Dirac's Algorithm for Constrained Systems from the perspective of Order Theory

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Abstract

Dirac based his theory of constrained systems on Linear Algebra foundations. It is a brackets-algebraic consistency procedure with multiple outcomes, including new constraints dropping out and redeclaring brackets becoming necessary (Dirac brackets). This procedure has not yet been edited, however, to caution about and remove scaffolding structures that turn out to not in general be brackets-algebraically consistent. We perform this task here.

Our main innovation is moreover demonstration that substantial progress can be made from placing Dirac's Algorithm on Linear Algebra *and* Order Theory foundations. For chains, lattices, posets, and digraphs abound therein For instance, in the simpler version of the algorithm, its iterations form a chain, of which its Dirac brackets updating steps are a subchain. Its consistent algebraic structures, meaningful notions of weak equality, of appended Hamiltonians and of observables form bounded lattices. Many key notions – such as Dirac's extended Hamiltonian or Dirac observables – are identified as extrema of these lattices, cementing their permanence. Others are however revealed to be but simplest examples of middles. By this, e.g. Kuchař observables are in general to be replaced by a poset of algebraicially-consistent middling A-observables.

In the harder – path dependent – (previously called bifurcating or branching) version of the algorithm, moreover, iterations and Dirac brackets types become digraph-valued. What previously was a lattice of consistent constraint subalgebraic structures now becomes a competing lattice, described overall by a semi-lattice, with weak equalities, appended Hamiltonians and observables following suit. Order Theory conceptualization thus *remains both lucid and under control* within this harder case.

Such Order-Theoretic considerations furthermore transcend to extended variants and to Temporal Relationalism implementing variants. And to the *Generalized Lie Algorithm*: a vast generalization of scope in which to apply Dirac's insights from constrained dynamical systems *to wherever Lie Theory is applicable*.

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1 Introduction

We reconsider *Dirac's Algorithm* [10, 16, 19, 32, 34, 47, 72, 78, 90, 94] for constrained systems [8, 11, 33] from the joint points of view of not just *Brackets-Algebraic consistency* but Order Theory [56, 64, 98] as well.[1] While this does not include all possible types of constraints (Sec 2), Dirac's modelling assumptions (Sec 3) cover many of those constraints which arise in Fundamental [18, 19, 47] and in particular in Background-Independent [48, 49, 63, 72, 89, 90, 93, 94] Physics. In common with various of his other best works [6, 7], Dirac grounded his Algorithm on Linear-Algebraic foundations. Given a set of constraints of this type, Dirac *formed Poisson brackets* between them to see *whether* they are consistent. Any combination of six outcomes ensue [19, 47, 90, 94], iteration by iteration. This includes production of *new constraints as integrabilities*, as well as replacing the incipient Poisson brackets by *Dirac brackets* [10, 19].

1.1 Local treatment

We first consider Dirac's Little Algorithm (Sec 4). This covers four possible outcomes and is based on the first 37 pages of [19]. Interpreting this from an Order-Theoretic point of view, the iterations form a chain. (n)-ary (and the cumulative $[n]$-ary) constraints are introduced as generalizations of Anderson and Bergmann's [11] primary to secondary distinction between constraints.

Good and bad termination conditions for this chain are discussed in Sec 5, by which some candidate theories pass while others fail. Dirac-type Algorithms are powerful through acting as *Selection Principles* in this way. Consistent theories' first-class constraints $\mathscr{F}\mathrm{irst}$ possess bounded lattices $\mathfrak{L}_{\mathscr{F}\mathrm{irst}}$ of consistent subalgebraic structures $\mathfrak{C}$SAS. (n)-ary constraints are not necessarily among these, and thus fade into insignificance in the realm of Algebra.

Dirac's weak notion of equality – up to linear functions of constraints – can be specialized constraint type by constraint type. So can Dirac's multiplier-appended Hamiltonians. But the only algebraically-meaningful such are those associated with constraint subalgebraic structures, so these two notions form bounded lattices isomorphic to $\mathfrak{L}_{\mathscr{F}\mathrm{irst}}$.

[1] Terms associated with Order Theory are picked out by underlining.

Finally, algebraically-meaningful notions of observables $\mathcal{O}$ form the corresponding dual lattice $\mathfrak{L}_{\text{Obs}}$ [71, 74, 75] (anti-isomorphic to $\mathfrak{L}_{\mathcal{F}\text{irst}}$, meaning with order reversed).

Survivors so far include a modification of *Dirac's extended Hamiltonian*, and *Dirac observables*; these are picked out as exceptionally significant by being the nontrivial terminal elements of their corresponding lattices. Casualties so far include that theories do not in general have Dirac's primed, starred or total Hamiltonians be algebraically meaningful. Nor do they in general possess Kuchař's notion of observables, which corresponds to GR's constraints blockwise-forming the 3-chain with this notion as its middle. This simple case of middle is in general replaced by the poset middle of our dual lattice: the notions of *A-observables* [67, 71, 72].

We next consider Dirac's Full Local Algorithm, meaning including the possibility of second-class constraints and thus of Dirac brackets used to eliminate these. This corresponds to the second half of Chapter 2 in [19]. We introduce the term *so-far-first-class* to cover the following subtlety. That as new constraints are discovered iteration by iteration, already-known constraints may cease to pass the first-class test of closing under Poisson brackets with the others. Using that first-class constraints kill one phase space pair of degrees of freedom each while second-class ones only kill one-half, we also rename these as *one-killer* $\mathcal{O}$**ne** and *half-killer* $\mathcal{H}$**alf** constraints respectively.

From an Order-Theoretic point of view, the iterations producing half-killer constraints and thus prompting notions of Dirac brackets form a subchain of the chain of iterations. Thus there is in general not just one notion of Dirac bracket but an *isomorphic subchain* of such. This generalizes Dirac's 2-chain of incipient Poisson bracket followed by a Dirac bracket. Our top element is the Dirac bracket corresponding to removing all half-killer constraints found up to the point of termination. By the projective nature of the Dirac bracket – projecting out irreducible half-killer constraints – removing these iteration by iteration coincides with the operation of removing all of these in one go.

1.2 Global treatment

This subject manifests two global directions in particular.

1) Brackets algebras produce **anomaly terms**: obstructions. Some are fatal (another bad termination condition). Some are not. And some can be eliminated by fixing constants associated with a given theory: a case of strong vanishing. The underlying type of globality here is *characteristic classes* [46].

2) **Path dependence**:

$$\{ F, G \} = C \times D \tag{1}$$

has $C = 0$ and $D = 0$ as distinct developable options. In earlier literature, this has previously been termed 'bifurcation' [47] and 'branching' [76]; [94] explains why path dependence is a truer name. By this, what were hitherto chains of iterations (and subchains thereof such as that of Dirac brackets) become digraphs. Path dependence's global connotations include diversity in *cohomology* and position-dependent-object generalizations of fibration (as per [94] e.g. *general bundles* [57, 52] or *copresheaves* [69]).

Different path choices can furthermore manifest different termination conditions. Within a single given model, those with good termination conditions can differ as regards which constraint subalgebraic structure of one-killer constraints they produce. This is a scenario in which competing lattices emerge. Taking all of these at once, the overall structure is a bounded semi-lattice. This corresponds to joint inclusion of certain generators not being possible by algebraic incompatibility. A simple and well-known paradigm for this is how in flat space affine (and then projective) generators are incompatible with conformal ones.

1.3 Scope of this work

Our innovation of looking at Dirac's Theory of constrained systems from an Order-Theoretic point of view is well-vindicated by the above-listed assortment of Order-Theoretic structures found therein. *We strongly suggest that all viable future treatises on Dirac's Theory of constrained systems shall be placed on joint Order-Theoretic and Linear-Algebraic foundations.*

2 Some preliminary notions of constraint

2.1 Whether or not constraints are time-dependent

Early studies of *constraints* in Mechanics often assumed these to be of the form

$$\mathcal{c}(\boldsymbol{Q}) \;=\; 0 \;. \tag{2}$$

Boltzmann coined the name *scleronomic constraints* for these. This is in contrast with *rheonomic constraints* [33, 8],[2]

$$\mathcal{c}(\boldsymbol{Q}, t) \;=\; 0 \;, \tag{3}$$

which have such as dissipative-system and irreversible-phenomenon modelling applications.

Once in the phase space setting, these generalize to

$$\mathcal{c}(\boldsymbol{Q}, \boldsymbol{P}) \;=\; 0 \tag{4}$$

and

$$\mathcal{c}(\boldsymbol{Q}, \boldsymbol{P}, t) \;=\; 0 \;. \tag{5}$$

We just call (2, 4) and (3, 5) *time-independent* and *time-dependent* respectively.

Notation 1 It has also become customary [33] to write $\boldsymbol{Q}, \boldsymbol{P}$ jointly as $\boldsymbol{\eta}$: *phase space coordinates* alias *symplectic coordinates.*

Notation 2 We pick out state objects, such as configurations $\boldsymbol{Q}$ and momenta $\boldsymbol{P}$ (and consequently $\boldsymbol{\eta}$) in copper. We emphasize constraints by use of leading small calligraphic font and, for now, highlighting in ruby. This notation is part of a larger system [92] that rather clarifies the Nature of Dynamical Law [93, 94] as locally being made in [85, 89, 90, 91] the image of Lie Theory.

2.2 Whether or not constraints are integrable equalities

Prior to the widespread advent of working in phase space, Routh and then Chaplygin considered

$$\mathcal{c}\left(\boldsymbol{Q}\,,\;\dot{\boldsymbol{Q}}\right) \;=\; 0 \;\; \text{or} \;\; \underline{\underline{\boldsymbol{A}}} \cdot \mathrm{d}\underline{\boldsymbol{Q}} \;+\; \underline{\boldsymbol{B}}\mathrm{d}t \;=\; 0 \tag{6}$$

such that both (2) is returned, or non-integrability prevents this. Hertz coined the names *holonomic* and *anholonomic constraints* for these two cases. Anholonomic constraints occur in models as simple as a rolling rigid object [33]. By now we know that holonomic integrability, or otherwise, is controlled by a Pfaffian [37] diagnostic. We also have (Berry-type [41, 59]) phase consequences and robotically-relevant examples, for which the falling cat is a charming toy model.

The above designation works just as well with or without explicit time dependence (the case without rendering the DE autonomous). It also works just as well with or without momentum dependence. Noting that in the case with momentum dependence, the extra complication of acceleration-dependent

$$\dot{\boldsymbol{P}} \;\sim\; \ddot{\boldsymbol{Q}} \;, \tag{7}$$

rather than just velocity-dependent, constraints is present.

Another longstanding possibility is that constraints are not equalities but inequalities,

$$\mathcal{c}\left(\boldsymbol{Q}\right) \;\geq\; 0 \;, \tag{8}$$

Marbles rolling down balls, or rolling around inside hollow balls, or a pendulum's string ceasing to be taut, are simple examples [33].

Again, this consideration – equality versus inequality constraints – composes with all of the above conceptualizations. We thus have a total of 40 notions of constraint.

[2] We assume that the Reader is familiar with the Principles of Dynamics to the level covered by these books, and with one or both of Chapters 1-2 of [19] and Chapters 1-4 of [47].

3 Constraints in Fundamental and Background-Independent Physics

3.1 Modelling assumptions

Remark 1 Among these, most occurrences in Fundamental and Background-Independent Physics are of the form (2) or (4), whether holonomic or not.

Remark 2 At a greater level of generality, we allow for (3, 5) to involve times other than the original conception's t^{Newton} Much of Fundamental Physics, and Background-Independent Physics in particular, contains no primary notion of time for the Universe as a whole [48, 49, 72]. We thus stay away from (3, 5). This does not affect (an)holonomic possibilities, since these can be phrased in terms of

$$\boldsymbol{c}\,(\boldsymbol{Q}, \mathrm{d}\boldsymbol{Q}) \;=\; 0 \tag{9}$$

This is provided that $\boldsymbol{c}$ is

$$\textit{homogeneous} \text{ in } \; \mathrm{d}\boldsymbol{Q} \tag{10}$$

(and thus is

$$\text{homogeneous in } \; \dot{\boldsymbol{Q}} \;=\; \mathrm{d}\boldsymbol{Q}/\mathrm{d}t \tag{11}$$

when meaningful). This is a common case in Fundamental Physics. Then the $\underline{\boldsymbol{B}}\,\mathrm{d}t$ term in (6)* drops out. Thus we cut down to eight possibilities: (2, 4) (an)holonomic (in)equality constraints.

Remark 3 Fundamental dynamical considerations tend moreover to lack in configurational region restrictions, halving this. Dirac then made one modelling assumption: that just equality constraints are involved. This is not entirely accurate for a full theory of Fundamental Physics [43, 44, 72]. For instance, spatial metric positivity in GR – the determinant condition

$$|h| \;>\; 0 \tag{12}$$

– is a such. Also,

$$\textit{Area} \;>\; 0 \tag{13}$$

a model analogue thereof in some 3-body problems. Finally, Dirac neither assumed nor did anything as regards the type of integrability issue that discerns between holonomic and anholonomic constraints.

Remark 4 We are thus left contemplating 2 of the above 40 possibilities: (3) with integrability unmentioned. This is what we mean by constraints throughout the rest of the current Article, bar a single well-signposted comment otherwise. A plethora of further more specialized notions of constraints ensue. The rest of the current Article assesses these, in particular by adjoining Order-Theoretic considerations to long-standing Linear-Algebraic ones.

3.2 A somewhat unexpected subtlety

In the Principles of Dynamics [33, 8, 31], passage from the *Lagrangian formulation* – $\boldsymbol{Q}, \dot{\boldsymbol{Q}}$ variables – to the *Hamiltonian formulation* – $\boldsymbol{Q}, \boldsymbol{P}$ variables – is widely known to consist of applying a *Legendre transformation* [33, 17, 31].

Structure 1 It is rather less widely known that [19, 47] this passage can be nontrivial. For the corresponding *Legendre matrix*

$$\underline{\underline{TMV}} \;\overset{T}{=}\; \underline{\underline{\mathbf{Leg}}} \;:=\; \frac{\partial^2 L}{\partial \underline{\dot{\boldsymbol{Q}}} \partial \underline{\dot{\boldsymbol{Q}}}} \left(\;=\; \frac{\partial \underline{\boldsymbol{P}}}{\partial \underline{\underline{\dot{\boldsymbol{Q}}}}} \right) \tag{14}$$

is in general non-invertible. The momenta $\boldsymbol{P}$ thus cannot in general be *independent* functions of the velocities $\dot{\boldsymbol{Q}}$.

Higher-Order Naming Remark We use $\overset{T}{=}$ to signify mathematical equality where moreover the leading left-hand-side giving a *truer naming* of the previously-used right-hand-side.

Naming Remark As our first example, our setting's Lagrangian-to-Hamiltonian variables instance of Legendre transformation is more truly named *total velocities-to-momenta transformation*, and TVM is its matrix. The condition controlling whether this nontriviality occurs is whether the corresponding determinant ('TMViant' $\overset{T}{=}$ 'Legendriant')

$$|\mathrm{TMV}| \;\overset{T}{=}\; |\mathrm{Leg}| \;=\; 0 \;\text{ at least somewhere}\;. \tag{15}$$

See [94] for detailed consideration of what kind of somewhere is involved.

Notation 1 As a Fundamental Physics article, working on curved spaces is generally assumed, so upstairs and downstairs distinction of tensorial indices is significant. Configurations are copper vectors and their conjugate momenta are copper covectors. See [92] for the colouring nota

Remark 2 The Euler–Lagrange equations of motion can indeed be rearranged to reveal the explicit presence of the TVM matrix, as follows.

$$\underline{\underline{\mathrm{TMV}}}\,\underline{\ddot{\boldsymbol{Q}}} \;=\; \frac{\partial^2 L}{\partial\overline{\dot{\boldsymbol{Q}}}\partial\overline{\dot{\boldsymbol{Q}}}}\,\underline{\ddot{\boldsymbol{Q}}} \;=\; \frac{\partial L}{\partial\overline{\boldsymbol{Q}}} \;-\; \frac{\partial^2 L}{\partial\overline{\boldsymbol{Q}}\partial\overline{\dot{\boldsymbol{Q}}}}\,\overline{\dot{\boldsymbol{Q}}}\;. \tag{16}$$

The above noninvertibility thus additionally means that the system's accelerations $\ddot{\boldsymbol{Q}}$ are not uniquely determined by $\boldsymbol{Q}, \dot{\boldsymbol{Q}}$.

3.3 Primary versus secondary constraints

Remark 2 Anderson and Bergmann [11] subsequently classified constraints in the following way (which the main sources [19, 47] continued to use).

Definition 1 A constraint arising from the above non-invertibility of the momentum–velocity relations is termed *primary*.

Definition 2 A constraint requiring input from the variational equations of motion is termed *secondary*.

Notation We shall refer to these of course as **𝓅rimary** and **𝓈econdary**.

4 Dirac Algorithm. I. Dirac's Little Algorithm

Given a candidate set of constraints $\boldsymbol{c}$, one is to determine whether they make sense by examining their Poisson brackets

$$\{\,\underline{c}, \underline{c}'\,\}\;. \tag{17}$$

We do this since these are a natural part of the underlying phase space 𝔓hase's structure (and thus are also cast in copper). At the local level, five types of condition can ensue from the computed-out right-hand-sides of these brackets. See [12, 16] for original articles alongside reviews and commentaries in [19, 34, 47, 72, 78, 82, 90, 94]. For the current article's purposes, the below Finite-Theory (as opposed to Field Theory) treatment suffices.

Definition 1 *Dirac's Little Algorithm* [19] consists of evaluating Poisson brackets between a given input set of constraints so as to determine whether these are consistent and complete. The first four possible types of outcome below – a) to d) – are allowed in this setting, as per the first chapter-and-a-half of [19].

Remark 1 In particular, the current section assumes that all constaints involved close under Poisson-brackets. Constraints behaving thus are widely referred to as *first-class*. We for now denote these by **𝓕irst** (we revisit this in Sec 7.1 with some subtleties about first-class notions and corresponding notations).

4.1 Outcome a) constraints old and new

i) One possible outcome for the right-hand-side of the Poisson brackets is zero. In this case, the constraints commute:

$$\{\,\underline{c}, \underline{c}'\,\} \;=\; 0\;. \tag{18}$$

ii) A more general possibility is that the Poisson brackets produce a linear function of the previously-known constraints. This returns a Poisson algebra: a mathematical structure covered in e.g. [53, 68].

iii) It is also possible for *structure functions* $\boldsymbol{C}(\boldsymbol{\eta}, \boldsymbol{c})$ to arise in place of structure constants:

$$\{\,\underline{c}, \underline{c}'\,\} \;=\; \overline{\underline{\underline{C}}}(\boldsymbol{\eta}\,,\; \boldsymbol{c}) \cdot \underline{c}''\;. \tag{19}$$

where the $\boldsymbol{c}$ themselves are constants. This gives the *Poisson algebroid* [53] generalization of Poisson algebra.

Naming Remark We furthermore term the portmanteau of Poisson algebras and Poisson algebroids a *Poisson algebraic structure.*

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Remark 1 Dirac furthermore considered being zero up to a linear function of constraints as a weak notion of zero. We denote this by

$$\approx \quad , \tag{20}$$

to the (usual) strong notion of equality being written as

$$= \quad . \tag{21}$$

Let us furthermore denote the portmanteau equality of these two cases by

$$`=' \quad . \tag{22}$$

We shall be discussing notions of weak (and consequently portmanteau) equality in further detail below.

Remark 2 Dirac [10, 15, 19] furthermore allowed for the possibility of new constraints $\boldsymbol{c}^{\text{new}}$ being discovered in the process.

Remark 3 Dirac also formalized how discovering integrabilities in general requires proceeding recursively. So having found new constraints $\boldsymbol{c}^{\text{new}}$, the *mutual*

$$\{\, \boldsymbol{c}, \boldsymbol{c}^{\text{new}} \,\} \tag{23}$$

and *self*

$$\{\, \boldsymbol{c}^{\text{new}}, \boldsymbol{c}^{\text{new}} \,\} \tag{24}$$

Poisson brackets need to be investigated. For they might themselves produce further constraints. A priori, the right-hand-sides of (24) themselves belonging to the Poisson algebraic structure is an involution condition. Via the local form of Frobenius' Theorem [66, 94], this furthermore becomes an integrability condition.

4.2 Outcome b) identities

Dirac also envisaged the necessity of including the possibility of brackets producing identities, i.e. equations reducing to

$$0 \ = \ 0 \, . \tag{25}$$

4.3 Outcome c) inconsistencies

A particularly significant insight of Dirac's was realizing the need to include *inconsistencies* – i.e. equations reducing to

$$0 \ = \ 1 \tag{26}$$

– among the outcomes of the Dirac Algorithm. This permits us to capitalize on the algorithm having the capacity to act as a *selection principle*. This points to using the language of 'candidate sets', cast in ruby. These only become theoretically bona fide sets of constraints – highlighted in fireopal – if they pass the test set by the algorithm; see Sec 5 for details. That the Lagrangian

$$L \ = \ \dot{Q} \ + \ Q \ , \tag{27}$$

gives (26) as its Euler–Lagrange equations suffices to show that inconsistencies are indeed a possibility in the Principles of Dynamics.

4.4 Outcome d) specifier equations

Dirac [10, 15, 19, 47] additively appends constraints to Hamiltonians H using Lagrange multipliers $\boldsymbol{\Lambda}$ to form total-Hamiltonian-type objects,

$$H_{\text{bare}} \ := \ H \ \longrightarrow \ H_{\text{total}} \ + \ \boldsymbol{\Lambda} \cdot \boldsymbol{c} \ . \tag{28}$$

The Dirac Algorithm is then capable of placing restrictions on the form of these a priori free multipliers $\boldsymbol{\Lambda}$. Such relations are called *specifier equations*, schematically

$$\mathcal{S}\text{peci}(\boldsymbol{Q}, \boldsymbol{P}, \boldsymbol{\Lambda}) \ = \ 0 \ \text{ to be solved for } \ \boldsymbol{\Lambda} \tag{29}$$

These are distinct entities from constraints, which contain no mention of any appending multipliers.

4.5 What we mean by 'total-Hamiltonian-type'.

Structure 1 *Dirac's 'starred' Hamiltonian* [19] is the result of the following appending. Of a formalism of a theory's primary constraints $\boldsymbol{\mathcal{P}}\mathbf{rimary}$ using a-priori-arbitrary phase space functions $\boldsymbol{A}(\boldsymbol{\eta})$ as multipliers. The Author [90] thus also termed this *arbitrary-primary Hamiltonian*. I.e.

$$H_{\boldsymbol{A}\boldsymbol{\mathcal{P}}} \;:=\; H^* \;:=\; H \;+\; \overline{\boldsymbol{A}} \cdot \underline{\boldsymbol{\mathcal{P}}\mathbf{rimary}} \;, \tag{30}$$

Here the first expression is my notation to the second expression being Dirac's traditional notation.

Structure 2 *Dirac's primed Hamiltonian* is

$$H_{\boldsymbol{p}\boldsymbol{\mathcal{P}}} \;\stackrel{T}{=}\; H' \;:=\; H \;+\; \overline{\boldsymbol{P}} \cdot \underline{\boldsymbol{\mathcal{P}}\mathbf{rimary}} \;. \tag{31}$$

On this occasion, the first expression's notation stands for my *particular-primary Hamiltonian* naming, signifying 'with particular-solution multipliers appending primary constraints'.

Structure 3 *Dirac's total Hamiltonian* [19] itself is the result of appending these with *unknown functions* $\boldsymbol{u}(\boldsymbol{\eta})$. I.e.

$$H_{\boldsymbol{u}\boldsymbol{\mathcal{P}}} \;\stackrel{T}{=}\; H_{\text{total}} \;:=\; H \;+\; \overline{\boldsymbol{u}} \cdot \underline{\boldsymbol{\mathcal{P}}\mathbf{rimary}} \;. \tag{32}$$

We are now here superceding Dirac's notation H_{total} (or H_{T}) by the Author's $H_{\boldsymbol{u}\boldsymbol{\mathcal{P}}}$ standing for *unknown-primary Hamiltonian*. This signifies 'with unknown multipliers appending primary constraints'.

Structure 4 *Dirac's extended Hamiltonian* is

$$H_{\mathcal{F}} \;\stackrel{T}{=}\; H_{\text{E}} \;:=\; H \;+\; \overline{\boldsymbol{u}} \cdot \underline{\boldsymbol{\mathcal{P}}\mathbf{rimary}} \;+\; \overline{\boldsymbol{a}} \cdot \underline{\boldsymbol{\mathcal{S}}\mathbf{econdary}} \;, \tag{33}$$

This is for arbitrary functions $\boldsymbol{a}(\boldsymbol{\eta})$ and first-class secondary constraints $\boldsymbol{\mathcal{S}}\mathbf{econdary}$. We are here finally superceding Dirac's notation H_{E} (or H_{extended}) by the Author's $H_{\mathcal{F}}$ standing for 'the Hamiltonian with first-class constraints appended', Therein, we leave it implicit that what multipliers can be solved for, are.

4.6 Example of working iteration by iteration

Method 1 If further constraints $\mathcal{F}\boldsymbol{\mathcal{S}}\text{econdary}$ arise in the ith iteration, then these are fed into the $(i+1)$th iteration.

Step A) Define $\boldsymbol{\mathcal{c}}[i+1]$ as one's initial $\mathcal{F}\boldsymbol{\mathcal{P}}\text{rimary}$ alongside the subset $\mathcal{F}\boldsymbol{\mathcal{S}}\text{econdary}(i)$ of the candidate theory's formulation's $\mathcal{F}\boldsymbol{\mathcal{S}}\text{econdary}$ that have been discovered so far. Here,

$$\mathcal{F}\boldsymbol{\mathcal{S}}\text{econdary}[i] \;=\; \coprod_{j=1}^{i} \mathcal{F}\boldsymbol{\mathcal{S}}\text{econdary}(j) \;. \tag{34}$$

Step B) Form a system

$$0 \;\approx\; \boldsymbol{\mathcal{c}}(i\dot{+}1) \;=\; \{\, \boldsymbol{\mathcal{c}}(i+1), H_{\boldsymbol{u}\boldsymbol{\mathcal{c}}(i+1)} \,\} \;=\; \{\, \boldsymbol{\mathcal{c}}(i+1), H \,\} \;+\; \{\, \boldsymbol{\mathcal{c}}(i+1), \underline{\boldsymbol{\mathcal{c}}(i+1)} \,\} \cdot \overline{\boldsymbol{u}} \;\approx\; 0 \;. \tag{35}$$

Remark 2 (35) is a linear system. Just as in introductory courses on ODEs, the general solution thus splits into

$$\boldsymbol{u} \;=\; \boldsymbol{P} \;+\; \boldsymbol{C} \;, \tag{36}$$

for particular solution $\boldsymbol{P}$ and complementary function $\boldsymbol{C}$. By definition, $\boldsymbol{C}$ solves the corresponding homogeneous equation

$$\overline{\boldsymbol{C}} \cdot \{\, \underline{\boldsymbol{\mathcal{c}}}, \boldsymbol{\mathcal{P}}\mathbf{rimary} \,\} \;\approx\; 0 \;. \tag{37}$$

Furthermore, $\boldsymbol{C}$ has the factorization structure

$$\overline{\boldsymbol{C}} \;=\; \underline{\boldsymbol{s}} \cdot \overline{\underline{\boldsymbol{R}}} \;. \tag{38}$$

Here the $\boldsymbol{s}$ are the totally arbitrary coefficients of the independent solutions. Also, $\boldsymbol{R}$ is a mixed-index (and thus in general rectangular rather than square) matrix. Its second and first indices run over primary constraints and the generally-distinct set of independent solutions respectively. The general solution obtained thus is then to be substituted into the extended Hamiltonian, updating it.

Structure 1 The iterations of a given model's Dirac Algorithm form a chain as indexed by the number of iterations,

$$\mathbb{0}\text{-}i \;=\; \text{i}(0) \;,\; \ldots \;,\; \text{i}(P) \;=\; i\text{-}\mathbb{1} \tag{39}$$

4.7 More on specifiers

Definition 1 Let us call an iteration in the Dirac Algorithm *purely-constraining* if it produces new first-class constraints but no specifiers.

Definition 2 It is *purely-specificatory* if it produces specifiers but no new first-class constraints.

Definition 3 It is *mixed* if both occur.

Remark 1 Appended Hamiltonians can each be split into an arbitrary-multiplier block and a specified multiplier part,

$$H_{[n]} \ = \ H \ + \ a_{[N]} \cdot \boldsymbol{c}_{a[N]} \ + \ p_{[N]} \cdot \boldsymbol{c}_{p[N]} \ . \tag{40}$$

This split does not in general align with the iterations the constraints in question arise from. The primary step cannot have specifiers as there are a priori no multipliers that can be specified. The secondary step can specify primary multipliers. The tertiary step can specify primary or secondary multipliers, and so on.

Structure 1 We use $H_{P[n]}$ for specification up to the $(n - 1)$th step included and $H_{S[n]}$ if the nth step's specification has been included. These start being able to be distinct in the secondary step.

4.8 Chains

Notation We also use

$$\mathrm{m}(1) \ , \quad \ldots \ , \quad \mathrm{m}(N-1) \tag{41}$$

to denote the middle of a finite chain that is itself indexed by 0 to N.

A finite chain has one of each of the following (as do bounded lattices, while posets can have multiple of whichever.

A *top*, alias *unit* alias *terminal* alias *maximal* alias *total* element, denoted by $\mathbb{1}$.

A *bottom* alias *zero* alias *initial* alias *minimal* also known in some contexts as *trivial* element denoted by $\mathbb{0}$.

A nontrivial chain has furthermore at least one *middling* alias *partial* element.

5 Termination conditions

Given some initial candidate set of constraints $\boldsymbol{c}$, we assess their brackets, by the (for now Little) Dirac Algorithm. This proceeds iteratively until one of the following *termination conditions* is attained [88, 85]. This is indeed termination in the sense of the above chains being finite and thus having terminal elements. This indeed partly motivates my choice of words 'termination condition'.

Termination condition 0) Hitting an immediate inconsistency, by at least one inconsistent equation arising [19].

Termination condition I) Combinatorially-critical chain. Here iterations of the Dirac Algorithm produce a chain of new objects down to the point of combinatorial triviality: leaving the candidate theory with no degrees of freedom.

Termination condition II) Chain to inconsistency. This places further conditions past the point of no degrees of freedom, plunging one's candidate theory into inconsistency.

Termination condition III) Arriving at an iteration that produces no new objects while retaining some degrees of freedom. This concluding iteration of the Dirac Algorithm produces no new constraints or specifier equations, indicating that all such have been found. Re-running the algorithm past this point cannot find any new equations.

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Remark 1 It is III) that is the termination condition which renders a candidate theory successful at this stage: consistent and nontrivial.

Remark 2 III) may come with further qualifications, such as retaining a *Temporally-Relationally*-nontrivial [72, 89] number of degrees of freedom. This means a minimum of two for Finite Theories. This is so that one can vary as a function of the other, in Background-Independent settings in which there is consequently no background time for a sole surviving variable to be a function of. Or of retaining at least some local (as opposed to global) degrees of freedom in the case of a Field Theory.

Remark 3 If termination is of the successful kind, then the final output is a Poisson algebraic structure. We here furthermore celebrate by re-declaring our chain (39) as a fireopal entity.

$$\mathrm{i}(0) \ \text{ to } \ \mathrm{i}(P) \ . \tag{42}$$

6 Further application of Order Theory

6.1 Order structures used

See [98] for a basic account of these. Fig 1 provides our notation and a few features of these.

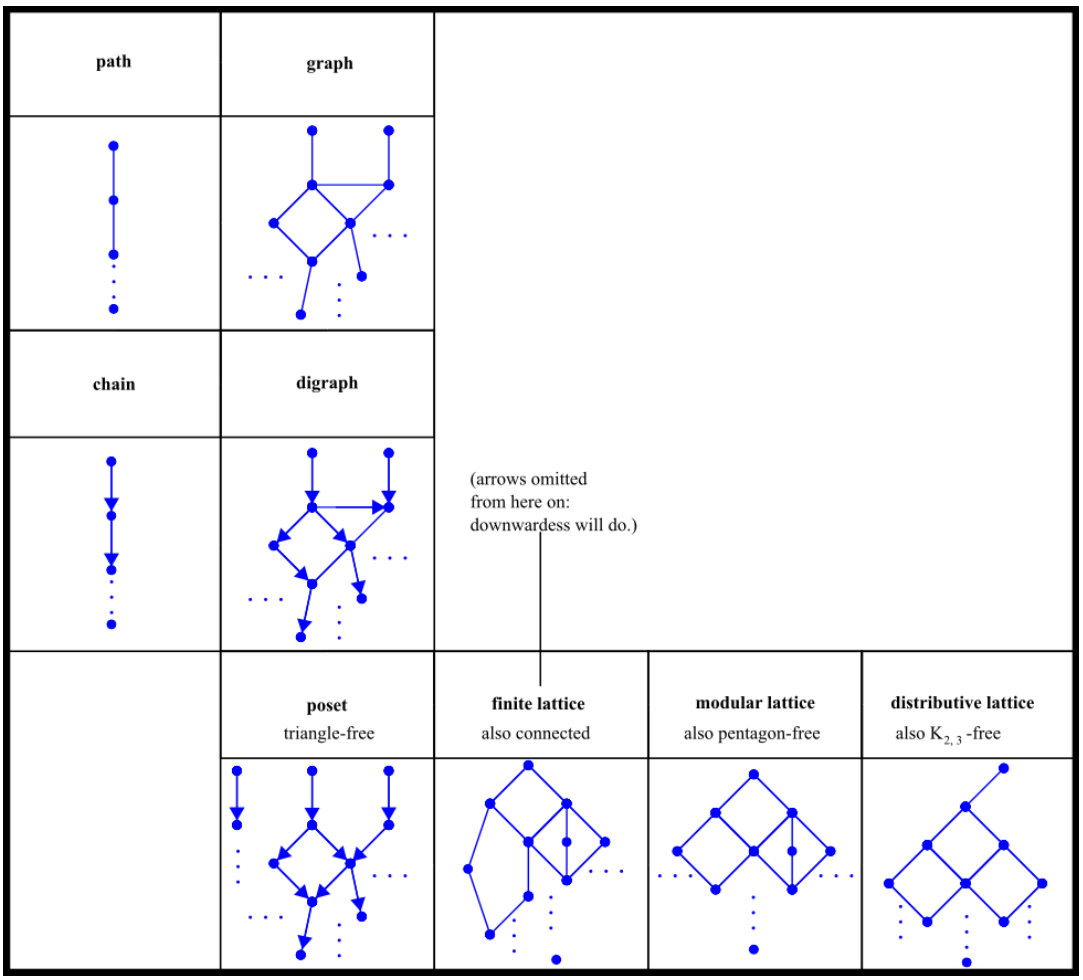

Figure 1:

6.2 Lattice of constraint subalgebraic structures

Structure 1 The end-product of a successful candidate theory's passage through the Dirac Algorithm is a *constraints algebraic structure* $\mathfrak{First}$ of first-class constraints $\mathcal{F}\mathrm{irst}$. This takes the form

$$\{ \underline{\mathcal{F}\mathrm{irst}}, \underline{\mathcal{F}\mathrm{irst}} \} \ = \ \underline{\underline{\overline{F}}} \cdot \underline{\mathcal{F}\mathrm{irst}} \ . \tag{43}$$

The $\boldsymbol{F}$ here can be the structure constants of a Poisson algebra, or a Poisson algebroid's structure functions, $\boldsymbol{F}(\boldsymbol{Q}\,,\ \boldsymbol{P})$.

Structure 2 This possesses a bounded lattice [99, 56, 64] $\mathfrak{L}_{\mathfrak{C}}$ of constraint subalgebraic structures $\mathfrak{C}$SAS. The join and meet operations here are just union and intersection of the generating sets of constraints involved.

Remark 1 Order-Theoretic consideration of constraint algebraic structures has already been outlined elsewhere [71].

These updates consist of more examples, further systematics, and justification of its restriction to 'geometrically-significant', 'tensorial' or 'Representation-Theoretically aligned' constraint blocks.

Remark 2 Bounded lattices also have unique tops and bottoms. The bottom element is the empty algebra (no generators and thus no relators either):

$$\mathbb{0}\text{-}\mathfrak{C}\text{SAS} \;=\; id \;=\; \langle \emptyset \, , \; \emptyset \rangle \tag{44}$$

The top element is the full first-class

$$\mathfrak{C}\text{AS} \;=\; \mathfrak{First} \; . \tag{45}$$

Notation Let us use z_m to denote any middle element of the constraint algebraic structure. The m here runs over $\mathfrak{First}$'s middle.

Remark 2 Finite lattices' middles are no more than posets, since removing top and bottom elements is well capable of disconnecting us out of having a lattice. Or of leaving us with more than one top or bottom element.

Remark 3 $\mathfrak{L}_{c}$ is mathematically stable. So subsequent steps in the Background-Independent Principles of Mechanics build on this rather than on any solely ruby extras.

6.3 Algebraic closure motivated modification of theory of appended Hamiltonians

Structure 1 Firstly, at the candidate level, I comment that one more accurately has a chain of such objects $H_{\mathcal{F}}$ as one progresses down Dirac's Algorithm. This is indexed by the number of iterations of the Algorithm, and makes use of the notion of so-far first class constraints. This Remark – new to the current Article – is built upon further in Sec 8.2.

Structure 2 At the confirmed level, we comment that $H_{\mathcal{F}}$, should be reinterpreted with no distinction between primary and secondary constraints as regards how multipliers are treated. This is because

$$\boldsymbol{\mathcal{P}}\textbf{rimary} \;\; \text{and} \;\; \boldsymbol{\mathcal{S}}\textbf{econdary} \text{ can be integrabilities of each other [47] .} \tag{46}$$

Primary-to-secondary distinction is thus not in general algebraically-meaningful. While (46) has been known for at least 30 years, its implication that $H_{\mathcal{F}}$ needs a new meaning is, as far as we are aware, also new to the current article. This new meaning is

$$H_{A\mathcal{F}} \;=\; \;=\; H \;+\; \underline{\alpha} \cdot \underline{\boldsymbol{\mathcal{F}}\textbf{irst}} \; , \tag{47}$$

$$H_{P\mathcal{F}} \;=\; H_{\text{E}} \;=\; H \;+\; \underline{\boldsymbol{\Pi}} \cdot \underline{\boldsymbol{\mathcal{F}}\textbf{irst}} \; , \tag{48}$$

where the Lagrange multipliers α arbitrary while $\boldsymbol{\Pi}$ are precisely as unfree as the model's specifier equations $\mathcal{S}$peci render them.

Non-Structures 3 to 5 Observation (46), alongside letting algebraic consistency direct us, also means that the following are to be discarded as not in general algebraically-meaningful. Dirac's starred, primed and total Hamiltonians. Thus, out of the notions of appended Hamiltonian considered so far, the modified $H_{\mathcal{F}}$ alone survives into the new world of theory-independent algebraic consistency.

Structure 6 One can furthermore posit a lattice of appended Hamiltonians covering appending by all constraint subalgebraic structures $\mathfrak{L}_{\boldsymbol{\mathcal{F}}\textbf{irst}}$ of the first-class constraint algebraic structure $\mathfrak{First}$. Here, the incipient alias bare Hamiltonian H is the bottom element. Our modified $H_{\mathcal{F}}$ is the top element. The middling elements constitute a new family of appended Hamiltonians,

$$H \;\longrightarrow\; H_l \;=\; H \;+\; \overline{\boldsymbol{M}} \cdot \boldsymbol{z}_l \; , \;\; l \;\in\; \mathfrak{L}_{\boldsymbol{\mathcal{F}}\textbf{irst}} \; . \tag{49}$$

Remark 1 In those formulations of those theories for which the primary constraints do constitute a subalgebra

$$\mathfrak{Primary} \;\leq\; \mathfrak{First} \; , \tag{50}$$

$$H_{\boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{P}}} \;:=\; H' \tag{51}$$

does remain meaningful. It is furthermore distinct from H and $H_{\mathcal{F}}$ if this subalgebra is nontrivial and proper. However, passage from this to the family $H_{\mathfrak{L}_{\boldsymbol{\mathcal{F}}\textbf{irst}}}$ is another instance of m(1) to a.

6.4 Example 1: Gauge Theory

In (for now pure Yang–Mills) Gauge Theory, the triviality condition on the momenta conjugate to the 0-component of the potential,

$$\pi_0 \; = \; 0 \tag{52}$$

is primary, while the Gauss constraint $\mathcal{G}\mathbf{auss}$ is secondary. Then

$$H_{\mathcal{F}\text{gauge}} \; = \; H \; + \; \boldsymbol{\lambda} \cdot \boldsymbol{\pi}_0 \; + \; \boldsymbol{\Lambda} \cdot \mathcal{G}\mathbf{auss} \; . \tag{53}$$

The constraints here enter the 2-chain

$$id \; \leq \; \mathcal{G}\mathbf{auss} \; = \; \mathcal{O}\mathbf{lin} \; . \tag{54}$$

See Fig 4.a) for subsequent objects.

6.5 Example 2: General Relativity

In (for now vacuum) GR-as-Geometrodynamics in the ADM formulation [18], the triviality conditions on the momenta conjugate to the lapse α and shift $\boldsymbol{\beta}$,

$$\pi_\alpha \; = \; 0 \; , \quad \boldsymbol{\pi}_{\boldsymbol{\beta}} \tag{55}$$

are primary constraints. In the ADM formulation, both the *Hamiltonian constraint*

$$\mathcal{H} \; = \; 0 \tag{56}$$

and the *momentum constraint*

$$\boldsymbol{\mathcal{M}} \; = \; 0 \tag{57}$$

are secondary.

Remark 1 GR's

$$H_{\text{bare}} \; = \; 0 \; : \tag{58}$$

a tell-tale sign of a Temporally Relational theory.

Remark 2 The above primary constraints are often glossed over to the extent of not even being written down in the appended Hamiltonian.

Remark 3 The appended Hamiltonian is often called total, but really meets the definition of extended, i.e. it is a H_{E} that I now term $H_{\mathcal{F}}$. Thus also the lapse and shift multipliers corresponding to $\mathcal{H}$ and $\boldsymbol{\mathcal{M}}$ respectively are $\boldsymbol{a}$ here rather than $\boldsymbol{u}$. In (partly-)relational formulations, however, $\mathcal{H}$ arises at the primary level, so the corresponding emergent lapse multiplier is now a $\boldsymbol{u}$.

Remark 4 Formulation-independently, in GR $\boldsymbol{\mathcal{M}}$ forms a consistent subalgebra $\mathfrak{Mom}$ but $\mathcal{H}$ does not. This is since it has the momentum constraint $\boldsymbol{\mathcal{M}}$ as an integrability [22].

Remark 5 The geometrodynamical formulation of GR thus has the minimal chain-with-nontrivial-middle of constraint subalgebraic structures,

$$id \; < \; \mathfrak{Mom} \; < \; \mathfrak{Dirac} \; = \; \mathfrak{Flin}(GR, Gdyn) \; . \tag{59}$$

See Fig 4.b) for subsequent objects.

6.6 Example 3: Relational Particle Mechanics

(Temporally and Euclideanly) Relational Particle Mechanics (RPM) [36, 60, 72] has the following primary constraints. The momenta associated with its translational and rotational auxiliaries

$$\boldsymbol{p}_a \; = \; 0 \; , \quad \boldsymbol{p}_\theta \; = \; 0 \; . \tag{60}$$

The *equation of time* (mathematically but not conceptually equivalent to conservation of energy)

$$\mathcal{E}\text{nergy} \; = \; 0 \; . \tag{61}$$

It has as secondary constraints the *zero-total -momentum* and *-angular-momentum* of the whole universe-model *constraints*,

$$\boldsymbol{\mathcal{P}} \; = \; 0 \; , \quad \boldsymbol{\mathcal{L}} \; = \; 0 \; . \tag{62}$$

Remark 1 Both $\mathcal{P}$-and-$\mathcal{L}$ and $\mathcal{E}$nergy are consistent subalgebras.

Remark 2 This corresponds to RPMs being able to separately support Temporal and Configurational Relationalisms. In contrast, GR cannot have Temporal Relationalism without Configurational Relationalism as consequence of Remark 4 of 6.5. For RPM has no corresponding integrability issues [60].

Remark 3 So sticking to this temporal versus spatial constraint block structure, the minimal nontrivial bounded poset is realized, nontrivial here meaning that it is not a chain. This has 4 points arranged into a diamond with the obvious four convex quadrilateral lines as ordering-relation edges. This is moreover indeed also the smallest nontrivial example of a lattice. See Fig 4.c) for subsequent objects.

6.7 Dual lattice of Observables algebraic structures

Structure 0 *Canonical observables* $\boldsymbol{\mathcal{O}}$ are at the most primary level [72, 83, 91] functions over phase space $\mathfrak{P}$hase. These form *function spaces* $\mathfrak{O}$AS which are a fortiori *function algebras* [40].

Structure 1 In the presence of constraints, it is well-known that [9, 48, 49] canonical observables need to commute with these. Let us further qualify that the constraints in question are first-class, $\boldsymbol{\mathcal{F}}$**irst**, so

$$\{\, \boldsymbol{\mathcal{F}}\mathbf{irst}, \boldsymbol{\mathcal{O}} \,\} \ \text{`='} \ 0 \ . \tag{63}$$

Dirac's conception of canonical observables [9] is thus arrived at.[3]

Structure 2 Each constraint algebraic structure $\mathfrak{C}$AS admits a corresponding observables algebraic structure $\mathfrak{O}$AS of observables $\boldsymbol{\mathcal{O}}$ commuting with it.

The lattice of constraint subalgebraic structures $\mathfrak{L}_{\mathfrak{C}}$ admits a dual lattice of observables algebraic structures, $\mathfrak{L}_{\mathfrak{O}}$.

Its bottom element is the function algebra

$$\mathbb{0}\text{-}\mathfrak{O} \ = \ \mathfrak{U}\text{nres} \tag{64}$$

of unrestricted observables $\boldsymbol{\mathcal{U}}$.

Its top element is the function algebra

$$\mathfrak{O}\text{-}\mathbb{1} \ = \ \mathfrak{D}\text{irac} \tag{65}$$

of Dirac observables $\boldsymbol{\mathcal{D}}$. This corresponds to the entire constraint algebraic structure of first-class constraints, $\mathfrak{F}$**irst**. Writing this in sapphire removes its confuseabiliy with the GR-specific Dirac algebroid of constraints.

6.8 Example 1: Gauge Theory

This has just unrestricted and Dirac notions of observables. The Dirac notion here coincides with the notion of gauge observables, $\boldsymbol{\mathcal{G}}$. Overall, one has the 2-chain[4]

$$\boldsymbol{\mathcal{U}} \ > \ \boldsymbol{\mathcal{G}} \ = \ \boldsymbol{\mathcal{D}} \tag{66}$$

of notions of observables. These form the chain of observables algebraic structures (Fig 4.a)

$$\mathfrak{U}\text{nres} \ > \ \mathfrak{G}\text{auge} \ = \ \mathfrak{D}\text{irac} \ . \tag{67}$$

6.9 Example 2: General Relativity

In the case of GR, there are additionally algebraically-meaningful *Kuchař observables* [50] $\boldsymbol{\mathcal{K}}$, which commute with the GR momentum constraint $\boldsymbol{\mathcal{M}}$ but not with the GR Hamiltonian constraint $\mathcal{H}$. The geometrodynamical formulation of GR thus exhibits the minimal chain with nontrivial middle of notions of observables. I.e.

$$\text{m}(1) \ = \ \boldsymbol{\mathcal{K}} \tag{68}$$

forming

$$\mathfrak{O}\text{SAS}(1) \ = \ \mathfrak{K}\text{uchař} \ . \tag{69}$$

[3]Implicitly with strong versus weak ambiguity. Assume the strong case occurs throughout the current article; see [74, 75, 83, 91] for the weak case.

[4]Notice the use of the dual order.

Which reside respectively in the middle of the following chains.

$$\boldsymbol{U} > \boldsymbol{K} > \boldsymbol{D} \tag{70}$$

of notions of observables.

$$\mathfrak{Unres} > \mathfrak{K}\text{uchař} > \mathfrak{Dirac}\ . \tag{71}$$

of observables algebras (Fig 4.b).

6.10 On the general conceptual form of replacing Kuchař by A observables

In [67, 71], this was generalized to one notion of *A observables* $\boldsymbol{A}$ per constraint subalgebraic structure.

The reasoning employed in so doing is along the following lines, which will on occasion be reused below.

1) Acknowledge algebraic consistency's cruciality, and specifically seek it out.

2) On the one hand, discard what notions turn out not to be algebraically consistent in a formulation-independent and theory-independent manner.

3) On the other hand, use Order Theory to find the totality of algebraically-consistent cases. This may be subjected to further geometrical meaningfulness, i.e. restriction to 'tensorial' or, more accurately, 'Representation-Theoretically meaningful' blocks of constraints.

We thus quite often obtain a passage of the following m(1) to a type.

m(1) is a single middle element that occurs in existing literature, linking up a 3-chain between an even more well-known notion and a trivial notion. m(1) is not however algebraically meaningful for the general theory, and can be formulation-dependent as well.

A) is the general bounded lattice's middle, which is algebraically meaningful for all formulations of all theories. What a consists of varies with which formulation of which theory one considers. But each model has an algebraically-meaningful A, by which a can always meaningfully be theorized about. In some models, a is empty. In some models, a is precisely m(1). In some models, a includes m(1) as a strict subcase. Finally, in some models m(1) is not part of a at all.

It is also possible for m(1) to be replaced by some other Order-Theoretic middle, such as a chain, poset or digraph. Questioning whether notions in the literature are m(1)'s that can be replaced by a's amounts to becoming *Order-Theoretically aware*. This bears some comparison with some of Dirac's best work [6, 7, 10, 12, 16, 19] involving becoming *Linear-Algebraically aware*. The suggestion then is that Linear Algebra has by now been applied to its logical conclusion in all existing branches of Mathematics and Physics. But Order Theory, or Combinatorics more generally, has not, and yet may offer comparable, complementary insights to those gained by 'Linear-Algebraicizing everything' in the 1930s through to the 1970s. A more general suggestion is that 'Combinatorializing everything' will be a profitable line of research over the next decade or two.

6.11 Example 3: RPM

Here one has a minimal nontrivial poset, a fortiori a minimal nontrivial lattice, of notions of observables.[5] The A observables in this case are, firstly, both Kuchař and gauge observables which commute with $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{L}}$. Secondly, Chronos observables $\boldsymbol{C}$ which commute with $\mathcal{E}$nergy (Fig 4.c).

[5] One has a bit more than this if one treats the translations and rotations as separate blocks; see [71, 97].

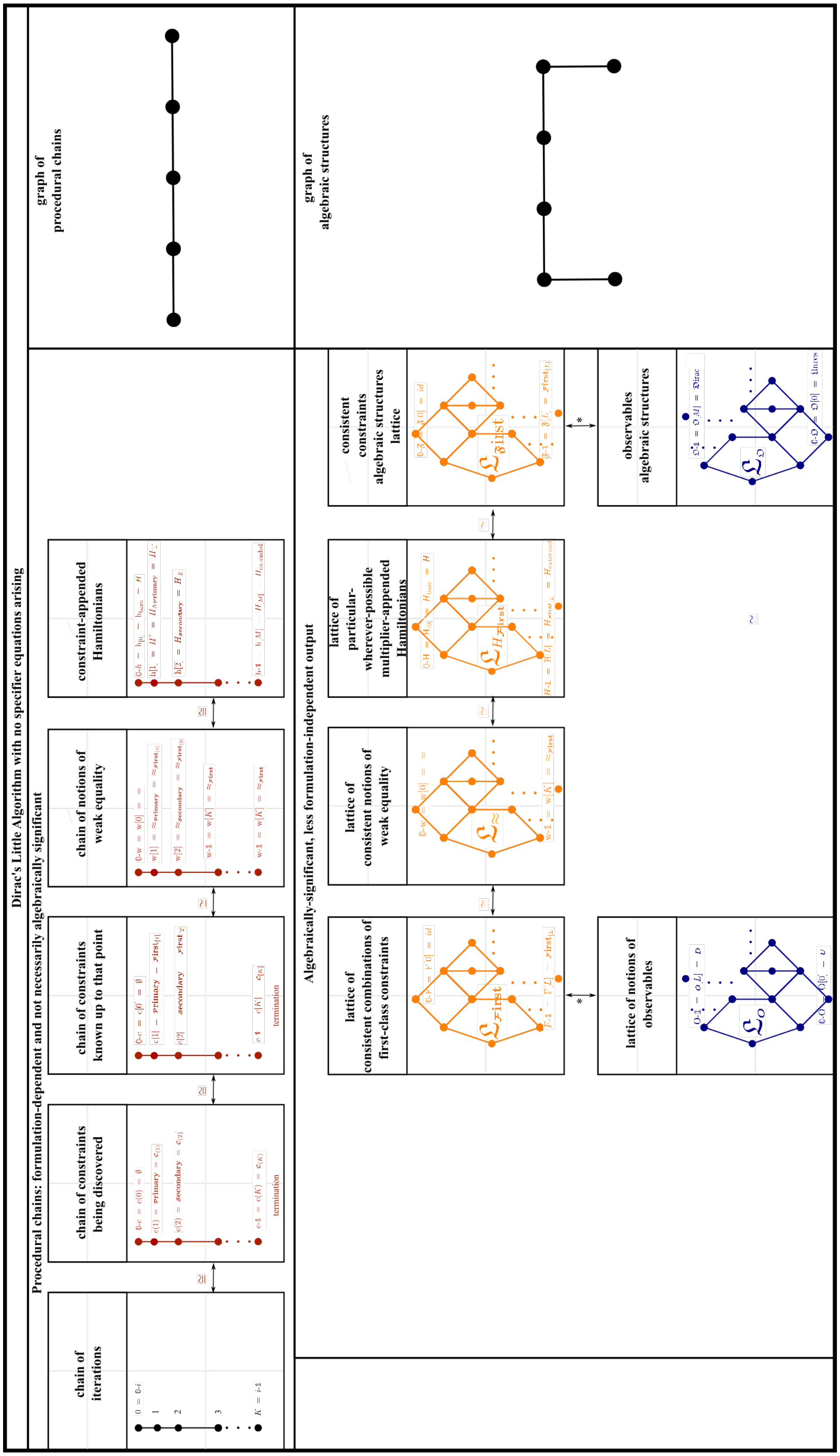

Figure 2:

7 Dirac Algorithm II. Full local version

7.1 (So-far-)first-class and second-class constraints

A classification of [10, 16, 19, 47, 90, 94] constraints is as follows.

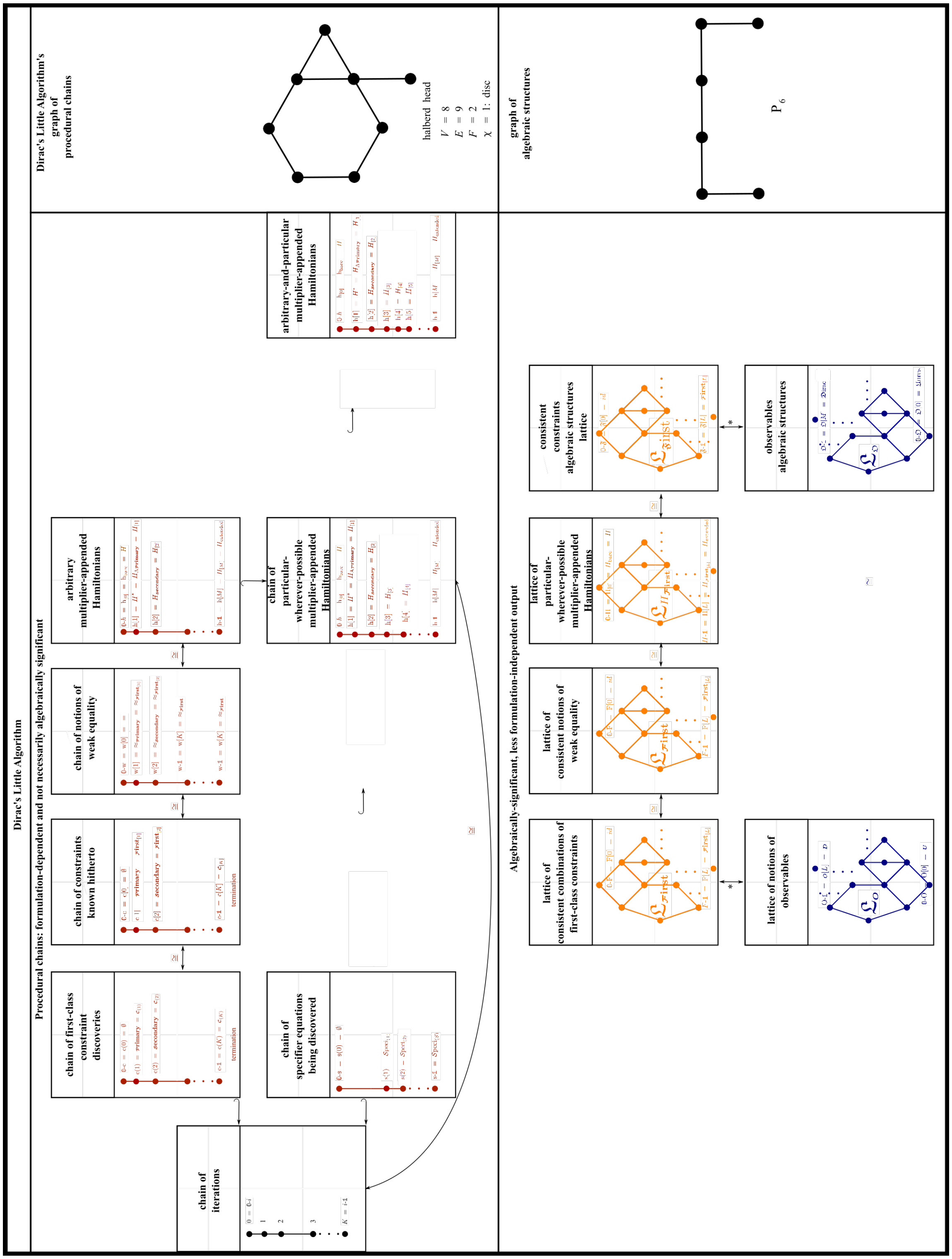

Figure 3:

Definition 1 A constraint is *so-far first-class* $\mathcal{SF}$**irst** if all its brackets with all known constraints close.

Remark 1 For now, 'brackets' means 'incipient-Poisson-brackets', though this notion will be generalized

Definition 2 It is *first-class* $\mathcal{F}$**irst** if this remains the case having applied the Dirac Algorithm until successful termination.

Definition 3 A constraint is *second-class* $\mathcal{S}$**econd** if it is not so-far first-class.

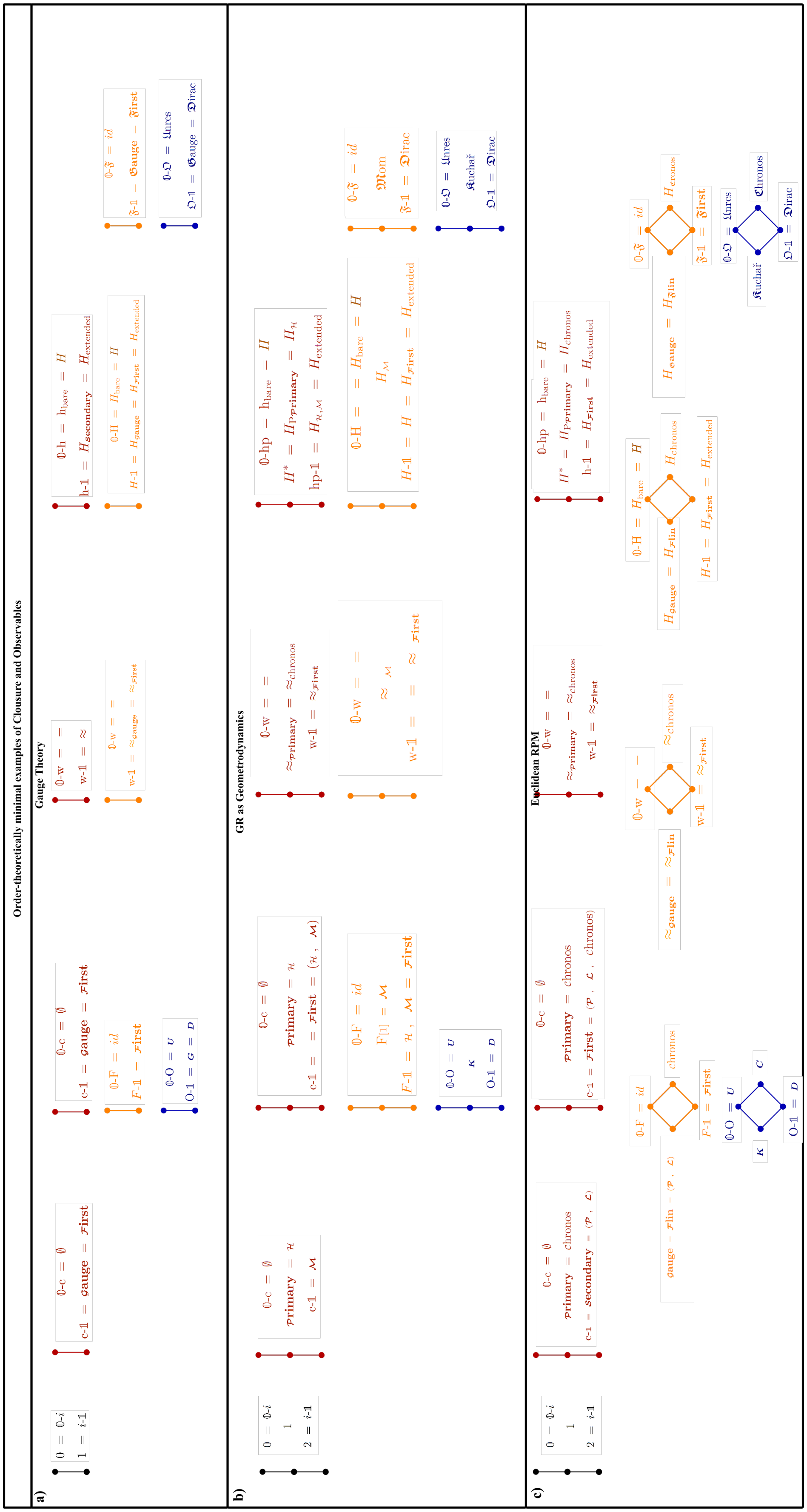
Order-theoretically minimal examples of Closure and Observables
a)
Gauge Theory
b)
GR as Geometrodynamics
c)
Euclidean RPM

Figure 4:

Remark 2 This definition by exclusion corresponds to some other constraint appearing at some iteration of the Dirac Algorithm with which our constraint of interest does not brackets-close. There is no 'so-far' to this notion, for this occurring at some iteration of Dirac's Algorithm is not mutable by what occurs in subsequent iterations.

Remark 3 Definition 1 and 2's notions are not the same since new constraints may arise that do not brackets-close with our constraint of interest. I.e. first-classness has a holistic character: depending on the entire output to termination of Dirac's Algorithm. This is as opposed to us having any capacity to make a permanent judgement at some fixed iteration of Dirac's Algorithm. All that can be done in the latter position in the algorithm is accord a constraint 'so-far first-class' status.

7.2 Commentary

Remark 1 Our notation also captures the following subtlety. Denizens of candidate theories' closure-undemonstrated iterations are cast in ruby, while those of actual closure-confirmed theories are highlighted in fireopal. $\mathcal{SF}\mathbf{irst}$ is one of the former, while $\mathcal{F}\mathbf{irst}$ is one of the latter. They do not have the same underlying uncoloured symbol because $\mathcal{F}\mathbf{irst}$ is also possible. This corresponds to a constraint ending up as first-class within a system that does not however satisfactorily pass Closure. This line of thinking also explains why $\mathcal{S}\mathbf{econd}$ is cast in ruby.

Diagnostic 1 For the purpose of counting degrees of freedom, first-class constraints use up 2 each. whereas second-class constraints use up only 1 [47].

Remark 2 Due to common misconceptions, it is worth pointing out that first-class constraints are not necessarily gauge constraints. In particular, Dirac's Conjecture that [19] these two notions coincide fails by counterexample, as discussed in e.g. [47, 72, 82]. So in general

$$\mathfrak{F}\mathbf{irst} \neq \mathfrak{G}\mathbf{auge} \; . \tag{72}$$

♠ ♡ ♣ ◇

Remark 3 A further conceptual and algebraic classification of second-classness is as follows.

Definition 1 A constraint (or block of constraints $\mathcal{C}_{\mathrm{B}}$) is *self-second-class* if its brackets with itself

$$\{\, \mathcal{C}_{\mathrm{B}}, \mathcal{C}_{\mathrm{B}} \,\} \;\; \text{do not close} \; . \tag{73}$$

Definition 2 A constraint (block) is *mutually-second-class* if some brackets between this constraint (block) and previously-found constraints $\mathcal{C}_{\mathrm{Prev}}$

$$\{\, \mathcal{C}_{\mathrm{B}}, \mathcal{C}_{\mathrm{Prev}} \,\} \;\; \text{do not close} \; . \tag{74}$$

Remark 4 By this, these previously-known constraints turned out to be just *so-far first-class*.

Remark 5 It is possible for a given constraint (block) to exhibit both self- and mutual-second-classness.

7.3 One- and half-killer renaming of first- and second-class

This is based on the above diagnostic, and on viewing phase space as more central than configuration space. It preserves Dirac's original numeration *in its denominators*.

$$\mathcal{O}\mathbf{ne} \stackrel{T}{=} \mathcal{F}\mathbf{irst} \; , \tag{75}$$

so consequently

$$\mathfrak{O}\mathrm{lin} \stackrel{T}{=} \mathfrak{F}\mathbf{irst} \; , \tag{76}$$

$$\mathfrak{L}_{\mathfrak{O}\mathrm{ne}} \stackrel{T}{=} \mathfrak{L}_{\mathfrak{F}\mathrm{irst}} \; , \tag{77}$$

$$\approx_{\mathfrak{O}\mathrm{ne}} \stackrel{T}{=} \approx_{\mathfrak{F}\mathrm{irst}} \; , \tag{78}$$

$$H_{\mathcal{O}\mathbf{ne}} \stackrel{T}{=} H_{\mathcal{F}\mathbf{irst}} \stackrel{T}{=} H_{\mathrm{E}} \; . \tag{79}$$

Also

$$\mathcal{H}\mathbf{alf} \stackrel{T}{=} \mathcal{S}\mathbf{econd} \; , \tag{80}$$

the consequences of which are in the next two subsections.

7.4 Outcome e) Half-killer constraints handled by rebracketing. i) Dirac's case.

If half-killer constraints arise at some iteration in the Dirac Algorithm, then we can proceed by removing them as follows.

Remark 1 Half-killer is not in general an invariant property under taking linear combinations of constraints. Linear Algebra then dictates that the invariant notion is, rather, *irreducibly half-killer constraints* [19, 47], $\mathcal{I}$.

Proposition 1 (Dirac) [19]. Irreducibly half-killer constraints can be factored in by replacing the incipient Poisson bracket by a *Dirac bracket*

$$\{\ \_\ ,\ \_\ \}_{\mathrm{D}} := \{\ \_\ ,\ \_\ \} - \{\ \_\ ,\underline{\mathcal{I}}\}\cdot\{\overline{\mathcal{I}},\underline{\mathcal{I}}'\}^{-1}\cdot\{\overline{\mathcal{I}}',\ \_\ \}\ . \tag{81}$$

Remark 2 The -1 here denotes the inverse of the given matrix. Each $\cdot$ contracts the underlined objects immediately adjacent to it.

Remark 3 Overall, the totality of [47] $\mathcal{I}$ encountered along a Dirac Algorithm are being *projected out*.

Remark 4 We now need to update Sec 2's material to stipulate that one-killer constraints are to close under the Dirac bracket. The current subsection assumes that we only need to do this on one occasion. This is not generally the case, however, as is clarified below.

Remark 5 Geometrically, the Dirac bracket is still a Poisson bracket [26] on a furtherly reduced state space.

Motivation Removal of half-killer constraints is especially relevant since many standard quantum procedures are based on just one-killer constraints remaining by that stage. This usually entails classical removal of any other nontrivial consistent entities which feature in the original formulation.

7.5 ii) Chains of notions of Dirac brackets

Remark 1 More generally, irreducibly half-killer constraints $\mathcal{I}$ can arise [47] at each iteration of the Algorithm. A sequence of intermediary brackets is then required.

Proposition 2 (Anderson) Chain of Dirac brackets. Suppose that the pth iteration of Dirac's Algorithm is the qth iteration in which irreducibly half-killer constraints $\mathcal{I}$ arise. Then we pass from the $(q-1)th$ Dirac bracket to the qth Dirac bracket as follows.

$$\{\ \_\ ,\ \_\ \}_{\mathrm{d}_{(q)}} := \{\ \_\ ,\ \_\ \}_{\mathrm{d}_{(q-1)}} - \{\ \_\ ,\underline{\mathcal{I}}_{\mathrm{d}(q)}\}_{\mathrm{d}_{(q-1)}}\cdot\{\underline{\mathcal{I}}_{\mathrm{d}_{(q)}},\underline{\mathcal{I}}_{\mathrm{d}_{(q)}}\}_{\mathrm{d}_{(q-1)}}{}^{-1}\cdot\{\underline{\mathcal{I}}_{\mathrm{d}_{(q)}},\ \_\ \}_{\mathrm{d}_{(q-1)}}\ . \tag{82}$$

Definition 1 For a model whose Dirac algorithm terminates, suppose that the Pth iteration is the Qth and last in which irreducibly half-killer constraints arise. Then the Qth Dirac bracket is the top Dirac bracket

$$\{\ \_\ ,\ \_\ \}_{\mathrm{d}(Q)} = \{\ \_\ ,\ \_\ \}_{\mathrm{D}\text{-}\mathbb{1}}\ . \tag{83}$$

Remark 2 From our Order-Theoretic point of view, the following repackaging also makes sense.

Definition 2 The incipient Poisson bracket is the bottom Dirac bracket,

$$\{\ \_\ ,\ \_\ \}_{\mathrm{d}(0)} = \{\ \_\ ,\ \_\ \}_{\mathbb{0}\text{-}\mathrm{D}}\ . \tag{84}$$

Definition 3 All other Dirac brackets involved in a given model's chain of Dirac brackets are middling.

Remark 3 Proposition 2 means that at each iteration in which new ireducible half-killer constraints arise, we project out a functionally-independent set thereof. Definitions 1 and 3 mean that on the first iteration in which this occurs, we have passage from the incipient Poisson bracket to a Dirac bracket that is a first nontrivial Dirac bracket and in general just a partial Dirac bracket. Definitions 2 and 3 mean that on the last iteration in which this occurs, we have passage from the last partial Dirac bracket Poisson bracket to the terminal and total Dirac bracket. On the last occasion that this occurs, we have passage from whatever bracket one has at that point to the final Dirac bracket.

<u>Proof</u> of Proposition 2. This is based on the simple geometrical result that the product of two projection operators is itself a projection operator:

$$P_i\,P_j = P_{i,j}\ . \tag{85}$$

This result is furthermore (at least) finitely productive. I.e.

$$\prod_{k=1}^{n} P_{i_k} \ = \ P_{i_1,\ldots i_n} \ . \tag{86}$$

Proposition 1 (Dirac) then provides the $q = 1$ case, as well as the inductive step from $q - 1$ to q. □

The chain of Dirac brackets

$$d(0), ..., d(Q) \tag{87}$$

that a given formulation of a given theory requires is chain-isomorphic to

$$h(0), ..., h(Q) \ : \tag{88}$$

the subchain of the chain of iterations (39) consisting of the zeroth iteration and whichever iterations give rise to new irreducible half-killer constraints. For consistent theories, we reissue all of this in fireopal.

Naming Remark 1 On the one hand, all of top, unit, terminal, and maximal, bottom, zero, initial, and terminal, and middling, are Order-Theoretic descriptors. Among which, unit and zero are notationally useful, while top and bottom may be the most pedagogically helpful. On the other hand, trivial, partial and total are to be viewed as words describing the extent to which the requisite amount of projection has been carried out.

Remark 4 In the Full Local Dirac Algorithm setting setting, we thus replace each mention of (incipient (Poisson)) bracket in Sec 2 with the Dirac bracket that is appropriate to the iteration in question. Our final outcome – if a candidate theory is successful – is a Hamiltonian to which all one-killer constraints found have been appended, H_{F} Any irreducible half-killer constraint content that arose have been factored out by use of a chain of strictly-increasingly projected Dirac brackets.

7.6 Simple examples

Example 1 Dirac's Little Algorithm's regime concerns a regime in which only one-killer constraints arise. Therein, so-far-one-killer constraints are never ousted by being subsequently discovered to be half-killer. This also turns out to mean that one can stick with the incipient Poisson bracket. In Order-Theoretic terms, Dirac's Little Algorithm's chain of Dirac brackets is the trivial chain, i.e. the point. Here,

$$\{\ \_\ ,\ \_\ \}_{\mathrm{D}\text{-}\mathbb{1}} \ = \ \{\ \_\ ,\ \_\ \}_{\mathbb{0}\text{-}\mathrm{D}} \ = \ \{\ \_\ ,\ \_\ \} \ , \tag{89}$$

which we can shorten to

$$\emptyset \ = \ \mathbb{0}\text{-D} \ = \ \text{D-}\mathbb{1} \ . \tag{90}$$

Example 2 The case Dirac discusses in the second half of Chapter 2 of [19] has as its chain of Dirac brackets the chain with trivial middle, i.e. the (directed) 2-path. This is the smallest instance with distinct bottom and top: the incipient Poisson bracket and this case's sole notion of Dirac bracket respectively. Here

$$\emptyset \ = \ \mathbb{0}\text{-D} \ < \ \text{D-}\mathbb{1} \ = \ \text{D} \ . \tag{91}$$

Example 3 The (locally) general case has a Dirac brackets chain with nontrivial middle. Its top element is now the top Dirac bracket (on some occasions also called just 'Dirac bracket'). Its nontrivial middle consists of *partial Dirac brackets* $\mathrm{d}(1)$ to $\mathrm{d}(Q-1)$. Chains of length 3 are minimal for possession of a nontrivial middle, i.e. some member of the chain that is neither a top nor a bottom element. Here,

$$\emptyset \ = \ \mathbb{0}\text{-D} \ = \ \mathrm{d}(0) \ < \ \mathrm{d}(1) \ < \ \mathrm{d}(2) \ = \ \text{D-}\mathbb{1} \ . \tag{92}$$

7.7 On the diversity of notions of weak equality

Remark 1 'Equality up to a linear function of constraints' can be further qualified by which type of constraints is involved. This has both candidate-theory and confirmed-theory versions.

Structure 1 For a confirmed theory, the meaningful notions of constraint to employ are the constraint algebraic structure's subalgebraic structures. As [71, 97] lay out, these notions of constraint differ on a theory-by-theory basis as well as a formulation-by-formulation basis. For each formulation of each theory, these moreover form a lattice. Thus the notions of weak equality supported by each formulation of a mature, confirmed theory themselves form a lattice isomorphic to this. Thus each fireopal incarnation of the weak equality symbol should really carry a lattice-valued subscript:

$$\approx_{\,l}\ , \quad l \in \mathfrak{L}_{\mathcal{O}\mathrm{ne}} \ . \tag{93}$$

The bottom element corresponds to the trivial subalgebra – bereft of any generators, in the current context one-killer constraints – and thus has already been encountered in the guise of strong equality. The top element corresponds to the whole constraint algebraic structure. This is probably what Dirac had particularly in mind, especially within the context of Dirac's Little Algorithm. For in this regime, the matter of whether to include half-killer constraints in the definition of weak equality is moot. The middle elements correspond to all the nontrivial proper constraint subalgebraic structures that a given formulation of a given theory possesses.[6]

Remark 2 Another distinction to be made is between linear functions of the constraints and any other functions thereof. One instance in which this greater generality appears is in subsection 8.2.

Structure 2 The most general fireopal portmanteau equality symbol then refers to whichever model's full lattice of constraint algebraic structures:

$$\text{`='}_{\,l}\ , \quad l \in \mathfrak{L}(\mathfrak{c}\mathrm{AS}) \ . \tag{94}$$

With possible further distinction as to whether a linear or general function is involved. In the case of Field Theory, moreover, whether a function or a *functional* of constraints is involved gives yet another source of distinction; GR's constraint algebroid [12, 23, 30] specifically requires a differential functional [72]. The most general ruby portmanteau equality symbol is looser, in possibly including subsets of the known constraints that later turn out to be algebraically untenable.

8 Dirac Algorithm III. Full global version

8.1 Outcome f) Topological obstruction terms

Structure 1 A more general outcome is the inhomogenous-linear form

$$\{\,\underline{c}, \underline{c}'\,\} \ = \ \underline{\underline{\overline{G}}}[\boldsymbol{\eta}\,,\ c]\,\underline{c} \ + \ \underline{\underline{\overline{N}}}[\boldsymbol{\eta}\,,\ c]\,\underline{c}^{\mathrm{new}} \ + \ \underline{\boldsymbol{\Theta}}[\boldsymbol{\eta}\,,\ c] \ . \tag{95}$$

The extra linear terms $\boldsymbol{c}^{\mathrm{new}}$ here arise as integrabilities, whereas the zeroth-order terms $\boldsymbol{\Theta}$ can underlie topological obstructions.

Further outcomes thus supported are as follows.

1) Perhaps the zeroth-order terms are tame.

2) Perhaps $\boldsymbol{\Theta}$ exceeds what can be supported by the theory. In this case, a 'hard' obstruction is realized, killing off candidate theories rather than just modifying them. This points to candidate theories being eliminable by topological means rather than by running out of degrees of freedom. This situation is widely known in the case of *anomalies*.[7]

3) Perhaps the constants $\boldsymbol{c}$ can be fixed so that $\boldsymbol{\Theta}$ disappears. This is termed *strong avoidance of obstructions*. It is a local strategy so as to avoid a global problem.

If such are kept, we have in general a *constraints, specifiers and obstructions algebraic structure*. These are of variable mathematical tractability and physical viability. If too hard and not strongly fixable, the strategy employed is usually to abandon ship.

Example 1 (Tame) The fundamental bracket

$$\{\,Q, P\,\} \ = \ 1 \tag{96}$$

right-hand-side can be regarded as a third object in its own right (a central extension).

Example 2 (Hard) Higher derivatives of the Dirac delta function

$$\delta^{(n)}(x, x') \tag{97}$$

[6]'Proper' here means to the exclusion of the full algebraic structure itself.

[7]Dirac knew about this [19], commenting on needing to be lucky to avoid such at the quantum level. However, no counterpart of it entered his own formulation of the classical-level Algorithm bearing his name.

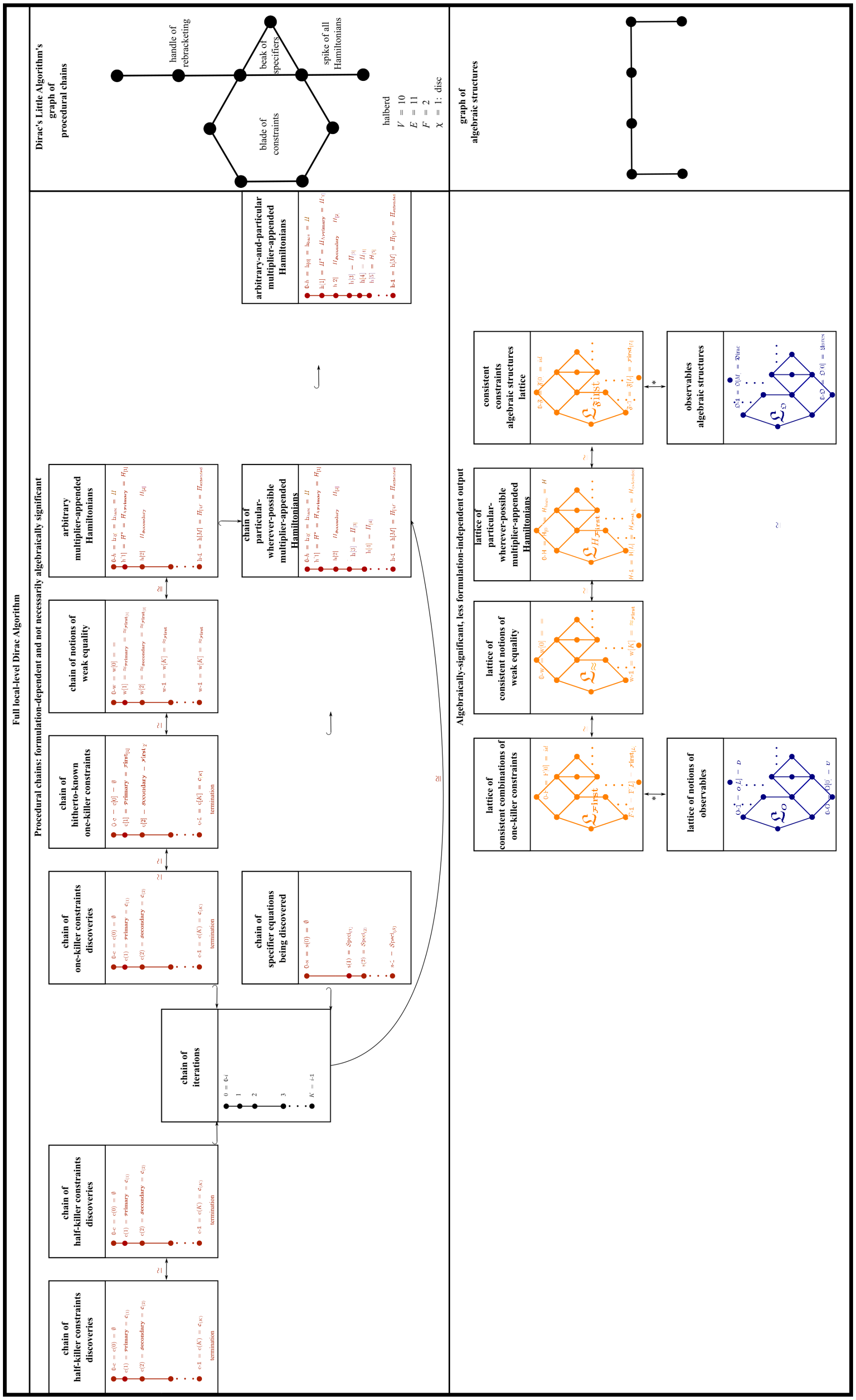

Figure 5:

are well-known to arise on the right-hand-side in certain models of QFT.

Example 3 (Strong fixing). This gives one explanation [58, 65] for how DeWitt's coefficient is fixed if one sets about formulating Geometrodynamics from first principles.

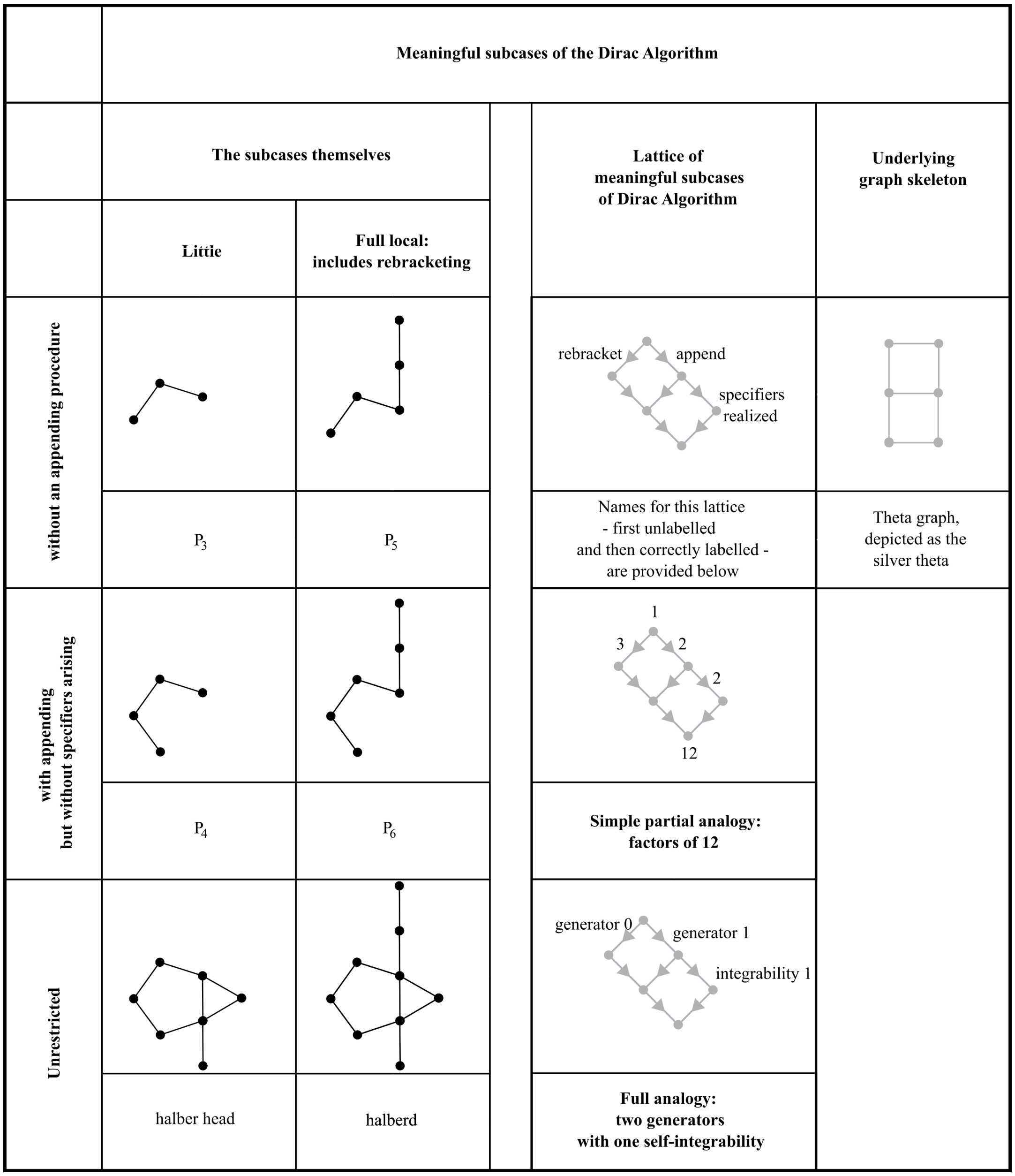

Figure 6:

Example 4 (Strong fixing) This is also how the particular dimensions 26 and 10 arise in Bosonic- and Super-string Theories respectively.

8.2 Path choice along the Dirac Algorithm

Naming Remark 1 Henneaux and Teitelboim [47] called this *bifurcation.* I do not, because the present situation does not carry Dynamical Systems connotations that this name has acquired. I then used *branching.* But I then found examples of loops, by which this botanical and a fortiori Graph-Theoretic name ceased to be suitable. So I now use *path choice*, with reference to paths within a general connected graph.

Remark 1 First-classness can differ from path to path.

Remark 2 The realization of final notion of Dirac bracket can vary from path to path.

Remark 3 The type of termination condition can be distinct for each path.

Remark 4 Specifier-, Dirac-bracket- and obstruction-producing steps form meaningful subgraphs, as do any combination thereof.

Remark 5 Dirac's inconsistency insight leads to the possibility of *competing lattices*. This is via the possiblity of mutually-inconsistent pairs of constraints, in which case we may be able to arrange for one of the two to be absent. Each such choice then leads to a different lattice. The superposition of these various lattices is what we mean by a competing lattice. These are instances of posets, which are not themselves lattices when a nonzero amount of competing is present. Fig 7.a) exhibits the simplest nontrivial example of a such.

Remark 6 Geometrical examples of this most immediately serve the purpose of illustrating this. Namely, affine and special-projective generators are incompatible with special-conformal generators in the flat space setting. the corresponding top elements are Projective Geometry and Conformal Geometry respectively. Indeed, the Generalized Lie Algorithm (GLA) approach obtains this in terms of affine and special-projective generators both being incompatible with special-conformal generators [76, 96].

Remark 7 In the path choice and competing lattice set-up, one CAS is obtained per terminus.

Remark 8 Each path in general projects onto a different flag of subspaces Loops are possible, in the form of reaching the same subspace via two different intermediaries. This includes the possibility of a triangle, by which the structure in question is not a poset.

Example 1

$$H = \frac{1}{2}p_1^2 + q_2p_1 + q_3p_1p_4 + q_1p_4 \ . \tag{98}$$

The first iteration gives

$$p_1p_4 = 0 \ , \tag{99}$$

$$p_1 = 0 \ . \tag{100}$$

Two possible ways of solving this system are

$$p_1 = 0 = p_4 \tag{101}$$

and

$$p_1 = 0 \ \text{ by itself} \ . \tag{102}$$

The second iteration gives termination in the first case, and yet

$$p_4 = 0 \tag{103}$$

in the second case. Taking the second case one iteration further attains termination. A common terminus is thus reached, directly down one path and via one intermediary down the other, thus forming a triangle (Fig 7.b-c).

Remark 9 In Dirac's Little Algorithm, everything stated was under the aegis of all objects involved being one-killer, and of no topological obstruction terms or path choices occuring.

8.3 Path dependence's interplay with types of constraint

Remark 1 Only upon Dirac's Algorithm terminating do we have a bona fide *maximally* extended $H_{\mathcal{F}} = H_{\text{E}}$.

Remark 2 Each path choice thus gives connected digraphs.

Remark 3 Suppose that path choice has entered contention. Then we can clarify that Remark 1's maximality varies pathwise rather than being global over all paths.

Remark 4 Sec 6's statements are incomplete because of path choice. What we need to know is the entire output down the given path under consideration.

This is the precise sense in which one-killerness is global (a more mathematically-concrete adjective than 'holistic').

Remark 5 Suppose that path choice enters contention. Then we have a different chain peer path choice.

Remark 5 Path dependence can alter cohomology. Strong vanishing options can also alter characteristic class content.

Remark 6 Path dependence's interplay with types of appended Hamiltonian.

8.4 Path dependence's interplay with chains of Dirac brackets

Remark 1 Each path can have a distinct notion of final Dirac bracket.

Remark 2 The competing lattice of types of Dirac's bracket can moreover include loops. By this, two paths that split off can meet again.

9 Conclusion

The current article expounds on the Order-Theoretic meaning of the Dirac Algorithm's path dependence and its interplay with the other applications of Order Theory mentioned above. In particular, firstly, iterations pass from being *(sub)chains* to *(sub)digraphs*. Secondly, top elements are now in general nonunique, and each carries its own *competing lattice*. At least for finite lattices, this can also be encapsulated in its totality by a *poset* (a fortiori a *semi-lattice*). See Fig 5 for the general local case and Fig 7 for the general global case.

Meaningful updates of HTBook's material should verily be expressed in Order-Theoretic terms, example by example. Indeed, the whole subject should probably be rebuilt on joint Linear Algebra and Combinatorics foundations. Interest then shifts from what physical theories one considers to populating each smaller lattice with at least one physical example. Or with a geometrical example, as extension 4) below acommodates.

Everything covered in the current article carries over to the following.

1) Field Theory (including GR and Field Theory coupled thereto). Such an extension has as a concise language the Portmanteau Calculus [72, 81] for Finite-and-Field Theory. (This can be taken to be ordinary–Banach Calculus [37] at the sketch level, but ordinary–tame-Fréchet Calculus [38] when considered in detail.

2) To reduced treatments[25, 72, 45, 93], and to Extended Dirac Algorithms [47] such as the BRST [27, 28, 29, 47, 54] and BV approaches [35, 39, 54], whose alternative/extra Order-Theoretic underpinnings we shall shortly detail elsewhere [99]).

3) To Dirac-type Algorithms (DtAs) such as the TRi-Dirac Algorithm [65, 72, 77, 82, 90] (Temporal Relationalism implementing [70, 72]).

4) To the Generalized Lie Algorithm (GLA) [3, 85, 88, 90, 94] enlargement of scope of Dirac's Theory of Closure from constrained Dynamics to Lie Theory *in whatever context*.

The Generalized Lie Algorithm, and its Dirac(-type) subcase, have moreover been argued to be central to Background-Independent formulations of Natural Law, which, among other things, properly resolve [72, 89, 90, 91, 84, 85, 88, 96] the Problem of Time [20, 21, 48, 49, 63, 72] at least at the local and classical level.

5) In the Nijenhuis [14, 68], Gerstenhaber [68] and Nambu [62] brackets contexts (see [86, 87] for partial treatments of Closure Algorithms here).

Finally, in whatever meaningful combination of 1) to 5).

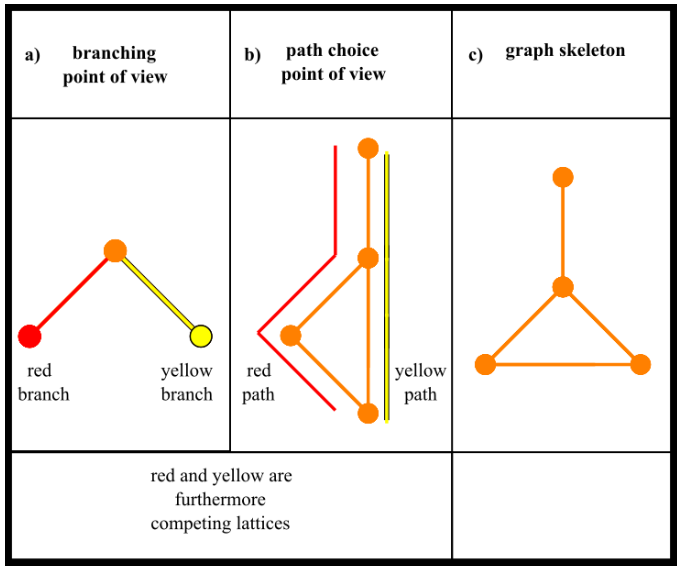

Figure 7:

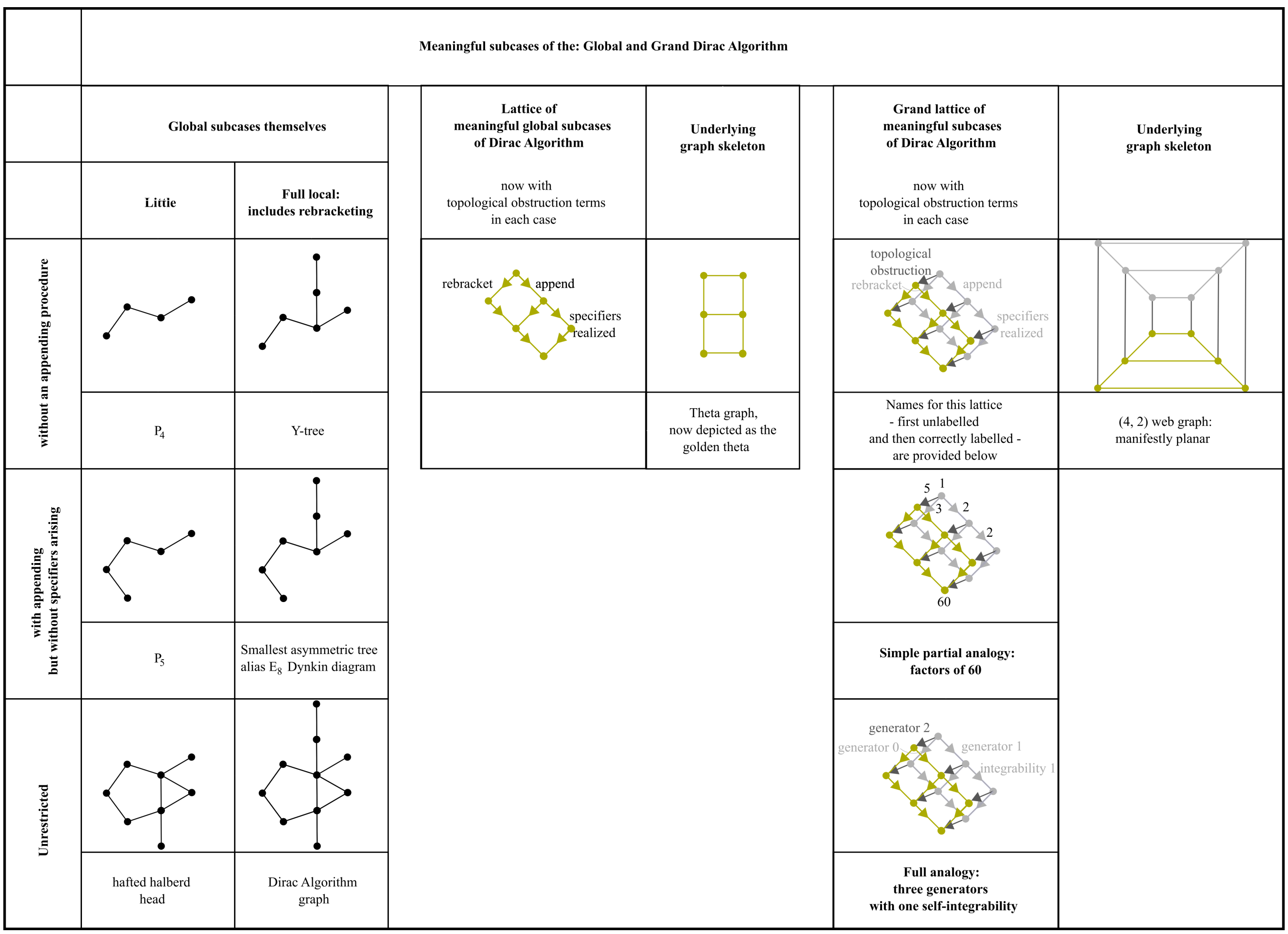

Figure 8:

Acknowledgments

My condolences to Donna and Niall's family. Niall was one of my mentors when I was young. Niall was exceptionally helpful toward me and many other people new to physics and mathematics. One of the main areas I first looked at alongside Niall in those days was Dirac's theory of constrained systems. I would have greatly liked to show Niall my work extending the technical foundations of this area. As things are, I have now instead written up this work at https://wordpress.com/post/conceptsofshape.space/814 and dedicated it to Niall. This dedication starts with what I write in Niall ó Murchadha's condolences book at https://rip.ie/cb.php?dn=472205 , yet it will grow over time as I think about Niall: mentor and friend.

By this point in time, I am happy to say that I too am and have for quite some time been supporting a number of more newly-arrived intellectuals and budding mentors. In a wide range of areas of the theory of STEM (Science, Technology, Engineering and Mathematics), of Conceptual Thinking, and more occasionally in the area of mentoring itself as well. Various of my life experiences have caused of me to do this; the way in which I do it, moreover, is itself one of the ways in which having had the pleasure of knowing Niall has had a strong and positive influence on my life. With many thanks also to Don Page, Chris Isham and Julian Barbour, and to one of the main two ways in which Britgrav (British Gravitational Conferences) are and have been run, for which I thank Carsten Gundlach in particular as well. This referring to the way of running Conference Series that puts giving a platform to newer researchers foremost. The main format, moreover, in which I deal with more newly-arrived intellectuals and budding mentors is the Queen Mary Style Discussion Group. As a first-year graduate Student, I agreed to run that for a couple of years, including quite often leading or co-leading discussions there. This is the model for the Applied Topology Discussion Group and the Applied Combinatorics Discussion Group that I have been running for the past year. This is a matter of working with newer people in areas that we strongly believe shall be fruitful for the next few decades, thus covering what most of these people will explore and make great use of. Various series of lecture notes, and books, shall be forthcoming on these matters.

Niall was a Professor of Physics in a style making considerable use of Mathematics. Niall had time for newer researchers to an extent that other Professors did not. Niall valued conceptual thinking, to get an understanding of problems prior to deciding whether it was worth investigating them rigorously and writing them up in a Mathematical Physics style. His recent conceptual thinking could be found in a book he carried around with him. This being of those books which are purchased as a series of blank pages for oneself to fill in. Like a diary, but bigger and without dates all over the place. This is a practice that I have myself adopted (my version thereof is electronic). This is one of the ways in which I remember Niall on a daily basis.

Aside from Dirac's approach to constrained systems, Niall and I discussed General Relativity's Initial-Value Problem, and its associated Conformal Geometry mathematics. This was the subject of his Ph.D. with Professor Jimmy York Jr. We furthermore discussed the theory of Partial Differential Equations, in particular of elliptic equations. Some harder versions of this occur in said GR IVP, yet this is a more widely-applicable subject as well. We also discussed notions of energy and of mass in General Relativity. By now, I have large lever-arch files of notes on each of the topics in this paragraph. (And multiple such on Dirac's approach and my generalization of it to all situations in which Lie Theory is applicable.) Some of these lever-arch folders may become books, or large parts of books on such as Geometry, the Principles of Dynamics, or Mathematical Relativity. When they do (or I get invited to write large yet not book-length reviews on these areas), Niall will certainly be among those who are most thanked in their prefaces. 'All situations in which Lie Theory is applicable' includes how I locally-solved the classical version of the Problem of Time in 2019 (with published and accepted accounts dated 2021), and also how this resolution's conterpart in pure Geometry gives new Foundations of Geometry. The Barbour-Foster-ó Murchadha paper's approach is part of this local classical resolution of the Problem of Time, and my two papers in collaboration with Niall are Conformal Geometry variants of this work. This is the 'Reconstruction' part of the Problem of Time, which is one of its five primary parts. Among other things, the Applied Topology and Applied Combinatorics work I have been dedicating half of my time to since 2014 is reshaping our understanding of the Global Problem of Time (the current article is the third to be publicly released in this regard). The Global Problem of time moreover a place where some of the above Mathematical Relativity and PDE topics discussed with Niall plays a prominent role as well.

Niall didn't just have time for newer researchers. Niall believed in us, and had confidence in us before we had self-confidence. For this, I am personally very thankful. I would not have had the confidence to undertake the above-mentioned works without Niall's supportiveness in those crucial early years.

Paragraphs 2 to 5 above are drafts I will update periodically, after plenty of slow thinking and contemplation. I also thank participants at the Global and Combinatorial Methods for Fundamental Physics Summer School 2021 for discussions, my strongest friend, and the person who stood their ground.

v2 Update (2024). This article's contents were developed in 2017 to 2019 with the Socratic support of S. Sánchez [73]. It marks the first of several times that S. Sánchez danced with Dirac at my behest. S. Sánchez is the wolf's share of the 'more newly-arrived intellectuals and budding mentors' alluded to in the original Acknowledgments. She gave me time-delayed Imprimatur by which she was not mentioned in the original Acknowledgments, but is now.

A rather general position is this. *Conceptual Analysis [64] is far less about product graphs and product orders and far more about breaking these [73]. So as to form sharper and more pertinent conceptualization. Whose graphs and orders are on occasion more interesting than the original products as well.*

The current Article gives an example of this. Namely, that one way to go one significant development further than Henneaux and Teitelboim's [47] update of Dirac's theory of contrained systems, is to Order-Theoretically classify all examples of constraint-algebraic structures. Actively seeking out in particular those which break graph, poset and lattice regularity criteria. Part VIII of my second book [95] gives a distinct example, in which I go after 'all the arenas of basic Combinatorial objects in this manner! This is the vanguard of the substantial Applied Combinatorics work alluded to in the original Acknowledgments.

As to the 'lever-arch folders' which 'may become books, or large parts of books on such as Geometry, the Principles of Dynamics, or Mathematical Relativity'. I can for now now give the update that two books on the Structure of Geometry are being written. As is a book on the Lie-Theoretic reworking of the Principles of Dynamics, and a further book on the portion of Local Lie Theory that this makes use of.

v1 of this Article was written in a prototype of (A, K and my version of) Testarossa. As such, the extent of its colour-coding is not fully mature. This will be remedied in the forthcoming v3. This also means that I thank A and K for discussions.

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