Quadrilaterals in Shape Theory. II.

Alternative Derivations of Shape Space: Successes and Limitations

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Abstract

We recently derived that triangleland is $\mathbb{S}^2$ by just considering the eigentheory of Heron's formula for the triangles' area, A . We now show that none of Brahmagupta's, or Bretschneider's first or second area formulae extend this derivation. The underlying situation is that N-a-gonland is $\mathbb{CP}^{N-2}$. Among which $\mathbb{CP}^1 = \mathbb{S}^2$ recovers the triangleland sphere. This generalization follows from e.g. Kendall's extremization or the generalized Hopf map.

We further explain the above non-extension in terms of total area no longer being a shape quantity for quadrilaterals. Its place has been taken by the square root of sums of squares of subsystem areas, S We further clarify what shape quantities, A and S are in Representation-Theoretic terms. $Isom\left(\mathbb{CP}^{N-2}\right)$ gives $su(N-1)$, with the corresponding shape quantities constituting an adjoint rep. This algebra has a geometrically-distinguished $so(N-1)$ of democracy transformations. The first two Casimirs supported by which are A^2 and S^2 . They are mathematically analogous to total AM squared J^2 in 2- and $3-d$ respectively.

Casson showed that 2-d triangleland's $\mathbb{S}^2$ separately generalizes as follows. To $\mathbb{S}^{d(d+1)/2-1}$ d-simplexlands at the topological level. For these, d-volume provides a shape quantity, which can be calculated by the Cayley–Menger formula. Whose first two nontrivial instances are the Heron and della Francesca–Tartaglia formulae. Even-d is required so as to have an eigentheory; even here, just dimension counts suffice to preclude on-sphere conditions dropping out. Overall, triangleland enjoys a large number of dimensional coincidences that neither N-a-gons nor d-simplices extend.

The current Article is thus a useful check on how far the least technically involved derivation of the smallest nontrivial shape space can be taken. This is significant since Kendall's Shape Theory is a futuristic branch of mathematics. With substantial applications in both Statistics (Shape Statistics) and Theoretical Physics (Background Independence: of major relevance to Classical and Quantum Gravitational Theory).

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1 Introduction

Diagonalizing Heron's formula for triangle area was recently demonstrated [86] to suffice as first principles to derive both of the following.

1) Kendall's Little Theorem [32, 36, 51] from *Kendall's Shape Theory* (see also [48, 55, 62, 64, 67, 71, 76, 80, 78, 79, 82, 83, 84, 86, 87, 91, 85, 88, 92, 90, 101, 105, 93, 103, 106]) that the space of triangles modulo similarities is a sphere,

$$\mathfrak{S}\text{hape}(2, 3) = \mathbb{S}^2 . \tag{1}$$

At both the Topological and Metric-Differential-Geometric levels of structure.

2) The Hopf map [16, 38, 58, 30, 53, 59, 45]

$$\eta : \mathbb{S}^3 \longrightarrow \mathbb{S}^2 . \tag{2}$$

Whose mathematics recurs in Dirac's monopole [17] the 3-body problem [22, 33, 46], and other applications [39, 44, 52, 59, 77]. And which is a first useful nontrivial example of various Differential-Geometric and Topological structures.

We outline this demonstration in Sec 2. See Appendices A and B for more about 1) and Appendix C for more about 2). In each case with valuable generalizations. With Appendix D descending to the extra-special case of $\mathbb{S}^2$.

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The current Article considers the extent to which these results generalize to Shape Theory with other spatial dimensions d and point-or-particle numbers N . Generalizations away from triangleland in 2-d are of two distinguished kinds.

A) N-a-gons, for which *Kendall's Theorem* ([32, 51] and the Appendix) gives that

$$\mathfrak{S}(2, N) = \mathbb{CP}^{N-2} . \tag{3}$$

B) d-simplexes, for which Casson [51] showed that at the topological level,

$$\mathfrak{S}(d, d+1) = \mathbb{S}^{d(d+1)/2 - 1} . \tag{4}$$

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The minimum nontrivial case of A) are quadrilaterals, as treated in Sec 3. For these, we show that Bretschneider's second area formula is sequentially more satisfactory than Bretschneider's first and Brahmagupta's for this application. But still falls short of requirements to provide a derivation of the topology and geometry of the space of quadrilaterals.

We further explain this non-extension in Sec 4. In terms of area's mass-weighted version α as normalized by the moment of inertia (MoI) ι ,

$$\mathcal{A} := \frac{\alpha}{\iota} ,$$

not being a shape quantity [67, 74] for quadrilaterals. Its place has been taken by [49, 74]

$$\mathcal{S} := \frac{\sqrt{\sum_{A=1}^{3} \alpha_A^2}}{\iota} . \tag{5}$$

This sum is over the 3 two-Jacobi-vector subsystems supported by the quadrilateral. We further clarify the reason that S features in the theory of quadrilaterals in Representation-Theoretic terms in Sec 5.

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We next consider whether there is a Casson's Theorem generalization of the Heron proof of 'Kendall's Little Theorem' in Secs 6 and 7. For the 3-simplexes alias tetrahaedrons, volume both has a general formula in terms of separations. I.e. the della Francesca–Tartaglia volume formula [4, 5, 20, 89]. And provides a shape quantity

$$\mathcal{V} \ := \ \frac{\mathcal{V}}{\iota^{3/2}} \tag{6}$$

(for $\mathcal{V}$ the mass-weighted volume). We show that this does not however support eigenvectors. By which the Heron derivation of the triangleland shape space does not generalize to tetrahaedronland either.

In Sec 7, we extend this analysis to the general d-simplexes. For which the d-volume is given by the Cayley–Menger generalization [7, 13, 15, 20, 34, 89] of the della Francesca–Tartaglia volume formula. d-volume moreover provides a shape quantity

$$\mathcal{V}_d \ := \ \frac{\mathcal{V}_d}{\iota^{d/2}} \ . \tag{7}$$

Eigenvectors can now be defined for the even-d cases. The dimension count does not however work out for this to give on-sphere conditions corresponding to Casson's result for the topology of the spaces of d-simplexes. Finally Secs 4 and 7 point to a long string of dimensional coincidences behind the Heron derivation.

2 Outline of Triangleland shape space from Heron's formula

2.1 Notation for the arbitrary triangle

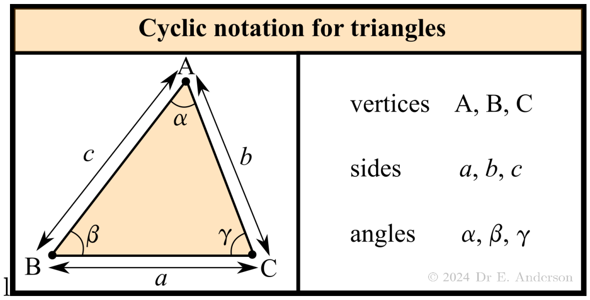

Figure 1:

Notational Remark 1 Consider an arbitrary triangle $\triangle\,\mathrm{ABC}$, denoted as in Fig 1. Using $s^I \;\;\stackrel{\triangle}{=}\;\; a^I$, $I = 1$ to 3 to denote a, b, c will also be useful for us.

Definition 1 The *semi-perimeter* is

$$s \;:=\; \frac{a + b + c}{2} \;=\; \frac{1}{2}\sum_{I=1}^{3} a^I \; . \tag{8}$$

Notational Remark 2 When unambiguous, let us use the shorthand $Area(\mathrm{ABC}) = Area$ for the area of $\triangle\,\mathrm{ABC}$.

2.2 Heron's formula

Theorem 1 (Heron's formula)

$$Area \;=\; \sqrt{s\,(s - a)\,(s - b)\,(s - c)} \; . \tag{9}$$

Remark 1 Squaring, introducing the tetra-area variable

$$T \;:=\; 4\,Area \; , \tag{10}$$

and expanding out, we obtain the following.

Proposition 1 (Expanded version of Heron's formula) [11]

$$T^2 \;=\; \sum_{\text{cycles}} \left(2\,a^2\,b^2 - a^4\right) \;=\; \sum_{\text{cycles}} A\,(2\,B - A) \; . \tag{11}$$

Remark 1 Using basic structures from Linear Algebra, a further reformulation is as follows.

Proposition 2 (Quadratic form version of Heron's formula) [Buchholz 1992]

$$T^2 \;=\; ||\boldsymbol{S}||_{\boldsymbol{H}}{}^2 \;=\; \underline{\boldsymbol{S}} \cdot \underline{\overline{\boldsymbol{H}}} \cdot \overline{\boldsymbol{S}} \; . \tag{12}$$

This '*Heron–Euler–Buchholz quadratic form*' is in terms of the *squared-sides* 3-vector

$$\boldsymbol{S} \;:=\; \begin{pmatrix} A \\ B \\ C \end{pmatrix} \;:=\; \begin{pmatrix} a^2 \\ b^2 \\ c^2 \end{pmatrix} \; . \tag{13}$$

And the '*Heron–Euler–Buchholz matrix*' [40]

$$\boldsymbol{H} \; := \; \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \; . \tag{14}$$

Naming Remark 1 A first truer name for Heron's formula is *side-data triangle area formula*. From which first truer names for the above quadratic form and matrix derive as well. Euler is mentioned as the first person to give the solution in integers [6] to the system now encoded by this matrix.

2.3 Diagonalizing Heron's formula gives Kendall's Little Theorem

Remark 1 Set

$$0 \; = \; \det \left(\, \boldsymbol{H} \; - \; \lambda \, \mathbb{I} \, \right) \; . \tag{15}$$

This yields eigenvalues $\lambda \; = \; 1$ with multiplicity 1 , and $\lambda \; = \; -2$ with multiplicity 2 .

The corresponding orthonormal eigenvectors are, respectively, as follows.

$$\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \; , \; \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \; , \; \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \; . \tag{16}$$

Remark 2 The diagonalizing geometrical variables are thus as follows.

$$\widetilde{A} \; = \; \frac{A \; + \; B \; + \; C}{\sqrt{3}} \; , \;\; \widetilde{B} \; = \; \frac{A \; - \; B}{\sqrt{2}} \; , \quad \text{and} \;\; \widetilde{C} \; = \; \frac{A \; + \; B \; - \; 2\,C}{\sqrt{6}} \; . \tag{17}$$

In terms of these, Heron's formula also takes the following form.

Proposition 3 (Diagonal Heron formula) [Anderson 2017]

$$T \; = \; \sqrt{\widetilde{A}^2 \; - \; 2 \left(\widetilde{B}^2 \; + \; \widetilde{C}^2 \right)} \; . \tag{18}$$

Remark 3 Introduce rescaled *ratio variables* [84] whose denominator is proportional to the moment of inertia.

$$X \; := \; \sqrt{2}\,\frac{\widetilde{B}}{\widetilde{A}} \; = \; \frac{\sqrt{3}\,(\,A \; - \; B\,)}{A \; + \; B \; + \; C} \; , \;\; Y \; := \; \frac{\mathrm{T}}{\widetilde{A}} \; , \;\; Z \; := \; \sqrt{2}\,\frac{\widetilde{C}}{\widetilde{A}} \; = \; \frac{A \; + \; B \; - \; 2\,C}{A \; + \; B \; + \; C} \; . \tag{19}$$

Then (18) becomes

$$X^2 \; + \; Y^2 \; + \; Z^2 \; = \; 1 \; . \tag{20}$$

I.e. the on-$\mathbb{S}^2$ condition.

Remark 4 We have thus found an alternative route to the following.

Corollary 1 ('Kendall's Little Theorem') The space of triangles modulo similarities is $\mathbb{S}^2$.

Remark 5 So the diagonalized normalized version of Heron's formula is the same as the triangleland sphere condition. To get the whole sphere, investigating coordinate ranges and their significance as triangles reveals that we need to consider vertex-labelled mirror-images-distinct triangles.

2.4 ... and the Hopf map

Corollary 2 X , Y and Z as the 3 Hopf quantities.

<u>Proof</u> *Relative Jacobi coordinates* drop out of a second diagonalization involving relative separation variables [82, 96]. Pick the *mass-weighted* [37] and *moment of inertia (MoI) normalized* [82, 97] version of these $\overline{\nu}^i$. With $i = 1, 2$ for triangles. Then X , Y and Z take the following standard form for the Hopf quantities.

$$X = 2\,\underline{\nu}_1 \cdot \overline{\nu}_2 \ . \tag{21}$$

$$Y = 2\,\underline{\nu}_1 \times \overline{\nu}_2 \ . \tag{22}$$

$$Z = {\nu_2}^2 - {\nu_1}^2 \ . \tag{23}$$

Exercise 1 Check the standard result that the expressions (21-23) add up to 1 .

Remark 2 Heron's formula thereby provides not only 'Kendall's Little Theorem' but also the Hopf quantities. And thus also the corresponding Hopf map. Which is the smallest and most commonly encountered of the various Hopf maps, η , of (2) and Fig 8 This is via the Hopf quantities' relation to this map through composition with some simpler maps as per Appendix 8.

A Heron–Jacobi–Hopf–Kendall unification has thus been attained. By which both Kendall's Little Theorem and the Hopf map can be derived from Heron's formula using just high school mathematics. This is interesting enough for the current Article to ask whether this simplest known approach to Kendall and Hopf generalizes in the quadrilateral or d-simplex setting.

Remark 3 Triangleland's shape quantities [65, 67, 84] coincide with the Hopf quantities. A Geometrical interpretation for the Hopf quantities in the current context is as follows. Y is the mass-weighted tetra-area per unit moment of inertia. Z is an *ellipticity*: a purely relative-ratio quantifier of whether the triangle is tall or flat. X is an *anisoscelesness*: a relative-angle-dependent measure of departure from isoscelesness. Anisoscelesness and ellipticity can moreover also be interpreted as the two eigenvectors that the Heron map $\boldsymbol{H}$ possesses in addition to the total moment of inertia.

Remark 4 Since the -2 eigenvalue has multiplicity 2 , the corresponding eigenspace is 2-d . As such, there are other basis choices for its eigenvectors. One can however then ask the question of whether there exists any basis for which the diagonalized Heron's formula takes on a geometrically-standard form. Then ellipticity and anisoscelesness are picked out. One could also pin down this particular basis by asking for the eigenvectors to be the simplest possible functions of invariants. Then Jacobi coordinates pick out anisoscelesness, with orthonormality then fixing the form to be taken by the last eigenvector to be ellipticity. A further 2 distinct arguments for this basis choice are given in [98].

3 Quadrilateral area formulae

3.1 Notation for the arbitrary quadrilateral

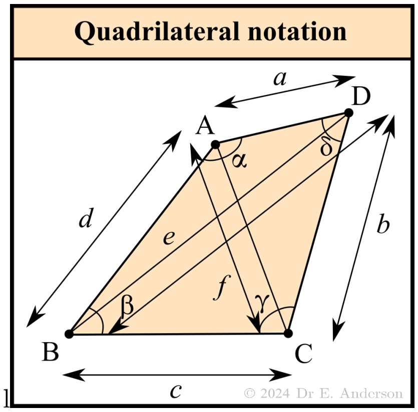

Figure 2:

We provide this in Fig 2. Now sides a^I have I run from 1 to 4 . And yet relative separations s^A have A run from 1 to 6 .

3.2 Sides-data cyclic-quadrilateral area formula (Brahmagupta)

Theorem 1 Sides-data cyclic-quadrilateral area formula[1] For a cyclic quadrilateral,

$$Area = \sqrt{(s - a)(s - b)(s - c)(s - d)} \ . \tag{24}$$

Remark 1 Setting $d = 0$, this returns Heron's formula.

Proposition 4 In expanded form,

$$\mathrm{T}^2 = 4\,(a\,b + c\,d)^2 - \left(a^2 - c^2 + b^2 - d^2\right)^2 = {a_{(2)}}^2 - 2\,a_{(4)} + 8\,a_{\Pi} \ . \tag{25}$$

Where

$$a_{(p)} := \sum_{I = 1}^{4} (a^I)^p \ . \tag{26}$$

for whichever power p . And

$$a_{\Pi} := \prod_{I = 1}^{4} a^I \ . \tag{27}$$

Caveat 1 Unlike for Heron's formula, the second expanded form precludes dependence on $S^I := (s^I)^2$ alone. This feature simplified repackaging Heron's formula as a quadratic form in squared variables [86]. So Brahmagupta's area formula not having this feature poses a complication.

Caveat 2 Brahmagupta's area formula is limited to cyclic quadrilaterals.

Remark 2 In starting to move around this limitation, we observe that for a general quadrilateral, SSSS – four sides – is not sufficient data [99]. Considering the rhombi suffices to reach this conclusion. Thus there can exist no area formula for general quadrilaterals without bringing in some further datum.

[1]This was discovered in 7th century A.D. India by Brahmagupta [3, 23, 31, 43]. And is usually referred to as *Brahmagupta's area formula*.

3.3 Bretschneider's first area formula

Remark 1 One alternative in bringing in further data to get around Caveat 2 is to involve angles between sides. This leads to the following.

Theorem 3 (Bretschneider's first area formula) [8, 43, 31] (1842) For a convex but elsewise general quadrilateral,

$$Area \;=\; \sqrt{(s - a)(s - b)(s - c)(s - d) - a\,b\,c\,d\cos^2\left(\frac{\alpha + \gamma}{2}\right)} \; . \tag{28}$$

Remark 2 As indicated in Fig 2, α and γ are opposite angles. Thus for a cyclic quadrilateral, their sum is π by an elementary Theorem of Euclid's. In this case,

$$\cos\frac{\pi}{2} \;=\; 0 \tag{29}$$

then accounts for the last term of Bretschneider's first area formula vanishing in Brahmagupta's.

Naming Remark 1 The new term admits the following Geometrical interpolation.

$$(\,\text{acyclicness}\,) \;:=\; a\,b\,c\,d\cos^2\left(\frac{\alpha + \gamma}{2}\right) \;: \tag{30}$$

the correction term to Brahmagupta's area formula.

Corollary 1 The area of a quadrilateral with given sides is maximized when it is cyclic.

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Remark 3 Two reasons why Bretchneider's first area formula does not provide a useful extension of Heron's formula for Shape-Theoretic use are as follows.

Caveat 3 It involves conceptually heterogeneous angular information, as a multiplicative factor on one summand.

Caveat 1′ $a\,b\,c\,d$ does not depend on the squares of the sides.

3.4 Relative-separation data

Remark 1 Caveats 1′ and 3 are moreover remedied by making use instead of *diagonal length data*. This is homogeneous with side length data through both constituting *separation data*. Indeed, in separational alias Lagrangian and dual constellational approaches [82] – an accurate description of Shape Theory – no distinction is to be made between sides and diagonals. So modelling purely in terms of unqualified separations serves here.

Remark 2 Heron's formula can be thought of not only as the side-data triangle area formula, but also as the *relative-separation-data triangle area formula*. This rests on these two notions coinciding for triangles. For $N = 3$ does not support any distinction between sides and relative separations. But the general N-a-gon possesses *diagonals* as well as its N sides. Giving a total of

$$\#(\,\text{relative separations}\,) \;=\; \binom{N}{2} \;=\; \frac{N\,(N - 1)}{2} \; .$$

Remark 3 To have equality between side and relative-separation counts,

$$N \;=\; \frac{N\,(N - 1)}{2} \;\Rightarrow\; 0 \;=\; N\,(N - 3) \;\Rightarrow\; N \;=\; 0 \;\text{ or }\; 3 \; .$$

So, in particular, this counting coincidence for triangles is gone forever for $N \geq 4$: diagonals are a persistent feature.

3.5 Relative-separation-data convex-quadrilateral area formula (RSCQAF)

Remark 1 Quadrilaterals possess the following relative-separation data extension of Heron's formula.

Theorem 4 (Relative-separation-data convex-quadrilateral area formula) [18].[2] For a convex but elsewise arbitrary quadrilateral,

$$Area \; = \; \sqrt{(s - a)(s - b)(s - c)(s - d) - \frac{1}{4}(ac + bd + ef)(ac + bd - ef)} \, . \tag{31}$$

Remark 2 This reduces to the side-data cyclic-quadrilateral area formula (24) iff the quadrilateral is cyclic. For the two subcases involved are nested by cyclic $\Rightarrow$ convex. Leaving us needing to consider the RSCQAF's extra term's two factors' vanishing conditions. The second factor being zero amounts to **Ptolemy's Theorem** [2] [23]. I.e. that

$$ef \; = \; ac \; = \; bd \quad \Leftrightarrow \quad \text{ABCD} \;\; \text{is cyclic} \, . \tag{32}$$

While the first factor can only be zero if the following holds (**Esercise 2**!). The quadrilateral is T or O : a triple or maximum coincidence-or collision respectively. Both of which are trivially cyclic.

Remark 3 The corresponding relative-separations formulation for the cyclator is thus as follows. Namely,

$$\text{acyclicness} \; = \; \frac{1}{4} \, (ac + bd + ef)(ac + bd - ef) \, . \tag{33}$$

Proposition 5 The *expanded version of RSCQAF* is

$$T^2 \; = \; e^2 f^2 \; - \; \left(a^2 - b^2 + c^2 - d^2\right)^2 \; = \; EF \; - \; (A - B + C - D)^2 \, . \tag{34}$$

Remark 4 RSCQAF thus generalizes Heron's in a second way, matching Proposition 1's form. It consequently admits the following further formulation matching Proposition 2.

Proposition 6 [Anderson 2018] The tetra-area squared of a quadrilateral is given by the following '*RSCQ quadratic form*'.

$$T^2 \; = \; ||\boldsymbol{S}||_{\boldsymbol{R}}{}^2 \; = \; \underline{\boldsymbol{S}} \cdot \overline{\underline{\boldsymbol{R}}} \cdot \overline{\boldsymbol{S}} \, . \tag{35}$$

Where $\boldsymbol{S}$ is now the following $(\,\text{separation}\,)^2$ 6-vector.

$$\boldsymbol{S} \; := \; \begin{pmatrix} A \\ B \\ C \\ D \\ E \\ F \end{pmatrix} \; := \; \begin{pmatrix} a^2 \\ b^2 \\ c^2 \\ d^2 \\ e^2 \\ f^2 \end{pmatrix} \, . \tag{36}$$

Which lives in *separationspace* [96]. And 6×6 '*RSCQ matrix*' alias '*Bretschneider matrix*'

$$\boldsymbol{R} \; := \; \left(\begin{array}{cccc|cc} -1 & 1 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 \\ -1 & 1 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{array}\right) \, . \tag{37}$$

Exercise 3 Derive (34) and that the matrix in (35) is (37).

[2] This is often referred to as *Coolidge's area formula*, though in fact Bretschneider [8, 12] already had another formula Linear-Algebraically equivalent to this. See [102] for further explanation of this point.

3.6 The corresponding eigentheory

Remark 1 Setting

$$(\boldsymbol{R} - \lambda \mathbb{1})\boldsymbol{x} = 0 , \tag{38}$$

the RSCQ matrix's eigenvalues and eigenvectors are as follows. We can solve this separately for its 2×2 and 4×4 block. Obtaining the following eigenspectrum.

$$\lambda = 0 \text{ with multiplicity } 3 , \quad \text{and -2 , } 2 , \ 4 \text{ each with multiplicity } 1 . \tag{39}$$

The corresponding eigenvectors are

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} , \ \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} , \ \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} , \tag{40}$$

for -2 , 2 and 4 respectively. And e.g.

$$\frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} , \ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \ \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} , \tag{41}$$

for the 3-d 0-eigenspaces.

Remark 2 These eigenvectors correspond to the following geometrical quantities.

$$\frac{1}{\sqrt{2}} \left(e^2 + f^2 \right) = \frac{1}{\sqrt{2}} \sum_{\Delta = 1}^{2} D^{\Delta} = \frac{1}{\sqrt{2}} \text{ (diagonal squares sum)} \tag{42}$$

$$\frac{1}{\sqrt{2}} \left(e^2 - f^2 \right) = \frac{1}{\sqrt{2}} \text{ (diagonal squares difference) } = \frac{1}{\sqrt{2}} \text{ (diagonal ellipticity) .} \tag{43}$$

$$\frac{1}{2} \sum_{I = 1}^{4} (-1)^I A^I = \frac{1}{2} \left(-a^2 + b^2 - c^2 + d^2 \right) = \frac{1}{2} \text{ (alternating side squares sum) } =$$
$$\frac{1}{\sqrt{2}} \text{ (difference of adjacent sides' ellipticities) .} \tag{44}$$

$$\frac{1}{2} \left(a^2 + b^2 + c^2 + d^2 \right) = \frac{1}{2} \sum_{I = 1}^{4} A^I \frac{1}{2} \text{ (sides}^2 \text{ sum) .} \tag{45}$$

$$\frac{1}{\sqrt{2}} \left(a^2 - c^2 \right) = \frac{1}{\sqrt{2}} \text{ (ellipticity of an opposite-sides pair) .} \tag{46}$$

$$\frac{1}{\sqrt{2}} \left(b^2 - d^2 \right) = \frac{1}{\sqrt{2}} \text{ (ellipticity of the other opposite-sides pair) .} \tag{47}$$

Where D^A takes values E, F . And the first I in (44) is a power while the second is a tensor index.

Exercise 4$^-$ Derive Remark 1's results and Remark 2's interpretations.

3.7 Discussion

Remark 1 Brahmagupta's area formula is only for cyclic quadrilaterals. Bretschneider's first and second area formulae extend this to convex quadrilaterals. the second – the RSCQAF – presents some conceptual advantages over the first, in particular though being based on pure-separations data.

Remark 2 As we shall see in the next section, it is easy to go from 6 separations to 5 ratio quantities on $\mathbb{S}^5$. So our objective is to find a further condition so as to pass to 4 fully non-redundant variables. For the triangle, diagonalizing Heron's formula succeeded in doing this. Is diagonalizing the RSCQAF capable of doing the same?

Remark 3 Observe however that the RSCQAF differs from Heron's area formula in not making equable use of all separations. Rather, the diagonals enter it in a different manner to the sides. This is tied to Bretschneider's convexity requirement, which uniquely allots the diagonals to be the interior separations.

Remark 4 In contrast, for $N \geq 4$, Kendall's Shape Theory is really about *complete* N-a-gons, the minimum nontrivial example of which is the complete quadrilateral. By which no diagonals-sides distinction can be made. Each generic complete quadrilateral encodes various quadrilaterals at once, among which convex, crossed and reflex ones are represented. Bretschneider's formulae's convexity restriction is not then a problem, since one could apply it to convex representatives of complete quadrilaterals.

Caveat 4 The RSCQAF is not invariant under change of representative.

Caveat 5 The RSCQAF does not make equable use of separations.

Remark 5 These are two interlinked reasons why the RSCQAF may not be well-adapted to Kendall's Shape Theory.

4 Shape quantities for quadrilaterals

4.1 Preshape space

Definition 1 The *dilatational group* comprises the translations and dilations in the following semidirect product [54] form.

$$Dilatat(d) \;=\; Tr(d) \;\rtimes\; Dil\;. \tag{48}$$

Remark 1 Identifying the mathematical form of each of these constituent groups's underlying sets,

$$Dilatat(d) \;=\; \mathbb{R}^d \;\rtimes\; \mathbb{R}_+\;. \tag{49}$$

Definition 2 Kendall's *preshape space* [32, 51, 97] is the result of quotienting constellationspace (104) by the dilatational group.

$$\mathfrak{P}\text{reshape}(\,d,\,N\,) \;=\; \frac{\mathfrak{q}(\,d,\,N\,)}{Dilatat(d)}\;. \tag{50}$$

Remark 2 This further works out to be [51, 97]

$$\mathfrak{P}\text{reshape}(\,d,\,N\,) \;=\; \frac{\mathbb{R}^{d\,N}}{\mathbb{R}^d \;\rtimes\; \mathbb{R}_+} \;=\; \frac{\mathbb{R}^{d\,n}}{\mathbb{R}_+} \;=\; \mathbb{S}^{d\,n\,-\,1}\;. \tag{51}$$

Which is a sphere at both the topological and metric levels. We thus arrive at $\mathbb{S}^3$ for triangles and $\mathbb{S}^5$ for quadrilaterals.

4.2 Preshape, and 1-d shape, quantities

Remark 1 In 1-d, since there are no continuous rotations, preshape space is furthermore equivalent to shape space. I.e.

$$\mathfrak{S}\text{hape}(\,1,\,N\,) \;=\; \mathfrak{P}\text{reshape}(\,1,\,N\,) \;=\; \mathbb{S}^{n\,-\,1}\;. \tag{52}$$

Remark 2 In 1-d, the n relative Jacobi scalars ρ^i are Euclidean invariants. As in Sec 2, let us normalize these using

$$\rho \;:=\; \sqrt{\iota}\;, \tag{53}$$

which furthermore plays the role of [82] preshape space radius. This leaves us with n quantities [3]

$$\nu^i \;:=\; \frac{\rho^i}{\rho} \tag{54}$$

Automatically subject to the on- $\mathbb{S}^{n\,-\,1}$ condition,

$$\sum\nolimits_{i\;=\;1}^{n} (\nu^i)^2 \;=\; 1\;. \tag{55}$$

Remark 3 This working furthermore generalizes for d-dimensional space at the level of preshape space [32, 82], to the following.

$$\overline{\nu}^i \;:=\; \frac{\overline{\rho}^i}{\rho}\;. \tag{56}$$

Subject to the on-$\mathbb{S}^{d\,n\,-\,1}$ condition. Namely,

$$\sum_{\Gamma\,=\,1}^{d\,n} ||\,\nu^\Gamma\,||^2 \;=\; \sum_{i\,=\,1}^{n}\sum_{a\,=\,1}^{d} ||\,\nu^{i\,a}\,||^2 \;=\; 1\;. \tag{57}$$

Where $||\;||$ is the relativespace $= \mathbb{R}^{d\,n}$ norm.

[3] These can also be found in the Molecular Physics literature, under the names *internal rotations* and *democracy transformations* [49]. With ρ referred to as the *hyperradius*; see [97] for truer names.

4.3 Nontrivial Euclidean invariants

Remark 1 For $d \geq 2$, $\underline{\rho}^i \cdot \overline{\rho}^j$ are Euclidean invariants. There are

$$\frac{n(n+1)}{2} \tag{58}$$

distinct such. We can furthermore package them into the *Euclidean matrix* $\boldsymbol{E}$. Whose indices $i = 1$ to n run over the relative Jacobi vectors' labels, with its components then taking the following form.

$$E^{ij} := \left(\underline{\rho}^i \cdot \overline{\rho}^j \right) . \tag{59}$$

Example 1 For $N = 3$, $\boldsymbol{E}$ has 3 independent elements. And 2 invariants. Firstly,

$$\mathrm{tr}(\boldsymbol{E}) = \rho^2 = \iota . \tag{60}$$

Secondly,

$$\det(\boldsymbol{E}) = {\rho_1}^2 {\rho_2}^2 - (\underline{\rho}_1 \cdot \overline{\rho}_2)^2 = \left| \underline{\rho}_1 \times \overline{\rho}_2 \right|^2 \propto \alpha^2 . \tag{61}$$

The second equality here is Lagrange's identity, whereas α is the mass-weighted area of the triangle.

Example 2 For $N = 4$, $\boldsymbol{E}$ has 6 independent elements.

And 3 invariants, as follows.

$$\mathrm{tr}(\boldsymbol{E}) = \rho^2 = \iota . \tag{62}$$

$$\mathrm{II}(\boldsymbol{E}) = \sum_{\text{cycles}} |\rho_1 \times \rho_2|^2 \propto \sum_{\text{cycles}} {\alpha_{12}}^2 . \tag{63}$$

And

$$\det(\boldsymbol{E}) = [\, \overline{\rho}_1 \,,\, \overline{\rho}_2 \,,\, \overline{\rho}_3 \,]^2 \propto \mathcal{V}^2 . \tag{64}$$

$[\ ,\ ,\ ]$ denotes scalar triple product. The first equality in the last equation is a generalization of Lagrange's identity. This third invariant is however just zero in 2-d.

4.4 A first few nontrivial shape invariants

Remark 1 Normalizing the Euclidean matrix by a dimensionally-matching power of ι gives the *similarity matrix* $\boldsymbol{S}$. Whose indices run over the relative Jacobi vectors' labels. While its components are take

$$S^{ij} := \left(\underline{\nu}^i \cdot \overline{\nu}^j \right) . \tag{65}$$

Here trace is just a number, N.

Example 1 For $N = 3$, the similarity matrix has just 1 other invariant. I.e.

$$\det(\boldsymbol{S}) = |\underline{\nu}_1 \times \overline{\nu}_2|^2 \propto \frac{\alpha^2}{\iota^2} . \tag{66}$$

Example 2 For $N = 4$, the similarity matrix possesses 2 other invariants, as follows.

$$\mathrm{II}(\boldsymbol{S}) = \sum_{\text{cycles}} |\underline{\nu}_1 \times \overline{\nu}_2|^2 \propto \sum_{\text{cycles}} \frac{{\alpha_{12}}^2}{\iota^2} . \tag{67}$$

And

$$\det(\boldsymbol{S}) = [\, \nu_1 \,,\, \nu_2 \,,\, \nu_3 \,]^2 \propto \frac{\mathcal{V}^2}{\iota^3} . \tag{68}$$

Remark 2 In general, such matrix elements are not geometrically independent. They can be interpreted as supplying an associated linear space containing conserved quantities.

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Remark 3 These are not the only possibilities. The matrix $\boldsymbol{A}$ with relative Jacobi vector label index components

$$A^{ij} \ = \ (\underline{\rho}^i \ \times \ \overline{\rho}^j) \tag{69}$$

has Equiareal-Geometric [35] significance. Being antisymmetric, this has

$$\frac{n\,(n \ - \ 1)}{2} \tag{70}$$

independent components. For $N \ = \ 3$, it has only 1 component, whereas for $N \ = \ 4$, it has 3 . In all cases,

$$\mbox{tr}(\boldsymbol{A}) \ = \ 0 \tag{71}$$

by the antisymmetry of $\boldsymbol{A}$.

Example $1'$ For $N \ = \ 3$, the only other invariant is as follows.

$$\mbox{det}(\boldsymbol{A}) \ = \ \left| \underline{\rho}_1 \ \times \ \overline{\rho}_2 \right|^2 \ = \ \alpha^2 \ . \tag{72}$$

Example $2'$ Whereas for $N \ = \ 4$, there are 2 . Firstly,

$$\mbox{det}(\boldsymbol{A}) \ = \ \left| \begin{array}{ccc} 0 & -\left(\underline{\rho}_1 \ \times \ \overline{\rho}_2\right)_\perp & -\left(\underline{\rho}_1 \ \times \ \overline{\rho}_3\right)_\perp \\ \left(\underline{\rho}_1 \ \times \ \overline{\rho}_2\right)_\perp & 0 & -\left(\underline{\rho}_2 \ \times \ \overline{\rho}_3\right)_\perp \\ \left(\underline{\rho}_1 \ \times \ \overline{\rho}_3\right)_\perp & \left(\underline{\rho}_2 \ \times \ \overline{\rho}_3\right)_\perp & 0 \end{array} \right| \ . \tag{73}$$

Secondly,

$$\mbox{II}(\boldsymbol{A}) \ = \ \left| \underline{\rho}_1 \ \times \ \overline{\rho}_2 \right|^2 \ + \ \left| \underline{\rho}_2 \ \times \ \overline{\rho}_3 \right|^2 \ + \ \left| \underline{\rho}_3 \ \times \ \overline{\rho}_1 \right|^2 \ = \ \sum_{k \, > \, l} \underline{\rho}^k \ \times \ \overline{\rho}^l \ = \ \sum_{k \, > \, l} \alpha_{k\,l} \ . \tag{74}$$

5 Repackaging in terms of $SU(3)$ and $SU(N)$ Representation Theory

5.1 Shape Quantities

Remark 1 The complex formulation

$$\boldsymbol{U} \ = \ \boldsymbol{E} \ + \ i\,\boldsymbol{A} \tag{75}$$

gives an n^2 of quantities. These can be split into 1 scale variable, $\rho^2 \ = \ \iota$. And

$$n^2 \ - \ 1 \ = \ N\,(\,N \ - \ 2\,) \tag{76}$$

shape quantities.

In 2-d , the latter pick out an adjoint representation of the following shape space isometry group [57].

$$Isom\left(\mathbb{CP}^{n \ - \ 1}\right) \ \ = \ \ \frac{SU(n)}{C_n} \ . \tag{77}$$

For C_n the cyclic group of order n . This $n^2 \ - \ 1$ of shape quantities consists of the components of $\boldsymbol{S}$. And those of $\boldsymbol{A}$ as normalized by ι :

$$\nu_i \ \ \times \ \ \nu_j \ .$$

Example 1 For triangles, this returns the 3 Hopf quantities: anisoscelesness, normalized area and ellipticity [86, 84].

Example 2 For the quadrilateral, these comprise 3 anisoscelesnesses, 3 areas, and 1 ellipticity. Alongside 1 linear combination (LC) of 2 ellipticities. These last 2 quantities are diagonal. In Particle Physics parlance, they are respectively the '3-component of the isospin' $\mathcal{I}_3$ and the 'hypercharge' $\mathcal{Y}$ [47].

Caveat 6 For the triangle, the shape space is of dimension 2 . While many other associated spaces have coincident dimension 3 . For instance, the 3-d space of sides coincides with the space of relative separations. In contrast, the relationalspace and the space formed by the shape quantities coincide to form another 3-d space. From the first pair, 1 relation on side data – Heron's formula – at least counts out right to send us to the shape space. While diagonalizing Heron puts the on-sphere relation on the second pair's space.

But for quadrilaterals, this quartet of spaces have dimensions $4, 6, 5$ and 8 . So many manoeuvres that worked for triangles do not even compatibly count out any more.

Let us next consider how all of this quartet's counts grow with N .

$$\#(\,\text{independent relational coordinates}\,) \ = \ \dim(\,\mathfrak{R}(\,2,\,N\,)\,) \ = \ 2\,N \ - \ 3\ . \tag{78}$$

$$\begin{aligned} \#(\,\text{shape quantities}\,) \ &= \ \dim\left(Isom\left(\mathbb{CP}^{N \ - \ 2}\right)\right) \ = \ \dim(\,SU(n)\,) \\ &= \ n^2 \ - \ 1 \ = \ (\,N \ - \ 1\,)^2 \ - \ 1 \ = \ N\,(\,N \ - \ 2\,)\ . \end{aligned} \tag{79}$$

So we require

$$N \ = \ \frac{N\,(\,N \ - \ 1\,)}{2} \ = \ 2\,N \ - \ 3 \ = \ N\,(\,N \ - \ 2\,)\ .$$

Or at least some subset of these equalities, if we are to keep *some* of the triangle case's benevolent features.

$N = 3$ is indeed the unique solution for the whole set. By Subsec 3.4's reconceptualization, the relative separations are to supplant the sides, so we can drop the first count. $N = 3$ remains the unique solution of the whole of this smaller equation set. The only partial solutions afforded are $N = 0, 1$ and 2 . We leave it to the Reader to figure out which pairs each of these corresponds to. But these are all trivial models. So in particular, for $N \geq 4$, *none* of the triangle case's above benevolent features are available.

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Proposition 7 [Littlejohn and Reinsch 1995]

$$\frac{(N - 1)(N - 2)}{2} \tag{80}$$

of the N-a-gon's shape quantities form an equiareally-significant $SO(n)$ restricted representation.

Example 1 For triangles, this $SO(2)$ comprises the area variable.

Example 2 For quadrilaterals, this $SO(3)$ comprises the 3 area variables.

Remark 3 The N-a-gon has symmetric nondiagonal anisoscelesnesses partnering the antisymmetric areas. Numbering

$$\#(\text{anisoscelesnesses}) = \binom{N - 1}{2} = \frac{(N - 1)(N - 2)}{2} \tag{81}$$

Remark 4

$$\#(\text{ellipticities}) = \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{N - 1}{2} \right\rfloor . \tag{82}$$

And

$$\#(\text{nontrivial LCs of ellipticities}) = \left\lfloor \frac{n - 1}{2} \right\rfloor = \left\lfloor \frac{N - 2}{2} \right\rfloor . \tag{83}$$

Where $\lfloor \ \rfloor$ is the floor function. And which respectively generalize to $SU(3)$'s isospin 3-component $\mathcal{I}_3$ and hypercharge $\mathcal{Y}$.

Remark 5 The quadrilateral shape quantities are representable by Gell–Mann λ-matrices [21]. And the N-a-gon shape quantities by general-N λ-matrices.

5.2 Casimirs

Structure 1 Crucially, $SU(p)$ possesses a total of $p - 1$ Casimirs.[4] So via its $SU(N - 1)$ content, the full isometry group of N-a-gonland possesses $N - 2$ Casimirs. Among which,

$$\left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{N - 1}{2} \right\rfloor . \tag{84}$$

are associated with the 'democratic-transformation' $SO(n)$ subgroup's restricted representation.

Proposition 8 In 2-d for $N = 4$, $\text{II}(\boldsymbol{S})$ commutes with all the other shape quantities.

Remark 1 This can be readily deduced from Littlejohn and Reinsch's account [46], so it has been known since at least 1995.

Proposition 9 [Anderson 2018] $\text{II}(\boldsymbol{S})$ is moreover a Casimir. Specifically that of the $SO(3)$ democracy subgroup corresponding to the $\boldsymbol{A}$. It is thereby mathematically analogous to total angular momentum in 3-d space.

Notational Remark 1 By which let us subsequently refer to this $\text{II}(\boldsymbol{S})$ as $\mathcal{J}^2$!

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Remark 2 This gives further Representation-Theoretic insight into the quantity which has replaced $(\text{tetra-area})^2$ in passing from triangles to quadrilaterals. And which has replaced volume in passing from 3-d to 2-d 4-body problems [49]. Thus clarifying the content of the quantity called a 'remarkable extra commuting quantity' in in the Molecular Physics literature [46].

Remark 3 Observe furthermore that this $\mathcal{J}^2$ is *not* some function of the area of the whole quadrilateral but rather the sum of squares of its triangle subsystems' areas.

Caveat 7 For triangles, normalized total area is among the shape variables. In contrast, for quadrilaterals, normalized total area is not even among the shape quantities. This state of affairs moreover persists for all larger N-a-gons. So *normalized total area is only a shape quantity in the case of triangles*. This undermines the RSCQAF having Shape-Theoretic significance in parallel to Heron's formula's

Caveat 8 $(\text{tetra-area})^2$ *is moreover the triangle system's sole Casimir*. This suggests pivoting research direction as follows. Generalize Heron's formula via viewing its subject – area – as a quantity whose key property is that it normalizes to give a Casimir. I.e. not by pursuing N-a-gons's area formulae but rather by pursuing their Casimirs! Starting with quadrilaterals' $\mathcal{J}^2$; we leave this to a subsequent Article [104].

Aside 1 Likewise, $\mathcal{J}^2$ might be more interesting to extremize than normalized area in generalizing [82, 83, 84]'s Calculus considerations.

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Remark 8 The Veronese embedding provides a further relation between shape quantities [27, 74]. This Projective-Geometric technique moreover generalizes to the Veronese–Whitney embedding for higher N-a-gons. This is already well-known in the Shape Statistics literature [78, 91].

Caveat 9 So, while Heron's formula for the area of a triangle suffices to understand triangleland's topology and geometry, the above projective structure, beyond the scope of area formulae, first appears in the corresponding study of quadrilaterals.

[4] See e.g. [61] for an outline of what these are.

6 Tetrahaedra and the della Francesca–Tartaglia volume formula

6.1 Notation for the arbitrary tetrahaedron

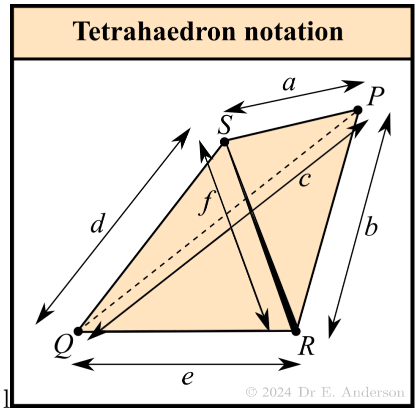

Figure 3: Tetrahaedron notation.

Notational Remark 1 Consider an arbitrary tetrahaedron denoted as in Fig 2. We set this up such that a , b , c meet at the vertex P . And d , e , f concur pairwise at the other vertices Q , R and S .

6.2 The della Francesca–Tartaglia volume formula

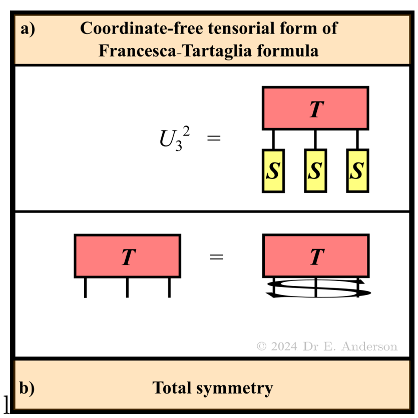

Figure 4:

Remark 1 In terms of this, the standard [4, 5, 89] della Francesca–Tartaglia volume formula simplifies to

$$ {U_3}^2 \; = \; \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & a^2 & f^2 & d^2 \\ 1 & a^2 & 0 & b^2 & c^2 \\ 1 & f^2 & b^2 & 0 & e^2 \\ 1 & 0 & c^2 & e^2 & 0 \end{vmatrix} \; = \; \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & A & F & D \\ 1 & A & 0 & B & C \\ 1 & F & B & 0 & E \\ 1 & 0 & C & E & 0 \end{vmatrix} \; . \qquad (85) $$

Proposition 10 (Expanded form of della Francesca–Tartaglia volume formula) [Anderson 2018] This can furthermore be rewritten as follows.

$$ {U_3}^2 \;=\; T_{i\,j\,k}\, S^i\, S^j\, S^k \;, \tag{86} $$

Where

$$ U_3 \;:=\; 12\,\sqrt{2}\,Vol \tag{87} $$

for volume Vol . $\boldsymbol{S}$ is again given by (36). $\boldsymbol{T}$ is the totally-symmetric *Tartaglia 3-tensor*,

$$ T_{i\,j\,k} \;=\; T_{(\,i\,j\,k\,)} \tag{88} $$

Whose components in our standard basis are as follows.

$$ \boldsymbol{T} \;=\; \begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & -2 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 1 & -2 & -1 & 1 & -2 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 1 & 1 & -2 & -2 & 1 & 1 \\ -1 & 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -1 & 1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & -2 & -2 & 1 & 1 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 & -1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 & -1 & 0 & 0 & 1 \\ 1 & -2 & 1 & 1 & -2 & 0 \\ -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & -2 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 1 & 0 \end{pmatrix} \\ \begin{pmatrix} -2 & 1 & 1 & 1 & 1 & -2 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & -1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{pmatrix} \;. \tag{89} $$

Exercise 5$^-$ Derive (89).

Remark 2 See Fig 4.a) and b) for coordinate-free renditions of (86) and (88), in vertical Penrose birdtracks.

Remark 3 Straightforwardly, there are no isotropic tensors of rank 3 in 6-d , so questions of diagonalization are moot.

Notational Remark 1 The $\mathcal{V}$ used in the Introduction is the mass-weighted analogue of the Vol used above.

7 d-simplices and their Cayley–Menger d-volume formulae

7.1 Introduction

Remark 1 We finally extend consideration to d-simplexes. The Heron and della Francesca–Tartaglia formulae here extend to the arbitrary-d *Cayley–Menger formula.*

Definition 1

$$U_d \ := \ k_d \ \times \ (\, d\text{-Volume} \,) \ := \ \left(2^{d/2} \, d \, ! \right) \ \times \ (\, d\text{-Volume} \,) \ . \tag{90}$$

Remark 2 In terms of this, the standard [7, 13, 15, 20, 89] *Cayley–Menger d-volume formula* simplifies to

$$U_d{}^2 \ = \ \text{abs} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & (\,0,\,1\,)^2 & \ldots & (\,0,\,n\,)^2 \\ 1 & \bullet & 0 & & \\ 1 & \vdots & \bullet & 0 & (\,n \, - \, 1,\,n\,)^2 \\ 1 & \bullet & \ldots & \bullet & 0 \end{vmatrix} \ =: \ \text{abs} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & A_{0\,1}{}^2 & \ldots & A_{0\,n}{}^2 \\ 1 & \bullet & 0 & & \\ 1 & \vdots & \bullet & 0 & A_{n \, - \, 1,\,n}{}^2 \\ 1 & \bullet & \ldots & \bullet & 0 \end{vmatrix} \ . \tag{91}$$

Where $(\,a,\,b\,)$ stands for distance between points a and b . And the heavy dot denotes that each lower triangular component coincides with the corresponding explicitly provided upper triangular component by symmetry.

Remark 3 The first value of k_d supporting a nontrivial Cayley–Menger formula is $k_2 \ = \ 4$. Tetra-area is thus not only a Heron quantity and the numerator of a Hopf quantity but also the first nontrivial Cayley–Menger quantity.

Remark 4 The Cayley–Menger formula can also be rewritten as

$$U_d{}^2 \ = \ C_{i_1 \, \ldots \, i_d} \, S^{i_1} \ldots S^{i_d} \ . \tag{92}$$

Where

$$\boldsymbol{S} \ := \ \begin{matrix} A_1 \\ \vdots \\ A_q \end{matrix} \ = \ \begin{matrix} a_1{}^2 \\ \vdots \\ a_q{}^2 \end{matrix} \tag{93}$$

the $(\,\text{separations}\,)^2$ q-vector. For

$$q \ := \ C(\,N,\,2\,) \ = \ \frac{N\,(\,N \, - \, 1\,)}{2} \ = \ \frac{d\,(\,d \, + \, 1\,)}{2} \ : \tag{94}$$

the number of separations for $N \ = \ d \, + \, 1$ point-or-particles. And

$$C_{i_1 \, \ldots \, i_d} \ = \ C_{(\,i_1 \, \ldots \, i_d\,)} \ : \tag{95}$$

the totally-symmetric *Cayley–Menger* d-tensor. (14) and (85) provide the explicit form for the first two nontrivial examples of this.

Remark 5 See Subfigs 5.a) and b) for coordinate-free renditions of (92) and (95) respectively.

Remark 6 In forming an infinite series of totally symmetric tensors, these bear some analogy to multipole expansion tensors [50]. Those are however spatial tensors, whereas Cayley–Menger tensors are configuration space tensors. It follows that the multipole expansion resides within a fixed dimension (usually $d \ = \ 3$ in applications). In contrast, Cayley–Menger tensors increase in dimension according to (94)'s count.

Exercise 6 Use a computer to obtain the components of $\boldsymbol{C}$ for the 4- and 5-simplexes.

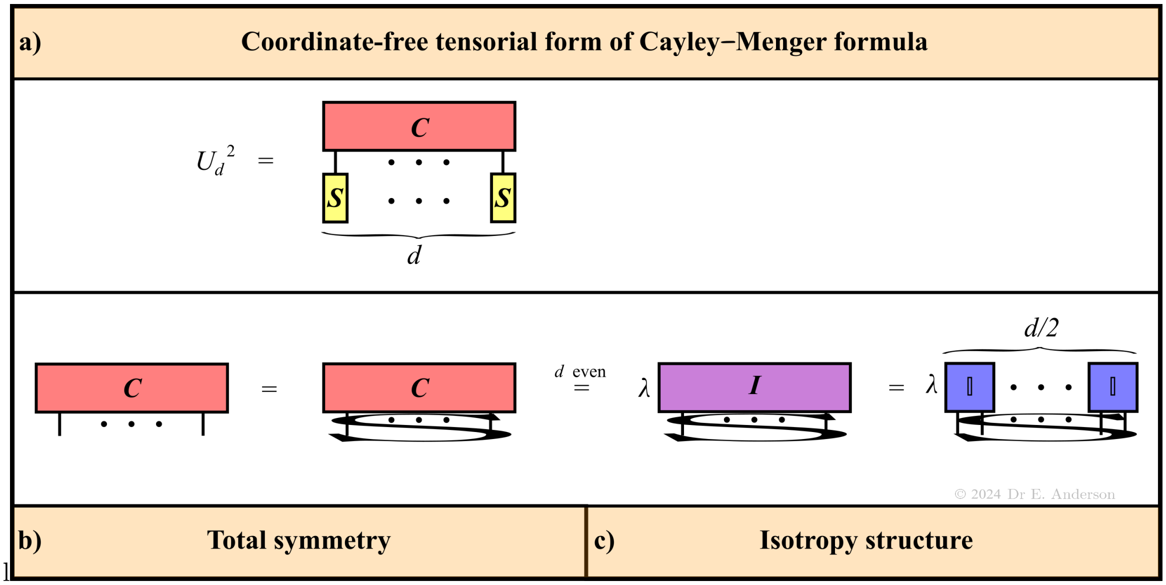

Figure 5:

7.2 Eigentheory, where possible

Remark 7 Let us next diagonalize the Cayley–Menger tensors. I.e. solve

$$C_{i_1 \dots i_d} = \lambda \, I_{i_1 \dots i_d} . \tag{96}$$

Where $I_{i_1 \dots i_d}$ is the d-dimensional matchingly totally-symmetric isotropic tensor. See Subfig c) for a coordinate-free rendition, where $\mathbb{I}$ is the identity-matrix coordinate-free notation for the Kronecker delta tensor.

Remark 8 This gives separate odd- and even-d cases.

A) For odd-d, there are no rank-d isotropic tensors in dimension d, so (96) is moot.

B) For even-d (> 0), there are, so we do have an eigenvalue problem. Which produces

$$\left(\frac{d(d+1)}{2} \right)^{d/2} \tag{97}$$

eigenvalues. This is to be compared with the Casson sphere corresponding to

$$\frac{d(d+1)}{2} \tag{98}$$

quantities summing to 1. [ι remains available as a scale variable along the Casson diagonal.]

So, for $d > 0$ even, the following is required.

$$\left(\frac{d(d+1)}{2} \right)^{d/2} = \frac{d(d+1)}{2} . \tag{99}$$

I.e.

$$\left(\frac{d(d+1)}{2} \right)^{d/2 - 1} = 1 . \tag{100}$$

Which can only be solved for $d \in 2\mathbb{N}$ by

$$d = 2 . \tag{101}$$

Thus, even just combinatorially, the extent of the Cayley–Menger–Casson coincidence is just the Heron–Kendall–Casson coincidence that yields the triangleland sphere.

8 Conclusion

It was recently shown [86] that for triangle constellations in 2-d, the form taken by the corresponding shape space – a sphere – can be derived from Heron's formula. This is a fourth derivation of this result: 'Kendall's Little Theorem'. The other three are the two outlined in the Appendix: Kendall's extremization [51] and using the Hopf map. And setting up an indirectly-formulated similarity mechanics action for the problem to be subjected to Lagrangian-level reduction [62] (or similar [67, 81]). The Hopf map itself is moreover derived from Heron's formula in the process. Its three coordinate functions – mass-weighted tetra-area, anisoscelesness and ellipticity in the Shape-Theoretic context – arise as follows. As the subject of Heron's formula and as the two non-unit eigenvectors of the Heron–Buchholz matrix respectively.

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In the current Article, however, we show that this fourth derivation is a one-off, as follows.

1) Triangleland benefits from there being 3 of each of the following. Sides, relative separations, independent relational coordinates and shape quantities. But for quadrilaterals, there are 4 6 5 and 8 of these! Nor does any higher N-a-gon have any numerical coincidences between these.

2) At the level of shape space geometries, triangleland is simplified by $\mathbb{CP}^1 = \mathbb{S}^2$. Which good fortune does not repeat for quadrilaterals or for any higher N-a-gons. One is here left needing to deal with $\mathbb{CP}^{N-2}$: the general Kendall Theorem (Appendix B). So some methods of arriving at, and comprehending, this 2-d space are 'spherical' rather than 'projective' and so fail to generalize to projective cases. Also scaled triangleland is then the cone over $\mathbb{S}^2$, and thus $\mathbb{R}^3$ again.[5] While scaled quadrilateralland is a distinct space: the cone over $\mathbb{CP}^{N-2}$.

3) At the level of area formulae, Brahmagupta's is too specialized and Bretschneider's first uses heterogeneous data. Neither of these can be expressed solely in terms of separation^2 variables. Bretschneider's second area formula uses just separation data, as is clear from its truer name relative-separations-data convex-quadrilateral area formula (RSCQAF). And depends purely on separations^2 . But it does not make equable use of separations, by distinguishing sides and diagonals.

4) Unlike the Heron–Euler–Buchholz matrix, the RSCQ matrix is not invertible. In more detail, it has zero as an eigenvalue with multiplicity 3. So its diagonal form inter-relates just 4 quantities. Nor does the ensuing condition directly help us in finding that quadrilateralland is $\mathbb{CP}^2$. There is also a mismatch between a formula for convex quadrilaterals and a shape theory about all quadrilaterals.

5) For triangles, shape space is 2-d and separationspace $=$ sidespace is 3-d . So 1 relation suffices to take us from separationspace to shape space. But for quadrilaterals, shape space is 4-d and separationspace is 6-d . So using an area formula to descend from separationspace to shape space is under-determined.

6) Total area (mass-weighted and MoI-normalized to $\mathcal{A}$) is among the shape variables for the triangle. But not for the quadrilateral. For the triangle, $\mathcal{A}$ is furthermore a Hopf quantity – and its square is the system's sole Casimir.

To generalize this, we need to consider the democratic $so(N-1)$ subalgebra [49] of $su(N-1)$. Itself arising from $Isom\left(\mathbb{CP}^2\right)$ [26]. In the N-a-gon context, an adjoint rep of $su(N-1)$ is realized by the shape quantities [74]. And the successor of $\mathcal{A}^2$ as $so(N-1)$ Casimir for an N-a-gon is as follows. The sum of squares of areas of constituent triangle subsystems, again mass-weighted and MoI-normalized, $\mathcal{S}^2$. So now $\mathcal{S}^2$ manages to be a Casimir without $\mathcal{S}$ being a shape quantity.

[5] Albeit this is not flat, though it is conformally flat [33].

Thus following the Casimirs, triangle area's natural successor for quadrilaterals is, rather, the minimum nontrivial example of $\mathcal{S}$. Involving 3 constituent triangle subsytems. We leave detailed study of this for a future occasion [104]. This item 6) also renders moot that the quadrilateral area formula considered in the current Article are not an exhaustive set [12, 95]. Though the Linear Algebra that drops out of considering this extended set is interesting enough to be the subject of another article, [102].

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Triangeland $\mathbb{S}^2$ is also the first nontrivial d-simplexland alias d-basisland [92]. I.e. the

$$N = d + 1 \tag{102}$$

diagonal, which Casson showed to be topologically

$$\mathbb{S}^{d(d+1)/2 - 1} . \tag{103}$$

So 'Kendall's Little Theorem' may furthermore be viewed as the unique case in which both Kendall's Theorem and Casson's Theorem apply. Reasons for non-extension of the Heron proof of Kendall's Little Theorem to higher simplexlands are as follows.

1) The next case along on the diagonal of simplexlands (Fig 7) is tetrahaedronland. This admits the della Francesca–Tartaglia volume formula. As is most easily seen from recasting Heron's formula as a determinant and generalizing dimensionally. This can be reformulated in terms of a (new, as far as the Author is aware) 'totally symmetric Tartaglia 3-tensor' in the 6-d separation space supported by tetrahaedral constellations. This does not however admit eigenvalues and eigenvectors since there are no isotropic tensors of rank 3 in 6-d .

2) More generally, the entire simplexland diagonal is populated by the d-volume formulae of Cayley–Menger [7, 13, 15, 20, 34, 89]. That these can be reformulated in terms of the (also new, as far as the Author is aware) infinite series of Cayley–Menger totally-symmetric d-tensors. That mass-weighted d-hypervolume divided by a matching power of the moment of inertia is a suitable shape quantity for each d . All Cayley–Menger formulae make equable use of separation data, in which sense they are superior to the RSCQAF. The even-d cases among these furthermore possess isotropic tensors of the right dimension-and-rank so as to have eigenvalue problems. These eigenvalue problems, however, do not have the right dimensionality to produce on-$\mathbb{S}^{d(d+1)/2-1}$ conditions. Other than in the $d = 2$ case in which the Heron derivation of 'Kendall's Little Theorem' is recovered.

3) That the Hopf map generalizes along the N-a-gonlands plays a deep underpinning role in these being more geometrically understood than the diagonal of simplexlands. This is with particular reference to Kendall's Theorem being a *Metric Differential-Geometric* as well as Topological result, whereas Casson's Theorem is solely Topological. This, and accumulated knowledge about the metric geometry [26, 29] and associated linear methods [74] for $\mathbb{CP}^{N-2}$, renders N-a-gonlands far more amenable to physical study than d-simplexlands. There is one sense in which this is unfortunate: that tetrahadreons – of the 4-body problem in 3-d – are more directly relevant to nature than N-a-gons in 2-d . And one sense in which this is fortunate is as follows. That N-a-gons provide a 'shape representation' [60] for Quantum-Information-Theoretic qu-N-its [39, 52, 77].

♠ ◇ ♣

All the other three derivations of Kendall's Little Theorem mentioned in the Introduction do generalize to N-a-gons. With the Hopf case proceeding via one of the usual generalizations of the Hopf map, as per Fig 8.c-d). The N-a-gon versions of these derivations are covered likewise in the Appendices and in and [51, 62, 67, 81].

A Suite of configuration spaces for N-body problems

Remark 1 *Carrier space* $\mathfrak{C}^d$ is some at-least-provisional model for the structure of space. It is referred to as *absolute space* in the case of physical space. Though e.g. Euclidean Geometry and Probability Theory also have notions of carrier space notions, such as 'the Euclidean plane' and 'sample space'; see [82] for further discussion. Indeed, $\mathbb{R}^d$ is the space most often cast in the role of carrier space, with Euclidean Geometry's plane being the 2-d subcase of this. Employing a manifold $\mathfrak{m}^d$ instead adds considerable scope to what can be modelled.

Definition 1 *Constellationspace* is the product of N copies of this carrier space. I.e.

$$\mathfrak{q}\left(\mathfrak{m}^d, N\right) = \bigtimes_{I=1}^{N} \left(\mathfrak{m}^d\right)^N . \tag{104}$$

Each point of which models a figure formed by N points on $\mathfrak{m}^d$: a *constellation*. Or, if these points are materially realized, a figure formed by N particles (classical, nonrelativistic).

Remark 2 Constellationspace (see [67, 82] for reviews) is a simple instance of *configuration space* [19, 28].

Remark 3 In *Kendall's Shape Theory* [32, 36, 51] constellationspace is considered for $\mathfrak{m}^d = \mathbb{R}^d$ with the group of similarities $Sim(d)$ furthermore quotiented out. The corresponding *shape spaces* – less trivial, *reduced*, configuration spaces – are thus of the following form.

$$\mathfrak{S}(d, N) = \frac{\mathfrak{q}(d, N)}{Sim(d)} = \frac{\times_{I=1}^{N} \mathbb{R}^d}{Sim(d)} = \frac{\mathbb{R}^{dN}}{Sim(d)} . \tag{105}$$

Kendall's work remains rather more familiar in the Shape Statistics literature [32, 36, 48, 51, 55, 71, 80, 78]. Though related work has also appeared in other fields e.g. Mechanics and Molecular Physics [46, 49, 56, 75, 62, 67, 92]. And in e.g. [62, 67, 66, 70, 73, 81] modelling some aspects of General Relativity's Background Independence [25, 24, 41, 42, 72, 81, 94].

Theorem 0 (Kendall's Little Theorem) The shape space of vertex-labelled mirror-images-distinct triangles in $\mathbb{R}^2$ is [32, 36, 51]

$$\mathfrak{S}(2, 3) = \mathbb{S}^2 . \tag{106}$$

For some context, we provide a small lattice of geometrical groups acting on flat space in Fig 6.a). With the corresponding quotients in Subfig b) comprising a suite of intermediary configuration spaces. Among these intermediaries, preshape space was already outlined in Sec 4.1.

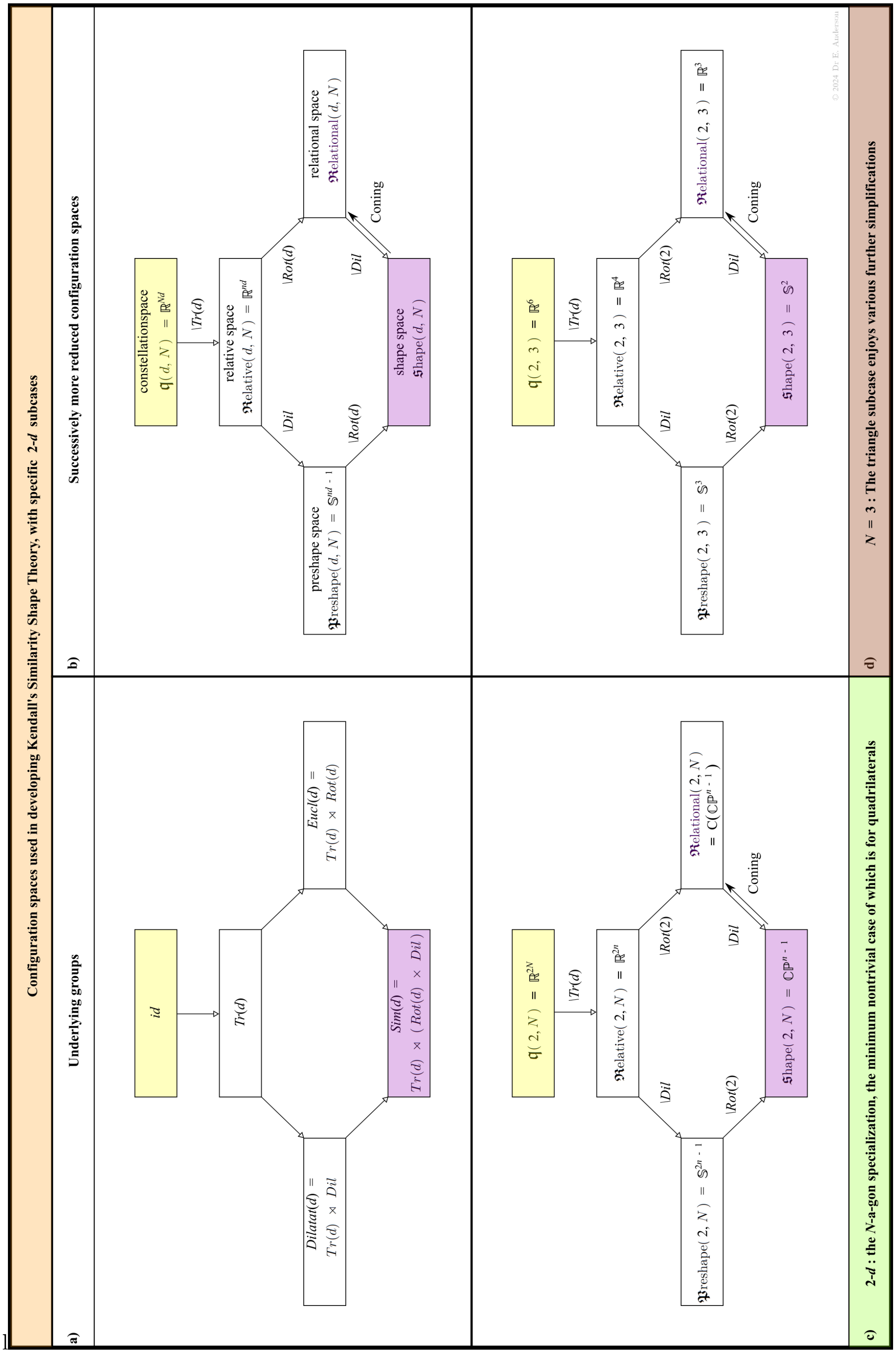
Configuration spaces used in developing Kendall's Similarity Shape Theory, with specific 2-d subcases
a)
Underlying groups
id
Tr(d)
Dilatat(d) = Tr(d) × Dil
Eucl(d) = Tr(d) × Rot(d)
Sim(d) = Tr(d) × (Rot(d) × Dil)
b)
Successively more reduced configuration spaces
constellationspace q(d, N) = ℝ^Nd
\Tr(d)
relative space ℜelative(d, N) = ℝ^nd
\Dil
\Rot(d)
preshape space 𝔓reshape(d, N) = 𝕊^nd - 1
relational space ℜelational(d, N)
shape space 𝔖hape(d, N)
Coning
c)
q(2, N) = ℝ^2N
ℜelative(2, N) = ℝ^2n
𝔓reshape(2, N) = 𝕊^2n - 1
ℜelational(2, N) = C(ℂℙ^n - 1)
𝔖hape(2, N) = ℂℙ^n - 1
\Tr(d)
\Dil
\Rot(2)
Coning
2-d : the N-a-gon specialization, the minimum nontrivial case of which is for quadrilaterals
d)
q(2, 3) = ℝ^6
ℜelative(2, 3) = ℝ^4
𝔓reshape(2, 3) = 𝕊^3
ℜelational(2, 3) = ℝ^3
𝔖hape(2, 3) = 𝕊^2
\Tr(d)
\Dil
\Rot(2)
Coning
N = 3 : The triangle subcase enjoys various further simplifications
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Figure 6:

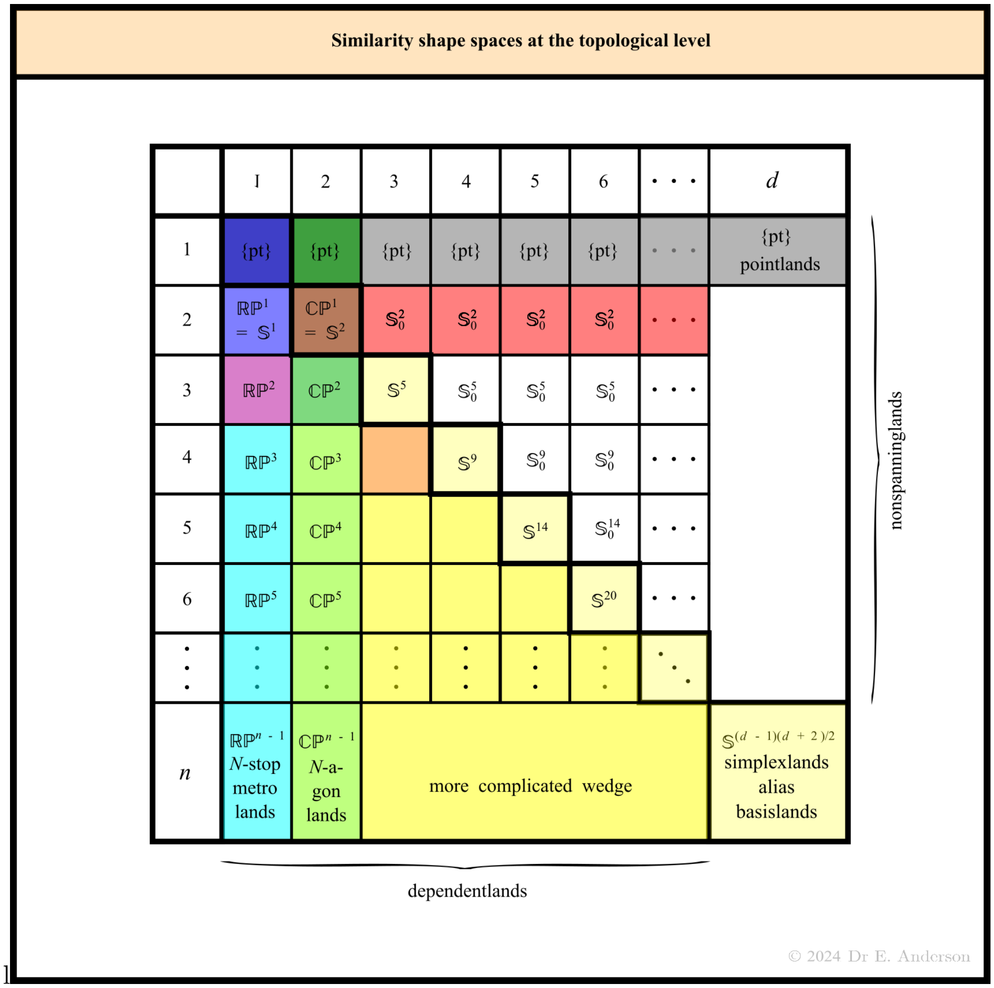

Figure 7:

Remark 4 Fig 7 gives the (d, N) grid of shape spaces at the topological level. Therein, $\mathbb{S}_0{}^k$ denotes the k-dimensional hemisphere.

B Kendall's geometric extremization proof

Proposition 1 Preshape space is straightforwardly a sphere (51). This carries furthermore the extrinsically-defined chordal metric. Or the topologically-equivalent [51] intrinsically-defined great circle metric,

$$D(\mathrm{P},\,\mathrm{Q}) \;=\; \arccos(\mathrm{P},\,\mathrm{Q})\;. \tag{107}$$

For $\mathrm{P},\,\mathrm{Q}$ arbitrary points on the sphere.

Proposition 2 2-d shape space $\mathfrak{S}(2,\,N)$ then carries the following quotient metric.

$$D(\,Q(\mathrm{P},\,\mathrm{Q})\,) \;=\; \min_{R\,\in\,Rot(2)} D(\mathrm{P},\,R(\mathrm{Q})\,) \;=\; \min_{R\,\in\,SO(2)} \arccos(\mathrm{P},\,R(\mathrm{Q})\,)\;. \tag{108}$$

Kendall shows furthermore that this takes the following form by carrying out a basic Calculus extremization.

$$\cos D(\,Q(\boldsymbol{z}),\,Q(\boldsymbol{w})\,) \;=\; \frac{|\,(\,\boldsymbol{z}\,\cdot\,\boldsymbol{w}\,)_{\mathrm{C}}\,|}{||\boldsymbol{w}||_{\mathrm{C}}||\boldsymbol{z}||_{\mathrm{C}}}\;. \tag{109}$$

For

$$(\,\boldsymbol{w}\,\cdot\,\boldsymbol{z}\,)_{\mathrm{C}} \;:=\; \sum_{i\,=\,1}^{n} z_A\,\bar{w}^A\;. \tag{110}$$

Where the bar denotes complex conjugate. And $||\;||_{\mathrm{C}}$ is the corresponding norm. See [51, 82] for what $\boldsymbol{z}$ is in the N-a-gon context.

Remark 1 A small perturbation

$$\boldsymbol{w} \;=\; \boldsymbol{z} \;+\; \delta\boldsymbol{w} \tag{111}$$

brings about the following small change.

$$\delta D^2 \;=\; \sin^2\delta D \;+\; O(\,(\delta D)^4\,) \;=\; 1 \;-\; \cos^2 D(\,Q(\boldsymbol{z}),\,Q(\,\boldsymbol{z}\;+\;\delta\boldsymbol{z}\,)\,) \;+\; O(\,(\delta\boldsymbol{z})^4\,)\;. \tag{112}$$

Thus using (109)

$$=\;1\;-\;\frac{|\,(\,\boldsymbol{z}\,\cdot\,(\,\boldsymbol{z}\;+\;\delta\boldsymbol{z}\,)_{\mathrm{C}}\,|^2}{||\boldsymbol{z}||_{\mathrm{C}}{}^2\,||\,\boldsymbol{z}\;+\;\delta\boldsymbol{z}\,||_{\mathrm{C}}{}^2} \tag{113}$$

So expanding

$$=\;\frac{||\boldsymbol{z}||_{\mathrm{C}}\,||\delta\boldsymbol{z}||_{\mathrm{C}}\;-\;|\,(\,\boldsymbol{z}\,\cdot\,\delta\boldsymbol{z}\,)_{\mathrm{C}}\,|^2}{||\boldsymbol{z}||_{\mathrm{C}}{}^4}\;+\;O(\,(\delta\boldsymbol{z})^4\,)\;. \tag{114}$$

Finally take the limit as $\delta\boldsymbol{z}\;\longrightarrow\;0$ to obtain the natural *Fubini–Study metric* in standard *homogeneous coordinates*. I.e.

$$\mathrm{d}s^2 \;=\; \frac{||\boldsymbol{z}||_{\mathrm{C}}\,||\mathrm{d}\boldsymbol{z}||_{\mathrm{C}}\;-\;|\,(\,\boldsymbol{z}\,\cdot\,\mathrm{d}\boldsymbol{z}\,)_{\mathrm{C}}\,|^2}{||\boldsymbol{z}||_{\mathrm{C}}{}^4}\;. \tag{115}$$

Changing homogeneous coordinate patch when necessary, it can be ascertained that this recovers the entirety of $\mathbb{CP}^{N\,-\,2}$.

Remark 2 A second proof follows from the Hopf map, in the extended $\mathcal{H}_{\mathrm{C}}$ sense of Fig 8.a-b).

C Hopf maps

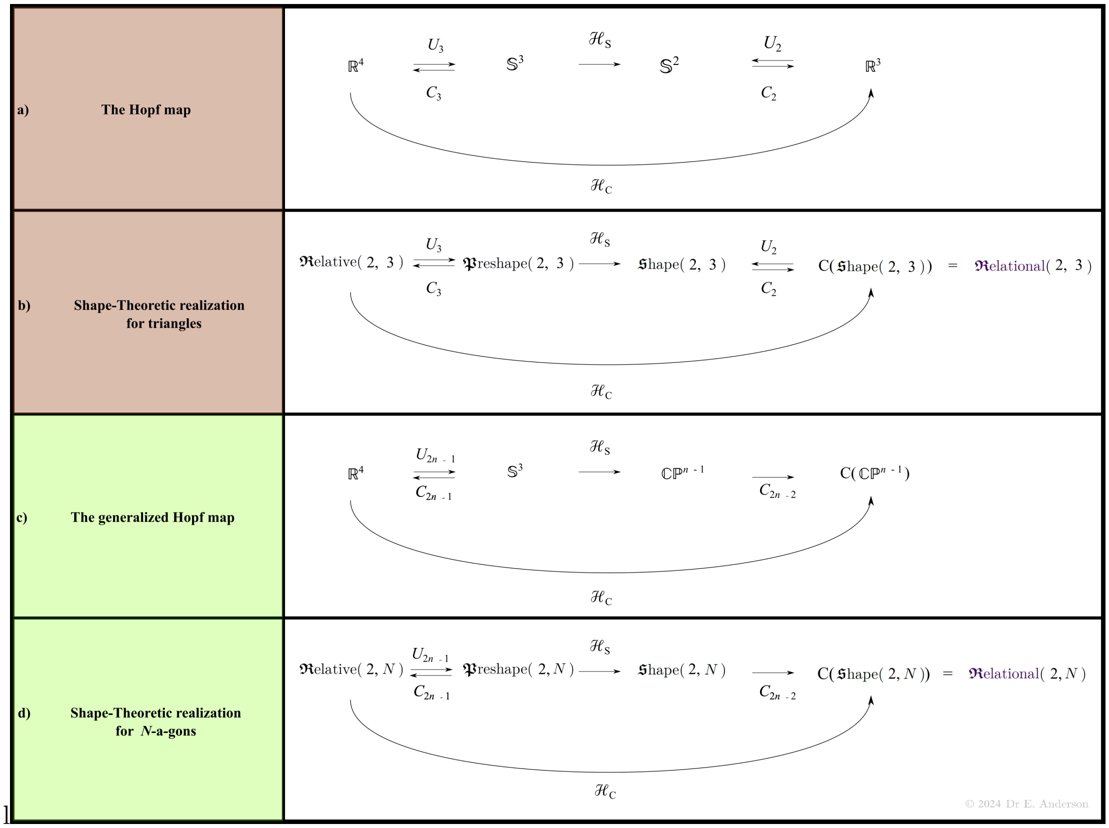

Figure 8:

Remark 1 The map that Hopf emphasized [16] is the *Hopf spheres map* $\mathcal{H}_{\mathrm{S}}$ (Fig 8).

Remark 2 Our figure includes also the map from the ambient $\mathbb{R}^4$ for the $\mathbb{S}^3$. To the less obviously realized ambient $\mathbb{R}^3$ for the $\mathbb{S}^2$. Which we term the *Hopf Cartesian map* $\mathcal{H}_{\mathrm{C}}$.

Remark 3 The U_k also displayed are *unit vector maps* to the corresponding $\mathbb{S}^k$.

Remark 4 Finally, the C_k are *cone maps* from $\mathbb{S}^k$.

Remark 5 See [38, 58] for brief pedagogical expositions of the Hopf map, [17, 22, 33, 46, 39, 44, 52, 59, 68, 77] for applications, and [59, 53, 30, 45] for more advanced theorization.

Remark 6 In deriving the generalized Kendall's Theorem in the Hopf manner, the Fubini–Study metric arises by projection along the fibres: now a Bundle-Theoretic construction.[6]

Remark 7 All of this Appendix generalizes to the

$$\mathbb{S}^{2\,n\,-\,1} \longrightarrow \mathbb{CP}^{n\,-\,1} \tag{116}$$

Hopf map (Subfig c) for N-a-gons (Subfig d) providing a proof for the general case of Kendall's Theorem.

[6]See furthermore e.g. [38, 77] as regards obtaining the Fubini–Study metric from Kähler first principles. There is for now no known Shape-Theoretic first principle for N-a-gonland's Kähler potential, thus providing us with a new research question [100]

D The triangleland subcase

Remark 1 Regardless of which route we derived the generalized Kendall Theorem by, for $N = 3$'s single z variable, the Fubini–Study metric collapses to the following.

$$\mathrm{d}s^2 = \frac{|\mathrm{d}z|^2}{(1 + |z|^2)^2} . \tag{117}$$

Remark 2 Then using the polar coordinates representation

$$z = \mathcal{R}\,\mathrm{e}^{i\,\Phi} , \tag{118}$$

we obtain the following expression for the metric.

$$\mathrm{d}s^2 = \frac{\mathrm{d}\mathcal{R}^2 + \mathcal{R}^2\mathrm{d}\Phi^2}{4\,(1 + \mathcal{R}^2)^2} . \tag{119}$$

This is readily identified as the spherical metric in stereographic polar coordinates.

Remark 3 Finally using the venerable substitution

$$\mathcal{R} = \tan\frac{\Theta}{2} , \tag{120}$$

we recover the the following up to a constant of proportion. The standard spherical coordinates form of the metric,

$$\mathrm{d}s^2 = \mathrm{d}\Theta^2 + \sin^2\Theta\,\mathrm{d}\Phi^2 . \tag{121}$$

This makes sense within the previous Appendices' context by the accidental topological-and-geometrical relation

$$\mathbb{CP}^1 = \mathbb{S}^2 . \tag{122}$$

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