; \underline{\underline{\boldsymbol{F}}} \, . \tag{68}$$

Reading that $\underline{\underline{\boldsymbol{F}}}$ is self-similar, now in the matrix sense of 'similar'. By which [36] medians' $\underline{\underline{\boldsymbol{O}}}$'s involutivity and $\boldsymbol{F}$-commutativity have been replaced by $\underline{\underline{\boldsymbol{Q}}}$ being similar and $\underline{\underline{\boldsymbol{F}}}$ being *doubly-similar*!

**Geometrical Endnote 1** Our two weakened conditions can furthermore be jointly packaged as follows.

$$\left.\begin{array}{ccc} \underline{\underline{\boldsymbol{W}}}^{\mathrm{T}} \cdot \underline{\underline{\mathbb{I}}} \cdot \underline{\underline{\boldsymbol{W}}} & = & \underline{\underline{\mathbb{I}}} \\ \underline{\underline{\boldsymbol{W}}}^{\mathrm{T}} \cdot \underline{\underline{\boldsymbol{F}}} \cdot \underline{\underline{\boldsymbol{W}}} & = & \underline{\underline{\boldsymbol{F}}} \end{array}\right\} \, . \tag{69}$$

The first equation here is by insertion of the identity into our first weakened condition. The second equation here is by the first weakened condition converting our second weakened equation into this form. In our joint packaging, the two equations are more closely analogous. Their interpretation is that we are seeking $\underline{\underline{\boldsymbol{W}}}$ under which both the Euclidean metric ( $3$-$d$, on the arena of Cevian triples) and the fundamental triangle matrix are invariants.

**Naming End Note 2** Finally, Theorem 2 for invertible Cevians involves in contrast tensor *transformation laws* and for the *general-linear* and *special-linear* notions of tensor for a) and b) respectively.

## 5.2 Pointers

**Pointer 1** The *concurrent Cevians* also play a distinguished role, as per Article II. These form a 2-parameter family

$$\mathfrak{Concur}(3) \subset \mathfrak{Ceva}(3) .$$

Some new particular cases of equi-Cevians, and their Heron's formulae, are included.

**Pointer 2** How $\mathfrak{Inv}(3)$ , $\mathfrak{Sing}(3)$ , $\mathfrak{Equi}(3)$ and $\mathfrak{Concur}(3)$ are realized within the arena of all triples of Cevians $\mathfrak{Ceva}(3)$ is considered in Article III. In the process, further unique properties of the sides and especially the medians are revealed. Both Article II and Article III have interplay with objects familiar from Routh's Theorems [7, 16, 15]. In the process, a further such object gets added to this Routhian repertoire, alongside a new '4-areas Theorem' for Affine Geometry [48].

**Open Question 1** Does the system

$$\underline{\overline{\boldsymbol{U}}}^{\mathrm{T}} \cdot \underline{\overline{\boldsymbol{U}}} = \underline{\overline{\mathbb{I}}} \tag{70}$$

admit further solutions? This is a compact way of formulating our system (37, 47).

**Open Question 2** Does the system

$$\underline{\overline{\boldsymbol{U}}} \cdot \underline{\overline{\boldsymbol{F}}} \cdot \underline{\overline{\boldsymbol{U}}}^{\mathrm{T}} = \underline{\overline{\boldsymbol{F}}} \tag{71}$$

admit further solutions? Like the previous system, this consists of 6 equations of up to 12th order in 3 unknowns.

**Open Question 3** Does the coupled version of the previous two Open Questions admit further solutions? This can now be written as

$$\left.\begin{array}{ccc} \underline{\overline{\boldsymbol{U}}}^{\mathrm{T}} \cdot \underline{\overline{\mathbb{I}}} \cdot \underline{\overline{\boldsymbol{U}}} & = & \underline{\overline{\mathbb{I}}} \\ \underline{\overline{\boldsymbol{U}}}^{\mathrm{T}} \cdot \underline{\overline{\boldsymbol{F}}} \cdot \underline{\overline{\boldsymbol{U}}} & = & \underline{\overline{\boldsymbol{F}}} \end{array}\right\} . \tag{72}$$

Now constituting a system of 12 equations of up to 12th order in 3 unknowns: now over-determined by a factor of 4 . In particular, each such would also possess a Heron's formula which directly manifest $\boldsymbol{F}$ ...

**Pointer 3** Article IV reveals where the reciprocal-altitudes-Heron formula [28] is hiding within arenas of Cevians. That many cases of geometrical interest couple a nonlinear equation to the linear Stewart system is also entertained there. With the significance of the linear Stewart system surviving some nonlinearities but not others.

**Pointer 4** See also [37, 41] for quadrilateral counterparts of the current Series' robustness test.

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# References


[1] Apollonius of Perga (3rd and 2nd Centuries B.C.E.).

[2] Heron, alias Hero, of Alexandria, *Metrica* (60 C.E.).

[3] King Hud of Zaragoza alias al-Mu'taman (11th Century).

[4] G. Ceva, *De Lineis Rectis (On Straight Lines)* (1678).

[5] M. Stewart, *Some General Theorems of Considerable Use in the Higher Parts of Mathematics* (Sands, Edinburgh 1746).

[6] For an account of Euler's contribution, see e.g. L.E. Dickson, *History of the Theory of Numbers* (Carnegie, Washington 1920).

[7] E.J. Routh, *A Treatise on Analytical Statics* (C.U.P., Cambridge 1891).

[8] E.W. Hobson, *A Treatise on Plane Trigonometry*, (C.U.P., Cambridge 1891).

[9] E.W. Hobson, *A Treatise on Plane Trigonometry*, (C.U.P., Cambridge 1918 edition, reprinted by Dover, Mineola, N.Y. 1967)

[10] H. Hopf, "Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche", ("Concerning the Images of $\mathbb{S}^3$ on $\mathbb{S}^2$ "), Math. Ann. (Berlin) Springer **104** 637 (1931).

[11] Hazel Perfect, *Topics in Geometry* (Pergamon, London 1963).

[12] H.S.M. Coxeter and S.L. Greitzer, *Geometry Revisited* (Mathematical Association of America, Washington, D.C. 1967).

[13] A.S. Posamentier and C.T. Salkind, *Challenging Problems in Geometry* (MacMillan, N.Y. 1970; Dover, N.Y. 1996).

[14] J-P. Serre, *Linear Representations of Finite Groups* (Springer–Verlag, New York, 1977). Part I provides an excellent introduction.

[15] Z.A. Melzak, *Invitation to Geometry* (Wiley, N.Y. 1983).

[16] H.S.M. Coxeter, *Introduction to Geometry* (Wiley, New York 1989).

[17] C. Marchal, *Celestial Mechanics* (Elsevier, Tokyo 1990).

[18] M. Nakahara, *Geometry, Topology and Physics* (Institute of Physics Publishing, London 1990).

[19] D.W. Lyons, "An Elementary Introduction to the Hopf Fibration", Math. Magazine **76** 87 (2003), arXiv:2212.01642.

[20] H.K. Urbantke, "The Hopf Fibration Seven Times in Physics", J. Geom. Phys. **46** 125 (2003).

[21] R.H. Buchholz, "Perfect Pyramids", Bull. Austral. Math. Soc. **45** 353 (1992).

[22] D.C. Kay, *College Geometry* (Harper Collins, 1994).

[23] R.G. Littlejohn and M. Reinsch, "Gauge Fields in the Separation of Rotations and Internal Motions in the $N$-Body Problem", Rev. Mod. Phys. **69** 213 (1997).

[24] D.G. Kendall, D. Barden, T.K. Carne and H. Le, *Shape and Shape Theory* (Wiley, Chichester 1999).

[25] J.R. Silvester, *Geometry Ancient and Modern* (Oxford University Press, New York 2001).

[26] See e.g. A. Benyi, "A Heron-type Formula for the Triangle", Math. Gazette **87** 324 (2003).

[27] W. Fulton and J. Harris, *Representation Theory. A First Course* (Springer, New York 2004).

[28] D.W. Mitchell, "A Heron-type Formula for the Reciprocal Area of a Triangle," Math. Gazette **89** 494 (2005).

[29] I. Matić, "Olympiad Training Materials, Geometric Inequalities", http://www.imomath.com/index.php?options=603&lmm=0 (2011).

[30] T. Frankel, *The Geometry of Physics: An Introduction* (C.U.P., Cambridge 2011).

[31] I.E. Leonard, J.E. Lewis, A.C.F. Liu and G.W. Tokarsky, *Classical Geometry. Euclidean, Transformational, Inversive and Projective* (Wiley, Hoboken N.J. 2014).

[32] E. Anderson, "The Smallest Shape Spaces. I. Shape Theory Posed, with Example of 3 Points on the Line.", arXiv:1711.10054. For the vastly updated version, see https://wordpress.com/page/conceptsofshape.space/1225 .

[33] "The Smallest Shape Spaces. III. Triangles in the Plane and in $3$-$d$ .", arXiv:1711.10115. For the updated version, see https://wordpress.com/page/conceptsofshape.space/1229 .

[34] "Two New Perspectives on Heron's Formula", arXiv:1712.01441 . For the updated version, see https://wordpress.com/page/conceptsofshape.space/1244 .

[35] "Shape (In)dependent Inequalities for Triangleland's Jacobi and Democratic-Linear Ellipticity Quantitities", arXiv:1712.04090. For the updated version, see https://wordpress.com/post/conceptsofshape.space/840 .

[36] The Naming Endnotes' middle paragraph comes from Geometrical discussions between E. Anderson and S. Sánchez in 2018. Their first and third paragraph use a level of detail about tensors that I first came into contact with in discussions with Jimmy York Jr.

[37] E. Anderson, "Quadrilaterals in Shape Theory. II. Alternative Derivations of Shape Space: Successes and Limitations", arXiv:1810.05282. For the updated version, see https://wordpress.com/page/conceptsofshape.space/1237 .

[38] "A brief Linear-Algebraic proof of Heron's Formula" (2024), https://wordpress.com/page/conceptsofshape.space/1295 .

[39] "The Fundamental Triangle Matrix" (2024), https://wordpress.com/page/conceptsofshape.space/1306 .

[40] "A New 'Physical' Proof of Apollonius' Theorem" (2024), https://wordpress.com/page/conceptsofshape.space/1353 .

[41] "Euler's Quadrilateral Theorem. I. A brief new Proof that is Physically Guaranteed to Generalize." (2024),
https://wordpress.com/page/conceptsofshape.space/1297 ;
"II. The Jacobi-K alias $4$-Path counterpart." (2024) https://wordpress.com/page/conceptsofshape.space/1299 ;
"III. Generalization to $N$-Body Problem's Path-Tree Eigenclusters." (2024)
https://wordpress.com/page/conceptsofshape.space/1301 ;
"IV. Generalization to the $5$-Body Problem's $3$ Eigencluster Shapes. (2024)
https://wordpress.com/page/conceptsofshape.space/1512 ;
"V. Generalization to the $6$-Body Problem's $3$ Eigencluster Shapes. (2024)
https://wordpress.com/page/conceptsofshape.space/1514 ;
"VI. Generalization to $N$-Body Problem's Star-Tree Eigenclusters." (2024)
https://wordpress.com/page/conceptsofshape.space/1516 ;
"Linear Algebra of Quadrilaterals. With particular emphasis on their Area Formulae" (2024)
https://wordpress.com/page/conceptsofshape.space/1303 .

[42] "The Fundamental Triangle Matrix Commutes with the Lagrange and Apollonius Matrices. With implications for deriving Kendall and Hopf Theorems." (2024), https://wordpress.com/page/conceptsofshape.space/1308 .

[43] "Lagrange Matrices: $3$-Body Problem and General" (2024), https://wordpress.com/page/conceptsofshape.space/1436 .

[44] "Only $2$ of the Fundamental Triangle, Lagrange and Apollonius Matrices are Independent. With ensuing Algebras, Irreducibles and Splits" (2024), https://wordpress.com/page/conceptsofshape.space/1310 .

[45] "Generalizing Heron's formula via Multi-Linear Algebra. II. Möbius, Jacobi and Routh for Concurrent Cevians." (2024), https://wordpress.com/page/conceptsofshape.space/1261 . Also referred to as Article II.

[46] "Generalizing Heron's formula via Multi-Linear Algebra. III. Cevian Arenas and Natural Variables." (2024), https://wordpress.com/page/conceptsofshape.space/1290 . Also referred to as Article III.

[47] "Generalizing Heron's formula via Multi-Linear Algebra. IV. Geometry can impart Sides-Cevians Nonlinearity." (2024), https://wordpress.com/page/conceptsofshape.space/1263 . Also referred to as Article IV.

[48] "A $4$-Areas Theorem for Affine Geometry", https://wordpress.com/page/conceptsofshape.space/1658 .

[49] "Hopf's Little Mathematics 32 times in Physics and Geometry" (2024), https://wordpress.com/page/conceptsofshape.space/1759 .

[50] "$N$-body Problems' Eigenclusterings: Linear Algebra and Order Theory" https://wordpress.com/page/conceptsofshape.space/1558 (2024).

[51] *The Structure of Flat Geometry*, *Widely-Applicable Mathematics. A. Improving understanding of everything by dual-wielding Combinatorics and Linear Algebra.* **3**, forthcoming 2025.