A 4-Areas Theorem for Affine Geometry

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Abstract

Equi-Cevians possess a third area ratio on top of the two that are accorded values by Routh's Theorems. By which the concurrency alias collinearity object can be written as a function of 4 areas. Or 3 area ratios, with the further benefit that such are affine invariants.

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1 [Transversals], [collinearity] and their Theorems

Notational Remark 1 Let A, B, C denote the vertices of a triangle $\triangle$. Thereupon, let $[T]$ denote the following projective-dual portmanteau. Of a triple of transversals (alias Menelians) T and co-transversals (alias Cevians) $\bar{T}$. Let [collinearity] denote the corresponding portmanteau of collinearity and concurrency.

Remark 1 $[T]$ then possesses the following [collinearity] criteria of Menelaus and of (Hud–)Ceva [1, 3, 7, 5].

$$\lambda\,\mu\,\nu \;=:\; \mathcal{C} \;=\; [\text{sign}] \;. \tag{1}$$

Notational Remark 2 Here sign $=\ -1$ and $\bar{\text{sign}}\ =\ 1$. And

$$\lambda \;:=\; \frac{\text{BL}}{\text{LC}} \;. \tag{2}$$

is the *co-transversal* (*Cevian*) *signed-ratio variable*. For L the point at which the co-transversal cuts the opposite side BC (possibly extended). Also μ, ν are the cycles of λ . Finally $\mathcal{C}$ is the [concurrency] object, alias Hud-Ceva and Menelaus object [10].

2 Routh's [Acollinearity] Theorems

Notational Remark 1 The *naïve* [*acollinearity*] is

$$[\mathfrak{a}\text{a}] \;:=\; \mathcal{C} \;-\; [\text{sign}] \;. \tag{3}$$

Where $\mathfrak{a}$a is the portmanteau of the acollinearity $\mathfrak{a}$i and the aconcurrency $\mathfrak{a}$u . Let us also denote the area of a triangle X by $A(X)$.

Remark 1 Both cyclic triples of $[T]$ form their own triangles. Routh's [Acollinearity] Theorems [3, 2] then give expressions for the area ratios

$$[\mathfrak{R}\text{a}] \;:=\; \frac{A([T])}{A(\triangle)} \;. \tag{4}$$

These Theorems can be jointly formulated as [10]

$$[\mathfrak{G}\text{a}] \;=\; [\mathfrak{a}\text{a}]\,[\mathfrak{C}\text{a}] \;. \tag{5}$$

Notational Remark 3 The *geometrical* [*acollinearity*] $[\mathfrak{G}\text{a}]$ is the portmanteau of $\mathfrak{G}\text{i} \;:=\; \mathfrak{R}\text{i}$ and $\mathfrak{G}\text{u} \;:=\; \sqrt{\mathfrak{R}\text{u}}$. And the *Routhian* [*acollinearity*] *scale-factor* $[\mathfrak{C}\text{a}]$ is the portmanteau of the following.

$$\mathfrak{C}\text{i} \;:=\; \prod_{\text{cycles}} \left(\,1 \;+\; \lambda\,\right)^{-1} \,, \quad \mathfrak{C}\text{u} \;:=\; \prod_{\text{cycles}} \left(\,1 \;+\; \lambda(1 \;+\; \mu\,)\,\right)^{-1/2} \;. \tag{6}$$

Remark 2 For what the original statements of these Theorems look like, and their conceptual repackaging into (5), see [10].

Remark 3 The above four Theorems are all part of Affine Geometry [6, 12]. Ratios of areas, such as $[\mathfrak{R}\text{a}]$, enjoy the further property of being *affine invariants* [13].

3 Equi-Cevians and their Heron's formula

Remark 1 A triple of co-transversals is *equi-Cevian* if

$$\lambda \;=\; \mu \;=\; \nu \;. \tag{7}$$

An equivalent statement in side-fraction variables,

$$\xi \;:=\; \frac{\mathrm{BL}}{\mathrm{BC}} \;\; \text{and cycles} \;\; \eta \;, \;\; \zeta \;, \tag{8}$$

is as follows.

$$\xi \;=\; \eta \;=\; \zeta \;. \tag{9}$$

Remark 2 Equi-Cevians possess the following analogue of Heron's formula.

$$T(\triangle) \;=\; \sigma \, ||\boldsymbol{E}||_{\boldsymbol{F}} \;. \tag{10}$$

Where T is the tetra-area. $\boldsymbol{F}$ is the fundamental triangle matrix [8], previously alias Heron matrix [4]. $\boldsymbol{E}$ is the vector of equi-Cevian lengths^2 . And

$$\sigma \;:=\; \left(1 \;-\; \xi \;+\; \xi^2\right)^{-1} \;. \tag{11}$$

For comparison, in this Linear-Algebraic formulation, Heron's formula takes the form

$$T(\triangle) \;=\; ||\boldsymbol{S}||_{\boldsymbol{F}} \;. \tag{12}$$

Where $\boldsymbol{S}$ is the vector of side lengths^2 subcase of the above, corresponding to either $\xi \;=\; 0$ or $\xi \;=\; 1$. For how this is equivalent to the usually stated square root form, and how in parallel the equi-Cevian case can also be cast in square root form, see [9].

4 The Equi-Cevian Area-Magnification Theorem

Remark 1 The equi-Cevians furthermore close to form their own triangle, E [11].

Remark 2

$$||\boldsymbol{E}||_{\boldsymbol{F}} \tag{13}$$

can thus also be interpreted as the tetra-area of E .

So

$$\mathfrak{Re} \;:=\; \frac{A(E)}{A(\triangle)} \;=\; \frac{T(E)}{T(\triangle)} \;=\; \frac{||\boldsymbol{E}||_{\boldsymbol{F}}}{\sigma\,||\boldsymbol{E}||_{\boldsymbol{F}}} \;=\; 1 \;-\; \xi \;+\; \xi^2 \;=\; \frac{1 \;+\; \lambda \;+\; \lambda^2}{(1 \;+\; \lambda)^2} \;=\; \left(\frac{\mathfrak{Ci}}{\mathfrak{Cu}}\right)^{2/3} \;. \tag{14}$$

Thus we have proven the following Theorem

Theorem 1

$$\mathfrak{Re} \;=\; \left(\frac{\mathfrak{Ci}}{\mathfrak{Cu}}\right)^{2/3} \;. \tag{15}$$

Remark 3 $\mathfrak{Ri}$, $\mathfrak{Ru}$ and $\mathfrak{Re}$ can also be interpreted as *area-magnification factors* in passing from the original triangle $\triangle$ to the T , $\bar{T}$ and E triangles respectively. By which, for equi-Cevians, Theorem 1 joins Routh's two Theorems in being an *area-magnification Theorem*.

Remark 4 Our new affine-invariant area ratio is furthermore purely a function of the ratio of the scale-factors that emerge from Routh's Theorems about two other affine-invariant area ratios.

5 The 4-Areas Theorem

Remark 1 We can therefore eliminate this ratio of scale factors so as to relate the four areas ocurring in the current Article to the aconcurrency, as follows.

Theorem 2 (4-Areas Theorem) For equi-Cevians,

$$\frac{A\left(\bar{T}\right)\,A(E)^3}{A(T)^2\,A(\triangle)^2} \;=\; \left(\frac{\mathcal{C}\;-\;1}{\mathcal{C}\;+\;1}\right)^2 . \tag{16}$$

Proof

$$\frac{A(E)^3}{A(\triangle)^3} \;=\; \frac{\mathfrak{Ci}^2}{\mathfrak{Cu}^2} \;=\; \frac{A(T)^2}{A(\triangle)^2}\,\frac{1}{(\,\mathcal{C}\;+\;1\,)^2}\,\frac{A(\triangle)}{A\left(\bar{T}\right)}\,(\,\mathcal{C}\;-\;1\,)^2 \;=\; \frac{A(T)^2}{A\left(\bar{T}\right)\,A(\triangle)}\left(\frac{\mathcal{C}\;-\;1}{\mathcal{C}\;+\;1}\right)^2 . \tag{17}$$

Finally concentrate all the area terms in the left-hand-side. □

Naming Remark 1 In Fluid Mechanics parlance, the right-hand-side is the square of the *Atwood number of the* [*acollinearity*] *quantity*. In its original setting, this is a density contrast between 2 layers of fluid. In the current context, it is the *contrast between the concurrency and its critical value*. Indeed, *contrast* is a useful notion in the Geometrical theory of ratios. By which *fluid* 2*-layer density contrast* is a truer name for Atwood number.

Remark 2 Setting

$$\mathcal{D} \;=\; \frac{\mathcal{C}\;-\;1}{\mathcal{C}\;+\;1} \quad\Rightarrow\quad \mathcal{C} \;=\; \frac{1\;+\;\mathcal{D}}{1\;-\;\mathcal{D}} .$$

Thus

$$\mathcal{C} \;=\; \frac{A(T)\,A(\triangle)\;+\;\sqrt{A\left(\bar{T}\right)\,A(E)^3}}{A(T)\,A(\triangle)\;-\;\sqrt{A\left(\bar{T}\right)\,A(E)^3}} . \tag{18}$$

6 Affine-Invariant Corollaries

Remark 1 Since area ratios are affine invariants, we end with the following rearrangements.

Corollary 1 (3-Area-Ratios Theorem)

$$\frac{\mathfrak{Ru}\,\mathfrak{Re}^3}{\mathfrak{Ri}^2} \;=\; \left(\frac{\mathcal{C}\;-\;1}{\mathcal{C}\;+\;1}\right)^2 . \tag{19}$$

Corollary 2 (Affine-Invariant formulation of [Acollinearity])

$$\mathcal{C} \;=\; \frac{\mathfrak{Ri}\;+\;\sqrt{\mathfrak{Ru}\,\mathfrak{Re}^3}}{\mathfrak{Ri}\;-\;\sqrt{\mathfrak{Ru}\,\mathfrak{Re}^3}} . \tag{20}$$

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