Only 2 of the Fundamental Triangle, Lagrange and Apollonius Matrices are Independent.

With ensuing Algebras, Irreducibles and Splits

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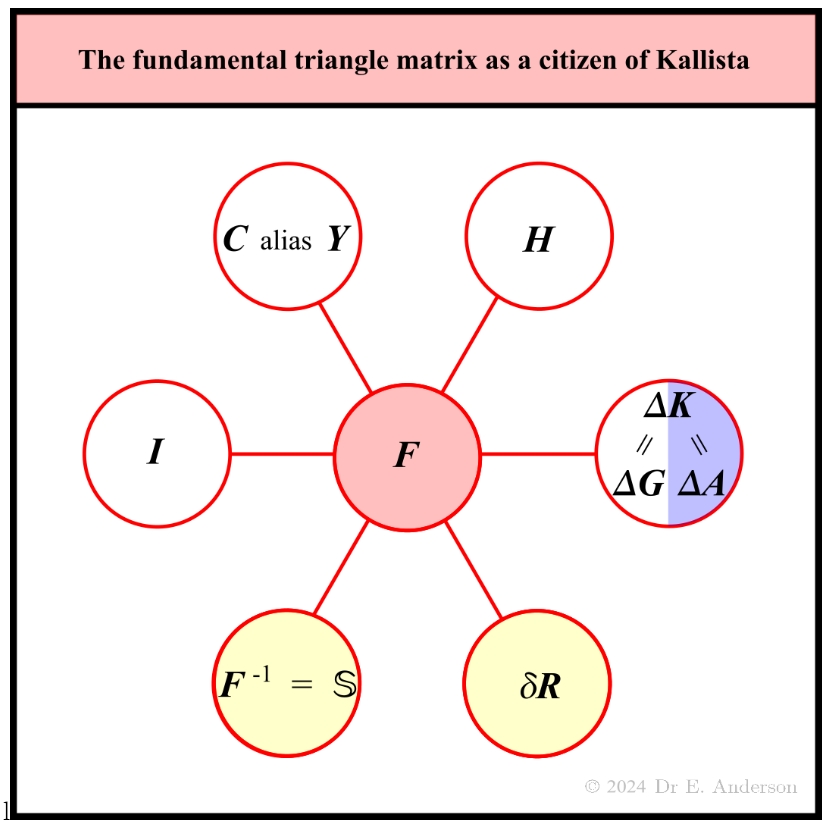
Abstract

Only two of the fundamental triangle $\boldsymbol{F}$, Lagrange $\boldsymbol{L}$ and Apollonius $\boldsymbol{O}$ matrices are independent. Picking $\boldsymbol{L}$ and $\boldsymbol{O}$ to work with gives a small finite algebra, with $\boldsymbol{F}$ as the difference between these. This induces a Geometric 'Apollonius–Lagrange' split of Heron's formula [2, 19], which is also an Algebraic involution–projection split.

We also break down our three matrices into blocks and then into irreducibles, working out the algebra formed in each case. $\boldsymbol{F}$ is furthermore the difference between the two irreducibles, yielding a further split formulation of Heron's formula. $\boldsymbol{F}^{-1}$ is additionally found to be proportional to one of the irreducibles. The corresponding split versions of the cycles of triangle inequalities, and cosine rules are also provided.

Six conceptually distinct routes to $\boldsymbol{F}$ have thus been found within Geometry, Algebra and Representation Theory.



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1 Introduction

We have so far contemplated three matrices from the Flat Geometry of triangles.

G.1) The *fundamental triangle matrix*

$$\boldsymbol{F} \ = \ \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \ . \tag{1}$$

As occurs in Article 1 [40] all of the triangle inequality cycle, the cosine rule cycle and Heron's formula. Under the respective notation $\boldsymbol{I}$, $\boldsymbol{C}$ alias $\boldsymbol{Y}$: pythagoreanness, and $\boldsymbol{H}$.

G.2) The *Lagrange matrix* ([8, 32, 33, 42] and Article 2: [44])

$$\boldsymbol{L} \ := \ \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \ = \ \boldsymbol{P} \ . \tag{2}$$

Which corresponds to placing an equal mass at each vertex of the triangle. Which is a projector [42] – hence the $\boldsymbol{P}$ notation – that arises in passing from vertex position vectors to relative separation vectors. By which *position-separation matrix* is another, in some sense truer, name for it.

G.3) The *Apollonius matrix* ([34, 1, 41] and Article 2)

$$\boldsymbol{O} \ := \ \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \ = \ \boldsymbol{J} \ . \tag{3}$$

Which is an involution – hence the notation $\boldsymbol{J}$ – that arises from passing from sides to medians. By which *sides–medians involution* is another, in some sense truer, name for it.

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We also showed in Article 2 [44] that these three matrices all commute, and furthermore fully share eigenspace structure. We next show (Sec 2) that only two of them are independent. Indeed,

$$\boldsymbol{F} \ = \ \boldsymbol{O} \ - \ \boldsymbol{L} \ = \ \boldsymbol{J} \ - \ \boldsymbol{P} \ .$$

constitutes a further Geometric route to $\boldsymbol{F}$ by the first equality, or an Algebraic one by the second. Furthermore (Sec 4), all 3 of our triangle matrices are linear combinations (LCs) of the following blocks.

B.1) The identity matrix

$$\mathbb{I} \ := \ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4}$$

B.2) The *matrix of 1's*

$$\mathbb{1} \ := \ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \ . \tag{5}$$

Sec 4's observations prompt a new notation $\boldsymbol{K}_{\pm}$ for G.2) and G.3) respectively. Furthermore,

$$\boldsymbol{F} \ = \ \boldsymbol{\Delta K} \ := \ \boldsymbol{K}_{+} \ - \ \boldsymbol{K}_{-} \ . \tag{6}$$

$\boldsymbol{K}_{\pm}$ close to form the algebra exhibited in Sec 3. The algebra generated by $\mathbb{I}$ and $\mathbb{1}$ is given in Sec 5. And that of $\boldsymbol{K}_{\pm}$ and $\boldsymbol{\Delta K}$ in Sec 8 Secs 7 and 6 serve to bring out two further properties of $\boldsymbol{L} \ = \ \boldsymbol{P}$. and 10 for one of $\boldsymbol{O} \ = \ \boldsymbol{L}$.

Representation Theory then prompts us to work in terms of irreducibles (Sec 12).

R.1) $\mathbb{I}$ is already a such.

R.2) But $\mathbb{1}$ is not. So we need to use instead its tracefree counterpart,

$$\mathbb{T} := \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} . \tag{7}$$

We next re-express everything in terms of $\mathbb{T}$ and $\mathbb{I}$. This gives another difference formulation,

$$\boldsymbol{F} = \boldsymbol{\delta R} := \mathbb{T} - \mathbb{I}$$

Where the 'R' denotes '(ir)reps'. The sense in which this is a weaker difference than (6) is explained in the Conclusion. The algebra generated by $\mathbb{I}$ and $\mathbb{T}$ is given in Sec 12.

While $\boldsymbol{F}$ is not irreducible, its inverse is (Sec 13):

$$\boldsymbol{F}^{-1} = \frac{1}{2}\mathbb{T} = \mathbb{S} . \tag{8}$$

For

$$\mathbb{S} := \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{9}$$

a subsequently useful rescaling of $\mathbb{T}$.

$\boldsymbol{F}$ has thus picked up 3 further conceptualizations with various accompanying accolades spelled out in the text, supporting its fundamental triangle matrix name.

Apollonius–Lagrange, block and irreducible split versions of Heron's formula are given in Secs 9 and 14 respectively. And for the triangle inequality and the cosine rule in Appendices A and B respectively. For useful comparison with elsewhere in the literature, the block preliminary to the irreducible split is also included in Sec 11 for Heron's formula and in Appendix B for the other two applications.

2 LD triangle matrices and the $K_\pm$, ΔK notation

Lemma 1 $\boldsymbol{F}$, $\boldsymbol{L}$ and $\boldsymbol{O}$ are linearly dependent (LD).

<u>Proof</u> Form the linear combination (LC)

$$0 = \boldsymbol{F} + 3\alpha\boldsymbol{L} + 3\beta\boldsymbol{O} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} + \alpha \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} + \beta \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} .$$

The diagonal components then give 3 copies of

$$2\alpha - \beta = 1 .$$

While the off-diagonal components comprise 6 copies of

$$2\beta - \alpha = -1 .$$

This is solved by

$$\alpha = \frac{1}{3} , \quad \beta = -\frac{1}{3} .$$

Since these are not both zeros, the definition of LD [10] is met. □

Remark 1 Our triangle matrices are thus related by, making the fundamental triangle matrix the subject,

$$\boldsymbol{F} = \boldsymbol{O} - \boldsymbol{L} = \boldsymbol{J} - \boldsymbol{P} . \tag{10}$$

Remark 2 So we will often need to pick two of the three matrices, though which two may differ with circumstances.

1) $\boldsymbol{F}$ and $\boldsymbol{O}$ are well-adapted [40], for an in-depth study of Heron's formula [34, 37, 44].

2) $\boldsymbol{F}$ and $\boldsymbol{L}$ are well-adapted [40, 44] for studying Smale's Little Theorem [15], Kendall's Little Theorem [17, 25] and the Hopf map [9].

3) Finally, the current Article argues for $\boldsymbol{J}$ and $\boldsymbol{P}$ to be the most lucid choice from an algebraic perpective (Sec 8).

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Lemma 2

$$\boldsymbol{O} + \boldsymbol{L} = \boldsymbol{J} + \boldsymbol{P} = \frac{1}{3}\mathbb{1} . \tag{11}$$

<u>Proof</u>

$$\boldsymbol{O} + \boldsymbol{L} = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \frac{1}{3}\mathbb{1} . \ \square$$

Corollary 1

$$\boldsymbol{O} = \boldsymbol{J} = \frac{1}{2}\left(\frac{1}{3}\mathbb{1} + \boldsymbol{F}\right) . \tag{12}$$

$$\boldsymbol{L} = \boldsymbol{P} = \frac{1}{2}\left(\frac{1}{3}\mathbb{1} - \boldsymbol{F}\right) . \tag{13}$$

<u>Proof</u> Combine Lemmas 1 and 2. □

Remark 3 In words, (12) and (13) show that $\boldsymbol{O}$ and $\boldsymbol{L}$ are reflection symmetric about

$$\frac{1}{6}\mathbb{1} .$$

Compare for instance how complex conjugates are reflection-symmetric about the real axis.

Corollary 2 (2-way basis change summary).

$$\begin{pmatrix} \boldsymbol{L} \\ \boldsymbol{O} \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 1 & -3 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} \mathbb{1} \\ \boldsymbol{F} \end{pmatrix} . \tag{14}$$

$$\begin{pmatrix} \mathbb{1} \\ \boldsymbol{F} \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 3 & 3 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \boldsymbol{L} \\ \boldsymbol{O} \end{pmatrix} . \tag{15}$$

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Notational Remark 1 Remark 4 motivates the following re-issuing of notation.

$$\boldsymbol{K}_+ := \boldsymbol{O} = \boldsymbol{J} , \tag{16}$$

$$\boldsymbol{K}_- := \boldsymbol{L} = \boldsymbol{P} . \tag{17}$$

Remark 4 In terms of this, let us consolidate the current Section's results so far into the following.

Proposition 1 i)

$$\boldsymbol{K}_\pm = \frac{1}{2}\left(\frac{1}{3}\mathbb{1} \pm \boldsymbol{F}\right) . \tag{18}$$

$$\mathbb{1} = 3\,(\boldsymbol{K}_+ + \boldsymbol{K}_-) . \tag{19}$$

ii)

$$\boldsymbol{F} = \boldsymbol{\Delta K} := \boldsymbol{K}_+ - \boldsymbol{K}_- . \tag{20}$$

$$\boldsymbol{K}_+ + \boldsymbol{K}_- = \frac{1}{3}\mathbb{1} . \tag{21}$$

Remark 5 In words, (6) is that the fundamental triangle matrix is the difference of the two reflection-symmetric quantities $\boldsymbol{K}_\pm$. Which further meets a weak definition of *proto-index* [36] , hence the notation $\boldsymbol{\Delta K}$.

3 The $\boldsymbol{K}_\pm$ algebra

Building up toward times tables for the first algebras arising from triangle matrices

a) Involution's group C_2

×	▯	K_+
▯	▯	K_+
K_+	K_+	▯

b) Projection's commuting monoid $Mon(2)$

×	▯	K_-
▯	▯	K_-
K_-	K_-	K_-

c) Apollonius-Lagrange commuting monoid $Mon(4)$

×	▯	K_+	K_-	$-K_-$
▯	▯	K_+	K_-	$-K_-$
K_+	K_+	▯	$-K_-$	K_-
K_-	K_-	$-K_-$	K_-	$-K_-$
$-K_-$	$-K_-$	K_-	$-K_-$	K_-

Figure 1:

Lemma 3 Any involution and the identity form the commutative group C_2 .

<u>Proof</u> Call our involution $\boldsymbol{I}$. Then the involution property

$$\boldsymbol{I}^2 = ▯$$

completes our algebra's times table; see Fig 1.a) for our study's particular case. The remainder being covered by the identity property

$$\boldsymbol{I}\, ▯ = ▯ = ▯\, \boldsymbol{I}$$

and

$$▯^2 = ▯ .$$

In each of our times tables, the entries determined by the identity property are marked in yellow. And the ones requiring further work in tangerine. The times table is furthermore symmetric about its lead diagonal, so our algebra is commutative.

Associativity is inherited from matrix multiplication in general. Both of our objects are self-inverse, so the inverse property is satisfied.

Our algebra is thus a commutative group. On 2 elements, this is forced to be isomorphic to C_2 . □

Lemma 4 Any projector $\boldsymbol{P}$ and the identity form the commutative monoid [27, 29]

$$Mon(2)$$

whose times table is in Fig 1.b) for our particular projector.

<u>Proof</u> The given table just summarizes what we already know.

$$\boldsymbol{P}^2 = \boldsymbol{P} \tag{22}$$

is the *idempotency property*. This table exhibits closure and commutativity as per above. Our algebra is not a group, however, since this table has repeat entries in its rows and columns. $\boldsymbol{P}$ is not invertible since (22) implies that it has ≥ 1 zero eigenvalue. Thus the inverse property is not enjoyed. Thus one has a commutative monoid. □

Theorem 1 The $\boldsymbol{K}_\pm$ algebra has the following relations. i)

$$\underline{\underline{\boldsymbol{K}}}_- \cdot \underline{\underline{\boldsymbol{K}}}_+ \; = \; -\underline{\underline{\boldsymbol{K}}}_- \; = \; \underline{\underline{\boldsymbol{K}}}_+ \cdot \underline{\underline{\boldsymbol{K}}}_- \; . \tag{23}$$

ii)

$$\boldsymbol{K}_+{}^2 \; = \; \mathbb{I} \; : \quad \text{involution} \; . \tag{24}$$

iii)

$$\boldsymbol{K}_-{}^2 \; = \; \boldsymbol{K}_- \; : \quad \text{idempotency} \; . \tag{25}$$

Furthermore, i) implies that

$$[\,\boldsymbol{K}_-\,,\,\boldsymbol{K}_+\,] \; = \; 0 \; : \quad \text{commutativity} \; . \tag{26}$$

Overall, we have a 4-element monoid, whose times table is in Fig 1.c), and which can be equipped with a Lie bracket under which everything commutes.

<u>Proof</u> The above relations are just a summary of results in Article 2.

The algebra of powers of these quantities contains just 4 objects. Its times table is as per Fig 1.c). Closure is thus attained, and commutativity and the identity property are manifested. Our algebra is not a group, however, since its times table has repeat entries in its rows and columns. $\boldsymbol{K}_-$ is a projector, thus singular, and so not in possession of an inverse. Thus the inverse property is not enjoyed by our algebra. Our algebra of powers is thus a 4-element *commutative monoid*,

$$Mon(4) \; .$$

It can furthermore be equipped with a Lie bracket, under which all the objects commute. □

Remark 1 Lemma 3 and 4's structures occur therein as submonoids, with the first being furthermore a group.

4 Triangle matrices' block formulation

Proposition 2 i)

$$\boldsymbol{K}_- \;=\; \mathbb{I} \;-\; \frac{1}{3}\mathbb{1} \;. \tag{27}$$

$$\boldsymbol{K}_+ \;=\; \frac{2}{3}\mathbb{1} \;-\; \mathbb{I} \;. \tag{28}$$

ii)

$$\boldsymbol{F} \;=\; \mathbb{1} \;-\; 2\,\mathbb{I} \;. \tag{29}$$

<u>Proof</u> i)

$$\boldsymbol{K}_- \;=\; \boldsymbol{L} \;=\; \frac{1}{3}\begin{pmatrix} 2 & \text{-}1 & \text{-}1 \\ \text{-}1 & 2 & \text{-}1 \\ \text{-}1 & \text{-}1 & 2 \end{pmatrix} \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;-\; \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \;=\; \mathbb{I} \;-\; \frac{1}{3}\mathbb{1} \;.$$

$$\boldsymbol{K}_+ \;=\; \boldsymbol{L} \;=\; \frac{1}{3}\begin{pmatrix} \text{-}1 & 2 & 2 \\ 2 & \text{-}1 & 2 \\ 2 & 2 & \text{-}1 \end{pmatrix} \;=\; \frac{2}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \;-\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;=\; \frac{2}{3}\mathbb{I} \;-\; \mathbb{1} \;.$$

ii)

$$\boldsymbol{F} \;=\; \begin{pmatrix} \text{-}1 & 1 & 1 \\ 1 & \text{-}1 & 1 \\ 1 & 1 & \text{-}1 \end{pmatrix} \;=\; \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \;-\; 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;=\; \mathbb{1} \;-\; 2\,\mathbb{I} \;. \;\Box$$

Exercise 1$^-$ Upgrade Corollary 2's 2-way basis change summary's notation to match Proposition 1's. Also write down the corresponding 2-way basis change between

$$\begin{pmatrix} \boldsymbol{K}_- \\ \boldsymbol{K}_+ \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathbb{I} \\ \mathbb{1} \end{pmatrix} \;.$$

Remark 1 Thus all three triangle matrices are thus just linear combinations of two extremely simple matrices: $\mathbb{1}$ and the identity $\mathbb{I}$.

Remark 2 With reference to Sec 2, Proposition 1's first two expressions are not however manifestly $\pm$ symmetric. This symmetry is also broken by $\boldsymbol{K}_+$ being an involution $\boldsymbol{J}$ while $\boldsymbol{K}_-$ is a projection $\boldsymbol{P}$.

5 The block algebra of $\mathbb{I}$ and $\mathbb{1}$

The powers of the matrix of 1's commuting monoid ***Mon*(∞)**

×	$\mathbb{I}$	$\mathbb{1}$	⋯	$3^u\mathbb{1}$	⋯	$3^v\mathbb{1}$	⋯
$\mathbb{I}$	$\mathbb{I}$	$\mathbb{1}$	⋯	$3^u\mathbb{1}$	⋯	$3^v\mathbb{1}$	⋯
$\mathbb{1}$	$\mathbb{1}$	$3\mathbb{1}$	⋯	$3^{u+1}\mathbb{1}$	⋯	$3^{v+1}\mathbb{1}$	⋯
⋮	⋮	⋮	⋮⋮⋮	⋮⋮⋮	⋮⋮⋮	⋮⋮⋮	⋮⋮⋮
$3^u\mathbb{1}$	$3^u\mathbb{1}$	$3^{u+1}\mathbb{1}$	⋮⋮⋮	$3^{2u+1}\mathbb{1}$	⋮⋮⋮	$3^{u+v+1}\mathbb{1}$	⋮⋮⋮
⋮	⋮	⋮	⋮⋮⋮	⋮⋮⋮	⋮⋮⋮	⋮⋮⋮	⋮⋮⋮
$3^v\mathbb{1}$	$3^v\mathbb{1}$	$3^v\mathbb{1}$	⋮⋮⋮	$3^{u+v+1}\mathbb{1}$	⋮⋮⋮	$3^{2v+1}\mathbb{1}$	⋮⋮⋮
⋮	⋮	⋮	⋮⋮⋮	⋮⋮⋮	⋮⋮⋮	⋮⋮⋮	⋮⋮⋮

Figure 2:

Theorem 2 The $\mathbb{I}$, $\mathbb{1}$ algebra has only 1 relation,

$$\mathbb{I}^2 \;=\; 3\,\mathbb{I} \;: \quad \text{powers are scaling} \;. \tag{30}$$

Consequently

$$\mathbb{I}^n \;=\; 3^{n\,-\,1}\,\mathbb{I} \;\;\forall\; n \;\in\; \mathbb{N} \;. \tag{31}$$

These form the infinite commutative monoid

$$Mon(\infty)$$

whose times table is in Fig 2.

<u>Proof</u>

$$\mathbb{1}^2 \;=\; \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \;=\; \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix} \;=\; 3\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \;=\; 3\,\mathbb{1}\;.$$

Next apply a trivial induction. The general element is thus

$$3^u\,\mathbb{1}\;.$$

The product of two such is

$$3^u\,\mathbb{1} \;\times\; 3^v\,\mathbb{1} \;=\; 3^{u\,+\,v\,+\,1}\,\mathbb{1}\;.$$

Closure, identity and commutativity immediately follow. While the corresponding times table has all elements distinct in each row and column, $\mathbb{1}$ is singular (of rank 1). Thus it has no inverse, nor does any multiple thereof. So again the inverse property fails, and so we have a commutative monoid. □

6 What projection is $\boldsymbol{L}$?

Remark 1 We already know this for its action mapping positions to separations: sending constellationspace $\mathbb{R}^6$ to relativespace $\mathbb{R}^4$. We now consider this instead for its action on (sides)^2 space. Observing that this is necessary so as to have our 3 triangle matrices acting on the same vector space, Without which the matrix products and commutators under consideration in the current Series would be interpretationally moot.

Theorem 3 For the 3-body problem with equal masses,

$$\boldsymbol{L} \;=\; \boldsymbol{P}_{\boldsymbol{n}^\perp} \;.$$

In words, the Lagrange matrix is the projector onto the complement of the normalized equal-entries 3-vector.

<u>Proof</u> Let $\boldsymbol{P}$ be a projector onto a subspace and $\boldsymbol{P}^\perp$ be the projector onto its complement subspace. Then

$$\mathbb{I} \;=\; \boldsymbol{P} \;+\; \boldsymbol{P}^\perp \;. \tag{32}$$

Let $\boldsymbol{P}_{\boldsymbol{v}}$ be a projector onto the direction picked out by the nonzero vector $\boldsymbol{v}$. Then

$$\underline{\underline{\boldsymbol{P}}}_{\boldsymbol{v}} \;=\; \underline{\boldsymbol{v}}\,\underline{\boldsymbol{v}} \;.$$

So

$$\underline{\underline{\boldsymbol{P}}}_{\boldsymbol{v}^\perp} \;=\; \underline{\underline{\mathbb{I}}} \;-\; \underline{\underline{\boldsymbol{P}}}_{\boldsymbol{v}} \;=\; \underline{\underline{\mathbb{I}}} \;-\; \underline{\boldsymbol{v}}\,\underline{\boldsymbol{v}} \;.$$

Finally, in the 3-d case, set $\boldsymbol{v} \;=\; \boldsymbol{n}$ as per the (II.20) subcase of (II.17). Then

$$\underline{\underline{\boldsymbol{P}}}_{\boldsymbol{n}^\perp} \;=\; \underline{\underline{\mathbb{I}}} \;-\; \underline{\boldsymbol{v}}\,\underline{\boldsymbol{v}} \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;-\; \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \;=\; \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \;=\; \underline{\underline{\boldsymbol{L}}} \;. \;\Box$$

Remark 2 This conceptualization also clarifies why $\mathbb{1}$ has entered our study, in the following sense.

$$\underline{\underline{\mathbb{1}}} \;=\; \underline{\mathbb{1}}\,\underline{\mathbb{1}} \;=\; \sqrt{3}\,\underline{\boldsymbol{n}}\,\sqrt{3}\,\underline{\boldsymbol{n}} \;=\; 3\,\underline{\boldsymbol{n}}\,\underline{\boldsymbol{n}} \;=\; 3\,\underline{\underline{\boldsymbol{P}}}_{\boldsymbol{n}} \;.$$

Remark 3 See [42] for extension to arbitrary masses, dimension, and point-or-particle number.

7 The Lagrange quadratic form is non-negative

Proposition 3 This is stated in Fig 3.

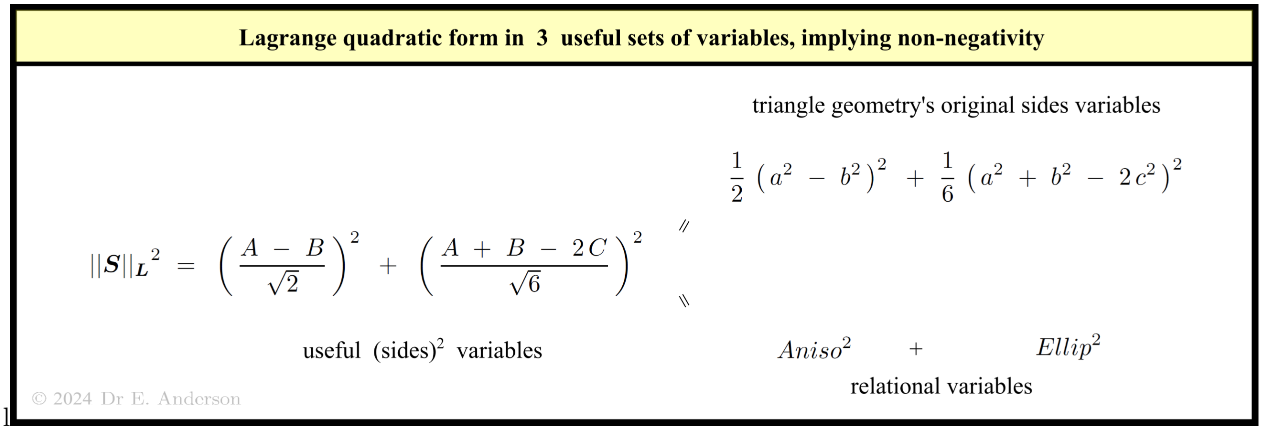

Figure 3:

<u>Proof</u> For the first equality,

$$6\,||\boldsymbol{S}||_{\boldsymbol{L}}{}^2 = 6\,\frac{2}{3}\sum_{\text{cycles}} A\,(A - B) =$$

$$4\left(A^2 + B^2 + C^2 - AB - BC - CA\right)$$

$$= \left(3A^2 - 6AB + 3B^2\right) + \left(A^2 + 2AB + B^2 + 4C^2 - 4AC - 4BC\right)$$

$$= 3(A - B)^2 + (A + B - 2C)^2 .$$

Next divide by 6 and finally factorize the remaining coefficients into two bracketed squares.

The subsequent upper equality follows trivially. For the subsequent lower equality, use the definitions of $Aniso$ and $Ellip$ in e.g. Article 1. □

Corollary 3 $||\ ||_{\boldsymbol{L}}$ is the $\mathbb{R}^2$ Euclidean norm on Aniso-Ellip space.[1]

Corollary 4

$$||\ ||_{\boldsymbol{L}}{}^2 \geq 0 .$$

Remark 2 In words, the Lagrange quadratic form is non-negative. We already established this in Article 2, by spectral analysis: independent of context such as which vector space the matrix acts upon. The second form can furthermore be interpreted as the eigen-vector expansion, the third eigen-vector being cut out by its eigenvalue being zero, as follows.

Corollary 5 (Eigenfunction expansion form)

$$||\boldsymbol{S}||_{\boldsymbol{L}}{}^2 = 0 \times R^2 + 1 \times \left(Aniso^2 + Ellip^2\right) . \tag{33}$$

[1] In e.g. [33], this is shown to be the collinear plane. Or, for some purposes, the collinear punctured plane: to the exclusion of the maximum coincidence-or-collision O. Constant values of the Lagrange quadratic form thus pick out concentric circles in this plane. Each of which is a copy of the shape space for 3 points-or-particles on a line [32]. And which, along with O, partition 3 points-or-particles on a line's relational space plane. Each copy corresponds to a fixed value of the radius of gyration, or, in mass-weighted configuration space, of the moment of inertia.

8 The $\boldsymbol{K}_\pm$, $\boldsymbol{\Delta K}$ algebra

Remark 1 Next suppose that we include $\boldsymbol{\Delta K}$ in our algebra. For instance, for an in-depth study of Heron's formula. Then each product with this, other than with the identity, gives a new element as follows.

Lemma 5

$$\boldsymbol{\Delta K}\,\boldsymbol{K}_+ \;=\; 2\,\mathbb{I} \;-\; \frac{1}{3}\,\mathbb{1}\;,$$

$$\boldsymbol{\Delta K}\,\boldsymbol{K}_- \;=\; \frac{2}{3}\,\mathbb{1} \;-\; 2\,\mathbb{I}\;,$$

$$\boldsymbol{\Delta K}\,(\,-\boldsymbol{K}_-\,) \;=\; 2\,\mathbb{I} \;-\; \frac{2}{3}\,\mathbb{1}\;,$$

$$(\,\boldsymbol{\Delta K}\,)^2 \;=\; 4\,\mathbb{I} \;-\; \mathbb{1}\;.$$

Exercise 2$^-$ We leave proof of Lemma 5 to the Reader

Remark 2 The first of these right hand sides already featured in Article 2 as the $\boldsymbol{Q}$ matrix.

Theorem 4 The algebra of arbitrary-power combinations of our 3 triangle matrices $\boldsymbol{K}_\pm$ and $\boldsymbol{\Delta K}$ is as follows. The infinite commutative monoid of the product form:

$$Mon(4) \;\times\; \mathbb{N}_0\;.$$

<u>Proof</u> Since everything commutes, the possible distinct products are of the following form.

$$\boldsymbol{K}_+{}^l\,\boldsymbol{K}_+{}^m\,(\,\boldsymbol{\Delta K}\,)^n \;\;,\;\; l,\,m\,n \;\in\; \mathbb{N}_0\;. \tag{34}$$

But

$$\boldsymbol{K}_+{}^l \;=\; \begin{cases} \mathbb{I} & l \text{ even} \\ \boldsymbol{K}_+ & l \text{ odd} \end{cases}.$$

And

$$\boldsymbol{K}_-{}^m \;=\; \begin{cases} \boldsymbol{K}_- & l \;\geq\; 1 \\ \mathbb{I} & l \;=\; 0 \end{cases}.$$

This leaves us with just

$$(\,\boldsymbol{\Delta K}\,)^n \;,\;\; \boldsymbol{K}_+\,(\,\boldsymbol{\Delta K}\,)^n \;,\;\; \boldsymbol{K}_-\,(\,\boldsymbol{\Delta K}\,)^n \;,\;\; \boldsymbol{K}_+\,\boldsymbol{K}_-\,(\,\boldsymbol{\Delta K}\,)^n\;.$$

Since

$$\boldsymbol{K}_+\boldsymbol{K}_- \;=\; -\,\boldsymbol{K}_-$$

as well, these become the following.

$$(\,\boldsymbol{\Delta K}\,)^n \;,\;\; \boldsymbol{K}_+\,(\,\boldsymbol{\Delta K}\,)^n \;,\;\; \pm\,\boldsymbol{K}_-\,(\,\boldsymbol{\Delta K}\,)^n\;.$$

Which is of the form

$$e\,(\,\boldsymbol{\Delta K}\,)^n \;\;\text{for}\;\; e \;\in\; Mon(4)\;.$$

We finally demonstrate in Sec 12 that the $(\boldsymbol{\Delta K})^n$ are all distinct. □

Remark 3 So the $\boldsymbol{K}_\pm$ form the smallest algebra for the triangle. While involving the fundamental triangle matrix $\boldsymbol{F} \;=\; \boldsymbol{\Delta K}$ renders the algebra infinite, but not much more complicated.

Remark 4 We could also consider the algebra of all powers of linear combinations of $\mathbb{1}$ and $\mathbb{I}$. This gives the product of 2 copies of the field or ring that the linear combinations are defined over. With the field $\mathbb{R}$ and the ring $\mathbb{Z}$ being of increasing relevance. The geometrical case nests within the $\frac{1}{3}$-$\mathbb{Z}$, isomorphic to $\mathbb{Z}$, so that $\boldsymbol{K}_\pm$ be included. Theorem 4's algebra isolates the Geometrical case, however.

9 Lagrange–Apollonius split version of Heron's formula

Theorem 5

$$T^2 \ = \ ||\boldsymbol{S}||_{\boldsymbol{K}_+}{}^2 \ - \ ||\boldsymbol{S}||_{\boldsymbol{K}_-}{}^2 \ . \tag{35}$$

Remark 1 In words, the tetra-area squared is the difference between the Apollonius and Lagrange quadratic forms, each acting on the sides-squared data.

<u>Proof</u>

$$T^2 \ = \ ||\boldsymbol{S}||_{\boldsymbol{H}}{}^2 \ = \ ||\boldsymbol{S}||_{\boldsymbol{F}}{}^2 \ = \ ||\boldsymbol{S}||_{\boldsymbol{\Delta K}}{}^2 \ = \ ||\boldsymbol{S}||_{\boldsymbol{K}_+ \, - \, \boldsymbol{K}_-}{}^2 \ = \ ||\boldsymbol{S}||_{\boldsymbol{K}_+}{}^2 \ - \ ||\boldsymbol{S}||_{\boldsymbol{K}_-}{}^2 \ .$$

The first step is by Heron's formula. The second to fourth just follow

$$\boldsymbol{H} \ = \ \boldsymbol{F} \ = \ \boldsymbol{\delta K} \ := \ \boldsymbol{K}_+ \ - \ \boldsymbol{K}_- \ .$$

The fifth step is by linearity. □

Corollary 6 This renders the following arrangement of the expanded Heron's formula Geometrically significant.

$$\begin{aligned} T^2 \ &= \ \frac{1}{3} \left[\ \sum_{\text{cycles}} A\,(\,4\,B \ - \ A\,) \ - \ 2 \sum_{\text{cycles}} A\,(\,A \ - \ B\,) \right] \\ &= \ \frac{1}{3} \ [\ (\ 4\,a^2\,b^2 \ + \ 4\,b^2\,c^2 \ + \ 4\,c^2\,a^2 \ - \ a^4 \ - \ b^4 \ - \ c^4 \ ) \\ &\quad - \ 2 \ (\ a^4 \ + \ b^4 \ + \ c^4 \ - \ a^2\,b^2 \ - \ b^2\,c^2 \ - \ c^2\,a^2 \ ) \ ] \end{aligned} \tag{36}$$

Remark 2 On the one hand, Geometrically, Theorem 5 reads as follows.

$$\underset{\substack{\text{Tetra-area} \\ \text{squared}}}{T^2} \ = \ \underset{\substack{\text{Heron} \\ \text{quadratic form}}}{||\boldsymbol{S}||_{\boldsymbol{H}}{}^2} \ = \ \underset{\substack{\text{Apollonius} \\ \text{quadratic form}}}{||\boldsymbol{S}||_{\boldsymbol{O}}{}^2} \ - \ \underset{\substack{\text{Lagrange} \\ \text{quadratic form}}}{||\boldsymbol{S}||_{\boldsymbol{L}}{}^2} \ .$$

Remark 3 On the other hand, Algebraically,

$$\underset{\substack{\text{Tetra-area} \\ \text{squared}}}{T^2} \ = \ \underset{\substack{\text{fundamental triangle} \\ \text{quadratic form}}}{||\boldsymbol{S}||_{\boldsymbol{F}}{}^2} \ = \ \underset{\substack{\text{involution} \\ \text{quadratic form}}}{||\boldsymbol{S}||_{\boldsymbol{J}}{}^2} \ - \ \underset{\substack{\text{projection} \\ \text{quadratic form}}}{||\boldsymbol{S}||_{\boldsymbol{P}}{}^2} \ .$$

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Corollary 7 For any admissible sides data, the following hold.

i) The Heron quadratic form is non-negative,

$$||\boldsymbol{S}||_{\boldsymbol{H}}{}^2 \ \geq \ 0 \ . \tag{37}$$

ii) The Apollonius form returns a larger value than the Lagrange form. I.e.

$$||\boldsymbol{S}||_{\boldsymbol{O}}{}^2 \ \geq \ ||\boldsymbol{S}||_{\boldsymbol{L}}{}^2 \ . \tag{38}$$

<u>Proof</u> i) Combine Theorem 5 with $T^2 \ \geq$ since for any such data, this is the square of a real quantity.

ii) Combine the Geometrical version of Theorem 5 with i) and trivially rearrange. □

Remark 4 This gives another weaker Order Theory sense in which the ± notation is merited: the $+$ piece is not smaller than the $-$ piece.

Corollary 8 The dimensionless version of Theorem 5 is as per Fig 4.

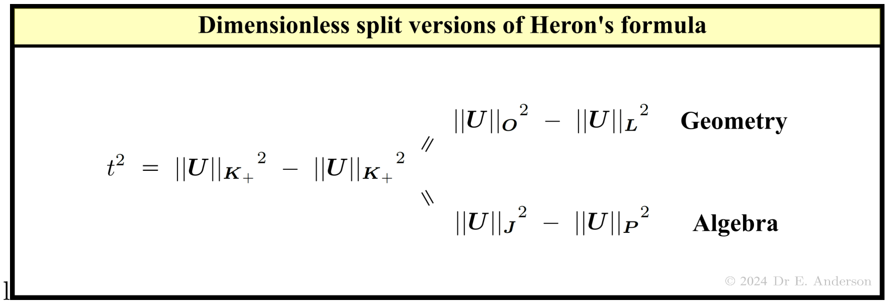

Figure 4:

Remark 5 Recollecting the radius of gyration (RoG) r and $R := r^2$ from Article 2, we are now making use of the following. Radius-of-gyration normalized tetra-area

$$t := \frac{T}{3\,R^2} \ .$$

And normalized $(\,\mathrm{sides}\,)^2$ vector

$$\boldsymbol{U} := \frac{1}{3\,R}\,\boldsymbol{S} = \frac{1}{A + B + C}\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \frac{1}{a^2 + b^2 + c^2}\begin{pmatrix} a^2 \\ b^2 \\ c^2 \end{pmatrix} \ .$$

Exercise 3$^-$ We leave proof of Corollary 8 to the Reader.

10 $\boldsymbol{J}$'s quadratic form is non-negative on admissible sides data

Proposition 2 For admissible sides data, the Apollonius quadratic form is non-negative:

$$||\boldsymbol{S}||_{\boldsymbol{O}}{}^2 \ = \ ||\boldsymbol{S}||_{\boldsymbol{J}}{}^2 \ = \ ||\boldsymbol{S}||_{\boldsymbol{K}_+}{}^2 \ \geq \ 0 \ .$$

<u>Proof</u> Concatenate Proposition 3 and Corollary 7 as follows.

$$||\boldsymbol{S}||_{\boldsymbol{K}_+}{}^2 \ \geq \ ||\boldsymbol{S}||_{\boldsymbol{K}_-}{}^2 \ \geq \ \mathit{Aniso}^2 \ + \ \mathit{Ellip}^2 \ \geq \ 0 \ . \ \Box$$

Remark 1 For arbitrary sides data, however, $\boldsymbol{T}^2$ can take either sign. So

$$||\boldsymbol{S}||_{\boldsymbol{K}_+}{}^2 \ < \ ||\boldsymbol{S}||_{\boldsymbol{K}_-}{}^2 \tag{39}$$

is possible. As is

$$||\boldsymbol{S}||_{\boldsymbol{K}_+}{}^2 \ < \ 0 \ . \tag{40}$$

This matches Article 2's spectral analysis.

Theorem 6 For admissible data, all three triangle matrices' quadratic forms are non-negative.

<u>Proof</u> This just collects up Proposition 1, Proposition 2 and Corollary 7.i). $\Box$

Exercise 4$^-$ Find examples of inadmissible data violating each possible combination of these non-negativities.

Remark 2 This has merit because two of them are indefinite for arbitrary sides data. There is thus a double spacelike condition enforced by sides data being admissible.

Remark 3 The eigenspectrum recovers Remark 1's signs but not Proposition 2's. Spectral quantifiers are thus capable of not knowing about data admissability criteria. Thus relying solely on these in analyzing the triangle matrices' quadratic forms would lead to errors.

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Proposition 3 The Apollonius and Lagrange quadratic forms coincide iff the triangle is degenerate.

<u>Proof</u> Corollary 7's inequality is now saturated. $\Box$

Remark 4 How is this non-negativity realized at the level of sides variables?

$$\begin{aligned} ||\boldsymbol{S}||_{\boldsymbol{J}}{}^2 \ &= \ 4 \left(a^2 b^2 \ + \ b^2 c^2 \ + \ c^2 a^2 \right) \ - \ \left(a^4 \ + \ b^4 \ + \ c^4 \right) \\ &= \ 4 a^2 b^2 \ + \ \left[\, 4 \left(a^2 \ + \ b^2 \right) \ - \ c^2 \, \right] c^2 \ . \end{aligned} \tag{41}$$

But also, for admissible sides data, the triangle inequality holds:

$$a \ + \ b \ \geq \ c \ \Rightarrow \ (a \ + \ b)^2 \ \geq \ c^2 \ \Rightarrow \ a^2 \ + \ b^2 \ \geq \ c^2 \ - \ 2 a b \ .$$

Thus

$$4 \left(a^2 \ + \ b^2 \right) \ - \ c^2 \ \ \geq \ \ 3 \left(a^2 \ + \ b^2 \right) \ - \ 2 \, a b \ = \ 3 (a \ - \ b)^2 \ + \ 4 \, a b \ . \tag{42}$$

Finally substitute (42) in (41) to obtain the following.

$$||\boldsymbol{S}||_{\boldsymbol{J}}{}^2 \ \ \geq \ \ 4 a^2 b^2 \ + \ \left[3 \, (a \ - \ b)^2 \ + \ 4 \, a b \right] c^2 \ \ \geq \ \ 0 \ .$$

Where the last inequality follows from squares and distances both being $\geq \ 0$.

11 Block split formulation of Heron's formula

Remark 1 From the block split of $\boldsymbol{H} = \boldsymbol{F}$, we recover the following quite well-known split.

$$T^2 = ||\boldsymbol{S}||_{\mathbb{1}}{}^2 - 2\,||\boldsymbol{S}||^2$$

$$= \sum_{\text{cycles}} A\,(\,A + 2\,B\,) - 2\sum_{\text{cycles}} A^2 = \left(\sum_{\text{cycles}} A\right)^2 - 2\sum_{\text{cycles}} A^2$$

$$= (\,A + B + C\,)^2 - 2\left(A^2 + B^2 + C^2\right) = \left(a^2 + b^2 + c^2\right)^2 - 2\left(a^4 + b^4 + c^4\right) . \quad (43)$$

Remark 2 So in the basic Algebra regime, the blocks are the cycle sum all squared and the cycle sum of squares.

Remark 3 Also

$$||\boldsymbol{S}||_{\boldsymbol{L}}{}^2 = ||\boldsymbol{S}||^2 - \frac{1}{3}\,||\boldsymbol{S}||_{\mathbb{1}}{}^2 = \sum_{\text{cycles}} A^2 - \frac{1}{3}\left(\sum_{\text{cycles}} A\right)^2$$

$$= \left(A^2 + B^2 + C^2\right) - \frac{1}{3}\,(\,A + B + C\,)^2 \,. = \left(a^4 + b^4 + c^4\right) - \frac{1}{3}\left(a^2 + b^2 + c^2\right)^2 .$$

Remark 4 We can also arrive at this as follows. The 3-d Euclidean norm of the $(\,\text{sides}\,)^2$ vector $\boldsymbol{S}$ decomposes with respect to projections according to

$$||\boldsymbol{S}||^2 = ||\boldsymbol{S}||_{\boldsymbol{P_n}}{}^2 + ||\boldsymbol{S}||_{\boldsymbol{P}_{\boldsymbol{n}}^{\perp}}{}^2 = \frac{1}{3}\,||\boldsymbol{S}||_{\mathbb{1}}{}^2 + ||\boldsymbol{S}||_{\boldsymbol{L}}{}^2 .$$

Remark 5 Finally

$$||\boldsymbol{S}||_{\boldsymbol{J}}{}^2 = \frac{2}{3}\,||\boldsymbol{S}||_{\mathbb{1}}{}^2 - ||\boldsymbol{S}||^2 = \frac{2}{3}\left(\sum_{\text{cycles}} A\right)^2 - \sum_{\text{cycles}} A^2$$

$$= \frac{2}{3}\,(\,A + B + C\,)^2 - \left(A^2 + B^2 + C^2\right) = \frac{2}{3}\left(a^2 + b^2 + c^2\right)^2 - \left(a^4 + b^4 + c^4\right) .$$

12 $\mathbb{T}$ and $\mathbb{I}$: the irreducible algebra

Structure 1 The irreducible blocks are, however, the identity matrix $\mathbb{I}$ and the tracefree matrix of 1's

$$\mathbb{T} := \mathbb{1} - \frac{\mathrm{tr}\,\mathbb{1}}{3}\mathbb{I} = \mathbb{1} - \frac{3}{3}\mathbb{I} = \mathbb{1} - \mathbb{I} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} . \tag{44}$$

Changing the subject,

$$\mathbb{1} := \mathbb{T} + \mathbb{I}$$

is the *tracefree-trace split* of the matrix of 1 's.

Lemma 6

$$\boldsymbol{J} = \boldsymbol{K}_{+} = \frac{1}{3}\,(2\mathbb{T} - \mathbb{I}) . \tag{45}$$

$$\boldsymbol{L} = \boldsymbol{K}_{-} = \frac{1}{3}\,(2\mathbb{I} - \mathbb{T}) . \tag{46}$$

$$\boldsymbol{F} = \boldsymbol{\Delta K} = \mathbb{T} - \mathbb{I} := \boldsymbol{\delta R} . \tag{47}$$

Exercise 5 Work out the two-way basis change between $\boldsymbol{L}$, $\boldsymbol{O}$ and $\mathbb{I}$, $\mathbb{T}$.

Remark 1 Thus there is a second significant difference that forms the fundamental triangle matrix. The first difference (10) has pieces which each have their own Geometrical significance. While the second difference has pieces which each have Representation-Theoretic significance, as the difference of two irreducibles.

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Theorem 7

$$\mathbb{T}^n = \frac{1}{3}\left[\left(2^n + (-1)^{n+1} \right) \mathbb{T} + 2\left(2^{n-1} + (-1)^n \right) \mathbb{I} \right] . \tag{48}$$

Proof

$$\mathbb{T}^2 = \mathbb{T} + 2\,\mathbb{I} .$$

So recursively

$$\alpha_{n+1}\,\mathbb{T} + \beta_{n+1}\,\mathbb{I} = \mathbb{T}^{n+1} = \mathbb{T}^n\,\mathbb{T} = (\alpha_n\,\mathbb{T} + \beta_n\,\mathbb{I})\,\mathbb{T}$$
$$= \alpha_n\,\mathbb{T}^2 + \beta_n\,\mathbb{T} = \alpha_n\,(\mathbb{T} + 2\,\mathbb{I}) + \beta_n\,\mathbb{T} = (\alpha_n + \beta_n)\,\mathbb{T} + 2\,\alpha_n\,\mathbb{I} . \tag{49}$$

Thus we have the following two coupled first-order linear difference equations.

$$\alpha_{n+1} = \alpha_n + \beta_n , \tag{50}$$

$$\beta_{n+1} = 2\,\alpha_n . \tag{51}$$

Shift the second down one value and substitute into the first to obtain the following second-order difference equation.

$$\alpha_{n+1} = \alpha_n + 2\,\alpha_{n-1} . \tag{52}$$

Corresponding initial conditions are

$$\alpha_1 = 1 = \alpha_2 .$$

Thus we have a *MUSoLDE (mostly-unary second-order linear difference equation)* problem [36]. The *EU-SoLDE* – *everywhere*-unary – problem is of course much more widely known as the *Fibonacci* problem [4, 23, 26]. So our problem gives one of the minimal deviations away from the Fibonacci problem.[2] More specifically, we have the *(2,2)-MUSoLDE problem*, since its deviation from EUSoLDE is that its second coefficient is a 2 .

[2]The most well-known nontrivially MUSoLDE numbers are the *Lucas numbers* [39].

Set

$$\alpha_n \;=\; m^n \;.$$

Then canceling off a factor of $m^{n\,-\,1}$ (52) becomes the following auxiliary equation.

$$0 \;=\; m^2 \;-\; m \;-\; 2 \;=\; (\,m \;+\; 1\,)\,(\,m \;-\; 2\,)\;. \tag{53}$$

Thus

$$m \;=\; -\,1 \;\text{ or }\; 2\;.$$

So the general solution is

$$\alpha_n \;=\; E\,(-1)^n \;+\; F\,2^n\;. \tag{54}$$

Our initial conditions give the following simultaneous linear equations for the hitherto arbitrary constants E and F.

$$1 \;=\; 2\,F \;-\; E\;,$$

$$1 \;=\; 4\,F \;+\; E\;.$$

Thus

$$F \;=\; \frac{1}{3} \;=\; -\,E\;.$$

And so our particular solution consists of the following *(2, 2)-MUSolDE numbers.*

$$\alpha_n \;=\; \frac{1}{3}\,\left[\,2^n \;+\; (-1)^{n\,+\,1}\,\right]\;. \tag{55}$$

Next, (51) gives that

$$\beta_n \;=\; \frac{2}{3}\,\left[\,2^{n\,-\,1} \;+\; (-1)^n\,\right]\;. \tag{56}$$

Finally, substituting (55) and (56) into the once down-shifted version of the opening equality of (49) in reverse, we obtain (48). □

13 $\boldsymbol{F}^{-1}$ is an irrep

Remark 1 Moreover, also

$$\mathbb{T}^{-1} \;=\; \frac{1}{2}\,\boldsymbol{\delta R}\,. \tag{57}$$

This can be rearranged as

$$\boldsymbol{F}^{-1} \;=\; (\boldsymbol{\delta R})^2 \;=\; \frac{1}{2}\mathbb{T} \;=\; \mathbb{S}\,. \tag{58}$$

Remark 2 (58) is an equation between irreducible quantities.

Remark 3 Including powers of $\boldsymbol{\delta R}$ can now be seen to amount to completing the $\mathbb{T}^n$ algebra to a group.

Lemma 7

$$[\,\mathbb{T}\,,\,\boldsymbol{F}\,] \;=\; [\,\mathbb{T}\,,\,\boldsymbol{J}\,] \;=\; [\,\mathbb{T}\,,\,\boldsymbol{L}\,] \;=\; 0\,.$$

Proof For, indeed,

$$\begin{aligned} &[\,\alpha\,\mathbb{T} \;+\; \beta\,\mathbb{1}\,,\,\gamma\,\mathbb{T} \;+\; \delta\,\mathbb{1}\,] \\ =\;& \alpha\,\gamma\,[\,\mathbb{T}\,,\,\mathbb{T}\,](\,\alpha\,\delta \;-\; \beta\,\gamma\,)\,[\,\mathbb{T}\,,\,\mathbb{1}\,] \;+\; \beta\,\delta\,[\,\mathbb{1}\,,\,\mathbb{1}\,] \;+ \\ =\;& \alpha\,\gamma\,0(\,\alpha\,\delta \;-\; \beta\,\gamma\,)\,0 \;+\; \beta\,\delta\,0 \;+ \;\;=\; 0\,. \end{aligned}$$

The first step is by linearity. And antisymmetry of the Lie bracket. The second is by anything commuting with itself and everything commuting with the identity.

Finally by (45, 46, 47), all commutators between of $\boldsymbol{J}$, $\boldsymbol{L}$ and $\boldsymbol{F}$ are covered by the above working. □

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Theorem 8 i)

$$(\boldsymbol{\delta R})^n \;=\; \frac{1}{3}\;\Big(\;[\;1 \;-\; (-2)^n\;]\;\mathbb{T} \;+\; \Big[\;1 \;-\; (-2)^{n\,+\,1}\;\Big]\;\mathbb{1}\;\Big)\;. \tag{59}$$

ii)

$$\mathbb{T}^{-n} \;=\; \left(\frac{\boldsymbol{\delta R}}{2}\right)^n \;=\; \frac{1}{3}\;\Big(\;\Big[\;2^{-n} \;+\; (-1)^{n\,+\,1}\;\Big]\;\mathbb{T} \;+\; \Big[\;2^{-n} \;+\; 2\,(-1)^n\;\Big]\;\mathbb{1}\;\Big)\;. \tag{60}$$

Proof i)

$$\begin{aligned} (\boldsymbol{\delta R})^n \;=\; (\,\mathbb{T} \;-\; \mathbb{1}\,)^n \;=\; \sum_{m\,=\,0}^{n}\binom{n}{m}\,(-1)^{n\,-\,m}\,\mathbb{T}^m\,\mathbb{1}^{n\,-\,m} \;=\; (-1)^n\sum_{m\,=\,0}^{n}\binom{n}{m}\,(-1)^m\,\mathbb{T}^m \\ =\; (-1)^n\sum_{m\,=\,0}^{n}\binom{n}{m}\,(-1)^m\frac{1}{3}\Big(\;\Big[2^m \;+\; (-1)^{m\,+\,1}\Big]\;\mathbb{T} \;+\; 2\;\Big[2^{m\,-\,1} \;+\; (-1)^m\Big]\;\mathbb{1}\;\Big) \\ =\; \frac{(-1)^n}{3}\left(\;\sum_{m\,=\,0}^{n}\binom{n}{m}\,\Big[\,(-2)^m \;+\; (-1)^{2\,m\,+\,1}\Big]\;\right)\;\mathbb{T} \\ +\;\left(\;\sum_{m\,=\,0}^{n}\binom{n}{m}\;\Big[\,-(-2)^{m\,-\,1} \;+\; (-1)^{2\,m}\Big]\;\right)\;2\,\mathbb{1} \\ =\; \frac{(-1)^n}{3}\left(\;\left[\;\sum_{m\,=\,0}^{n}\binom{n}{m}\,(-2)^m\,1^{n\,-\,m} \;-\; \sum_{m\,=\,0}^{n}\binom{n}{m}\,1^m\,1^{n\,-\,m}\;\right]\;\mathbb{T}\right. \\ \left.+\;\left[\;\sum_{m\,=\,0}^{n}\binom{n}{m}\,(-2)^m\,1^{n\,-\,m} \;+\; 2\,\sum_{m\,=\,0}^{n}\binom{n}{m}\,1^m\,1^{n\,-\,m}\;\right]\;\mathbb{1}\;\right) \end{aligned}$$

$$= \frac{(-1)^n}{3} \left([\,(1 - 2)^n - (1 + 1)^n\,]\,\mathbb{T} + [(1 - 2)^n + 2(1 + 1)^n\,]\,\mathbb{I} \right)$$

$$= \frac{(-1)^n}{3} \left([\,(-1)^n - 2^n\,]\,\mathbb{T} + \left[2^{n+1} + (-1)^n\right]\mathbb{I} \right)$$

$$= \frac{1}{3} \left([\,1 - (-2)^n\,]\,\mathbb{T} + \left[1 - (-2)^{n+1}\right]\mathbb{I} \right) .$$

Where step 1 is (47). In Step 2, the two matrices commute with each other, so the Binomial Theorem applies. Step 3 uses that any power of the identity is itself the identity, and then the identity property. While also taking out a common factor. And using that $\pm m$ have the same parity, so that $(-1)^{-m}$ can be replaced by $(-1)^m$

Step 4 substitutes in (45), which Step 5 then expands. Step 6 then cancels off even powers of -1 . And makes 4 judicious uses of $0 = 1 - 1$. Which enable Step 7 to make 4 uses of the Binomial Theorem in reverse. Steps 8 and 9 are just tidying up.

ii)

$$\mathbb{T}^{-n} = \left(\frac{\boldsymbol{\delta R}}{2}\right)^n = \frac{1}{2^n}\,\frac{1}{3} \left([\,1 - (-2)^n\,]\,\mathbb{T} + \left[1 - (-2)^{n+1}\right]\mathbb{I} \right) .$$

The first step is by (57). The second is by i). Finally, the last form written here is equivalent to the statement ii) by expansion and then cancellation. □

Theorem 9 The algebra of all powers of $\boldsymbol{K}_\pm$ and $\mathbb{T}$ that exist is as follows. The infinite commutative monoid of product form

$$Mon(4) \times \mathbb{Z} .$$

If $\boldsymbol{K}_-$ is excluded, then one is left with the group

$$C_2 \times C_\infty .$$

If $\boldsymbol{K}_+$ is instead, then one has

$$Mon(2) \times C_\infty .$$

And if $\mathbb{T}$ is, then one is back to

$$Mon(4) .$$

Proof All objects listed commute by Article 2 and Lemma 6. So we can represent this as

$$\boldsymbol{K}_+^l \boldsymbol{K}_-^m \mathbb{T}^n \text{ for } l, n \in \mathbb{Z} \text{ and } m \in \mathbb{N}_0 .$$

But the first two factors collapse as in Theorem 4's proof. And the last factor is parametrized by the integers as per Theorem 7. And is thus C_∞ .

The abovelisted submonoids arise in parallel to how Fig 1.c)'s times table contains those of Subfig a) and b). □

14 Irreducible split of Heron's formula

Theorem 10 (Irreducible split formulations of Heron's formula) i) Tetra-area squared as the difference of the tracefree and trace parts' norms.

$$T^2 \;=\; ||\boldsymbol{S}||_{\mathbb{T}}{}^2 \;-\; ||\boldsymbol{S}||_{\mathbb{I}}{}^2 \;\;=\; ||\boldsymbol{S}||_{\mathbb{T}}{}^2 \;-\; ||\boldsymbol{S}||^2 \;. \tag{61}$$

ii) The fundamental triangle matrix norm in terms of its inverse norm and the Euclidean norm.

$$||\boldsymbol{S}||_{\boldsymbol{F}}{}^2 \;=\; 2\,||\boldsymbol{S}||_{\boldsymbol{F}^{-1}}{}^2 \;-\; ||\boldsymbol{S}||^2 \;. \tag{62}$$

iii) The Euclidean norm split into fundamental-triangle and inverse-fundamental triangle parts.

$$||\boldsymbol{S}||^2 \;=\; 2\,||\boldsymbol{S}||_{\boldsymbol{F}^{-1}}{}^2 \;-\; ||\boldsymbol{S}||_{\boldsymbol{F}}{}^2 \;. \tag{63}$$

iv) Tetra-area squared as the difference of two cycle sums.

$$\begin{aligned} T^2 \;&=\; \left(\sum_{\text{cycles}} A^2 \;-\; 2 \sum_{\text{cycles}} A\,B \right) \\ &=\; \left(\, A^2 \;+\; B^2 \;+\; C^2 \,\right) \;-\; \left(\, 2\,(A\,B \;+\; B\,C \;+\; C\,A\,)\, \right) \\ &=\; \left(\, a^4 \;+\; b^4 \;+\; c^4 \,\right) \;-\; \left(\, 2\left(a^2\,b^2 \;+\; b^2\,c^2 \;+\; c^2\,a^2\,\right)\, \right) \;. \end{aligned} \tag{64}$$

<u>Proof</u> i)

$$T^2 \;=\; ||\boldsymbol{S}||_{\boldsymbol{F}}{}^2 \;=\; ||\boldsymbol{S}||_{\boldsymbol{F}}{}^2 \;=\; ||\boldsymbol{S}||_{\boldsymbol{\delta R}}{}^2 \;=\; ||\boldsymbol{S}||_{\mathbb{T}\,-\,\mathbb{I}}{}^2 \;=\; ||\boldsymbol{S}||_{\mathbb{T}}{}^2 \;-\; ||\boldsymbol{S}||_{\mathbb{I}}{}^2 \;=\; ||\boldsymbol{S}||_{\mathbb{T}}{}^2 \;-\; ||\boldsymbol{S}||^2 \;.$$

The first step is by Heron's formula. The second to fourth just follow

$$\boldsymbol{H} \;=\; \boldsymbol{F} \;=\; \boldsymbol{\delta R} \;:=\; \mathbb{T} \;-\; \mathbb{I} \;.$$

The fifth step is by linearity. The last is that $||\;||_{\mathbb{I}}$ is just a longer way of expressing $||\;||$, since $\mathbb{I}$ is furthermore the Euclidean metric in Cartesian coordinates.

ii) Apply (57) to the previous.

iii) Change the subject of the previous.

iv) Expand using the explicit form of the matrices in the first equation of i). □

Remark 1 iv) is just the obvuious split of the standard cycle-summarized expanded Heron's formula (I.11) This reflects that the two cycle sums immediately present in this formulation *already are* irreps! I.e.

$$\sum_{\text{cycles}} A^2 \;:=\; \sum_{i\,=\,\text{cycles}} A_i{}^2 \;\;\text{is an irrep}\;.$$

And

$$\sum_{\text{cycles}} A\,B \;:=\; \sum_{\substack{i,\,j\;\in\;\text{cycles}\\ i\,\neq\,j}} A_i\,A_j \;\;\text{is an irrep}\;.$$

End Remark A summary of the various cycle sum expansions given in the current Series is provided as Fig 5.

Types of split of the triangle's 3 quadratic forms					
		Split			
Form	**Unsplit**	**Algebraic or Geometric**	**Block**	**Irreducible**	**Eigenvector expansion**
$T^2 = \|\|S\|\|_F{}^2$ **Heron-fundamental**	$\sum_{\text{cycles}} A(2B - A)$	$\frac{1}{3}\left[\sum_{\text{cycles}} A(4B - A) - 2\sum_{\text{cycles}} A(A - B)\right]$	$\left(\sum_{\text{cycles}} A\right)^2 - 2\sum_{\text{cycles}} A^2$	$2\sum_{\text{cycles}} AB - \sum_{\text{cycles}} A^2$	$R^2 - 2(Aniso^2 + Ellip^2)$
$\|\|S\|\|_J{}^2$ **Apollonius**	$\frac{1}{3}\sum_{\text{cycles}} A(4B - A)$		$\frac{2}{3}\left(\sum_{\text{cycles}} A\right)^2 - \sum_{\text{cycles}} A^2$	$\frac{1}{3}\left(4\sum_{\text{cycles}} AB - \sum_{\text{cycles}} A^2\right)$	$R^2 - (Aniso^2 + Ellip^2)$
$\|\|S\|\|_L{}^2$ **Lagrange**	$\frac{2}{3}\sum_{\text{cycles}} A(A - B)$		$\sum_{\text{cycles}} A^2 - \frac{1}{3}\left(\sum_{\text{cycles}} A\right)^2$	$\frac{2}{3}\left(\sum_{\text{cycles}} A^2 - \sum_{\text{cycles}} AB\right)$	$Aniso^2 + Ellip^2$

Figure 5:

15 Conclusion

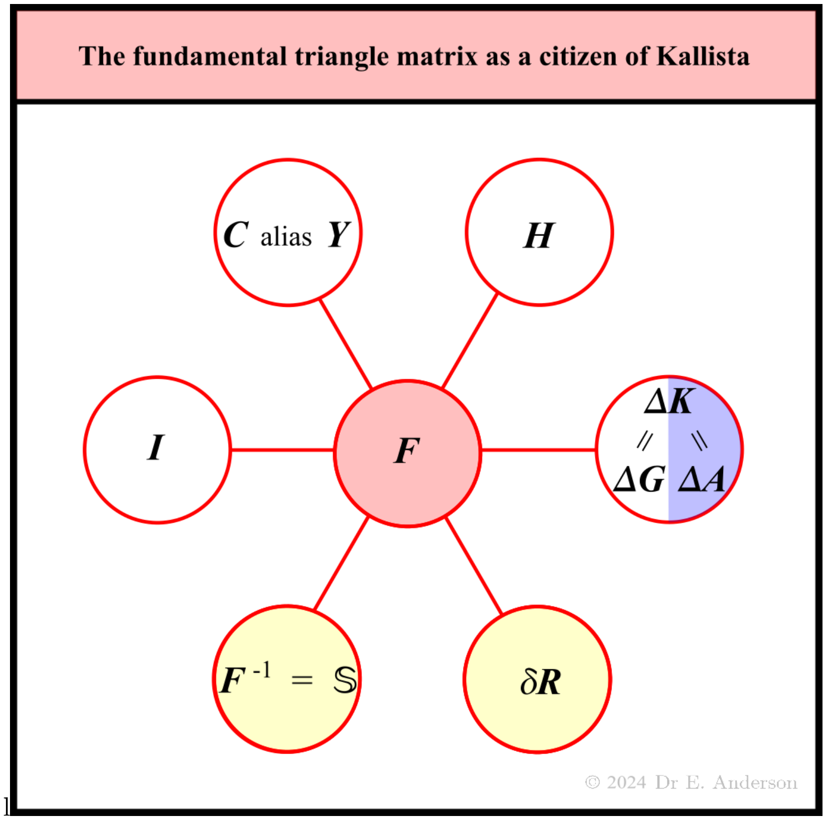

Figure 6:

Naming Remark 1 The name 'fundamental triangle matrix' [40] has grown in merit by three further routes to it being uncovered in the current Article. By which Fig 1.2 has expanded to Fig 6.

The names of $\boldsymbol{F}$'s previously documented [40] aspects are *triangle-inequality matrix* $\boldsymbol{I}$. *Cosine-rule matrix* $\boldsymbol{C}$ alias *apythagorean matrix* $\boldsymbol{Y}$. And *Heron((–Euler)–Buchholz) matrix* [21, 7, 38] $\boldsymbol{H}$.

Two of the current Article's three new names can be summarized by '*concurrently two differences matrix*'. The two differences being $\boldsymbol{\Delta K}$: the Geometrical Apollonius–Lagrange split difference. And $\boldsymbol{\delta R}$: the Representation-Theoretic irreducibles split difference. The third is that $\boldsymbol{F}^{-1}$ is itself one of the irreducibles, $\mathbb{S}$, giving the name *irreducible-inverse matrix*. The second and third names are Representation-Theoretic, which we colour-code in yellow in Fig 6.

Naming Remark 2 The *sides-medians matrix* $\boldsymbol{T}$ can be arrived at by cycling the sides-median relation. Which is most simply arrived at via Apollonius' Theorem: a basic Corollary of Pythagoras' Theorem. It is a subcase of the *sides–Cevians matrix*, which generalization is most simply arrived at via Stewart's Theorem [6, 14, 31], itself a simple Corollary of the cosine rule. Whence the respectively aliases 'Apollonius matrix' $\boldsymbol{O}$ and 'Stewart matrix' $\boldsymbol{T}$.

The sides–medians matrix admits a scaling–involution split, by which its involution rescaling $\boldsymbol{J}$ constitutes a more mathematically interesting core. $\boldsymbol{J}$ occurs furthermore as the involution part of the Algebraic involution-projection split-difference formulation of $\boldsymbol{F}$. This Algebraic conceptualization produces exactly the same split-difference as the abovementioned Geometric conceptualization. Thereby, there is a second route to our object that makes no reference to medians. This gives a sense in which sides-medians involution is *not* a truer name. *Triangle involution matrix* covers both routes, with sides-medians involution then being the name of one of its two known aspects.

Naming Remark 3 The Lagrange matrix $\boldsymbol{L}$ can be arrived at by considering how the triangles's point-or-particle vertices' position vectors convert to its relative separation vectors. It is the equal-masses subcase of a more general Lagrange matrix, also denoted $\boldsymbol{L}$, meeting this description with arbitrary masses at each vertex. *Position-separation matrix* is thus in one sense a truer name for this matrix.

This matrix is a projector, denoted by $\boldsymbol{P}$. Which projects *out* the direction picked out by the equal-entries vector. Which signifies, for instance, the absolute origin or the centre of mass, depending on context [30, 32]. *Down from* constellationspace [11, 30, 32]. I.e. the configuration space containing, for triangles, 3 copies of the position in absolute space of the vertices. *Onto* relative configuration space [30, 24, 32]

$\boldsymbol{P}$ occurs furthermore as the projection part of the Algebraic involution–projection split-difference formulation of $\boldsymbol{F}$. In the context of Heron's formula, the action is now on $(\,\textit{sides}\,)^2$ space. By which what is projected out now is the $(\text{ radius of gyration })^2$ vector. Or upon mass-weighting, the moment of inertia vector.

Thereby, there is a second route to our object that makes no reference to position coordinates. This gives a sense in which 'positions-separations matrix' is *not* a truer name. *Triangle projection matrix* covers both routes, with 'position-separation matrix' then being the name of one of its two known aspects.

Notational Remark 1 $\boldsymbol{J}$ and $\boldsymbol{L}$ furthermore form a $\boldsymbol{K}_{\pm}$ pair, due to being reflection-symmetric about the identity. This notation just comes from K being the letter between J and L.

$$\boldsymbol{\Delta K} \;=\; \boldsymbol{K}_{+} \;-\; \boldsymbol{K}_{-}$$

then follows suit. It is this symmetry in the difference which renders it a stronger difference than the other two in the current Article. Which we honour with a $\boldsymbol{\Delta}$ in place of a $\boldsymbol{\delta}$.

For some applications, $\boldsymbol{J}$ and $\boldsymbol{P}$ are more appropriate, out of emphasizing the involutive and projective nature of these matrices. Using $\boldsymbol{G}$s in place of $\boldsymbol{K}$s would emphasize that the split is Geometrically meaningful (white). While $\boldsymbol{A}$s would emphasize that the split is Algebraically meaningful (blue).

♠ ◇ ♣

End-Remark 1 All in all, $\boldsymbol{F}$ is a citizen of Kallista.[3] The six conceptually distinct routes to this found in the current Series span all of Geometry, Algebra and Representation Theory: white, blue and yellow respectively in Fig 6.

End Remark 2 While a reflection-symmetric difference suffices to have a *proto-index* – a mathematical name for Δ differences – to be an index the two pieces should be counts. And there should be a Topology–Analysis bridge – and Index Theorem [20] – from this index count to an elsewise rather harder object to compute. $\mathbb{R}_0$-valued quadratic forms are not counts, however, and the Author has no reason to believe that the Geometric-split Heron's formula is such a bridge.

♠ ◇ ♣

[3]This qualifier for objects, Theorems, Theories... arrived at by multiple conceptually distinct routes is introduced and explained in [36].

Pointer 1 The current Series' analysis, and the medians–Heron formula in particular, are given a Cevian [5, 3, 16, 19, 22] robustness test in a second Series, [45, 46, 47]. We explain there that the general Stewart matrix is a 3-input function

$$\boldsymbol{T}(\,\xi,\,\eta,\,\zeta\,)\ .$$

With the medians corresponding to the unique subcase

$$\boldsymbol{O} \;=\; \boldsymbol{T}\left(\,\frac{1}{2}\,,\;\;\frac{1}{2}\,,\;\;\frac{1}{2}\,\right)\ .$$

Whenever $\boldsymbol{T}$ is invertible, moreover, it admits a scaling–unimodular split. With unimodular piece

$$\boldsymbol{U}(\,\xi,\eta,\,\zeta\,)\ .$$

Then

$$\underset{\text{sides–medians}}{\underset{\text{Apollonius :}}{\boldsymbol{O}}} \;=\; \underset{\text{involution}}{\boldsymbol{J}} \;=\; \underset{\text{sides–Cevians}}{\underset{\text{unimodular Stewart :}}{\boldsymbol{U}\left(\,\tfrac{1}{2}\,,\;\tfrac{1}{2}\,,\;\tfrac{1}{2}\,\right)}}\ .$$

Pointer 2 The arbitrary-masses Lagrange matrix also enters this study [46]. Only concurrent Cevians admit reformulation in terms of arbitrary masses. One now has just 2 input functions, which can be viewed as the 2 independent ratios that [13] 3 masses support. Thus, when we double-booked the shorthand $\boldsymbol{L}$, we really meant the following. That the equal-masses case is the

$$\boldsymbol{L}(\,1,\,1\,)$$

subcase of the general [42]

$$\boldsymbol{L}(\,\mathcal{M}_{\mathrm{A}},\,\mathcal{M}_{\mathrm{B}}\,)\ .$$

For mass ratios

$$\mathcal{M}_{\mathrm{A}} \;:=\; \frac{m_{\mathrm{A}}}{m_{\mathrm{C}}}\ ,\quad \mathcal{M}_{\mathrm{B}} \;=\; \frac{m_{\mathrm{B}}}{m_{\mathrm{C}}}\ .$$

Pointer 3 A third series [43] deals with quadrilateral and N-a-gon analogues of medians; the generalized Kendall Theorem and larger Hopf maps arise here.

Pointer 4 And a fourth (starting by substantially rewriting [38]) with quadrilateral area formula generalizations of Heron's formula [48].

Pointer 5 All four of these Series proceed largely by making an unprecedentedly large leap in the detailed application of Linear Algebra to the Flat Geometry N-a-gons.

Pointer 6 And more occasionally to d-simplices. For Heron's formula is also a first foundational result for *Distance Geometry* [12, 18, 35]. And its natural generalization here is, via the *della Francesca–Tartaglia's formula* for the volume of the tetrahaedron, the *Cayley–Menger determinant* for the d-volume of the general d-simplex.

Pointer 7 The abovemetioned four Series are the start of a systematic program to systematically Linear Algebraicize specific cases in Flat Geometry. Rather than just taking Linear Algebra as an over-arching alternative pillar [28] for Flat Geometry, which is then however seldom tied to specific notions and results that are widely familiar from Euclidean-type pillar studies.

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A Apollonius–Lagrange split of the triangle inequality and the cosine rule

Remark 1 For the triangle inequality,

$$0 \underset{\text{componentwise}}{\leq} \underline{\underline{\boldsymbol{F}}} \cdot \underline{\boldsymbol{s}} = \underline{\underline{\boldsymbol{O}}} \cdot \underline{\boldsymbol{s}} - \underline{\underline{\boldsymbol{L}}} \cdot \underline{\boldsymbol{s}} = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} .$$

Which reads

$$\frac{1}{3}(-a + 2b + 2c) \geq \frac{1}{3}(b + c - 2a) \quad \text{and cycles} \quad . \tag{65}$$

Remark 2 For the cosine rule,

$$2\,\underline{\boldsymbol{D}} = \underline{\underline{\boldsymbol{F}}} \cdot \underline{\boldsymbol{S}} = \underline{\underline{\boldsymbol{O}}} \cdot \underline{\boldsymbol{S}} - \underline{\underline{\boldsymbol{L}}} \cdot \underline{\boldsymbol{S}} = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} .$$

Which reads

$$2\,b\,c \cos\alpha = \frac{1}{3}(-A + 2B + 2C) - \frac{1}{3}(B + C - 2A) \quad \text{and cycles} \quad .$$

Thus

$$6\,\underline{\boldsymbol{b}} \cdot \underline{\boldsymbol{c}} = 6\,b\,c \cos\alpha = 4\,{m_a}^2 + \mathit{Ellip} \, . \tag{66}$$

So we have obtained a *dot-median-Ellip exchange equation* relating objects from 3 different conceptualizations. Namely, sides vectors, medians and the eigenvectors that our 3 triangle matrices share.

B Block and irreducible split of the triangle inequality and the cosine rule

Remark 1 The block split version of the triangle inequality is

$$0 \underset{\text{componentwise}}{\leq} \underline{\underline{\boldsymbol{F}}} \cdot \underline{\boldsymbol{s}} = \underline{\underline{\mathbb{1}}} \cdot \underline{\boldsymbol{s}} - 2\underline{\underline{\mathbb{I}}} \cdot \underline{\boldsymbol{s}}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} .$$

Which reads

$$(a + b + c) \geq 2a \quad \text{and cycles} \ . \tag{67}$$

I.e.

$$a \leq s \ \text{ and cycles} \ . \tag{68}$$

So every side's length is bounded by the semi-perimeter, s .

Remark 2 The block split of the cosine rule is

$$2\underline{\boldsymbol{D}} = \underline{\underline{\boldsymbol{F}}} \cdot \underline{\boldsymbol{S}} = \underline{\underline{\mathbb{1}}} \cdot \underline{\boldsymbol{S}} - 2\underline{\underline{\mathbb{I}}} \cdot \underline{\boldsymbol{S}} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} .$$

Which reads

$$2\underline{\boldsymbol{b}} \cdot \underline{\boldsymbol{c}} = 2bc\cos\alpha = (A + B + C) - 2A = 3R - 2A \quad \text{and cycles} \ . \tag{69}$$

So we now have a *dot-RoG-side*^{2} *exchange equation.*

Remark 3 The irreducible split version of the triangle inequality is

$$0 \underset{\text{componentwise}}{\leq} \underline{\underline{\boldsymbol{F}}} \cdot \underline{\boldsymbol{s}} = \underline{\underline{\mathbb{T}}} \cdot \underline{\boldsymbol{s}} - \underline{\underline{\mathbb{I}}} \cdot \underline{\boldsymbol{s}} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} .$$

Which reads

$$(b + c) \geq a \quad \text{and cycles} \ . \tag{70}$$

Remark 4 The irreducible split of the cosine rule is

$$2\underline{\boldsymbol{D}} = \underline{\underline{\boldsymbol{F}}} \cdot \underline{\boldsymbol{S}} = \underline{\underline{\mathbb{T}}} \cdot \underline{\boldsymbol{S}} - \underline{\underline{\mathbb{I}}} \cdot \underline{\boldsymbol{S}} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} .$$

Which reads

$$2\underline{\boldsymbol{b}} \cdot \underline{\boldsymbol{c}} = 2bc\cos\alpha = (A + B) - C \quad \text{and cycles} \ . \tag{71}$$

End Remark I.e. just the sign splits of the usual triangle inequality and cosine rule. In each of these linear realizations, the split pieces are not themselves cycles, unlike in the Heron formula's quadratic realization of Secs 9, 11 and 14.

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