Euler's Quadrilateral Theorem.

I. A brief new Proof that is Physically Guaranteed to Generalize

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Abstract

We give a brief new proof of Euler's Quadrilateral Theorem that is Physically guaranteed to generalize Firstly to the 4-body Problem in arbitrary dimension, as provided here in the context of Jacobi H eigenclusters. Secondly to other point-or-particle numbers N and eigencluster trees. Starting with the Jacobi K eigencluster in Article II and its N-path generalization in Article III.

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1 Introduction

1.1 Euler's Quadrilateral Theorem

Theorem 1 [Euler's Quadrilateral Theorem: convex case] Suppose that we are given a convex quadrilateral with sides a, b, c, d and diagonals e, f . Then

$$a^2 + b^2 + c^2 + d^2 = e^2 + f^2 + 4n^2 .$$

Where n is the *Euler length* between the midpoints of the two diagonals.

Naming Remark 1 The segment formed by n extends to form the *Newton line* [5, 33] later alias *Newton–Gauss line* [8, 10, 11, 28]. By which *Newton length* is a suitable alias for the above Euler length.

Remark 1 Geometrical [7] – including Parallelogram Law [26] and thus Corollary-of-Pythagoras – are standard. As are trigonometric (cosine rule) [26] and vectorial [24] proofs.

Naming Remark 2 The second of these highlights that the Newton length is an *aparallelogramness* [25], i.e. a measure of departure from a parallelogram. Indeed, for a parallelogram, the midpoints of the two diagonals coincide.

1.2 Two sets of sequentially simplifying variables

Notational Remark 1 We refer to the sides collectively as the a_I , $I = 1$ to 4 4-cycle. To the diagonals as the d_D , $D = 1$ to 2 2-cycle. And to the separations – comprising both of these – as the s_S , $S = 1$ to 6 6-cycle.

Definition 1 The quadrilateral's (*separations*)^2 *variables* are

$$A := a^2 \text{ and 6-cycles} .$$

Among which, we refer to the 4 (*sides*)^2 as the A_I . And to the 2 (*diagonals*)^2 as the D_D , Let us finally introduce the (Newton length)^2

$$N := n^2 .$$

Remark 2 Squared variables have already turned out to be useful in the theory of triangles [39, 42, 44, 36].

Corollary 1 (separations)^2 variables presentation)

$$4N = \sum_I A_I - \sum_D D_D . \tag{1}$$

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Definition 2 Whenever there is a sum of equal powers of objects $\mathcal{O}_Z$, the *pth-power sum variable* is

$$\mathcal{O}_p \ := \ \sum_Z \mathcal{O}_Z \ .$$

Notational Remark 2 This is an inequalities and Functional Analysis notational simplifier. We use $p = 1$ exclusively.

Corollary 2 (sum variables presentation)

$$4\,N \ = \ A_1 \ - \ D_1 \ . \tag{2}$$

1.3 Eulerian 4-Body Problem Theorem

Remark 3 While usually stated for convex quadrilaterals, Euler's Theorem in fact holds true in a much wider range of contexts. This includes for arbitrary, rather than convex, quadrilaterals: **Theorem 1*** **[Euler 1748]** [7]. And indeed for complete quadrilaterals, for which it is not specified which of the 6 separations are to be sides or diagonals. And also for tetrahaedrons [24]. It can consequently be taken to be a result about *constellations* [34] of 4 points, whether in 2-*d* or otherwise.

This means that this result's conceptual class (here meaning arena of validity [38])has expanded to the 4-body problem [16, 17, 20, 21]. It is furthermore then also compatible with Kendall's Shape Theory [14, 19, 22, 32, 34, 36, 37, 54]. And thus we have arrived at the following truer name.

Theorem 1′ [Eulerian 4-body Problem Theorem: cyclic-pair formulation] For opposite-(separations)^2 pairs (A, C) , (B, D) and (E, F) ,

$$4\,N \ = \ A \ + \ C \ + \ B \ + \ D \ - \ E \ - \ F \ \text{ and opposite-pairs 3-cycles} \ . \tag{3}$$

Remark 4 I.e. the following also hold.

$$4\,L \ = \ B \ + \ D \ + \ E \ + \ F \ - \ A \ - \ C \ . \tag{4}$$

$$4\,M \ = \ E \ + \ F \ + \ A \ + \ C \ - \ B \ - \ D \ . \tag{5}$$

Introducing 2 further *transversals* [11] L and M that complement the (Newton length)^2 to form a further opposite-pairs 3-cycle of quantities.

Remark 5 Opposite-(separations)^2 pairs are indeed well-defined for a quadruple of points in arbitrary dimension. As those pairs of separations which share no vertices. I.e. the 3 pairs of 2-path graphs P_2 supported.

Naming Remark 3 Which a Particle Physicist would call the ' s , t and u *channels*' [18] on which the *Mandelstam variables* are based. The connection being 4-point functions to 4-body problem vertices.

Notational Remark 3 We refer to (3, 4, 5) as the (1) , (2) and (3) cycle choices respectively. Collectively denoted by (a) .

Remark 6 Sides-diagonals splits are *not* invariant under 3-cycles. But a different such split can be made for each 3-cycle.

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Remark 7 The above cyclic triple of equations can be further condensed using Linear Algebra. Though perhaps less obviously, doing so with a $(\text{sides})^2$-pairs-to-Newton-lengths 3×3 matrix is more useful than with a $(\text{sides})^2$-to-Newton-lengths 6×3 matrix [53].

Theorem 1″ [Eulerian 4-Body Problem Theorem (cyclic Linear Algebra form)

$$\overline{\boldsymbol{T}} = \underline{\overline{\boldsymbol{E}}} \cdot \overline{\boldsymbol{O}} . \tag{6}$$

For *tetra-transversal* $(\,length\,)^2$ *vector*

$$\boldsymbol{T} := 4 \begin{pmatrix} N \\ M \\ L \end{pmatrix} . \tag{7}$$

Opposite $(\,sides\,)^2$ *sum vector*

$$\boldsymbol{O} := \begin{pmatrix} A + C \\ B + D \\ E + F \end{pmatrix} . \tag{8}$$

And *Euler matrix*

$$\boldsymbol{E} := \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} . \tag{9}$$

Remark 8 We furthermore recognize $\boldsymbol{E}$ as the 'fundamental triangle matrix' $\boldsymbol{F}$ [42, 46, 47]. That arises not only in Heron's formula but also from the cycle of cosine rules and even the cycle of triangle inequalities. Thus we have found a *ninth* route to this citizen of Kallista ([42, 47] document the above three and five more).

1.4 Motivation

Remark 9 The level of generality of the Eulerian 4-body Problem Theorem caused us to juxtapose thinking about it and the following.

Theorem 2 [Varignon's Quadrilateral Theorem, 1731] [6, 13] Given a quadrilateral, the midpoints of its sides form a parallelogram whose area is half that of the original figure.

Motivation 1 For this result also readily extends to other spatial dimensions. In the Varignon case, the reason is clear: there is a Physical proof based on the notion of centre of mass (CoM). Might then Euler's Quadrilateral Theorem also have a not only convexity-insensitive but also dimension-insensitive Physical proof? It does. Thus providing a first raison d'être for the current Note (Sec 3, after a preamble on Jacobi-H coordinates in Sec 2). See the Conclusion for yet further motivation.

2 Jacobi H-coordinates

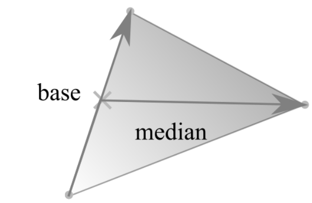
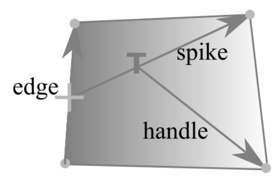
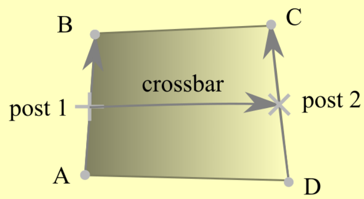
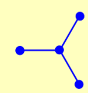
N	Clus-tering	Partition structure	a) Jacobi vectors	b) Corresponding tree graphs
3	(T)	(1 + 1) + 1		3-path P_3
4	K	((1 + 1) + 1) + 1		4-path P_4
	H	(1 + 1) + (1 + 1)		 Claw alias 3-star S_3

Figure 1:

Structure 1 Consider the 4-body problem with equal masses. Here, the relative-separation-diagonalizing *relative Jacobi vectors* [9, 16, 20, 29, 34, 37] alias *eigencluster vectors* [45] can be chosen in the following H-shape [see also Fig 1.4.H)].

$$\begin{array}{lcll}
\text{post 1} & : & \overline{\boldsymbol{R}}_1 := \overline{\boldsymbol{q}}^B - \overline{\boldsymbol{q}}^A & \\
\text{post 2} & : & \overline{\boldsymbol{R}}_2 := \overline{\boldsymbol{q}}^D - \overline{\boldsymbol{q}}^C & \\
\text{crossbar} & : & \overline{\boldsymbol{R}}_3 := \frac{\overline{\boldsymbol{q}}^C + \overline{\boldsymbol{q}}^D}{2} - \frac{\overline{\boldsymbol{q}}^A + \overline{\boldsymbol{q}}^B}{2} & = \frac{1}{2}\left(\overline{\boldsymbol{q}}^C + \overline{\boldsymbol{q}}^D - \overline{\boldsymbol{q}}^A - \overline{\boldsymbol{q}}^B\right) .
\end{array} \tag{10}$$

With corresponding *Jacobi* alias *cluster-hierarchy masses* (a subcase of reduced masses) as follows.

$$\begin{array}{lcl}
\text{post masses} & : & \frac{1}{\mu_1} = \frac{1}{\mu_2} = \frac{1}{1} + \frac{1}{1} = 2 \Rightarrow \mu_1 = \frac{1}{2} = \mu_2 \\
\text{crossbar mass} & : & \frac{1}{\mu_3} = \frac{1}{2} + \frac{1}{2} = 1 \Rightarrow \mu_3 = 1 .
\end{array} \tag{11}$$

Notational Remark 1 In the present context, Sec 1.3's cycle choices (a) are specifically H-clustering labelling choices $(\mathrm{H}\, a)$. Though we still use (a) as shorthand for this. The above is then, in further detail, the $(\mathrm{H}\,3) = (\mathrm{H}\, s)$ cluster, i.e. with the s-channel choice of pairing.

Structure 2 The *mass-weighted (mw) Jacobi coordinates* are then

$$\overline{\boldsymbol{\rho}}_i := \sqrt{\mu_i}\,\overline{\boldsymbol{R}}_i \, , \quad i = 1 \text{ to } 3 . \tag{12}$$

Remark 1 We do not require their explicit forms.

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Definition 1 The *partial moments of inertia (MoI)* are

$$\iota_i \ := \ {\rho_i}^2 \quad \text{(no sum)} \ . \tag{13}$$

I.e. the sums of the magnitudes of the previous. The *(total) MoI* ι is the sum of the partial MoI. Finally, the *radius of gyration* (RoG) r is the mass-unweighted counterpart of the MoI. I.e. such that

$$I \ = \ m\,R \ . \tag{14}$$

For

$$m \ := \ \sum_{I \, = \, 1}^{4} m_I \tag{15}$$

the *total mass* of the system ($= 4$ for us if our equal masses are taken to be unit masses). And

$$R \ := \ r^2 \ , \tag{16}$$

so as to have everything in squared variables.

Naming Remark 1 The Newton length alias Euler length, recurs as the *crossbar-length* in the Jacobi-H coordinates. 'Posts' and 'crossbar' being with reference to a set of Rugby goalposts [30]. This conceptualization is furthermore framework-deformable [41] and democratically valid [20]: for whichever cluster labelling. By which it applies to the whole cycle of 3 opposite-side pairs.

Naming Remark 2 *Transversals* [11] are in various senses a truer name for the role played by Newton lines in the current work. This is ultimately a projective notion, however, and thus has no metric content. So we need to specify something like 'halving transversals' *H-transversals* is quite a nice name for these within the specific 4-body problem context. Where H now refers both to the H-shaped eigencluster and to halving. But what if arbitrary-mass H-clusters come into play? Then ' H-medians', where H has reverted to just meaning the eigencluster and 'medians' deputizes for halving, may be more prudent...

Remark 2 The 4-body problem's Jacobi H or K ambiguity is rooted in $N = 4$ vertices being minimum for tree graphs to be nonunique. As per Fig 2.b). Subsequent cluster-shape ambiguities in the N-body Problem remain labelled by [15] $\mathfrak{Tree}$. I.e. the arena of unlabelled trees on arbitrary vertex number [40]. See [37] for cluster-shape detail up to $N = 6$.

3 Physical proof of Eulerian 4-Body Problem Theorem

3.1 The proof

Proof Our First Principle (A) is the partial MoI expansion of the MoI in Jacobi H-coordinates.

Our Second Principle (B) is the (Lagrange! [45]) democratic RoG formula [45].

Squared variables and sum variables sequentially save us symbols.

Latin $\longleftrightarrow$ Greek 'translations' (inter-conversions) must always remember to (un)deploy Jacobi mw factors.

ι_1 , α_1 , Δ_1 and whichever of their Latin counterparts feature are implicitly for the same $(a) = (\mathrm{H}\,a)$ cluster-labelling choice.

Everything else is in Fig 2.

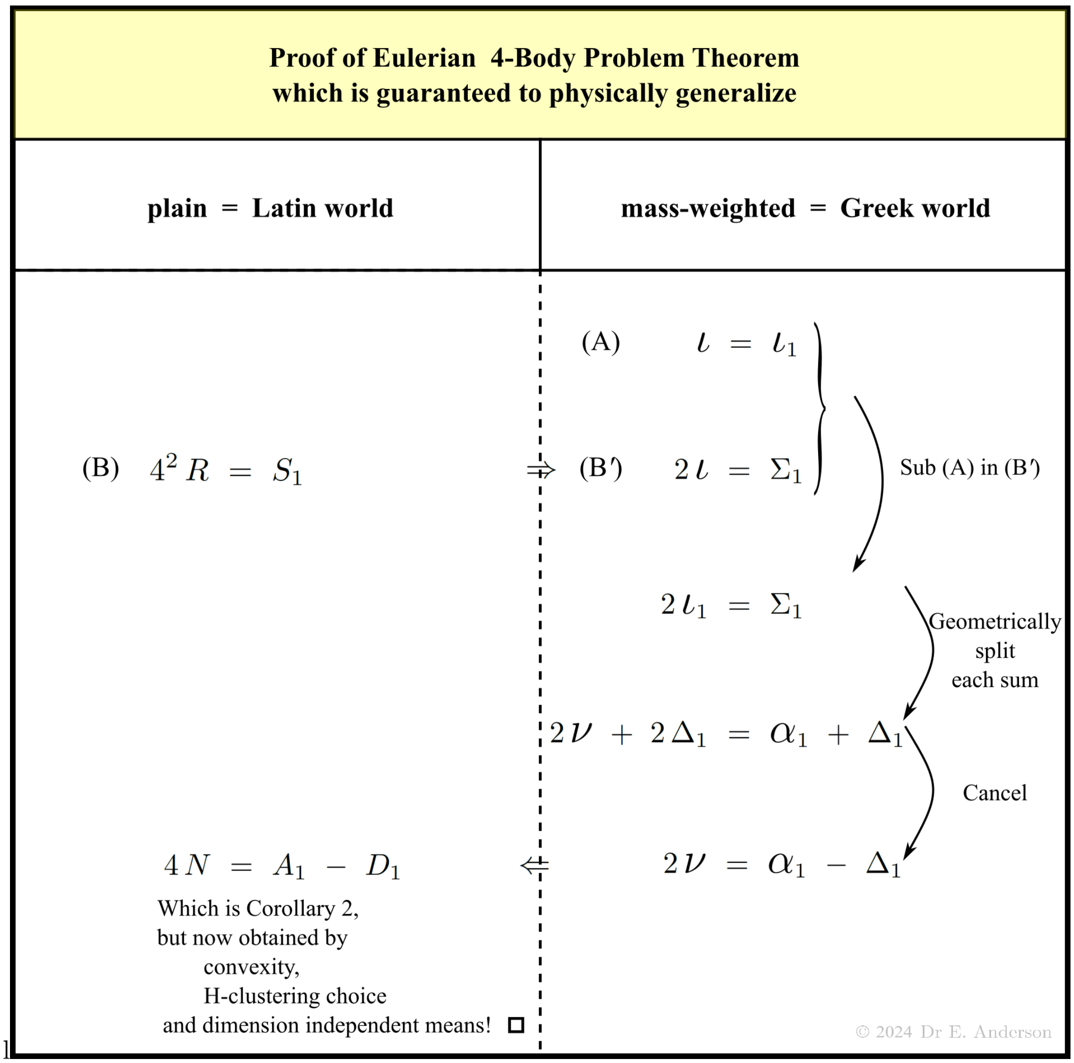

Figure 2:

3.2 Discussion

Question 1 What is the conceptual content of the factor of 4 in the Theorem?

Remark 1 From the mw Jacobi point of view, one factor of 2 is the above-mentioned ratio of Jacobi masses. While the other is from the mw Jacobi version of the RoG Lemma (B).

Remark 2 However, from a sides-data point of view – be it geometrical or Lagrangian – the simplification that the whole 4 then comes from the RoG Lemma occurs. So it is the ratio of the total mass of the system to the mass of each constituent particle. All the while resting on our assumption of equal particle masses.

Remark 3 Finally, returning to the mw Jacobi point of view, the second 2 is thus one half of the ratio of the total mass of the system to the mass of each constituent particle.

Remark 4 The average MoI of a transversal line segment is

$$\langle \mathcal{V} \rangle = \frac{\iota}{3}$$

While the average MoI of an inter-particle separation is also

$$\langle \Sigma \rangle = \frac{\iota}{3} .$$

There are however half as many Newton lines as separations, so each on average contributes 2 times as much. This gives another way of accounting for the second factor of 2 from the mw Jacobi point of view. In which guise it is a ratio of combinatorial counts rather than a ratio built from cluster-democratic Statics.

4 Summary and and outlook

Summary We have provided a new Physical proof for Euler's Quadrilateral Theorem. This uses, firstly, the centre of mass (CoM) family of concepts, including the mass-weighted Jacobi H-coordinates as very familiar from the N-body Problem literature [16, 20, 37]. Whose crossbar length [30] is a third incarnation of the Newton length and the Euler length. Secondly, the moment of inertia (MoI) and radius of gyration family of concepts. But these first and second moments notions manifestly extend to non-convex quadrilaterals, complete quadrilaterals, tetrahaedrons and 4 points on the line [27, 35].

Motivation 2 Our proof is more effective than constructs requiring multiple cases, or the trigonometric proof, which involves 5 uses of the cosine rule. It also includes the 1-d case, which Pythagoras based proofs leave out. It is comparable to the vectorial proof; we give a more detailed comparison between the two in [52].

Motivation 3 and Pointer 1 Both Apollonius's Theorem [1] for median lengths and Euler's Theorem for H-median lengths can be envisaged as Corollaries of [31, 54] Pythagoras' Theorem. And results rooted in CoM-and-MoI considerations [43, 54]. While the latter extend any vertex number, *all cluster shapes and dimensions* (including to 1-d [48], where Pythagoras' Theorem ceases to be contentful). This is the 'fanfare' alluded to in [43]. Whose consequent first few further representative Theorems are for the cases announced in our Abstract in [49, 50]. Some more are in [51, 52].

By this stage, the conceptual class is the N-body Problem. So we have found a $\mathfrak{T}$ree-*indexed family of Eulerian N-Body Problem Theorems* [38]. For unlabelled trees on arbitrary numbers of vertices. Among which Apollonius is the P_3 and minimum nontrivial statement. While Euler is the S_3 alias Claw : the minimum non-path tree. [49] gives the P_4 case and [50] the P_n case.

Remark 1 Contrasting with Motivation 1's Varignon's Theorem, there is precisely 1 Varignon Theorem per even-N-body problem. While the odd-N body problems each have precisely 1 Median Concurrency Theorem. Whose smallest nontrivial instance is *the Triangle Medians' Concurrency Theorem.*

Naming Remark 1 [*Transversal*] and the more metrically specific notion of *median* remain useful notions in this greatly enlarged conceptual class. Where [ ] indicates the *plain-or-dual portmanteau* [40]. Which allows for how e.g. the usual notion of median is a subspecies of Cevian [3, 4, 12, 13, 23, 31], which is a co-transversal. Triangle transversals being, rather, Menelians! [2, 12, 13, 23, 31]. [Transversal] is thus our final truer name for today.

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