Sides-ratio data versus the Triangleland Sphere

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Abstract

For triangles, we consider converting side-lengths data into medians-lengths and Swiss-army-knife angles between medians and the corresponding half-base sides. And the corresponding conversion of sides ratios into base-median ratios and Swiss-army-knife angles. These conversions produce the Jacobi coordinate quantities that clarify the structure of triangles' more reduced configuration spaces, culminating in Kendall's triangleland shape sphere.

We then ask what the spherical metric is in terms of sides-ratio data. We also give a brief account of the base choice- alias clustering choice-dependent sides-ratio flow on the triangleland sphere. Both of these workings are Metric- and Differential-Geometric in nature.

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1 Introduction

1.1 Median lengths and Swiss army knife angles

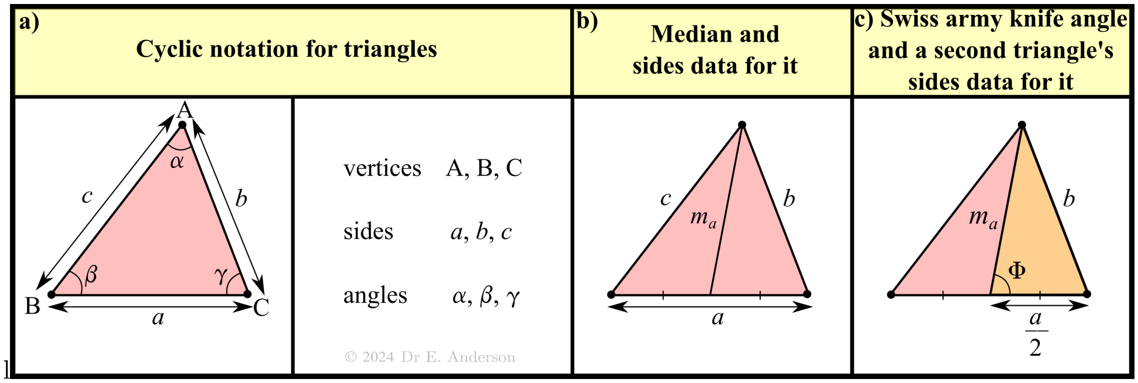

Figure 1:

Apollonius' Theorem [1, 20] is that the length of a median of a triangle is related to to the lengths of its 3 sides as follows.

$$m_a{}^2 \;=\; \frac{2\left(b^2 \,+\, c^2\right) \,-\, a^2}{4}\;. \tag{1}$$

This is with reference to Fig 1.a) and b)'s notation.

Remark 1 Median length turns out to be useful in studying the relational space [15, 19, 23, 25] and shape space [7, 10, 14, 16, 22, 23, 25, 29] of triangles in the manner explained in Appendix A. Wherein one additionally needs the '*Swiss army knife*' angle Φ between the median and the corresponding base side. Using the cosine rule in Fig 1.c)'s orange triangle,

$$a\,m_a\,\cos\,\Phi \;=\; 2\,\frac{a}{2}\,m_a\,\cos\,\Phi \;=\; \left(\frac{a}{2}\right)^2 \,+\, m_a^2 \,-\, c^2\;. \tag{2}$$

Remark 2 This Φ is strictly $\Phi^{(a)}$. I.e. the Swiss army knife angle of the (a)-clustering in which the side a is privileged by being given the role of base. One can of course cycle all of the above notation to consider the corresponding (b)- and (c)-clusterings.

Remark 3 Let us now combine (1) and (2). Firstly,

$$a\,m_a\,\cos\,\Phi \;=\; \frac{a^2}{4} \,+\, \frac{b^2}{2} \,+\, \frac{c^2}{4} \,-\, \frac{a^2}{4} \,-\, c^2 \;=\; \frac{b^2 \,-\, c^2}{2}\;. \tag{3}$$

Secondly, fully eliminating m_a ,

$$\cos\,\Phi \;=\; \frac{b^2 \,-\, c^2}{a\sqrt{2\,(\,b^2 \,+\, c^2\,) \,-\, a^2}}\;. \tag{4}$$

Remark 4 Also bringing in Appendix A's azimuthal spherical angle variable, (1) yields the following.

$$\tan\,\frac{\Theta}{2} \;=\; \mathcal{R} \;=\; \frac{2}{\sqrt{3}}\,\frac{m_a}{a} \;=\; \sqrt{\frac{2\,(\,b^2 \,+\,c^2\,) \,-\, a^2}{3\,a^2}}\;. \tag{5}$$

1.2 Side-ratio variables

Remark 1 The 3 side lengths support 2 independent side ratios [5]. A convenient choice for these is as follows.

Notation 1

$$\mathcal{B} \;:=\; \frac{b}{a} \;, \tag{6}$$

$$\mathcal{C} \;:=\; \frac{c}{a} \;. \tag{7}$$

Remark 2 For later use, these have range $\mathbb{R}_0$.

Remark 3 In terms of these, (4) becomes

$$\cos\Phi \;=\; \frac{\mathcal{B}^2 \;-\; \mathcal{C}^2}{\sqrt{2\;(\,\mathcal{B}^2 \;+\; \mathcal{C}^2\,)\;-\;1}} \;. \tag{8}$$

And (5) becomes

$$\tan\frac{\Theta}{2} \;=\; \mathcal{R} \;=\; \sqrt{\frac{2\;(\,\mathcal{B}^2 \;+\; \mathcal{C}^2\,)\;-\;1}{3}} \;. \tag{9}$$

Thus

$$\Phi \;=\; \arccos\left(\frac{\mathcal{B}^2 \;-\; \mathcal{C}^2}{\sqrt{2\;(\,\mathcal{B}^2 \;+\; \mathcal{C}^2\,)\;-\;1}}\right) \;. \tag{10}$$

And

$$\Theta \;=\; 2\,\arctan\left(\sqrt{\frac{2\;(\,\mathcal{B}^2 \;+\; \mathcal{C}^2\,)\;-\;1}{3}}\right) \;. \tag{11}$$

1.3 Outline of the rest of this Article

In Sec 2, we cast the triangleland sphere's standard spherical metric[1]

$$\mathrm{d}s^2 \;=\; \mathrm{d}\Theta^2 \;+\; \sin^2\Theta\,\mathrm{d}\Phi^2$$

in terms of the side ratios $\mathcal{B}$ and $\mathcal{C}$. The resulting expression is rather complicated. We simplify it by defining more adapted ratio variables.

In Sec 3, we form the *cluster-specific sides ratio flow* parametrized by $\mathcal{C}$ by eliminating $\mathcal{B}$ between (8) and (9). We furthermore sketch this flow. Pointing out the special roles played by the following features of our clustering. The equator of collinearity C, and the bimeridian of isoscelesness I. Alongside the binary coincidence-or-collision B and the most uniform collinear state U.

Appendix A takes us from constellation space down to Kendall's [10, 12, 16] shape space: the triangleland sphere. Appendix B then explains its C, I, B and U decor for the 3 choices of cluster. As well as the most distinguished roles played by equilateral triangles E and the maximum coincidence-or-collision O.

[1]See Appendix A for how Φ and Θ are related to the triangle's properties.

2 The triangleland metric in side-ratio coordinates

2.1 In the original side-ratio variables

Remark 1 One possible approach is to substitute (11, 10) the spherical metric (62) so as to cast the triangleland metric in side-ratio coordinates.

Structure 1 Let

$$\mathcal{X} \ := \ 2 \left(\mathcal{b}^2 \ + \ \mathcal{c}^2 \right) \ - \ 1 \ ,$$

$$\mathcal{Y} \ := \ \mathcal{b}^2 \ - \ \mathcal{c}^2 \ .$$

These cover the ranges $[-1, \infty)$ and $\mathbb{R}$ respectively. Also let

$$\mathcal{Z} \ := \ \sqrt{\mathcal{X}} \ .$$

This only covers $\mathbb{R}_0$.

Remark 2 In $\mathcal{Z}, \mathcal{Y}$ coordinates, the metric on the triangleland sphere takes the following form.

$$\Theta \ = \ 2 \arctan \frac{\mathcal{Z}}{\sqrt{3}} \ , \quad \text{and} \quad \Phi \ = \ \arccos \frac{\mathcal{Y}}{\mathcal{Z}} \ .$$

Lemma 1 In $\mathcal{Z}, \mathcal{Y}$ coordinates, the metric on the triangleland sphere takes the following form.

$$\mathrm{d}s^2 \ = \ \frac{4}{3 \left(1 \ + \ \frac{\mathcal{Z}^2}{3} \right)^2} \left[3 \, \mathrm{d}\mathcal{Z}^2 \ + \ \frac{(\ \mathcal{Z} \, \mathrm{d}\mathcal{Y} \ - \ \mathcal{Y} \, \mathrm{d}\mathcal{Z} \ )^2}{\mathcal{Z}^2 \ - \ \mathcal{Y}^2} \right] \ .$$

Lemma 2 In $\mathcal{X}, \mathcal{Y}$ coordinates, the metric on the triangleland sphere takes the following form.

$$\mathrm{d}s^2 \ = \ \frac{3}{\mathcal{X} \ (\mathcal{X} \ + \ 3)^2} \left[3 \, \mathrm{d}\mathcal{X}^2 \ + \ \frac{(\ \mathcal{Y} \, \mathrm{d}\mathcal{X} \ - \ 2 \, \mathcal{X} \, \mathrm{d}\mathcal{Y} \ )^2}{\mathcal{X} \ - \ \mathcal{Y}^2} \right] \ .$$

Proposition 1 In $\mathcal{b}, \mathcal{c}$ coordinates, the metric on the triangleland sphere takes the following form.

$$\mathrm{d}s^2 \ = \ \sigma^2 \mathrm{d}\underline{\boldsymbol{\mathcal{b}}} \cdot \underline{\underline{\boldsymbol{m}}} \cdot \mathrm{d}\underline{\boldsymbol{\mathcal{b}}} \ . \tag{12}$$

For

$$\underline{\underline{\boldsymbol{m}}} \ := \ \begin{pmatrix} \mathcal{b}^2 \, \mathcal{K} & \text{-} \mathcal{b} \, \mathcal{c} \, \mathcal{L} \\ \text{-} \mathcal{b} \, \mathcal{c} \, \mathcal{L} & \mathcal{c}^2 \, \mathcal{K} \end{pmatrix} \ . \tag{13}$$

$$\underline{\boldsymbol{\mathcal{b}}} \ := \ \begin{pmatrix} \mathcal{b} \\ \mathcal{c} \end{pmatrix} \ . \tag{14}$$

$$\sigma^{-2} \ = \ \frac{1}{96} \left[\ 2 \left(\mathcal{b}^2 \ + \ \mathcal{c}^2 \right) \ - \ 1 \ \right] \ \left[\ 2 \left(\mathcal{b}^2 \ + \ \mathcal{c}^2 \right) \ + \ 2 \ \right]^2 \ \left[\ 2 \left(\mathcal{b}^2 \ + \ \mathcal{c}^2 \right) \ - \ 1 \ - \ \left(\mathcal{b}^2 \ - \ \mathcal{c}^2 \right)^2 \ \right] \tag{15}$$

$$\mathcal{K} \ := \ - 3 \, \mathcal{b}^4 \ + \ 6 \, \mathcal{b}^2 \, \mathcal{c}^2 \ + \ 3 \, \mathcal{c}^4 \ + \ 2 \, \mathcal{b}^2 \ - \ 1 \ . \tag{16}$$

$$\mathcal{L} \ := \ 3 \, \mathcal{b}^4 \ + \ 2 \, \mathcal{b}^2 \, \mathcal{c}^2 \ + \ 3 \, \mathcal{c}^4 \ - \ 5 \, \mathcal{b}^2 \ - \ 5 \, \mathcal{c}^2 \ + \ 2 \ . \tag{17}$$

Remark 1 The coefficients here are 4th-order binomials over a common 10th-order binomial.

2.2 In squared side-ratio variables

Structure 2 The variables

$$\mathcal{B} \ := \ \mathcal{b}^2 \ = \ \frac{b^2}{a^2} \ , \tag{18}$$

$$\mathcal{C} \ := \ \mathcal{c}^2 \ = \ \frac{c^2}{a^2} \tag{19}$$

are still rather closely related to the original ratios, and simplify our metric. Their ranges remain $\mathbb{R}_0$.

Proposition 2 In $\mathcal{B}, \mathcal{C}$ coordinates, the metric on the triangleland sphere takes the following form.

$$\mathrm{d}s^2 \; = \; \phi^2 \, \mathrm{d}\underline{\boldsymbol{\mathcal{B}}} \cdot \underline{\underline{\boldsymbol{\mathcal{M}}}} \cdot \mathrm{d}\underline{\boldsymbol{\mathcal{B}}} \; . \tag{20}$$

Where

$$\phi^{-2} \; := \; \frac{1}{24} \left[\, 2 \, (\mathcal{B} \, + \, \mathcal{C} \,) \, - \, 1 \, \right] \, \left[\, 2 \, (\mathcal{B} \, + \, \mathcal{C} \,) \, + \, 2 \, \right]^2 \, \Big[\, 2 \, (\mathcal{B} \, + \, \mathcal{C} \,) \, - \, 1 \, - \, (\mathcal{B} \, - \, \mathcal{C} \,)^2 \, \Big] \; . \tag{21}$$

$$\boldsymbol{\mathcal{B}} \; := \; \begin{pmatrix} \mathcal{B} \\ \mathcal{C} \end{pmatrix} \; . \tag{22}$$

$$\boldsymbol{\mathcal{M}} \; := \; \begin{pmatrix} \mathcal{P} & \text{-}\mathcal{Q} \\ \text{-}\mathcal{Q} & \mathcal{P} \end{pmatrix} \; . \tag{23}$$

$$\mathcal{P} \; := \; - \, \mathcal{B}^2 \, + \, 2 \, \mathcal{B} \mathcal{C} \, + \, 3 \, \mathcal{C}^2 \, + \, 2 \, \mathcal{B} \, - \, 1 \; . \tag{24}$$

$$\mathcal{Q} \; := \; 3 \, \mathcal{B}^2 \, + \, 2 \, \mathcal{B} \mathcal{C} \, + \, 3 \, \mathcal{C}^2 \, - \, 5 \, \mathcal{B} \, - \, 5 \, \mathcal{C} \, + \, 2 \; . \tag{25}$$

Remark 1 We now have two ratios of quadratics by the same quintic. This is still however quite complicated. This Section's moral is that, while side-ratio variables seem natural at the level of studying triangles, using these variables unfortunately rather obscures what the simple geometry of the corresponding space of triangles is!

2.3 In apythagorean-anisoscelesness characteristic variables

Structure 3 The following simple characteristic combination of variables simplifies our metric in a different way from the form (62) that we obtained it from.

$$\mathcal{U} \; = \; 2 \, (\, \mathcal{B} \, + \, \mathcal{C} \,) \; = \; 2 \, \frac{b^2 \, + \, c^2}{a^2} \; , \tag{26}$$

$$\mathcal{V} \; = \; \mathcal{B} \, - \, \mathcal{C} \; = \; \frac{b^2 \, - \, c^2}{a^2} \; . \tag{27}$$

Proposition 3 In $\mathcal{U}, \mathcal{V}$ coordinates, the metric on the triangleland sphere takes the following form.

$$\mathrm{d}s^2 \; = \; \psi^2 \, \Big(\mathcal{G} \, \mathrm{d}\mathcal{U}^2 \, + \, \mathcal{H} \, \mathrm{d}\mathcal{V}^2 \Big) \; . \tag{28}$$

Where

$$\psi^{-2} \; := \; \frac{4}{3} \, (\mathcal{U} \, - \, 1 \,) (\mathcal{U} \, - \, 1 \, - \, \mathcal{V}^2 \,) (\mathcal{U} \, + \, 2 \,)^2 \; . \tag{29}$$

$$\mathcal{G} \; := \; - \, 2 \, \mathcal{U}^2 \, - \, 4 \, \mathcal{U} \mathcal{V} \, + \, 7 \, \mathcal{U} \, + \, 4 \, \mathcal{V} \, - \, 12 \; . \tag{30}$$

$$\mathcal{H} \; := \; 4 \, \Big(\, 4 \, \mathcal{U}^2 \, - \, 4 \, \mathcal{U} \mathcal{V} \, - \, 3 \, \mathcal{U} \, + \, 4 \, \mathcal{V} \, + \, 4 \, \Big) \; . \tag{31}$$

Remark 1 Thereby, we have succeeded in eliminating cross terms so as to attain a diagonal metric form.

Remark 2 Our characteristic variables receive the following interpretation. $\mathcal{U}$ is an *apythagorean ratio variable* for our cluster choice. Such that if our cluster's base is a hypotenuse, $\mathcal{U}$ returns the value 1 . While $\mathcal{V}$ is an *anisoscelesness ratio variable* for our cluster choice. Such that if our cluster's legs are of equal length – one of the equivalent formulations of isoscelesness – then our ratio returns the value 0 . Both of these ratio variables are rendered dimensionless by dividing by the cluster-dependent squared length of the base.

Remark 2 Our final characteristic variables are arrived at in two stages. Firstly, the obvious $\pm$ combinations are formed. Then one of them is rescaled so as to additionally simplify the common denominator.

3 The cluster-dependent side-ratio flow on the Triangleland Sphere

3.1 Setting up the flow using $\mathcal{C}$ as parameter

Remark 1 As compared to the abovementioned metric calculation, it is rather more straightforward to eliminate one of $\mathcal{B}$ or $\mathcal{C}$ between our equations (8) and (9). Say $\mathcal{B}$. This results in a family of curves on the triangleland sphere. Which family is then parametrized by the remaining squared side ratio, for us $\mathcal{C}$.

Theorem 1 (Cluster-dependent Side-Ratio Flow on Triangleland Sphere)

$$\cos \Phi = \frac{\sqrt{3}}{2} \tan \frac{\Theta}{2} + \varepsilon \cot \frac{\Theta}{2} . \quad (32)$$

Where

$$\mathcal{E} := \frac{1 - 4\,\mathcal{C}}{2\sqrt{3}} , \quad (33)$$

whose range is $\left(-\infty , \frac{1}{2\sqrt{3}}\right]$.

<u>Proof</u> The properly Jacobi mass-weighted version of (9) in $(\text{sides})^2$ is as follows.

$$\sqrt{2\,(\mathcal{B} + \mathcal{C}) - 1} = \sqrt{3}\,\mathcal{R} .$$

The $(\text{sides})^2$ version of (8) is as follows.

$$\cos \Phi = \frac{\mathcal{B} - \mathcal{C}}{\sqrt{2\,(\mathcal{B} + \mathcal{C}) - 1}} . \quad (34)$$

As a first combination of these two equations,

$$\cos \Phi = \frac{\mathcal{B} - \mathcal{C}}{\sqrt{3}\,\mathcal{R}} . \quad (35)$$

But also the first equation rearranges to

$$2\,(\mathcal{B} + \mathcal{C}) - 1 = 3\,\mathcal{R}^2 .$$

So

$$\mathcal{B} = \frac{3}{2}\mathcal{R}^2 - \frac{1}{2} - \mathcal{C} . \quad (36)$$

Substitute (34) in (36) to obtain the following.

$$\cos \Phi = \frac{\frac{3}{2}\mathcal{R}^2 + \frac{1}{2} - \mathcal{C} - \mathcal{B}\mathcal{C}}{\sqrt{3}\,\mathcal{R}} = \frac{3\left(\mathcal{R}^2 - 4\,\mathcal{C}\right) + 1}{2\sqrt{3}\,\mathcal{R}} .$$

Thus by (61),

$$\cos \Phi = \frac{3\left(\tan^2 \frac{\Theta}{2} - 4\,\mathcal{C}\right) + 1}{2\sqrt{3} \tan \frac{\Theta}{2}} .$$

Finally use the given definition of $\mathcal{E}$, and the definition of $\cot$.

The range of $\mathcal{E}$ immediately follows from its definition alongside sides ratios being real so their squares are positive. □

Corollary 1 i)

$$\Phi = \arccos\left(\frac{\sqrt{3}}{2} \tan \frac{\Theta}{2} + \mathcal{E} \cot \frac{\Theta}{2}\right) . \quad (37)$$

ii)

$$\Theta = 2 \arctan\left(\frac{\cos \Phi \pm \sqrt{\cos^2 \Phi - 2\sqrt{3}\,\mathcal{E}}}{\sqrt{3}}\right) . \quad (38)$$

Proof i) just uses the definition of arccos .

ii) uses instead the quadratic formula to solve for

$$\mathcal{R} \ = \ \tan \frac{\Theta}{2} \ . \ \Box$$

Remark 2 ii) is often computationally more useful, while i) has the benefit of no sign ambiguity.

3.2 Sketching the $\mathcal{C}$ parameter flow

Remark 1 Let us next sketch this flow. Firstly, it is reflection symmetric about the equator of collinearity C . This is due to cos being an even function.

Remark 2 For the maximum possible value

$$\mathcal{E} \ = \ \frac{1}{2\sqrt{3}} \ ,$$

$$\cos \Phi \ = \ 1 \ \ \Rightarrow \ \ \Phi \ = \ 0$$

is forced. This is collinear, with

$$\mathcal{R} \ = \ \frac{1}{\sqrt{3}} \ .$$

It corresponds to

$$\mathcal{c} \ = \ 0 \ , \ \ \mathcal{B} \ = \ 1 \ \ \Rightarrow \ \ c \ = \ 0 \ , \ \ a \ = \ b \ .$$

Thus it is the distinct (b)-clustering's binary coincidence-or-collision, B .

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Remark 3

$$\mathcal{E} \ = \ 0$$

corresponds to

$$\tan \frac{\Theta}{2} \ = \ \mathcal{R} \ = \ \frac{2}{\sqrt{3}} \cos \Phi \ .$$

Which is a circle in the stereographic plane.

Remark 4 For

$$\mathcal{E} \ < \ 0 \ ,$$

we just get a loop enclosing the previous circle, which solely realizes the positive root solution.

As $\mathcal{E} \longrightarrow -\infty$, these become arbitrarily large in the stereographic plane. But on the triangleland sphere, they become arbitrarily small circles around our variables' chosen (a)-cluster's own binary coincidence-or-collision B . And in the limit, just this point is realized.

Remark 5

$$0 \ < \ \mathcal{E} \ < \ \frac{1}{2\sqrt{3}}$$

requires both roots. Each such $\mathcal{E}$ has a critical wedge-angle at which one needs to switch roots. In the stereographic plane, the envelope of these endpoints forms a circle,

$$\mathcal{R} \ = \ \frac{1}{\sqrt{3}} \cos \Phi \ .$$

I.e. with half the radius of $\mathcal{E} \ = \ 0$'s circle and kissing it at the origin U .

Remark 6 Our two circles are not separatrices since the qualitative type of the flow does not care about how many roots were needed to cover it.

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Remark 7 The (b)-clustering's notion of isoscelesness is among the flowlines. This is a great circle and thus cuts the triangleland sphere into two equal pieces.

Remark 8 There is also a reflection symmetry about this great circle.

Remark 9 This great circle separates the other two clustering's B-points' 'basins'. These are only basins in a weak sense, since the points in question are centres.

Remark 10 We thus arrive at the sketch in Fig 2.

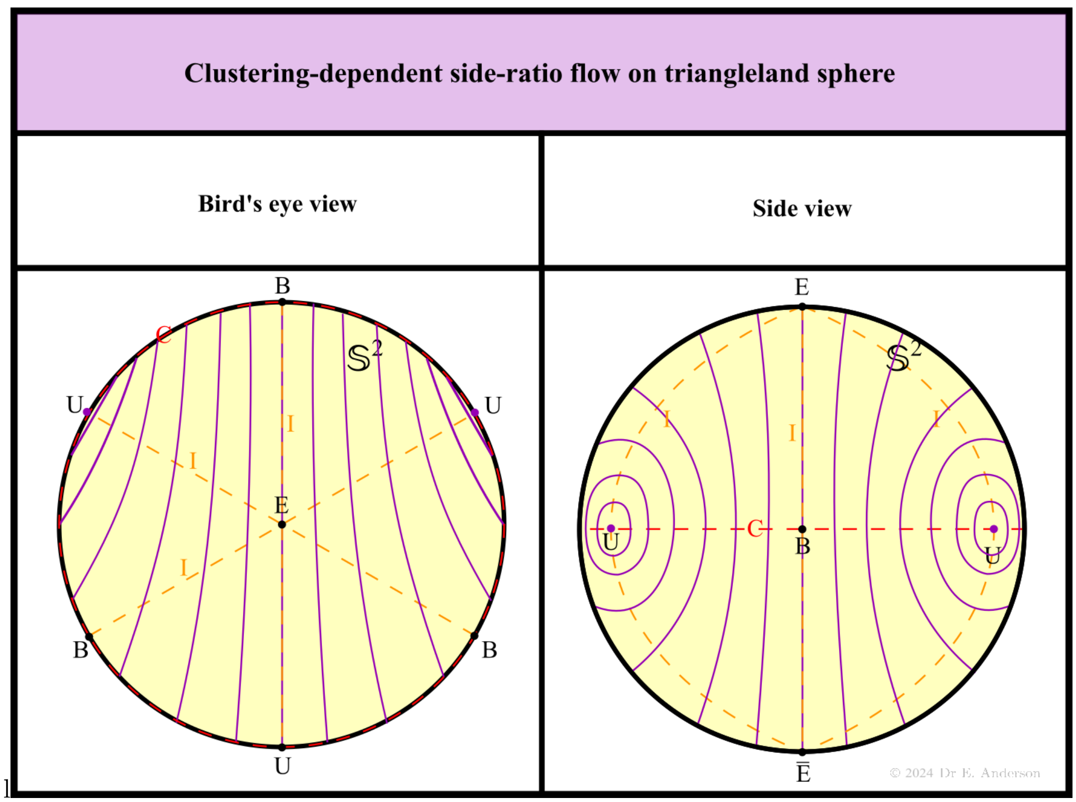

Figure 2:

3.3 The $\mathcal{B}$ parameter flow

Remark 1 This is qualitatively very similar indeed to the $\mathcal{C}$ flow. Now we have $-\cos\Phi$ in the quadratic formula's roots. So radius non-negativity selects the positive root exclusively, requiring this flow's analogue of $\mathcal{E}$ to be non-positive. Working out this flow thus does not encounter a root-joining problem. Cluster label permutations are needed to align the $\mathcal{B}$ and $\mathcal{C}$ flow's special points and lines.

Remark 2 It is also more natural to use $\mathcal{B}$ in the sense that it belongs to the same sub-triangle as the Φ and median in use.

4 Conclusion

We cast the triangleland sphere's standard spherical metric in terms of the Flat-Geometrically natural and yet Shape-Theoretically complicated side ratio variables. Sides-ratio squared variables somewhat simplify this expression. $\pm$ combinations of these – characteristic coordinates – simplify further. These comprise an apythagorean ratio and a non-dimensionalized version of anisoscelesness.

We also derived and sketched the cluster-dependent sides-ratio flow. The flow consists of two back-to-back centres at B points, with reflection symmetry about both I and C .

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Pointer 1 It would be more natural to form a cluster-independent, i.e. cycle-symmetric (Geometry) or democratically-invariant (Molecular Physics) side ratio flow. Such as the *minimum side-ratio flow*, where the minimum is over all 6 dependent side ratios supported. I.e.

$$\frac{a}{b} \quad \text{and cycles and reciprocals} \quad .$$

Only then would the third part of the current work be comparable to the maximum-angle flow of [28].

Pointer 2 See [31] for further novel considerations of Apollonius' Theorem.

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A From constellation space to Kendall's triangleland sphere

Remark 1 The *carrier space* $\mathfrak{C}^d$ is some at-least-provisional model for the structure of space [23, 29]. In the case of Physical space, it is referred to as *absolute space*. Though e.g. Euclidean Geometry and Probability Theory also have notions of carrier space notions, such as 'the Euclidean plane' and 'sample space'. Indeed, $\mathbb{R}^d$ is the space that has so far most often been cast in the role of carrier space, and the one we use below.

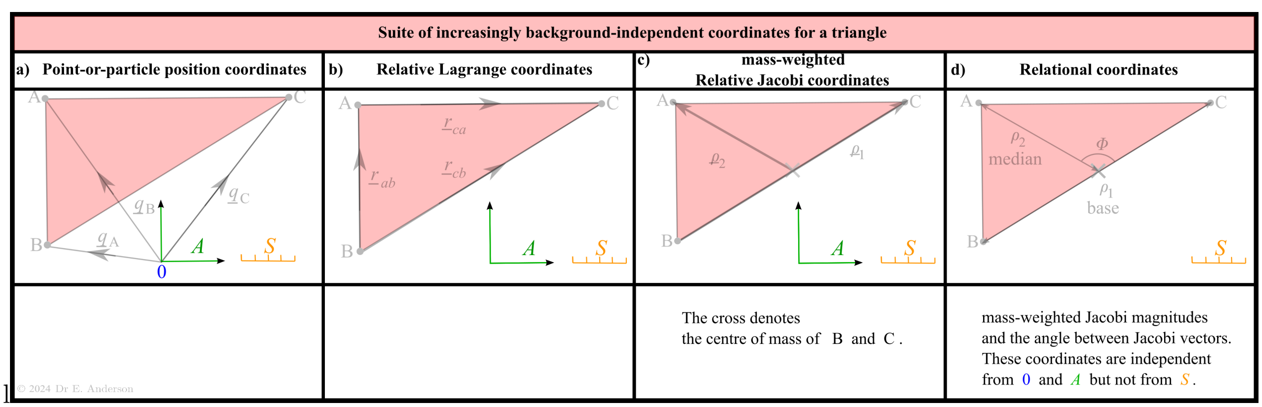

Figure 3:

Definition 1 *Constellation space* [23, 29] is the product of N copies of this carrier space. I.e.

$$\mathfrak{q}\left(\mathfrak{C}\mathrm{arrier}^d,\, N\right) \;=\; \mathop{\times}_{I\,=\,1}^{N} \mathfrak{C}\mathrm{arrier}^{d\,N} \; . \tag{39}$$

Remark 2 So for us

$$\mathfrak{q}(\, d,\, N\,) \;:=\; \mathfrak{q}\left(\mathbb{R}^d,\, N\right) \;=\; \bigtimes_{I\,=\,1}^{N} \mathbb{R}^d \;=\; \mathbb{R}^{N\,d}\;. \tag{40}$$

Each point of which models a figure formed by N points on $\mathfrak{C}\text{arrier}^d$: a *constellation.* Or, if these points are Physically realized, a figure formed by N particles (modelled here as classical, pointlike, nonrelativistic).

Notation 1 Let us use $\underline{\boldsymbol{q}}^I$ to denote the position coordinates of our N points-or-particles. For a triangle, these are depicted in Fig 3.a). Where the underline indicates spatial vectors and the point-or-particle label I runs from 1 to N .

Remark 3 Constellation space is a simple instance of *configuration space* [4, 8].

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Remark 4 In *Kendall's Shape Theory* [10, 12, 16, 23, 25, 22, 29] constellation space is furthermore subject to quotienting out the corresponding $d-$dimesional similarity group $Sim(d)$. A subsequent suite of less trivial, *reduced*, configuration spaces – are as follows.

Definition 2 *Relative space* [19, 23] is

$$\mathfrak{R}\text{elative}(\, d,\, N\,) \;:=\; \frac{\mathfrak{q}(\, d,\, N\,)}{Tr(d)}\;. \tag{41}$$

Definition 2.a The *relative separation vectors* between points-or-particles are

$$\underline{\boldsymbol{r}}^{IJ} \;=\; \underline{\boldsymbol{q}}^J \;-\; \underline{\boldsymbol{q}}^I\;. \tag{42}$$

Remark 5 For $N \;\geq\; 3$, these are not all independent. One can resolve this by picking a basis for these. In Fig 3.b), $\{\,\underline{\boldsymbol{r}}_{12},\, \underline{\boldsymbol{r}}_{23}\,\}$ is a such.

Remark 6 But the inertia quadric in such a basis is not diagonal. Diagonalizing it produces the *relative Jacobi vectors* $\underline{\boldsymbol{R}}^i$, $i \;=\; 1$ to $n \;:=\; N \;-\; 1$. Alongside the centre of mass (CoM) vector, which gets projected out. By which, freedom from the absolute origin O (blue in Fig 3) has been attained.

For the triangle with equal particle masses, one can take these to be

$$\underline{\boldsymbol{R}}_1 \;=\; \underline{\boldsymbol{q}}_{\mathrm{C}} \;-\; \underline{\boldsymbol{q}}_{\mathrm{B}} \;=\; \boldsymbol{a} \quad\text{and}\quad \underline{\boldsymbol{R}}_2 \;=\; \underline{\boldsymbol{q}}_{\mathrm{A}} \;-\; \frac{\underline{\boldsymbol{q}}_{\mathrm{B}} \;+\; \underline{\boldsymbol{q}}_{\mathrm{C}}}{2} \;=\; \underline{\boldsymbol{m}}_a\;. \tag{43}$$

Where $\underline{\boldsymbol{a}}$ and cycles are sides vectors and $\underline{\boldsymbol{m}}_a$ is a median vector. So, for a triangle, relative Jacobi vectors amount to a side vector and the corresponding median vector.

These relative Jacobi vectors have corresponding Jacobi reduced masses $\mu_1 \;=\; 1/2$ and $\mu_2 \;=\; 2/3$ respectively. One can then pass to mass-weighted relative Jacobi vectors $\underline{\boldsymbol{\rho}}^i$. For the triangle, these are

$$\underline{\boldsymbol{\rho}}_1 \;=\; \frac{1}{\sqrt{2}}\underline{\boldsymbol{R}}_1 \quad\text{and}\quad \underline{\boldsymbol{\rho}}_2 \;=\; \sqrt{\frac{2}{3}}\,\underline{\boldsymbol{R}}_2\;. \tag{44}$$

See Fig 3.c). By use of Jacobi coordinates, it is straightforward to establish that

$$\mathfrak{R}\text{elative}(\, d,\, N\,) \;=\; \mathbb{R}^{n\,d}\;. \tag{45}$$

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Definition 3 Kendall's *preshape space* [10, 16, 23, 29] is

$$\mathfrak{P}\text{reshape}(\, d,\, N\,) \;\;:=\;\; \frac{\mathfrak{q}(\, d,\, N\,)}{Dilatat(d)} \;\;=\;\; \frac{\mathfrak{q}(\, d,\, N\,)}{Tr(d) \;\rtimes\; Dil}\;. \tag{46}$$

Where $Dilatat(d)$ denotes the d-dimensional dilatational group. And $\rtimes$ denotes semidirect product of groups [17].

Definition 3.a

$$\underline{\boldsymbol{\nu}}^i \ := \ \frac{\underline{\boldsymbol{\rho}}^i}{\rho} \ := \ \frac{\underline{\boldsymbol{\rho}}^i}{I^{1/2}} \qquad (47)$$

Where I is the total moment of inertia (MoI) and ρ is the configuration space radial variable. So the $\underline{\boldsymbol{\nu}}^i$ are *MoI-normalized Jacobi coordinates.*

Remark 7 It is straightforward to establish that

$$\mathfrak{P}\text{reshape}(\,d,\,N\,) \ = \ \mathbb{S}^{n\,d\,-\,1} \ . \qquad (48)$$

For MoI-normalized mass-weighted relative Jacobi coordinates obey

$$1 \ = \ ||\boldsymbol{\nu}||_{\mathfrak{R}\text{elative}}{}^2 \ = \ \sum_{i\,=\,1}^{n} ||\boldsymbol{\nu}_i||^2 \ . \qquad (49)$$

And these have a total of $n\,d$ components, making this the on-$\mathbb{S}^{n\,d\,-\,1}$ condition. Now freedom from absolute scale (orange in Fig 3) has been attained.

Definition 4 Kendall's *shape space* [10, 14, 16, 22, 23, 24, 25, 26, 29, 30] is

$$\mathfrak{S}\text{hape}(\,d,\,N\,) \ := \ \frac{\mathfrak{q}(\,d,\,N\,)}{Sim(d)} \ = \ \frac{\mathfrak{q}(\,d,\,N\,)}{Tr(d) \ \rtimes \ (\,Rot(d) \ \times \ Dil,\,)} \ . \qquad (50)$$

Theorem A (Kendall's Theorem) The shape space of vertex-labelled mirror-images-distinct N-a-gons in $\mathbb{R}^2$ is [10, 12, 16]

$$\mathfrak{S}\text{hape}(\,2,\,N\,) \ = \ \mathbb{CP}^{N\,-\,2} \ = \ \mathbb{CP}^{n\,-\,1} \ . \qquad (51)$$

Both topologically and metrically, with standard Fubini–Study metric.

Remark 8 Now freedom from all of absolute origin, scale and axes (green in Fig 3) has been attained.

♠ ◇ ♣

Definition 5 *Relational space* [11, 15, 19, 23, 25, 29, 30]

$$\mathfrak{R}\text{elational}(\,d,\,N\,) \ := \ \frac{\mathfrak{q}(\,d,\,N\,)}{Eucl(d)} \ = \ \frac{\mathfrak{q}(\,d,\,N\,)}{Tr(d) \ \rtimes \ Rot(d)} \ . \qquad (52)$$

Where $Eucl(d)$ denotes the d-dimensional Euclidean group.

Remark 9 The following serve as adapted coordinates for triangleland's relationalspace.

1) The *Jacobi magnitudes*

$$\rho^i \ := \ ||\underline{\boldsymbol{\rho}}^i|| \ . \qquad (53)$$

Suppose that one rescales these by removing the square roots of the Jacobi relative masses. Then one has a base side, e.g. a and the corresponding median, then m_a .

2) Alongside the Swiss army knife angle

$$\Phi \ := \ \arccos\,(\,\underline{\nu}_1 \ \cdot \ \underline{\nu}_2\,) \ = \ \arccos\left(\frac{\underline{\rho}_1 \ \cdot \ \underline{\rho}_2}{||\underline{\rho}_1||\,||\underline{\rho}_2||}\right) \ = \ \arccos\left(\frac{\underline{\boldsymbol{R}}_1 \ \cdot \ \underline{\boldsymbol{R}}_2}{||\underline{\boldsymbol{R}}_1||\,||\underline{\boldsymbol{R}}_2||}\right)$$

$$= \quad \arccos\left(\frac{\frac{\underline{\boldsymbol{a}}}{2} \cdot \underline{\boldsymbol{m}}_a}{\left|\left|\frac{\underline{\boldsymbol{a}}}{2}\right|\right| \; ||\underline{\boldsymbol{m}}_a||} \right) \quad = \quad \arccos\left(\widehat{\frac{\underline{\boldsymbol{a}}}{2}} \cdot \widehat{\underline{\boldsymbol{m}}_a} \right) \ . \tag{54}$$

Where the hats denote unit-magnitude vectors.

Observe that this is rescaling-invariant, so Jacobi mass-scaling does not affect its form. Nor does scaling by $1/2$. Thus it is the very same angle as between the half-side base and the corresponding median, as in formula (2).

1) and 2) are depicted in Fig 3.a). Now freedom from absolute axes and origin have been attained, but not from absolute scale.

Proposition 1 Relationalspace is both metrically and toplogically the cone over shape space,

$$\mathfrak{R}\text{elational}(\, d, \, N \,) \;=\; \text{C}(\mathfrak{S}\text{hape}(\, d, \, N \,)) \ . \tag{55}$$

Remark 10 We fit togther the above suite of configuration spaces into Fig 4.b)'s lattice. This mirrors the lattice of subgroups in Subfig a). With the family of N-a-gons' specifics in Subfig c), and the triangle's in Subfig d).

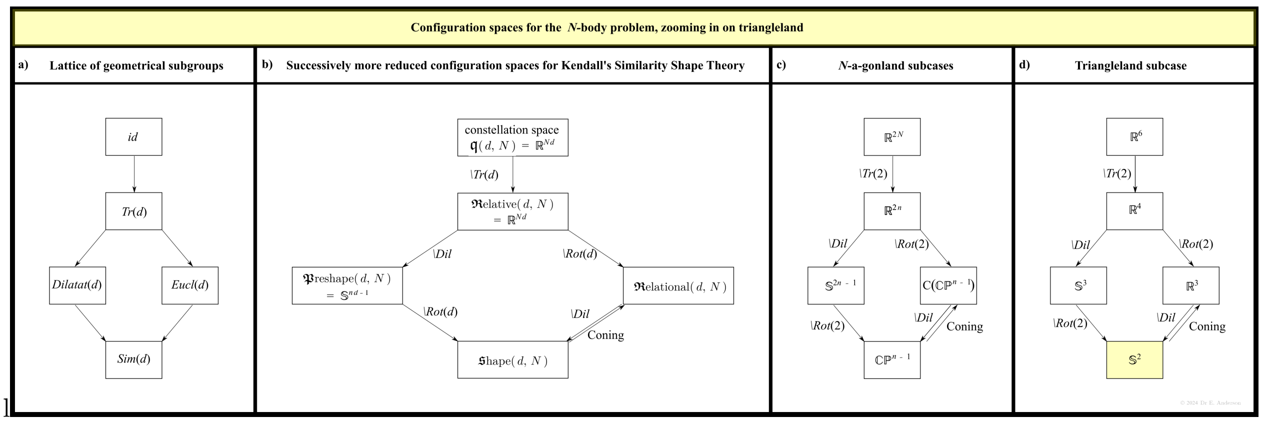

Figure 4:

♠ ◇ ♣

Remark 11 Kendall's work remains rather more familiar in the Shape Statistics literature [10, 12, 14, 16, 22]. See [10, 16] for a Geometrical proof. And [18, 19, 23] for conceptually-distinct proofs, whether by reducing Mechanics actions or by the generalized Hopf map. For related work in 'Mechanics, the N-body problem or Molecular Physics', see e.g. [6, 9, 11, 15, 21, 18, 19, 23, 26].

Theorem B (Kendall's Little Theorem) The shape space of vertex-labelled mirror-images-distinct triangles in $\mathbb{R}^2$ is [10, 12, 16]

$$\mathfrak{S}\text{hape}(\, 2, \, 3 \,) \;=\; \mathbb{S}^2 \ . \tag{56}$$

Remark 12 This follows immediately as a Corollary of Kendall's Theorem by the accidental topological-and-geometrical relation

$$\mathbb{CP}^1 \;=\; \mathbb{S}^2 \ . \tag{57}$$

The Hopf proof now involves the simplest Hopf map,

$$\mathbb{S}^3 \;\longrightarrow\; \mathbb{S}^2 \tag{58}$$

in its

$$\mathfrak{P}\text{reshape}(\, 2, \, 3 \,) \;\longrightarrow\; \mathfrak{S}\text{hape}(\, 2, \, 3 \,) \tag{59}$$

realization.

And additional proofs are now available, including that diagonalizing Heron's formula [27] suffices to get the topological part of the result. The topological part of both of the above Theorems was already known to Smale [7].

Remark 13 Some adapted coordinates on the triangleland shape sphere are as follows.

The *ratio of Jacobi magnitudes* is

$$\mathcal{R} \;:=\; \frac{\rho_2}{\rho_1} \;=\; \sqrt{\frac{\mu_2}{\mu_3}}\,\frac{R_2}{R_1} \;=\; \frac{2}{\sqrt{3}}\,\frac{m_a}{a}\;. \tag{60}$$

This plays the role of stereographic radius on the triangleland sphere. The venerable substitution

$$\mathcal{R} \;=\; \tan\frac{\Theta}{2} \tag{61}$$

in this context gets us to standard spherical angle coordinates. With the natural spherical metric

$$\mathrm{d}s^2 \;=\; \mathrm{d}\Theta^2 \;+\; \sin^2\Theta\,\mathrm{d}\Phi \tag{62}$$

on the triangleland sphere. With the above interpretations for its azimuthal angle Θ and polar angle Φ .

B Where some common types of triangle reside in the triangleland sphere

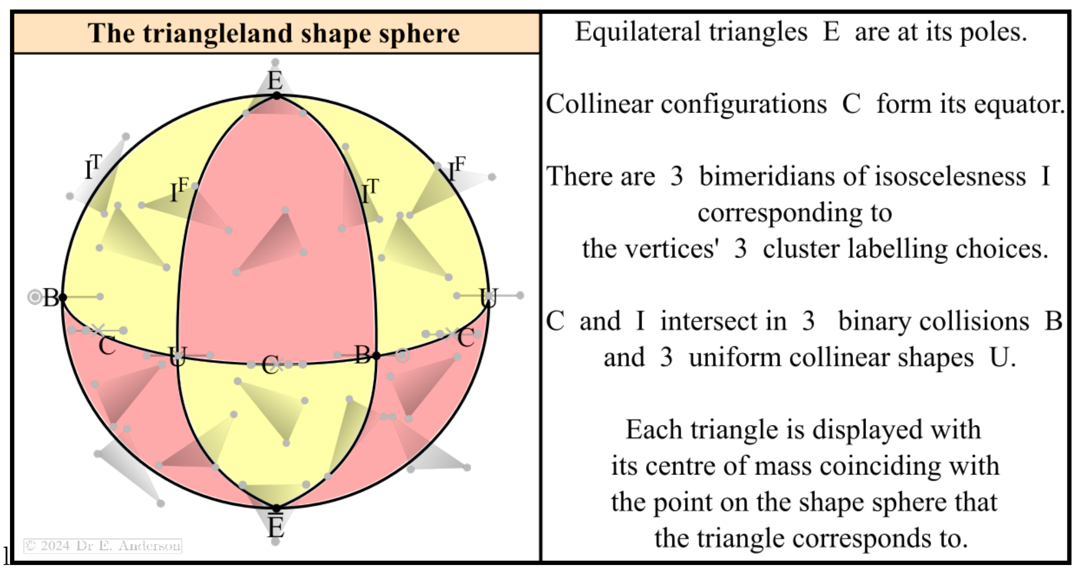

Figure 5:

Remark 1 Mirror-image distinct triangles support 2 distinct copies of the equilateral triangle, which we denote by E $\overline{\text{E}}$. Fig 5 then uses the democratic E$\overline{\text{E}}$ as principal axis. In contrast, Fig 2 uses the clustering specific UB as principal axis. The above Θ and Φ are cluster-dependent. To pass to E$\overline{\text{E}}$ as principal axis's spherical coordinates, we need to apply a $\pi/2$ rotation.

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