A brief Linear-Algebraic Proof of Heron's Formula

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Abstract

A brief new Linear-Algebraic proof of Heron's formula is offered. This method immediately yields a second area formula.

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Structure 1 The most standard formulation of Heron's formula [1, 2] is

$$Area \;=\; \sqrt{s\,(\,s\,-\,a\,)\,(\,s\,-\,b\,)\,(\,s\,-\,c\,)} \;=\; \sqrt{s \prod_{\text{cycles}} (\,s\,-\,a\,)} \;. \tag{1}$$

Where

$$s \;:=\; \frac{1}{2}\sum_{\text{cycles}} a \;=\; \frac{1}{2}\,(\,a\,+\,b\,+\,c\,)$$

is the *semi-perimeter* of the triangle.

Expanding, multiplying by 4 and then squaring,

$$T^2 \;=\; \sum_{\text{cycles}} A\,(\,2\,B\,-\,A\,) \;=\; \begin{pmatrix} A & B & C \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} \;. \tag{2}$$

Where the first form is the expanded Heron's formula, with $A \;=\; a^2$ and cycles. And the second is the '*Heron-Buchholz quadratic form*' [6]. Whose coordinate-independent form is

$$T^2 \;=\; \underline{\boldsymbol{S}} \cdot \underline{\underline{\boldsymbol{H}}} \cdot \underline{\boldsymbol{S}} \;. \tag{3}$$

For $(\,\text{sides}\,)^2$ vector $\boldsymbol{S}$. Useful [8] tetra-area variable T . And '*Heron-Buchholz matrix*' $\boldsymbol{H}$.

Structure 2 The cycle of cosine rules can be reformulated as follows.

$$2\begin{pmatrix} \underline{\boldsymbol{b}} \cdot \underline{\boldsymbol{c}} \\ \underline{\boldsymbol{c}} \cdot \underline{\boldsymbol{a}} \\ \underline{\boldsymbol{c}} \cdot \underline{\boldsymbol{b}} \end{pmatrix} \;=\; 2\begin{pmatrix} b\,c\,\cos\gamma \\ c\,a\,\cos\beta \\ a\,b\,\cos\alpha \end{pmatrix} \;=\; \begin{pmatrix} b^2\,+\,c^2\,-\,a^2 \\ c^2\,+\,a^2\,-\,b^2 \\ a^2\,+\,b^2\,-\,c^2 \end{pmatrix} \;=\; \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} \tag{4}$$

In coordinate-free form,

$$2\,\underline{\boldsymbol{D}} \;=\; \underline{\underline{\boldsymbol{C}}} \cdot \underline{\boldsymbol{S}} \;. \tag{5}$$

For *dot-product vector* $\boldsymbol{D}$. And *cosine rule cycle matrix* $\boldsymbol{C}$.

Remark 1 Observe that $\boldsymbol{C} \;=\; \boldsymbol{H}$. We consequently sense an opportunity here to find a Linear Algebraic proof of Heron's formula; see e.g. [3, 9, 5, 4, 7] for assorted other proofs.

<u>Proof</u> Apply $\underline{\boldsymbol{D}}\,\cdot$ to (5). So, using $||\;||$ to denote Euclidean norm,

$$2\,||\boldsymbol{D}||^2 \;=\; \frac{1}{2}\,\underline{\boldsymbol{S}} \cdot \underline{\underline{\boldsymbol{C}}}^2 \cdot \underline{\boldsymbol{S}} \;. \tag{6}$$

But

$$\boldsymbol{C}^2 \;=\; \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \;=\; \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix} \;. \tag{7}$$

Thus

$$\sum_{\text{cycles}} \Big(\,4\,(\,\underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{b}}\,)^2 \;+\; 2\,A\,B \;-\; 3\,A^2\,\Big) \;=\; 0. \tag{8}$$

Next, *Lagrange's identity* is

$$(\,\underline{\boldsymbol{a}} \cdot \underline{\boldsymbol{b}}\,)^2 \;=\; ||\underline{\boldsymbol{a}}||^2\,||\underline{\boldsymbol{b}}||^2 \;-\; ||\,\underline{\boldsymbol{a}} \;\times\; \underline{\boldsymbol{b}}\,||^2 \tag{9}$$

Combine (9), the cross-product area formula, and (8) to obtain

$$\sum_{\text{cycles}} \Big(\,4\,(\,A\,B\,-\,T\,)^2 \;+\; 2\,A\,B \;-\; 3\,A^2\,\Big) \;=\; 0 \;.$$

But this trivially rearranges to the expanded Heron's formula. □

Remark 2 To arrive at the most usually encountered formulation of Heron's formula, one just undoes three moves given above. I.e. one now takes the positive square root, divides by 4 and factorizes to obtain (1).

Remark 3 Readers unfamiliar with the name 'Lagrange identity' will however see that it is but a ready consequence of the geometrical formulation of dot- and 2-d-cross-product. Alongside the basic trigonometric identity $\sin^2\gamma \;+\; \cos^2\gamma \;=\; 1$.

Remark 4 It is of course standard that Heron's formula can be proven from the cosine rule. The above provides a Linear Alegebra version of such a proof. The first article in [10] uses our Linear Algebra approach to show that, conversely, Heron's formula implies the cycle of cosine rules.

Remark 5 We comment further on the multiple conceptual interpretations of the matrix $\boldsymbol{C} = \boldsymbol{H}$ in [10].

Remark 6 Finally, if we apply $\underline{\boldsymbol{S}} \cdot$ to (5) instead, the following drops out.

$$T^2 = \underline{\boldsymbol{S}} \cdot \underline{\underline{\boldsymbol{C}}} \cdot \underline{\boldsymbol{S}} = 2\,\underline{\boldsymbol{S}} \cdot \underline{\boldsymbol{D}} = 2 \left(\prod_{\text{cycles}} a \right) \sum_{\text{cycles}} a \cos\alpha$$

$$= 2\,a\,b\,c(a \cos \alpha + b \cos \beta + c \cos \gamma) \ . \qquad (10)$$

Where the first step is by Heron's formula, and the third is by evaluating the dot product and forming cycles.

End Remark Thus two area formulae for the triangle drop out of our simple Linear-Algebraic considerations. Our second area formula is less interesting, since it involves 6 data functions instead of 3 . We leave its more detailed study to another occasion.

References

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