Euler's Quadrilateral Theorem. II.

The Jacobi-K alias 4-Path counterpart.

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Abstract

We recently associated Apollonius' Theorem and Euler's Quadrilateral Theorem with the 3-path and 3-star = Claw tree graphs. Giving a proof that generalizes to all other eigencluster shapes (in $1:1$ correspondence with the unlabelled tree graphs). For all N-body problems in all dimensions.

We now provide the 4-path counterpart. Corresponding to the Jacobi-K eigencluster shape as opposed to Euler's Theorem's Jacobi-H eigencluster shape.

For the H, separations are supplemented by the Newton line segment, alias crossbar of the H. Whose length is a measure of aparallelogramness. In contrast, for the K, they are supplemented by the spike and the handle of the K: its second and third strokes. These names arise from viewing the K as an axe, with the 3-body subsystem it picks out as the blade. So to Euler's Theorem giving the crossbar length in terms of the separations, our new Theorem relates a sum of squares of the spike and the handle to the separations. This is a quantifier of departure from the central binary coincidence.

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1 Introduction

In [23], we gave a new proof of Euler's Quadrilateral Theorem [2, 11, 9, 10]. Which renders it clear that it is in fact a 4-body problem result (dimension independent). This proof is 'Physically guaranteed to generalize'. For it is built out of centre of mass (CoM) and moment of inertia (MoI) or radius-of-gyration (RoG) considerations.

[23] argues furthermore that my slightly earlier new proof of Apollonius' Theorem [21] is the P_3 version to [23]'s S_3 = Claw version of a working that holds for the following. *Any eigencluster shape for any point-or-particle number N in any dimension d.* This is the full extent to which I am aware that the above 'Physically guaranteed to generalize' applies.

Eigencluster shape is usually referred to in terms of types of Jacobi coordinates. Which are unlabelled tree graph valued [5, 16]. By which the 4-body problem is minimum for it to become ambiguous. Where the 4-path graph P_4 versus 3-star S_3 = Claw ambiguity manifests. Which corresponds to the Jacobi-K versus Jacobi-H coordinates ambiguity [3, 6, 8, 12, 15, 16] at the level of the 4-Body Problem. So Euler's Quadrilateral Theorem (and 4-body Theorem) can be viewed in terms of Jacobi- H coordinates.

It thus remains to show what further trees give in place of Apollonius' median-length Theorem and Euler's Aparallelogramness Theorem.[1] In the current Note, we do so for the 4-path tree P_4, which corresponds to the Jacobi-K coordinates' eigencluster shape. See Fig 1 for the 3 Jacobi coordinate systems, eigencluster shapes and trees mentioned in this Introduction. Sec 2 gets what we need about the Jacobi-K coordinates. Sec 3 then parallels [23]'s proof. Appendix A provides some supporting Linear Algebra.

[24] subsequently provide further examples of such Theorems.

[1] See [21], [23] and Fig 2 for discussion of these truer-name qualifiers.

2 Jacobi K-coordinates

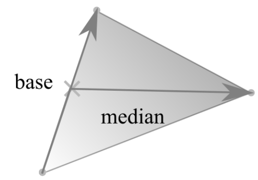
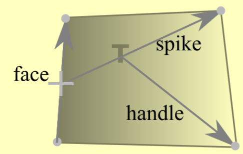
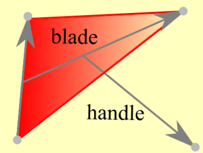
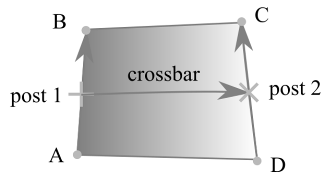
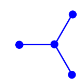
N	Clus-tering	Partition structure	a) Jacobi vectors	b) The axe conceptualization	c) Corresponding tree graphs
3	(T)	(1 + 1) + 1			 3-path P_3
4	K	((1 + 1) + 1) + 1			 4-path P_4
	H	(1 + 1) + (1 + 1)			 Claw alias 3-star S_3

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Figure 1:

Structure 1 Consider the 4-body problem with equal masses. Here, the relative-separation-diagonalizing *relative Jacobi vectors* [3, 6, 8, 12, 15, 16] alias *eigencluster vectors* [22, 23] can be chosen in the following K-shape [see also Fig 1.4.K)]. The $\underline{\boldsymbol{q}}$ are position coordinate vectors for our 4 points-or-particles.

$$\begin{array}{lclclcl} \text{edge} & : & \underline{\boldsymbol{R}}_g & := & \underline{\boldsymbol{q}}^B & - & \underline{\boldsymbol{q}}^A \\ \text{spike} & : & \underline{\boldsymbol{R}}_p & := & \underline{\boldsymbol{q}}^C & - & \frac{1}{2}\left(\underline{\boldsymbol{q}}^A + \underline{\boldsymbol{q}}^B\right) \\ \text{handle} & : & \underline{\boldsymbol{R}}_h & := & \underline{\boldsymbol{q}}^D & - & \frac{1}{3}\left(\underline{\boldsymbol{q}}^A + \underline{\boldsymbol{q}}^B + \underline{\boldsymbol{q}}^C\right) \\ . \end{array} \quad (1)$$

With corresponding Jacobi masses (a subcase of reduced masses) as follows.

$$\begin{array}{lcccccccccc} \text{edge mass} & : & \frac{1}{\mu_g} & = & \frac{1}{1} + \frac{1}{1} & = & 2 & \Rightarrow & \mu_g & = & \frac{1}{2} \\ \text{spike mass} & : & \frac{1}{\mu_p} & = & \frac{1}{2} + \frac{1}{1} & = & \frac{3}{2} & \Rightarrow & \mu_p & = & \frac{2}{3} \\ \text{handle mass} & : & \frac{1}{\mu_h} & = & \frac{1}{3} + \frac{1}{1} & = & \frac{4}{3} & \Rightarrow & \mu_h & = & \frac{3}{4} \end{array} \quad . \quad (2)$$

♠ ◇ ♣

Naming Remark 1 We name the the Jacobi-K's 3 strokes as follows. This is from the axe conceptualization that Kneller and I exhibited in [13], as per Fig 1.b).[2] The *edge* is the leading face of the axe – one of the 6 separations supported by the quadrilateral. The *spike* is the thickness co-transversal [25] from the CoM of the edge to the third point-or-particle. Which 3-body subsystem constitutes the blade of the axe. Finally, the *handle* is the co-transversal from the blade's triple CoM T to the final point-or-particle.

Notational Remark 1 Let us denote *face length*, *spike length* and *handle length* by

$$g \ , \ \ p \ , \ \ h \ .$$

This reflects that f is already booked as standard notation for one of the quadrilateral's 6 separations. While s is already in play for the totality of separations. So g stands for 'edge', being the only free letter therein. And p is the first free letter in 'spike'.

Remark 1 So to the quadrilateral's squared-length variables given in [23], we now add the following.

Definition 2 The $(\text{spike length})^2$

$$P \ := \ p^2 \ .$$

The $(\text{handle length})^2$

$$H \ := \ h^2 \ .$$

Of course, we call whichever $(\text{separation})^2$ that we allot leading edge to the $(\text{edge length})^2$

$$G \ := \ g^2 \ .$$

Definition 3 The corresponding *partial moments of inertia (MoI)* are, in order along the K's P_4 path's edges,

$$\iota_G \ , \ \ \iota_P \ , \ \ \iota_H \ .$$

Notational Remark 2 The 4-body problem supports

$$\binom{4}{2} \ = \ 6$$

choices of leading edge. And then

$$\binom{2}{1} \ = \ 2$$

choices to complete the blade 3-subsystem. Then everything is fixed. So there are

$$6 \ \times \ 2 \ = \ 12$$

possible labellings of Jacobi K-coordinates.

Which we index by $(\, S\, T\,)$. Or by $(\mathrm{K}\, S\, T\,)$ is Jacobi H's are also in play. For S an index running over separations. And T a quite rare occurrence of a 2-index in the theory of the 4-body problem.

[2] Though there we used face, thickness and handle.

3 The Jacobi-K 4-Body Theorem

Theorem 1 [Anderson 2018] For p and h the spike and handle strokes of a Jacobi- K with the vertex separation $g = a$ as leading edge, the following hold.

a) **('Euler–Jacobi–Jacobi-K')**

$$2 \, (\iota_P + \iota_H) = - \iota_A + \iota_B + \iota_C + \iota_D + \iota_E + \iota_F \tag{3}$$

b) **('Euler–Jacobi-K')**

$$\frac{2}{3} P + \frac{3}{4} H = \frac{1}{4} (\text{-}A + B + C + D + E + F) \, . \tag{4}$$

<u>Proof</u> The below refers to the method in Fig 2 of Article I.

Our First Principle is the partial MoI expansion of the MoI in Jacobi K-coordinates. I.e.

$$\iota = \iota_a + \iota_p + \iota_h \, . \tag{5}$$

Our Second Principle is the (Lagrange! [22]) democratic RoG formula (B) in Fig 2 of Article I.

a) Substitute the Greek (B) in (5), cancel terms and multiply by 2 .

b) Insert eq. (2)'s Jacobi masses to return to the Latin world. $\Box$

Remark 1 This form of b) directly exhibits the Jacobi masses involved ('manifest Jacobi masses form'). Multiplying both sides by 2 casts the left-hand side in terms of the Jacobi mass ratios.

Remark 2 b) can also be written more neatly as follows.

b′) (Rational form)

$$8 \, P + 9 \, H = 3 \, (\text{-}A + B + C + D + E + F) \, . \tag{6}$$

For all that this obscures that Jacobi masses are in play.

Remark 3 In the original variables of the quadrilateral, we have the following.

c)

$$\frac{2}{3} p^2 + \frac{3}{4} h^2 = \frac{1}{4} \left(\text{-}a^2 + b^2 + c^2 + d^2 + e^2 + f^2 \right) \, . \tag{7}$$

While the rational counterpart is as follows.

c′)

$$8 \, p^2 + 9 \, h^2 = 3 \left(\text{-}a^2 + b^2 + c^2 + d^2 + e^2 + f^2 \right) \, . \tag{8}$$

Remark 4 In terms of the sum of separations,

a) reads

$$2 \, (\iota_p + \iota_h) = \iota_1 - \iota_g \, . \tag{9}$$

b) reads

$$\frac{2}{3} P + \frac{3}{4} H = \frac{1}{4} (S_1 - G) \, . \tag{10}$$

b′) reads

$$8 \, P + 9 \, H = 3 \, (S_1 - G) \, . \tag{11}$$

c) reads

$$\frac{2}{3} p^2 + \frac{3}{4} h^2 = \frac{1}{4} \left(s_2 - g^2 \right) \, . \tag{12}$$

c′) finally reads

$$8 \, p^2 + 9 \, h^2 = 3 \left(s_2 - g^2 \right) \, . \tag{13}$$

4 Sphynxnopsis

Can you think of a better name for a *Wheelerian comparison table* of *Namings*? (Fig 2)

	Starting to name our family of Theorems		
Historical names albeit in tree lattice order!	Apollonius' Theorem [3rd and 2nd Centuries B.C.]	Theorem 1 [Anderson 2018]	Euler's Quadrilateral Theorem [1748]
Jacobi alias eigencluster shape names	Jacobi-(T) 3-Body Theorem alias (T)-eigencluster 3-Body Theorem	Jacobi-K 4-Body Theorem alias K-eigencluster 4-Body Theorem	Jacobi-H 4-Body Theorem alias H-eigencluster 4-Body Theorem
Functional computation names	Median-Length Theorem	Spike-and-handle length Theorem	Crossbar-length Theorem alias Newton-length Theorem, Euler-length Theorem and Newton-Gauss-length Theorem
Names based on higher purpose as quantifiers of deviation	3-Body Anuniformity Theorem	4-Body Acentralbinarycoincidenceness Theorem	Aparallelogramness Theorem
Graph-Theoretic tree-indexed names, with advantage of extending to whole treespace-valued set of Theorems	3-Body 3-Path Theorem alias 3-Body Tree Theorem	4-Path 4-Body Theorem	4-Body 3-Star Theorem alias 4-Body Claw Theorem

Figure 2:

Remark 1 Names for the current Article's Theorem's are highlighted in yellow. The penultimate one is explained in the Appendix.

Riddle 1 We leave what anuniformity may refer to as a riddle for the Reader (for Sphinxes are also Riddlers).

Remark 2 While [10] first envisaged the concept for the name highlighted in red, the name *Aparallelogramness Theorem* arose in discussions between Sánchez and I [18]. This red is accorded for the best name so far for the best theorem in the set. Some highlights are as follows. Since parallelograms have 4 vertices, in this case there is no need to say *4-body*. There are also many ways in which the parallelogram is more interesting than the other configurations which these Theorems quantify deviations from.

It should come as no surprise that Euler's case is the most interesting, and the one permitting the shortest name. Elsewhere Euler founded Affine Geometry as an axiomatization of Parallelism. And also founded the very Graph Theory that indexes our whole family of Theorems! We bow to the master.

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A Linear Algebra of the Euler–Jacobi-K 4-Body Problem Theorem

Remark 1 Theorem 1 – the Jacobi-K 4-body problem Theorem alias everything in column 2 of Fig 2 – is a new result.

So we here re-run the Linear Algebra analysis that we had already conducted upon first mention for the Euler–Jacobi-H 4-Body Problem Theorem alias everything in Column 3 of Fig 2.

Structure 1 The *Jacobi-K-matrix* is

$$\boldsymbol{K} := \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -1 \end{pmatrix} . \tag{14}$$

With this scaling,

$$\underline{\boldsymbol{\kappa}} = \underline{\underline{\boldsymbol{K}}} \cdot \underline{\boldsymbol{\Sigma}} . \tag{15}$$

For *separation partial moments of inertia* 6*-vector* $\underline{\boldsymbol{\Sigma}}$. And *combined spike-and-handle inertia vector* $\boldsymbol{\kappa}$ with components

$$\kappa_S = \iota_{P\,S} + \iota_{H\,S} = {\rho_{p\,S}}^2 + {\rho_{h\,S}}^2 \ \ (\text{ no sum }) \ . \tag{16}$$

Remark 2 Observe that this construct does not use all K-clusters at once. Rather, so as to obtain a square and thus well-determined system, for each inter-vertex separation we pick precisely 1 K which has this as its leading edge.

Structure 2 The corresponding *spike-and-handle-squares* 6$-$*vector* is

$$\boldsymbol{K} = \frac{2}{3}\boldsymbol{P} + \frac{3}{4}\boldsymbol{H} .$$

For (spike length)^2 6-vector $\boldsymbol{P}$ and for (handle length)^2 6-vector $\boldsymbol{H}$. Setting

$$\boldsymbol{K} = \boldsymbol{0}$$

then reveals our Theorem to be a quantifier of deviation from central binary collisions. Hence the penultimate name in column 2 of Fig 2.

Remark 3 Its eigenvalues are 1 with multiplicity 1 and -2 with multiplicity 5 . Its rank is 6 , which is the full rank supported, so the nullity is 0. With reference to [20]'s conceptualization, its Mathematicians' signature is 5 and its Physicists' signature is 4 , and its Physicists' signature-in-detail is $- - - - - +$.

The corresponding eigenvectors can be taken to be

$$\frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\1\\1\\1\\1\end{pmatrix} \ , \quad \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\\0\\0\\0\end{pmatrix} \ , \quad \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\\0\\0\\0\end{pmatrix} \ , \quad \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\0\\1\\0\\0\end{pmatrix} \ , \quad \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\0\\0\\1\\0\end{pmatrix} \ , \quad \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\0\\0\\0\\1\end{pmatrix} \ , \qquad (17)$$

The first is obligatory, while the remainder constitute a convenient pure-ellipticity [17] basis.

Remark 4 $\boldsymbol{K}$ is 6×6 , and invertible as implied by its full rank as can be read off its eigenspectrum. A nice application of this, not possible for the 3×3 'Newton–Euler–Jacobi'-H-matrix, shall be given in a subsequent Article.

Remark 5 We did not present eigentheory, rank or signature in [23] for the following reason. The Jacobi-H cycle matrix just returns the Heron [7, 17] alias fundamental triangle matrix [20]. For which e.g. [20] already provided such an analysis (eigenvalues and eigenvectors first appeared in [17]). [20] furthermore gave 6 technically distinct routes to this matrix within the theory of triangles, two of which each have two distinct conceptualizations. By which the name 'Fundamental Triangle matrix' is well justified.

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