Two new versions of Heron's Formula

Edward Anderson*

Abstract

Heron's formula for the area of a triangle given its sides has a counterpart given its medians instead, which carries an extra factor of $\frac{4}{3}$.

Firstly, we formulate these two results Linear-Algebraically, showing that they are related by a sides-to-medians involution $\boldsymbol{J}$. Which we find to furthermore commute with the 'Heron map' $\boldsymbol{H}$ that occurs in the expanded version of Heron's formula. Upon further casting these two results in terms of mass-weighted Jacobi coordinates, we find moreover that the factor of $\frac{4}{3}$ cancels out, so identical form has been attained. This motivates our 'Heron–Jacobi' version of Heron's formulae, for mass-weighted area in terms of mass-weighted sides and mass-weighted medians respectively.

Secondly, we show that diagonalizing the Heron map $\boldsymbol{H}$ provides new derivations of both the famous Hopf map and Kendall's Little Theorem that the space of triangles is a sphere. This occurs by the 'Heron–Hopf' version of Heron's formula simplifying down to none other than the on-sphere condition! Thus we establish that – both a valuable fibre bundle model, and a foundational theorem of Shape Theory: a widely-applicable Differential Geometry and Topology topic – arise together. As consequences of just Heron's formula and some elementary Linear Algebra.

This working also accounts for the extra factor of 4 in the Hopf coordinate that is elsewise equal to the mass-weighted area in the 3-body problem context. It finally offers a new interpretation of the Shape-Theoretic ellipticity and anisoscelesness which realize the other two Hopf quantities for triangles. They are eigenvectors shared by the Heron map $\boldsymbol{H}$ and the sides-to-medians involution $\boldsymbol{J}$.

Mathematics keywords: Applied Geometry, triangles, spaces of triangles, Shape Geometry, Shape Statistics, relative Jacobi coordinates, Hopf fibration.

This is v2 of arXiv:1712.01441, which it supercedes because I, as the author of both, assert that it does. Updates in this v2 are reasonably significant.

* Dr.E.Anderson.Maths.Physics *at* protonmail.com . Institute for the Theory of STEM.

1 Introduction

1.1 Notation

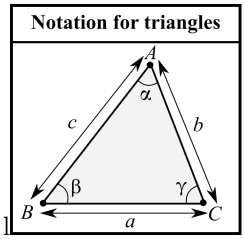

Figure 1:

Definition 1 Consider an arbitrary triangle $\triangle ABC$, denoted as per Fig 1. Using a_i, $i = 1$ to 3 to denote a, b, c will also turn out to be useful.

Definition 2 The *perimeter* P is

$$P := a + b + c = \sum_i a_i . \tag{1}$$

While the *semi-perimeter* is

$$\mathrm{s} := \frac{P}{2} = \frac{a + b + c}{2} = \frac{1}{2} \sum_i a_i . \tag{2}$$

Notational Remark 1 Let us use $\mathrm{Area}(ABC)$, or Area for short when unambiguous, to denote the area of $\triangle ABC$.

1.2 The usual Heron's formula

Theorem 1 (Heron's formula)

$$Area = \sqrt{\mathrm{s}(\mathrm{s} - a)(\mathrm{s} - b)(\mathrm{s} - c)} . \tag{3}$$

This is a classical result, known since the first century A.D. [1, 6]; see e.g. [15, 12, 19, 28] for some modern-era proofs.

Corollary 1 (Expanded Heron's formula)

$$Area = \frac{1}{4} \sqrt{\sum_{\mathrm{cycles}} (2\, a^2 b^2 - c^4)} . \tag{4}$$

1.3 Outline of the rest of the current paper

In Sec 2, we re-express (4) in Linear Algebra terms, involving what we term the 'Heron matrix', $\boldsymbol{H}$. In Sec 3, we recollect that side lengths control median lengths and vice versa. This is via a Corollary [31] of *Stewart's Theorem* [4, 11] (another classical result, now from the 18th century). We also recast this inter-relation in Linear Algebra terms. Showing furthermore that it can be formulated as an involution $\boldsymbol{J}$: the *sides-medians involution.* Perhaps surprisingly, $\boldsymbol{H}$ and $\boldsymbol{J}$ are furthermore shown to commute. This accounts for why the usual side's Heron formula and the medians' Heron formula ([21, 27], Sec 3) are very similar in appearance. They differ only by a relative factor of $\frac{4}{3}$.

We introduce 3-particle relative Jacobi coordinates in Secs 4 and 5. These are well-known to be useful in the N-body problem context [17, 24]. For the equal point masses case currently under consideration, medians and sides are coprimary as inputs to relative Jacobi coordinates. This gives the first reason – Jacobian motivation – why I am reappraising the theory of medians. Mre specifically, I am considering Jacobi mass-weighted sides-and-medians to be coprimary.

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We show moreover in Sec 6 that the Jacobi mass-weighted side and median forms of Heron's formulae – which we term *'Heron–Jacobi' formulae* – have *identical* form. Here the factor of $\frac{4}{3}$ gets absorbed and conceptually identified.

In Sec 7, we furthermore consider diagonalizing the Heron matrix $\boldsymbol{H}$. We observe this to give none other than a recovery of the famous *Hopf map* [7] (Appendix A). Which, in the present 2-d 3-body problem context [9, 14, 24, 33, 40, 42] (Appendix B), is also a way of obtaining *Kendall's Little Theorem* [13, 16, 25]. Which in turn states that that the space of all triangular shapes is a sphere. This is via the *'Heron–Hopf'* version of Heron's formula having reduced down to what is mathematically just the on-sphere condition.

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This approach furthermore builds upon the preceding use of Relative Jacobi coordinates. Which are thus useful in setting up Kendall's Shape Theory: a new subject of considerable promise [16, 23, 25, 33, 34, 35, 37, 38, 39, 36, 40, 41, 42, 43, 47, 44, 45, 48, 49, 51, 52].

This working also accounts for the extra factor of 4 in the Hopf coordinate that, in the 3-body problem context, is mass-weighted area up to this factor. It furthermore points to the other 2 Hopf coordinates – interpreted in [30, 33, 42] in the 3-body problem context as *ellipticity* and *anisoscelesness* – being comparably motivated to the much more well-known area variable. This is from the point of view that these 2 variables feature co-primarily with the area as a set of 3 Cartesian axes for the shape sphere's natural ambient $\mathbb{R}^3$. It additionally offers a new interpretation of the Shape-Theoretic ellipticity and anisoscelesness realizations of Hopf's other 2 quantities. Namely, these are 2 of the Heron map $\boldsymbol{H}$'s eigenvectors, which, by commutativity, are also shared with the sides–median involution map $\boldsymbol{J}$.

We consider Linear Algebra for the Hopf quantities in Sec 8. This permits us to show in Sec 9 that the ellipticity and anisoscelesness quantities are invariant in form under exchange of sides and medians. Up to signs allowed as part of changing Cartesian axes. This provides our second – now 'Hopfian' – motivation for coprimary treatment of sides and medians. Sec 10 finally summarizes what we term the *'Heron–Hopf'* and *'Heron–Kendall–Hopf'* forms of Heron's formula: the area-subject and symmetrical presentations respectively. Alongside giving two (almost) equivalent concomitant formula with ellipticity and anisoscelesness as their subjects respectively.

As complementary reading, see [25, 40, 42, 51] for accounts of Kendall's Shape Theory in general and the shape space of triangles in particular, And also [42, 52] for an outline of the Hopf map and its realization in the Shape Theory of triangles.

2 The Heron matrix

Lemma 1 The expanded form (4) of Heron's formula (3) can be recast in Linear Algebra terms as the following quadratic form in the squares $a_i{}^2$, so it is quartic in the a_i themselves.

$$(4 \times \text{Area})^2 = H_{i\,j}\, a_i{}^2\, a_j{}^2 \ . \tag{5}$$

For '*Heron matrix*' [20]

$$\boldsymbol{H} := \frac{1}{3} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} . \tag{6}$$

Naming Remark 1 A truer name for the formula habitually named after Heron is '*area from side data* formula for triangles'.

3 The sides–medians involution

Definition 1 The *medians* of a triangle are as per 2.a)-b). It will also be useful for us to use m_i , $i = 1$ to 3 to denote m_a , m_b and m_c .

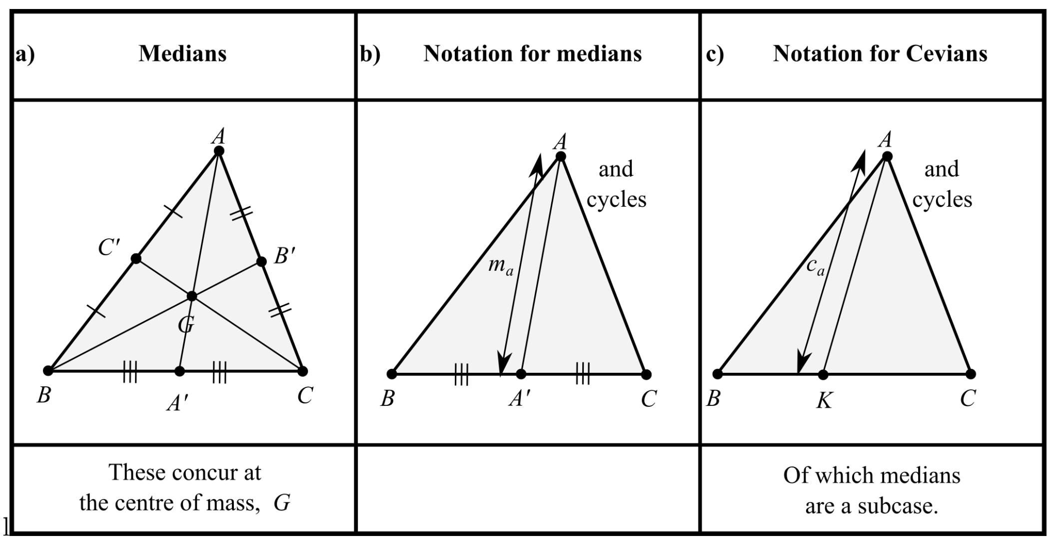

Figure 2:

Remark 1 By treating sides and medians as coprimary, we have more definitions (or at least accordances of equal significance) than in hitherto standard treatments of triangles. Starting with the following.

Definition 2 The *medimeter* M is

$$M := m_a + m_b + m_c = \sum_i m_i \ . \tag{7}$$

Whereas the *semi-medimeter* is

$$s := \frac{M}{2} \ . \tag{8}$$

Remark 2 The perimeter and medimeter can furthermore be viewed as first moments of sides and medians respectively. The second moments counterparts of each of these also enter the current paper, as follows.

Definition 3 The *second moment of sides* is

$$P_{\rm II} \ := \ a^2 \ + \ b^2 \ + \ c^2 \ = \ \sum_i {a_i}^2 \ =: \ \sum_i S_i \ . \tag{9}$$

While the *second moment of medians* is

$$M_{\rm II} \ := \ {m_a}^2 \ + \ {m_b}^2 \ + \ {m_c}^2 \ = \ \sum_i {m_i}^2 \ := \ \sum_i M_i \ . \tag{10}$$

S_i and M_i are components of subsequently useful vectors $\boldsymbol{S}$ and $\boldsymbol{M}$.

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Theorem 2 (Stewart's Theorem) Let $\triangle ABC$ be a triangle with L an arbitrary point on side BC . Then

$$AL^2 \ = \ \frac{LC}{BC} AB^2 \ + \ \frac{BL}{BC} AC^2 \ - \ BL\,LC \ . \tag{11}$$

Proof This follows in two lines from the cosine rule. □

Naming Remark 1 The segment $AL \ = \ c_a$ is in general a *Cevian* [2, 3, 10, 11, 26] (Fig 2.c). These enter Hud–Ceva's Theorem, which has Affine-Geometric significance. In Euclidean Geometry, Cevians have lengths as well, and the function of Stewart's Theorem is to compute these. A more functional name for Stewart's Theorem is thus *Cevian length Theorem*.

Remark 3 Medians are indeed a simple subcase of Cevians, hence the relevance of Stewart's Theorem to the current paper, as follows.

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Corollary 2 i) The median lengths' squares are given by

$$M_a \ = \ {m_a}^2 \ = \ \frac{2\,b^2 \ + \ 2\,c^2 \ - \ a^2}{4} \ \text{ and cycles } . \tag{12}$$

ii) The second moments of medians and of sides are related by

$$M_{\rm II} \ = \ \frac{3}{4} P_{\rm II} \ . \tag{13}$$

<u>Proof</u> i) This readily follows from Stewart's Theorem, as per worked problem 1 of [31].

ii) then follows immediately from both parts of Definition 6 upon summing i) over all cycles. □

Corollary 3 i) In Linear Algebra form,

$$\begin{pmatrix} {m_a}^2 \\ {m_b}^2 \\ {m_c}^2 \end{pmatrix} \ = \ \frac{1}{4} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \begin{pmatrix} a^2 \\ b^2 \\ c^2 \end{pmatrix} \ . \tag{14}$$

I.e.

$$\underline{\boldsymbol{M}} \ = \ \frac{1}{4} \underline{\underline{\boldsymbol{B}}} \cdot \underline{\boldsymbol{S}} \ . \tag{15}$$

Where

$$\boldsymbol{B} \ := \ \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \ . \tag{16}$$

Observe that this is a symmetric matrix,
ii) Inverting, we find that

$$\underline{\boldsymbol{S}} \;=\; \frac{4}{9}\,\underline{\underline{\boldsymbol{B}}} \cdot \underline{\boldsymbol{M}} \;. \tag{17}$$

Remark 4 That the same $\boldsymbol{B}$ appears in the inverted expression indicates that $\boldsymbol{B}$ is proportional to an *involution* $\boldsymbol{J}$. I.e. it is a matrix such that

$$\boldsymbol{J}^2 \;=\; \mathbb{1} \;: \quad \text{the identity matrix} \;. \tag{18}$$

We can thereby further tidy up Corollary 1's Linear Algebra formulation by identifying and using $\boldsymbol{J}$, as follows.

Corollary 4 i)

$$\underline{\boldsymbol{M}} \;=\; \frac{3}{4}\,\underline{\underline{\boldsymbol{J}}} \cdot \underline{\boldsymbol{S}} \;. \tag{19}$$

ii)

$$\underline{\boldsymbol{S}} \;=\; \frac{4}{3}\,\underline{\underline{\boldsymbol{J}}} \cdot \underline{\boldsymbol{M}} \;. \tag{20}$$

For *sides-medians involution*

$$\boldsymbol{J} \;:=\; \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \;=\; \frac{1}{3}\,\boldsymbol{B} \;. \tag{21}$$

Rescaling of course preserves the symmetric character of the matrix.

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Theorem 3 (Medians' Heron formula, alias 'area from median data' formula for triangles).

$$\text{Area} \;=\; \frac{4}{3}\sqrt{\text{s}\,(\,\text{s} \;-\; m_a\,)\,(\,\text{s} \;-\; m_b\,)\,(\,\text{s} \;-\; m_c\,)} \;=\; \frac{1}{3}\sqrt{\sum_{\text{cycles}}\,(\,2\,{m_a}^2\,{m_b}^2 \;-\; {m_c}^4\,)} \;. \tag{22}$$

Proof While traditional Geometric proofs of this are not uncommon [27], I give instead a striking Linear Algebra proof. First observe Lemma 1's Linear Algebra form of the square of Corollary 1's expanded Heron formula. Next substitute Corollary 3.ii) in,

$$(\,4 \times \text{Area}\,)^2 \;=\; \underline{\boldsymbol{S}} \cdot \underline{\underline{\boldsymbol{H}}} \cdot \underline{\boldsymbol{S}} \;=\; \left(\frac{4}{3}\,\underline{\underline{\boldsymbol{J}}} \cdot \underline{\boldsymbol{M}}\right) \cdot \underline{\underline{\boldsymbol{H}}} \cdot \left(\frac{4}{3}\,\underline{\underline{\boldsymbol{J}}} \cdot \underline{\boldsymbol{M}}\right) \;=\; \frac{16}{9}\,\underline{\boldsymbol{M}} \cdot \underline{\underline{\boldsymbol{J}}} \cdot \underline{\underline{\boldsymbol{H}}} \cdot \underline{\underline{\boldsymbol{J}}} \cdot \underline{\boldsymbol{M}} \;. \tag{23}$$

Making use of the symmetric matrix property in the last step.

Thus evaluating the matrix product,

$$\text{Area}^2 \;=\; \frac{1}{9}\,\underline{\boldsymbol{M}} \cdot \underline{\underline{\boldsymbol{H}}} \cdot \underline{\boldsymbol{M}} \;. \tag{24}$$

So reversing the expansion of Heron with m_i in place of a_i, (22) ensues. □

Remark 5 This proof contains an insight which traditional Geometric proofs miss. Namely, that the sides–medians involution matrix $\boldsymbol{J}$ and the 'Heron matrix' $\boldsymbol{H}$ *commute*,

$$[\,\boldsymbol{J},\; \boldsymbol{H}\,] \;=\; \mathbb{0} \;. \tag{25}$$

$$\underline{\underline{\boldsymbol{J}}} \cdot \underline{\underline{\boldsymbol{H}}} \;=\; \underline{\underline{\boldsymbol{H}}} \cdot \underline{\underline{\boldsymbol{J}}} \;. \tag{26}$$

Observe furthermore that

$$\underline{\underline{\boldsymbol{J}}} \cdot \underline{\underline{\boldsymbol{H}}} \ = \ \underline{\underline{\boldsymbol{K}}} \ = \ \underline{\underline{\boldsymbol{H}}} \cdot \underline{\underline{\boldsymbol{J}}} \tag{27}$$

for

$$\boldsymbol{K} \ := \ \frac{1}{3} \begin{pmatrix} 5 & \text{-1} & \text{-1} \\ \text{-1} & 5 & \text{-1} \\ \text{-1} & \text{-1} & 5 \end{pmatrix} \ . \tag{28}$$

Remark 6 It is because of this that the 'median-Heron' matrix $\boldsymbol{G}$ in the a priori conceptual form of (24),

$$Area^2 \ = \ \underline{\boldsymbol{M}} \cdot \underline{\underline{\boldsymbol{G}}} \cdot \underline{\boldsymbol{M}} \ , \tag{29}$$

is just proportional to the 'Heron matrix' itself, as follows.

$$\boldsymbol{G} \ = \ \frac{1}{9} \, \boldsymbol{H} \ . \tag{30}$$

Remark 7 In summary, the sides-Heron and medians-Heron formulae are as follows.

$$\sqrt{\mathrm{s}\,(\mathrm{s} \ - \ a\,)\,(\mathrm{s} \ - \ b\,)\,(\mathrm{s} \ - \ c\,)} \ = \ \text{Area} \ = \ \frac{4}{3} \sqrt{s\ (\,s \ - \ m_a\,)\ (\,s \ - \ m_b\,)\ (\,s \ - \ m_c\,)} \ . \tag{31}$$

4 Jacobi coordinates for the triangle

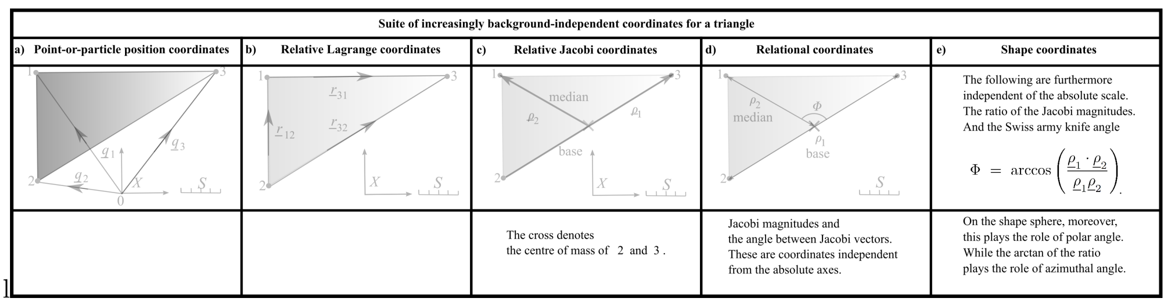

Figure 3:

Structure 1 Let us next consider our triangle's vertices to be equal-mass particles. With position vectors $\underline{q}_I$, $I \ = \ 1$ to 3 , relative to an absolute origin 0 and axes A (Fig 3.a).

Definition 1 The *inertia quadric* for N particles in any dimension is as follows.

$$I\left(\underline{q}_I\right) \ = \ \sum_{I\,=\,1}^{N} m_I \, q_I{}^2 \ = \ \sum_{I\,=\,1}^{N} q_I{}^2 \ . \tag{32}$$

Where the last equality is for equal masses, taken without loss of generality to be of unit size.

Remark 1 Translating the origin by some arbitrary amount $\underline{a}$,

$$I\left(\underline{q}_I, \underline{a}\right) \ = \ \sum_{I\,=\,1}^{N} m_I \, \left|\left|\underline{q}_I \ - \ \underline{a}\right|\right|^2 \ . \tag{33}$$

Furthermore, extremizing with respect to $\underline{a}$,

$$\underline{a} \ = \ \frac{1}{M} \sum_{I\,=\,1}^{N} m_I \, q_I \ =: \ \underline{a}_{\mathrm{CoM}} \ . \tag{34}$$

Where 'CoM' stands for 'centre of mass'. And where

$$M := \sum_{I = 1}^{N} \mathrm{m}_I \tag{35}$$

is the *total mass*. I.e. introducing an arbitrary $\underline{a}$ and extremizing thereover picks out the centre of mass position.

Then substituting for this back in (33), one obtains the following relative Lagrangian version of the inertia quadric.

$$I(\underline{r}_{IJ}) = \frac{1}{M} \sum_{I = 1}^{N} \sum_{\substack{J = 1 \\ I < J}}^{N} m_I m_J {r_{IJ}}^2 . \tag{36}$$

For *relative Lagrange coordinates*

$$\underline{r}_{IJ} := \underline{q}_J - \underline{q}_I \tag{37}$$

(see Fig 3.b)). Formulating in terms of these, by virtue of their being differences of position vectors $\underline{q}_I$, all reference to the absolute origin, 0, is cancelled out.

Remark 2 For $N \geq 3$, this object has disadvantages stemming from not all the $\underline{r}_{IJ}$ being independent. Non-diagonality ensues. This can of course be circumvented by diagonalization, which, in this context, amounts to passing to *relative Jacobi coordinates*. These are moreover no longer in general inter-particle separations, being rather the broader concept of inter-particle *cluster* separations. As we shall see below for the particular example of $N = 3$ in 2-d – the triangle – this generalization involves relative separations between subsystem centres of mass. [This concept includes inter-particle separations by the identity that particle positions coincide with that 1-particle subsystem's centre of mass.]

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Remark 3 To proceed for our particular example, we rewrite (36) as the following quadratic form.

$$I(\underline{r}_{IJ}) = L_{IJ} q_I q_J = \underline{\boldsymbol{q}} \cdot \underline{\underline{\boldsymbol{L}}} \cdot \underline{\boldsymbol{q}} . \tag{38}$$

Where $\boldsymbol{L}$ is the 'relative Lagrange matrix'.

Narrowing down consideration to $N = 3$,

$$\frac{1}{3} \left({r_{12}}^2 + {r_{13}}^2 + {r_{23}}^2 \right) . \tag{39}$$

For which we have the specific form

$$\boldsymbol{L} = \frac{1}{3} \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} . \tag{40}$$

Remark 4 The corresponding characteristic equation is as follows.

$$0 = \det(\boldsymbol{L} - \lambda \mathbb{1}) = \xi^3 - 3\xi - 2 . \tag{41}$$

Where

$$\xi := 2 - \lambda . \tag{42}$$

The Factor Theorem then gives that $\xi = -1$ solves, reducing the problem to a quadratic equation. Consequently,

$$(\xi - 2)(\xi + 1)^2 = 0 . \tag{43}$$

So $\xi = 2$ or -1. Thus the eigenspectrum is $\lambda = 0, 3$ with multiplicities 1 and 2 respectively.

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Remark 5 Corresponding orthonormal eigenvectors are, respectively,

$$\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} \ , \ \frac{1}{\sqrt{2}}\begin{pmatrix}0\\-1\\1\end{pmatrix} \ , \ \frac{1}{\sqrt{6}}\begin{pmatrix}2\\-1\\-1\end{pmatrix} \ . \tag{44}$$

Remark 6 The first of these corresponds to the eigenvalue 0 and is the centre of mass coordinate. This occurs no matter what N is, and contributes nothing to the diagonalized relative Jacobi form of the inertia quadric. The inertia quadric can thereby be considered to involve one coordinate vector less: $n = N - 1$ coordinate vectors. So we write it as

$$I\left(\widetilde{\underline{\boldsymbol{R}}}\right) = \widetilde{\underline{\boldsymbol{R}}} \cdot \widetilde{\underline{\underline{\boldsymbol{Y}}}} \cdot \widetilde{\underline{\boldsymbol{R}}} \ . \tag{45}$$

Where the $\widetilde{\underline{R}}$ are proportional to the conventional relative Jacobi coordinate vectors. Let us tilde everything for now so as to reserve the untilded version for the conventionally used proportions themselves.

Remark 7 For $N = 3$,

$$\widetilde{\boldsymbol{Y}} = \text{diag}(\,1,\,1\,) \ . \tag{46}$$

We thus arrive at

$$I\left(\widetilde{\underline{\boldsymbol{R}}}\right) = \widetilde{R}_1{}^2 + \widetilde{R}_2{}^2 \ . \tag{47}$$

The conventional scaling is

$$\underline{R}_1 := \underline{q}_3 - \underline{q}_2 \ , \ \underline{R}_2 := \underline{q}_1 - \frac{\underline{q}_2 + \underline{q}_3}{2} \ . \tag{48}$$

Which, as promised, is recognizable as consisting entirely of cluster separation vectors. The first is a fortiori an interparticle separation vector, whereas the second involves a 2-particle centre of mass (see Fig 3.c).

If these are used, the diagonal relative Jacobi separation matrix $\boldsymbol{Y}$ furthermore consists of the reduced masses of the clusters in question. I.e.

$$\boldsymbol{Y} = \text{diag}\left(\frac{1}{2}, \frac{2}{3}\right) \ . \tag{49}$$

This is indeed the standard definition of reduced mass. I.e. conceptually

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \ . \tag{50}$$

Which rearranges to the more computationally immediate form

$$\mu = \frac{m_1\,m_2}{m_1 + m_2} \ . \tag{51}$$

For equal masses, this gives the following, as claimed.

$$\mu_1 = \frac{1 \times 1}{1 + 1} = \frac{1}{2} \ . \tag{52}$$

And

$$\mu_2 = \frac{1 \times 2}{1 + 2} = \frac{2}{3} \ . \tag{53}$$

Naming Remark 1 The relative Jacobi separation matrix can thus be allotted a further, now conceptual, name: *reduced mass matrix*. This is with reference to the cluster subsystems picked out in the allocation of the particular Jacobi coordinates at hand. We mark this by replacing the

$\boldsymbol{Y}$ notation with $\underline{\underline{\mu}}$. Which we take to be a capital μ , standing for both 'mass' and 'diagonal' (paralleling how Λ is probably the most common notation for a diagonal matrix). I.e. the *Jacobi mass matrix*.

Remark 8 So we end up with a relative Jacobi inertia quadric of the form

$$I(\underline{\boldsymbol{R}}) = \underline{\boldsymbol{R}} \cdot \underline{\underline{\mu}} \cdot \underline{\boldsymbol{R}} . \tag{54}$$

Remark 9 For $N = 3$, the sole ambiguity in picking out cluster subsystems in forming Jacobi coordinates is which 2 points-or-particles to start with. So there are 3 possible clustering choices, corresponding to the second orthonormal eigenvector above being free to have its zero in whichever component.[1]

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Notational Remark 1 Let us denote the above choice by $\underline{R}^{(1)}$ alias $\underline{R}^{(a)}$. And the clusters with $\underline{r}_{31}$ and $\underline{r}_{12}$ as their first relative Jacobi coordinate by $\underline{R}^{(2)}$ alias $\underline{R}^{(b)}$ and $\underline{R}^{(3)}$ alias $\underline{R}^{(c)}$ respectively.

Notational Remark 2 Let us furthermore denote μ_1 by μ_{s} – *side Jacobi mass* – and μ_2 by μ_{m} : *median Jacobi mass*. This is *possible* since the μ_i are cluster choice independent. And *useful* by its replacing the 1 and 2 labels with more conceptually meaningful and memorable labels: s for side and m for median. We follow suit by calling the triangle model's first and second relative Jacobi vectors the *side* and *median* vectors. For all that these *are* cluster-dependent. I.e.

$$R_1^{(a_i)} = a_i , \tag{55}$$

$$R_2^{(a_i)} = m_i . \tag{56}$$

Notational Remark 3 Paralleling our use of $\boldsymbol{M}$ and $\boldsymbol{S}$, let us introduce $\boldsymbol{C}_a$ as the vector with components $C_a^{(i)\,2}$, for $a = 1, 2$. This $\boldsymbol{C}$ can be taken to stand for 'clustering'.

Corollary 5 i)

$$\underline{\boldsymbol{C}}_2 = \frac{3}{4} \underline{\underline{\boldsymbol{J}}} \cdot \underline{\boldsymbol{C}_1} . \tag{57}$$

Inverting to ii)

$$\underline{\boldsymbol{C}}_1 = \frac{4}{3} \underline{\underline{\boldsymbol{J}}} \cdot \underline{\boldsymbol{C}_2} . \tag{58}$$

<u>Proof</u> Substitute (48) into the Linear Algebra form of the sides–medians relation (19). □

5 Mass-weighted Jacobi coordinates

Remark 1 Our principal interest in the current paper concerns notions deriving from the following.

Structure 1 *Mass-weighted relative Jacobi coordinates* are given by the following.

$$\underline{\rho}_a := \sqrt{\mu_a}\, \underline{R}_a . \tag{59}$$

Where the a-index takes the values 1 and 2 .

Structure 2 *Mass-weighted relative Jacobi separations* are the magnitudes of the preceding,

$$\rho_a := \sqrt{\mu_a}\, R_a . \tag{60}$$

[1]For $N \geq 4$, there are further ambiguities. Which can be shown to result from $N \geq 4$ points supporting multiple shapes of tree graph (see e.g. [40, 47]). Jacobi coordinates [5] are widely used for instance in Celestial Mechanics [17] and in Molecular Physics [24].

Remark 2 Thus computationally,

$$\rho_1^{(a)} \;:=\; \sqrt{\mu_1}\, R_1^{(a)} \;=\; \frac{a}{\sqrt{2}} \;\text{ and cycles } . \tag{61}$$

Alongside

$$\rho_1^{(a)} \;:=\; \sqrt{\mu_1}\, R_1^{(a)} \;=\; \frac{m_a}{\sqrt{2}} \;=\; \frac{\sqrt{2\,b^2 \;+\; 2\,c^2 \;-\; a^2}}{2\,\sqrt{2}} \;\text{ and cycles } . \tag{62}$$

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Definition 1 The *mass-weighted inertia quadric* is

$$I\left(\underline{\rho}_a\right) \;=\; \sum_a {\rho_a}^2 \;=\; {\rho_1}^2 \;+\; {\rho_2}^2 \;, \tag{63}$$

Where the last equality is for $N \;=\; 3$. Computationally, this amounts to returning to the previous section's tilded formulation. So one motivation for the mass-weighted relative Jacobi coordinates is that they are what drops out of the Linea Algebra approach. Another ensues from the matrix in the quadric being the identity, alongside the following interpretation.

Structure 3 *Relative space* is the space of independent relative separations. This is furthermore equipped the standard flat metric. This is numerically equal to $\widetilde{\boldsymbol{Y}}$, but merits a new conceptual notation $\widetilde{\boldsymbol{R}}$, standing for 'relative-space'. And yet is computationally just the identity matrix, $\mathbb{1}$.

Remark 2 The Cartesian equivalence in this (mass-weighted notion of) relative space of these moreover constitutes the

'Jacobian' first motivation for treating medians and sides coprimarily.

Mass-weighted medians and mass-weighted sides are geometrically coprimary in (mass-weighted) relative space. These moreover drop out of a Linear-Algebraic treatment most directly, in obtaining Jacobi coordinates by diagonalization. Motivating relative Jacobi coordinates themselves has further parts to it. For, in addition to being useful in treating the N-body problem, they turn out to be coordinates in terms of which the shape space's own natural coordinates are simple [42].

Remark 3 The mass-weighted Jacobi separations are furthermore related to the more widely known *partial moments of inertia* I_a as follows.

$$\rho_a \;=\; \sqrt{I_a} \;\text{ i.e. }\; I_a \;=\; {\rho_a}^2 \;. \tag{64}$$

In particular, with clustering labels explicit,

$$I_1^{(a)} \;=\; {\rho_1}^2 \;=\; \mu_1\, {R_1}^2 \;=\; \frac{a^2}{2} \;\text{ and cycles } . \tag{65}$$

And

$$I_2^{(a)} \;=\; {\rho_2}^2 \;=\; \mu_2\, {R_2}^2 \;=\; \frac{{m_a}^2}{2} \;\text{ and cycles } . \tag{66}$$

♠ ◇ ♣

Definition 2 More familiarly, summing over disjoint partial moments rather than over clusters, the *total moment of inertia* is

$$I^{(a)} \;:=\; I_1^{(a)} \;+\; I_2^{(a)} \;. \tag{67}$$

Remark 4 So the total object is the sum of all disjoint partial contributions.

Lemma 2 (Democratic formula for the moment of inertia)[2]

$$I = \frac{a^2 + b^2 + c^2}{3} = \frac{1}{3}\sum_i {a_i}^2 = \frac{P_{\rm II}}{3} . \tag{68}$$

<u>Proof</u>

$$I^{(a)} = I_1^{(a)} + I_2^{(a)} = \frac{1}{2}a^2 + \frac{2}{3}{m_a}^2 = \frac{a^2}{2} + \frac{2}{3}\frac{2b^2 + 2c^2 - a^2}{4} = \frac{3a^2 + 2b^2 + 2c^2 - a^2}{6} , \tag{69}$$

from which the result follows. The first equality is (67). The second uses (65, 66) and the third uses Corollary 2. Finally, the last two steps are just elementary algebra. □

Remark 5 By this Lemma's right-hand-side's democracy invariance, we are entitled to rewrite (67) stripped of its left-hand side clustering dependence (a) as follows.

$$I := I_1^{(a)} + I_2^{(a)} \text{ or cycles} . \tag{70}$$

It is also clear from the I-ρ inter-relation and $I\left(\underline{\rho}_a\right)$ formula that total moment of inertia is another name for the inertia quadric. One could argue that $I\left(\underline{q}_I\right)$ and $I\left(\underline{r}_{IJ}\right)$ were *a priori* clustering-independent formulations. Whereupon $I\left(\underline{\rho}_a\right)$ introduced prima facie clustering dependent features. Further inspection, however, confirms the cluster-dependent labels on these to be spurious since labelling-independence can indeed be maintained in Jacobi coordinates. So I is inherently cluster-independent.

♠ ◇ ♣

Notational Remark There is already a notation available for squared quantities in this case: $\boldsymbol{I}_a$, $a = 1, 2$ for vectors with components $I_a^{(i)\,2}$.

Corollary 6
i)

$$\underline{\boldsymbol{I}}_2 = \underline{\underline{\boldsymbol{J}}} \cdot \underline{\boldsymbol{I}}_1 . \tag{71}$$

iv) This inverts to

$$\underline{\boldsymbol{I}}_1 = \underline{\underline{\boldsymbol{J}}} \cdot \underline{\boldsymbol{I}}_2 . \tag{72}$$

<u>Proof</u> i) Substitute (64) and (59) into the Linear Algebra form of the sides–medians relation (19), as follows.

$${\mu_2}^{-1}\,\underline{\boldsymbol{I}}_2 = \frac{3}{4}\,\underline{\underline{\boldsymbol{J}}} \cdot {\mu_1}^{-1}\,\underline{\boldsymbol{I}}_1 \Rightarrow \underline{\boldsymbol{I}}_2 = \frac{3}{4} \times \frac{4}{3}\,\underline{\underline{\boldsymbol{J}}} \cdot \underline{\boldsymbol{I}}_1 . \tag{73}$$

ii) Just invert, using the involution property. □

Remark 5 Observe that the mass-weighting cleans out the awkward numerical factor of $\frac{4}{3}$ in eqs. (57-58), revealing this to be as follows.

$$\frac{4}{3} = \frac{\mu_{\rm m}}{\mu_{\rm s}} = \frac{(\text{ Jacobi reduced mass of median })}{(\text{ Jacobi reduced mass of corresponding side })} . \tag{74}$$

Remark 6 This can also be accorded a democratic, i.e. clustering-independent interpretation as follows.

$$\frac{4}{3} = \frac{\mu_{\rm m}}{\mu_{\rm s}} = \text{geometric mean over all clusters}\left(\frac{(\text{ Jacobi reduced masses of medians })}{(\text{ Jacobi reduced masses of sides })}\right) =$$

$$\left(\prod_{i=1}^{3}\frac{\mu_{{\rm m}_i}}{\mu_{{\rm s}_i}}\right)^{1/3} . \tag{75}$$

[2]This is a moment of inertia if the $1/3$ is accorded units of mass: the reduced mass for the whole system. If this is not entertained, the formula is, rather, for the radius of gyration squared.

6 Consequent Heron–Jacobi formulae

Notation 1 Let us introduce the *mass-weighted semi-perimeter*, σ , and the *mass-weighted semi-medimeter*, ς .

Remark 1 Passing to mass-weighted Jacobi versions of Heron's formula requires furthermore knowing how the bounding quantities scale.

Lemma 3 i) *Mass-weighted semi-perimeter* is a mass-weighting side-vector,[3]

$$\sigma \ := \ \sqrt{\mu_{\mathrm{s}}}\,\mathrm{s} \ = \ \frac{\mathrm{s}}{\sqrt{2}} \ . \tag{76}$$

ii) *Mass-weighted semi-medimeter* is a mass-weighting median-vector,

$$\varsigma \ := \ \sqrt{\mu_{\mathrm{m}}}\, s \ = \ \sqrt{\frac{2}{3}}\, s \ . \tag{77}$$

iii) Area is a mass-weighting side–median bivector,

$$\mathcal{A}\mathrm{rea} \ = \ \sqrt{\mu_{\mathrm{s}}\,\mu_{\mathrm{m}}}\,\mathrm{Area} \ = \ \frac{\mathrm{Area}}{\sqrt{3}} \ . \tag{78}$$

<u>Proof</u> i)

$$s \ := \ \sum_i a_i \ = \ \sum_i R_1^{(i)} \ = \ \frac{1}{\sqrt{\mu_1}} \sum_i \rho_1^{(i)} \ = \ \frac{\sigma}{\sqrt{\mu_{\mathrm{S}}}} \ = \ \sqrt{2}\,\sigma \ . \tag{79}$$

ii)

$$\mathrm{s} \ := \ \sum_i m_i \ = \ \sum_i R_2^{(i)} \ = \ \frac{1}{\sqrt{\mu_2}} \sum_i \rho_2^{(i)} \ = \ \frac{\varsigma}{\mu_{\mathrm{m}}} \ = \ \sqrt{\frac{3}{2}}\,\varsigma \ . \tag{80}$$

iii)

$$\mathrm{Area} \ = \ \frac{1}{2}\,(\underline{\mathrm{R}}_1 \ \times \ \underline{\mathrm{R}}_2)_3 \ = \ \frac{1}{2\sqrt{\mu_1\,\mu_2}} \left(\underline{\rho}_1 \ \times \ \underline{\rho}_2\right)_3 \ = \ \frac{\mathcal{A}\mathrm{rea}}{\sqrt{\mu_1\,\mu_2}} \ = \ \frac{\mathcal{A}\mathrm{rea}}{\sqrt{\frac{1}{2}\,\frac{2}{3}}} \ = \ \sqrt{3}\,\mathcal{A}\mathrm{rea} \ . \ \Box \tag{81}$$

Theorem 4 (Mass-weighted area's *Heron–Jacobi formulae* in terms of each of mass-weighted sides and mass-weighted medians)

$$\sqrt{\sigma\left(\sigma \ - \ \rho_1^{(a)}\right)\left(\sigma \ - \ \rho_1^{(b)}\right)\left(\sigma \ - \ \rho_1^{(c)}\right)} \ = \ \mathcal{A}\mathrm{rea} \ = \ \sqrt{\varsigma\left(\varsigma \ - \ \rho_2^{(a)}\right)\left(\varsigma \ - \ \rho_2^{(b)}\right)\left(\varsigma \ - \ \rho_2^{(c)}\right)} \ . \tag{82}$$

<u>Proof</u> See Fig 4. $\Box$

Remark 2 Observe that the sides and medians versions are now identical in form. I.e. without any constant prefactor difference like the $\frac{4}{3}$ in the mass-unweighted medians case (Theorem 1) relative to the mass-unweighted sides case (Theorem 2). This amounts to explaining the $\frac{4}{3}$ factor discrepancy between the medians' Heron's formula and the standard sides' Heron's formula. As resulting from formulating Heron's formula for area rather than for mass-weighted area.

Pass to Jacobi coordinates. Therein, mass-weighted area is more natural for formulating Heron's formulae. Now the mass-weighted sides' version and the mass-weighted medians' version share identical form: without any such numerical factor. The numerical factor's significance is thus unmasked to be the same ratio of reduced masses as tidied up the preceding Linear Algebra, as per eq. (74). This is embodied in the Heron–Jacobi formulae (82). Compare with the less symmetrical, and thus less transparent mass-unweighted summary equation (31).

[3]See [44] for further exposition of 'side-vector', 'median-vector' and 'side-median bivector'.

Proof of Theorem 4

$$(4 \times \mathcal{A}\mathrm{rea})^2 \overset{(5,\,55,\,52,\,60)}{=} \underline{\boldsymbol{I}}_1 \cdot \underline{\underline{\boldsymbol{H}}} \cdot \underline{\boldsymbol{I}}_1 \overset{(72)}{=} \underline{\boldsymbol{I}}_2 \cdot \underline{\underline{\boldsymbol{J}}} \cdot \underline{\underline{\boldsymbol{H}}} \cdot \underline{\underline{\boldsymbol{J}}} \cdot \underline{\boldsymbol{I}}_2 \overset{(27)}{=} \underline{\boldsymbol{I}}_2 \cdot \underline{\underline{\boldsymbol{H}}} \cdot \underline{\boldsymbol{I}}_2$$

(81, 79, 3) $\|$ … $\|$ (80, 22)

$$16\,\sigma \left(\sigma - \rho_1^{(a)}\right)\left(\sigma - \rho_1^{(b)}\right)\left(\sigma - \rho_1^{(c)}\right) \qquad 16\,\varsigma \left(\varsigma - \rho_2^{(a)}\right)\left(\varsigma - \rho_2^{(b)}\right)\left(\varsigma - \rho_2^{(c)}\right)$$

Figure 4:

7 Diagonalizing the Heron matrix gives the Hopf Map and Kendall's Theorem

♠ ◇ ♣

Remark 1 Set

$$0 = \det(\boldsymbol{H} - \lambda \mathbb{1}) = \nu^3 - \nu + 2 \,. \tag{83}$$

For

$$\nu := -1 - \lambda \,. \tag{84}$$

Then the Factor Theorem gives that $\nu = -2$ solves, reducing the problem to a quadratic equation. Consequently,

$$(\nu - 1)^2 (\nu + 2) = 0 \,. \tag{85}$$

So the eigenvalues are $\nu = -2$, i.e. $\lambda = 1$ with multiplicity 1. And $\nu = 1$, i.e. $\lambda = -2$ with multiplicity 2.

Remark 2 Corresponding orthonormal eigenvectors are, respectively,

$$\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} , \quad \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix} , \quad \frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\-2\end{pmatrix} . \tag{86}$$

Structure 1 The diagonalizing variables are thus

$$\overline{a}^2 = \frac{a^2 + b^2 + c^2}{\sqrt{3}} \,, \tag{87}$$

$$\overline{b}^2 = \frac{a^2 - b^2}{\sqrt{2}} \,, \quad \text{and} \tag{88}$$

$$\overline{c}^2 = \frac{a^2 + b^2 - 2\,c^2}{\sqrt{2}} \,. \tag{89}$$

Remark 3 The Heron quadratic form (5) is hence equal to

$$\sum_i \Lambda_{i\,i}\, \overline{a}_i{}^2 \overline{a}_i{}^2 \,. \tag{90}$$

For *diagonalized Heron matrix*

$$\Lambda_{i\,j} = \mathrm{diag}(1, -2, -2) \,. \tag{91}$$

Which is related to the original Heron matrix $\boldsymbol{H}$ by conjugation with the usual orthogonal transformation matrix $\boldsymbol{P}$ as here formed by using (86) orthonormal eigenvectors as its columns. Thus we have derived that Heron's formula also takes the following form.

♠ ◇ ♣

Theorem 5 (Diagonal Heron formula)

$$Area \ = \ \frac{1}{4}\sqrt{\bar{a}^4 \ - \ 2\left(\bar{b}^2 \ + \ \bar{c}^2\right)} \ . \tag{92}$$

Corollary 8 Next multiply both sides through by $\frac{4}{\bar{a}_i{}^2}$ to obtain the following.

$$\frac{4 \ \times \ Area}{\bar{a}^2} \ = \ \sqrt{1 \ - \ \left[\sqrt{2}\left(\frac{\bar{b}}{\bar{a}}\right)^2\right]^2 - \left[\sqrt{2}\left(\frac{\bar{c}}{\bar{a}}\right)^2\right]^2} \ . \tag{93}$$

Remark 4 The $4 \ \times \ Area$ scaling present in the Hopf quantity has its 4 come from

$$Area \ = \ \frac{1}{4}\sqrt{(\text{ expanded Heron form })} \ . \tag{94}$$

So

$$(4 \ \times \ Area)^2 \ = \ (\text{ expanded Heron form }) \ . \tag{95}$$

♠ ◇ ♣

Definition 1 It is thus natural to finally define the rescaled *ratio variables*

$$Z \ := \ \sqrt{2}\left(\frac{\bar{c}}{\bar{a}}\right)^2 \ = \ \frac{a^2 \ + \ b^2 \ - \ 2\,c^2}{a^2 \ + \ b^2 \ + \ c^2} \ . \tag{96}$$

$$X \ := \ \sqrt{2}\left(\frac{\bar{b}}{\bar{a}}\right)^2 \ = \ \frac{\sqrt{3}\left(a^2 \ - \ b^2\right)}{a^2 \ + \ b^2 \ + \ c^2} \ . \tag{97}$$

And

$$Y \ := \ \frac{4 \ \times \ Area}{\bar{a}^2} \ . \tag{98}$$

Remark 5 The denominator of the ratio is proportional to the moment of inertia by Lemma 2. Y is moreover to be interpreted precisely as *mass-weighted area per unit moment of inertia*.

Remark 6 In terms of these rescaled ratio variables, Heron's formula has been reduced to just the following.

♠ ◇ ♣

Corollary 9 The rescaled ratio variables version of the diagonal Heron formula is

$$X^2 \ + \ Y^2 \ + \ Z^2 \ = \ 1 \ . \tag{99}$$

Which is mathematically just the on 2-sphere condition.

Remark 7 We furthermore identify $Y \ = \ 4 \ \times$ (mass-weighted area) as a Hopf quantity.

Remark 8 X and Z are also Hopf quantities, which, in the triangle context, can moreover be interpreted as follows [30, 33, 42]. Without normalizing, one has

$$Aniso \ = \ \frac{a^2 \ - \ b^2}{\sqrt{3}} \ . \tag{100}$$

And

$$Ellip \ = \ \frac{a^2 \ + \ b^2 \ - \ 2\,c^2}{3} \ . \tag{101}$$

One can readily check that these and $4 \times \mathcal{A}rea$ obey

$$Aniso^2 \ + \ Ellip^2 \ + \ (\,4 \ \times \ \mathcal{A}rea\,)^2 \ = \ I^2 \ . \tag{102}$$

Remark 9 Or, at the level of shape quantities, i.e. with normalization, one has

$$ellip \ = \ \frac{a^2 \ + \ b^2 \ - \ 2\,c^2}{a^2 \ + \ b^2 \ + \ c^2} \ . \tag{103}$$

And

$$aniso \ = \ \frac{a^2 \ - \ b^2}{\sqrt{3}\,(\,a^2 \ + \ b^2 \ + \ c^2\,)} \ . \tag{104}$$

One can also check that these and $4 \times area$ obey the on-sphere condition

$$aniso^2 \ + \ ellip^2 \ + \ (\,4 \times \alpha rea\,)^2 \ = \ 1 \ . \tag{105}$$

For normalized mass-weighted area

$$\alpha rea \ := \ \frac{\mathcal{A}rea}{I} \ . \tag{106}$$

Remark 10 Anisoscelesness and ellipticity can moroever now be interpreted as two of the eigenvectors of the Heron map $\boldsymbol{H}$.

Remark 11 Moreover, due to $\boldsymbol{H}$ and $\boldsymbol{J}$ commuting with each other, these maps share their eigenvectors. Such sharing is well-known in Quantum Mechanics, under the name of 'complete set of commuting observables' (CSCO), and in Methods of Mathematical Physics. Anisoscelesness and ellipticity are thus also eigenvectors of the sides–medians involution $\boldsymbol{J}$.

8 Median–sides interchange form invariance of diagonal Heron–Hopf formula

Corollary 10 In terms of the medians, i)

$$Ellip \ = \ -\frac{4}{3}\,\frac{{m_a}^2 \ + \ {m_b}^2 \ - \ 2\,{m_c}^2}{3} \ . \tag{107}$$

And

$$Aniso \ = \ -\frac{4}{3}\,\frac{{m_a}^2 \ - \ {m_b}^2}{\sqrt{3}} \ . \tag{108}$$

ii) At the level of shape quantities,

$$ellip \ = \ -\frac{{m_a}^2 \ + \ {m_b}^2 \ - \ 2\,{m_c}^2}{{m_a}^2 \ + \ {m_b}^2 \ + \ {m_c}^2} \ . \tag{109}$$

And

$$aniso \ = \ -\frac{{m_a}^2 \ - \ {m_b}^2}{\sqrt{3}\,(\,{m_a}^2 \ + \ {m_b}^2 \ + \ {m_c}^2\,)} \ . \tag{110}$$

Proof Use the below Lemma and the sides–medians involution. □

Lemma 4 (Democratic medians form of total moment of inertia)

$$I \ = \ \frac{4}{9}\sum_i {m_i}^2 \ = \ \frac{4}{9}\,M_{\mathrm{II}} \ . \tag{111}$$

Proof As in the proof of Lemma 2,

$$I = \frac{1}{2}a^2 + \frac{2}{3}{m_a}^2 . \quad (112)$$

But now substitute for a^2 using the sides-to-medians involution, as follows.

$$I = \frac{1}{2}\frac{4}{9}\left(2{m_b}^2 + 2{m_c}^2 - {m_a}^2\right) + \frac{2}{3}{m_a}^2 = \frac{4}{9}\left({m_a}^2 + {m_b}^2 + {m_c}^2\right) =$$

$$\frac{4}{9}\sum_i {m_i}^2 = \frac{4}{9}M_{\mathrm{II}} . \quad (113)$$

As desired, using Definition (10) in the last step. □

Remark 1 Expressing *ellip* and *aniso* in terms of medians instead does not affect the diagonality. It does flip the signs over. But this is part and parcel of the allowed conventions in setting up a Cartesian axis system. The Heron–Hopf formula is thus independent of whether one is conceptualizing in terms of sides or of medians. I.e. the Hopf quantities offer a third point of view that is side–median symmetric. This constitutes the

'Hopfian' second motivation for coprimary treatment of medians and sides.

9 Linear Algebra for Hopf Quantities

Structure 1 Let us further formulate anisoscelesness and ellipticity in Linear Algebra terms as follows.

$$Aniso = \underline{\boldsymbol{\mathcal{A}}} \cdot \underline{\boldsymbol{S}} = \underline{\boldsymbol{\mathcal{A}}} \cdot \underline{\underline{\boldsymbol{J}}} \cdot \underline{\boldsymbol{M}} = -\underline{\boldsymbol{\mathcal{A}}} \cdot \underline{\boldsymbol{M}} . \quad (114)$$

And

$$Ellip = \underline{\boldsymbol{\mathcal{E}}} \cdot \underline{\boldsymbol{S}} = \underline{\boldsymbol{\mathcal{E}}} \cdot \underline{\underline{\boldsymbol{J}}} \cdot \underline{\boldsymbol{M}} = -\underline{\boldsymbol{\mathcal{E}}} \cdot \underline{\boldsymbol{M}} . \quad (115)$$

For vectors

$$\underline{\boldsymbol{\mathcal{A}}} := \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \text{ and } \underline{\boldsymbol{\mathcal{E}}} := \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} . \quad (116)$$

Structure 2 Introduce furthermore $\boldsymbol{E}$ and $\boldsymbol{A}$ matrices for ellipticity squared and anisoscelesness squared. Then

$$Ellip^2 = \underline{\boldsymbol{S}} \cdot \underline{\underline{\boldsymbol{E}}} \cdot \underline{\boldsymbol{S}} . \quad (117)$$

And

$$Aniso^2 = \underline{\boldsymbol{S}} \cdot \underline{\underline{\boldsymbol{A}}} \cdot \underline{\boldsymbol{S}} . \quad (118)$$

Where

$$\boldsymbol{E} := \begin{pmatrix} 4 & -2 & -2 \\ -2 & 1 & 1 \\ -2 & 1 & 1 \end{pmatrix} , \quad (119)$$

And

$$\boldsymbol{A} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} . \quad (120)$$

We try this for squared quantities so that the vectors being acted upon have sides-squared components. And thus match the space that $\boldsymbol{H}$ and $\boldsymbol{J}$ act upon. We then find the following.

Theorem 6 All 3 of the $Hopf^2$ quantities' matrices commute

i) with each other:

$$[\,\boldsymbol{H}, \boldsymbol{E}\,] \;=\; [\,\boldsymbol{H}, \boldsymbol{A}\,] \;=\; [\,\boldsymbol{E}, \boldsymbol{A}\,] \;=\; \mathbb{0}\;. \tag{121}$$

ii) With the sides-median involution $\boldsymbol{J}$:

$$[\,\boldsymbol{H}, \boldsymbol{J}\,] \;=\; [\,\boldsymbol{E}, \boldsymbol{J}\,] \;=\; [\,\boldsymbol{A}, \boldsymbol{J}\,] \;=\; \mathbb{0}\;. \tag{122}$$

Proof Use Lemma 5 below. Itself established by mere matrix multiplication and the definitions of $\boldsymbol{E}$, $\boldsymbol{A}$ and eq. (28)'s $\boldsymbol{K}$. □

Lemma 5

$$\underline{\underline{\boldsymbol{H}}} \cdot \underline{\underline{\boldsymbol{A}}} \;=\; 2\,\underline{\underline{\boldsymbol{A}}} \;=\; \underline{\underline{\boldsymbol{A}}} \cdot \underline{\underline{\boldsymbol{H}}}\;, \tag{123}$$

$$\underline{\underline{\boldsymbol{H}}} \cdot \underline{\underline{\boldsymbol{E}}} \;=\; -\,2\,\underline{\underline{\boldsymbol{E}}} \;=\; \underline{\underline{\boldsymbol{E}}} \cdot \underline{\underline{\boldsymbol{H}}}\;, \tag{124}$$

$$\underline{\underline{\boldsymbol{A}}} \cdot \underline{\underline{\boldsymbol{E}}} \;=\; 0 \;=\; \underline{\underline{\boldsymbol{E}}} \cdot \underline{\underline{\boldsymbol{A}}}\;, \tag{125}$$

$$\underline{\underline{\boldsymbol{H}}} \cdot \underline{\underline{\boldsymbol{J}}} \;=\; \underline{\underline{\boldsymbol{K}}} \;=\; \underline{\underline{\boldsymbol{J}}} \cdot \underline{\underline{\boldsymbol{H}}}\;, \tag{126}$$

$$\underline{\underline{\boldsymbol{J}}} \cdot \underline{\underline{\boldsymbol{E}}} \;=\; -\,\underline{\underline{\boldsymbol{E}}} \;=\; \underline{\underline{\boldsymbol{E}}} \cdot \underline{\underline{\boldsymbol{J}}}\;, \tag{127}$$

$$\underline{\underline{\boldsymbol{J}}} \cdot \underline{\underline{\boldsymbol{A}}} \;=\; -\,\underline{\underline{\boldsymbol{E}}} \;=\; \underline{\underline{\boldsymbol{A}}} \cdot \underline{\underline{\boldsymbol{J}}}\;. \tag{128}$$

Corollary 11 (Matrix form of the Hopf on-sphere condition)

$$\boldsymbol{H} \;+\; \boldsymbol{A} \;+\; \boldsymbol{E} \;=\; \frac{1}{9}\,\boldsymbol{I}\;. \tag{129}$$

Where $\boldsymbol{I}$ is the following degenerate 'all unit entries matrix'.

$$\boldsymbol{I} \;:=\; \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\;. \tag{130}$$

So that $\frac{1}{9}\boldsymbol{I}$ is what occurs in the moment of inertia squared regarded as a quadratic form of squares.

10 Heron–Hopf–Kendall, Heron–Hopf and two concomitant formulae

Remark 1 Let us conclude the previous three sections as follows. Sec 7's workings readily imply the following Theorems.

Theorem 7 The diagonalized form of the mass-weighted Heron formula is as follows.

$$4 \;\times\; \alpha rea \;=\; \sqrt{1 \;-\; aniso^2 \;-\; ellip^2}\;. \tag{131}$$

This '*Heron–Hopf*' formula is moreover sides–to-medians symmetric.

Theorem 8 The most symmetrical presentation of the diagonalized mass-weighted Heron formula is as follows.

$$(\,4 \times \alpha rea)^2 \;+\; aniso^2 \;+\; ellip^2 \;=\; 1\;. \tag{132}$$

This '*Heron–Hopf–Kendall*' formula is mathematically just the on-sphere condition. Observe that this moreover amounts to a *recovery* of *Kendall's Little Theorem* that the shape space of all triangles in 2-d is a sphere (Appendix B).

Remark 3 The Heron–Hopf formula, as the area-subject Hopf formula in terms of ellipticity and anisoscelesness data, now has two concomitant formulae in many senses. These are, firstly, the ellipticity-subject Hopf formula in terms of anisoscelesness and area data, as follows.

$$ellip \ = \ \sqrt{1 \ - \ aniso^2 \ - \ (\,4 \ \times \ \alpha rea\,)^2} \ . \tag{133}$$

Secondly, the anisoscelesness-subject Hopf formula in terms of ellipticity and area data, as follows.

$$aniso \ = \ \sqrt{1 \ - \ ellip^2 \ - \ (\,4 \ \times \ \alpha rea\,)^2} \ . \tag{134}$$

A sense in which Heron–Hopf has a further quality that these other two formulae do not have is as follows. It is the only one among these whose subject is democratic i.e. clustering-independent [24, 42].

11 Conclusion

In the current paper, we considered coprimary treatment of the medians and sides of triangles, as motivated by Jacobi and Hopf mathematical structure. Both of these are structures entering the Shape Theory of the space of triangles.

♠ ◇ ♣

First we reformulated the medians–sides inter-relation in terms of an involution $\boldsymbol{J}$. We observed that the following two factors of $\frac{4}{3}$ have the same origin.

A) That in the sides–medians involution $\boldsymbol{J}$.

B) And that in the discrepancy between sides and medians versions of Heron's formula.

Moreover, in both A) and B), the factor of $\frac{4}{3}$ can be removed as follows. By passage to the mass-weighted Jacobi coordinate version. The mass-weighted sides and mass-weighted medians versions of Heron's formula for the now also mass-weighted area are thereby given identical form. We term these the *Heron–Jacobi formulae*. In the process, the factor of $\frac{4}{3}$ is identified to be the ratio of the Jacobi mass of the medians to that of the sides.

♠ ◇ ♣

Secondly, we point to the elsewise well-known Hopf coordinates diagonalizing the Heron map $\boldsymbol{H}$, a fact that appears to have hitherto escaped attention. Indeed, in this manner, diagonalizing Heron's map $\boldsymbol{H}$ provides us with a new derivation of both of the following.

1) The Hopf map.

2) That the shape space formed by the triangles is a sphere equipped with the standard spherical metric: Kendall's Little Theorem.

So at the level of the Hopf formulation of the triangle, Heron's formula becomes a '*Heron–Hopf* formula. Which furthermore coincides with the on-sphere condition that determines that the shape space of triangles is a sphere.

In its 3-body problem incarnation, the factor of 4 in the Hopf quantity that is the 'tetra-area' is moreover accounted for as follows. It is none other than the prefactor of 1/4 in the expanded version of Heron's formula.

The other two Hopf quantities are, in their 3-body problem incarnation, ellipticity and anisoscelesness. In the current paper, these receive the further enlightening interpretation as eigenvectors of the Heron map $\boldsymbol{H}$. These are furthermore eigenvectors of the sides–medians involution $\boldsymbol{J}$. Which follows by the commutation relation between $\boldsymbol{H}$ and $\boldsymbol{J}$ also established in the current paper.

More specifically, our *Heron–Hopf formula* is obtained by diagonalizing the expanded Heron form, while keeping mass-weighted area as the subject. As already mentioned above, this takes the mathematical form of an on-sphere condition. In the case in which this is represented symmetrically, let us coin the name *Heron–Hopf–Kendall formula* for it in honour of Kendall's iconic shape sphere of triangles. We finally argued that one can (almost) just as well interpret ellipticity or anisoscelesness as the subject. Giving two further concomitant forms of the Heron–Hopf formula.

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A The Hopf map

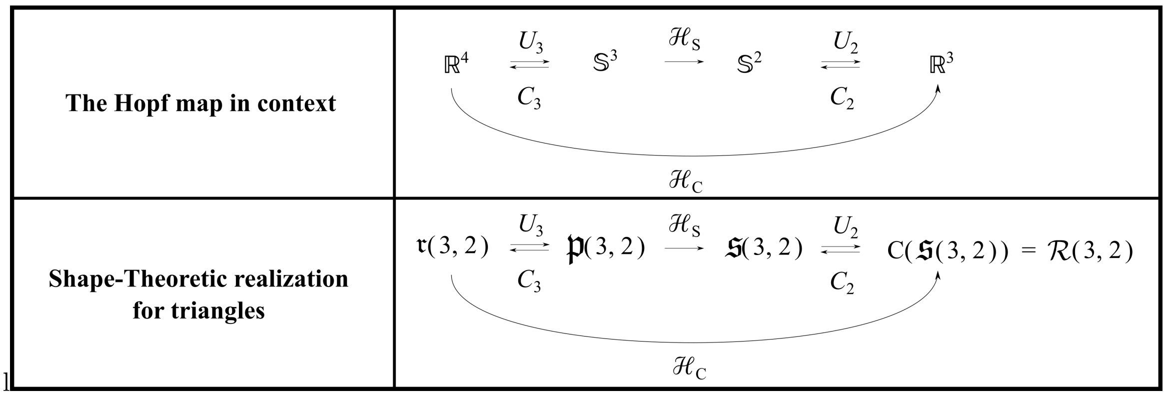

Figure 5:

Structure 1 The main Hopf map considered in the current Article is the simplest, as outlined in Fig 5.

The map that Hopf emphasized [7] is the *Hopf spheres map* $\mathcal{H}_{\mathrm{S}}$ Our figure includes also the map from the ambient $\mathbb{R}^4$ for the $\mathbb{S}^3$. To the less obviously realized ambient $\mathbb{R}^3$ for the $\mathbb{S}^2$. Which we term the *Hopf Cartesian map* $\mathcal{H}_{\mathrm{C}}$. The U_k also displayed are *unit vector maps* to the corresponding $\mathbb{S}^k$. Finally, the C_k are *cone maps* from $\mathbb{S}^k$.

Structure 2 The Cartesian directions of $\mathbb{R}^3$ moreover make equable use of those of $\mathbb{R}^4$ in the following way. [This is modulo signs and permutations as regards which Hopf quantities are suffixed X , Y and Z .]

$$Hopf_X \ := \ 2\,\underline{x} \cdot \underline{x}' \ . \tag{135}$$

$$Hopf_Y \ := \ 2\,(\,\underline{x} \ \times \ \underline{x}'\,)_3 \ . \tag{136}$$

And

$$Hopf_Z \ := \ x^2 \ - \ x'^2 \ . \tag{137}$$

Structure 3 Normalizing, the unit Cartesian directions in $\mathbb{R}^3$ are as follows.

$$hopf_i \ = \ \frac{Hopf_i}{x^2 \ + \ x'^2} \ . \tag{138}$$

These can be readily checked to obey the on-2-sphere condition,

$$\sum_{i = 1}^{3} hopf_i{}^2 \ = \ 1 \ . \qquad (139)$$

Structure 4 The Hopf spheres map can moreover be regarded as a principal fibre bundle. With base space $\mathbb{S}^2$. Fibre $\mathbb{S}^1 = U(1)$, also serving as structure group. And total space $\mathbb{S}^3$.

♠ ◇ ♣

Remark 1 Applications of this Hopf mathematics include the following.

Application 1) It provides a simple nontrivial example of fibre bundle and of fibration [18, 32, 22].

Application 2) It is theoretically realized in Physical space by the Dirac monopole [8].

Application 3) It is realized in configuration space for the 3-body problem (alongside variants reviewed in [42, 46]). This realization is moreover Shape-Theoretic and thus is further explained in Appendix B.

Application 4) It extends to two other special-dimensional cases as supported by the quaternions and octonians [22].

Application 5) It extends systematically to inter-relate spheres of odd dimension and complex-projective spaces of one dimension less (see Appendix C).

Application 6) Application 5) is furthermore realized in configuration space for the planar N-body problem. In what is again a Shape-Theoretic realization as outlined in Appendix D.

B Shape-Theoretic realization of the Hopf map, and Kendall's Little Theorem

Theorem 9 (Kendall's Little Theorem) Take triangles in the Euclidean plane with vertices distinctly labelled and mirror images held to be distinct. Then the shape space formed by these is topologically a sphere. It is furthermore equipped with the standard spherical metric (in the sense of Riemannian Metric Geometry).

Remark 1 This can be proven by each of the following means.

a) By Kendall's Geometrical construct [13, 16, 25].

b) By reduction of the corresponding Mechanics action [29].

c) From the Hopf map [42].

d) As per the body of the current Article, by reformulating Heron's formula so as to obtain Hopf's realization of the on-sphere condition. This uses *basic* Euclidean Geometry followed by a very natural piece of Linear Algebra. And kills two birds with one stone: both Kendall's Little Theorem and the usual Hopf map follow from Heron's formula!

Remark 2 A common rubric for a), b) and c) is outlined in Fig 6.a). Where $Tr(d)$ are translations, $Rot(d)$ are rotations and Dil are dilations. Specializing down to the case of triangles in Subfig b). For which

$$Rot(2) \ = \ SO(2) \ = \ U(1) \ .$$

Then in approach c), the bottom-left quotienting is Appendix A's Hopf map $\mathcal{H}_{\mathrm{S}}$. $\mathcal{H}_{\mathrm{C}}$ is also realized. The previous Appendix's $\underline{x}$ and $\underline{x}'$ are here realized by the mass-weighted relative Jacobi vectors $\underline{\rho}_1$ and $\underline{\rho}_2$.

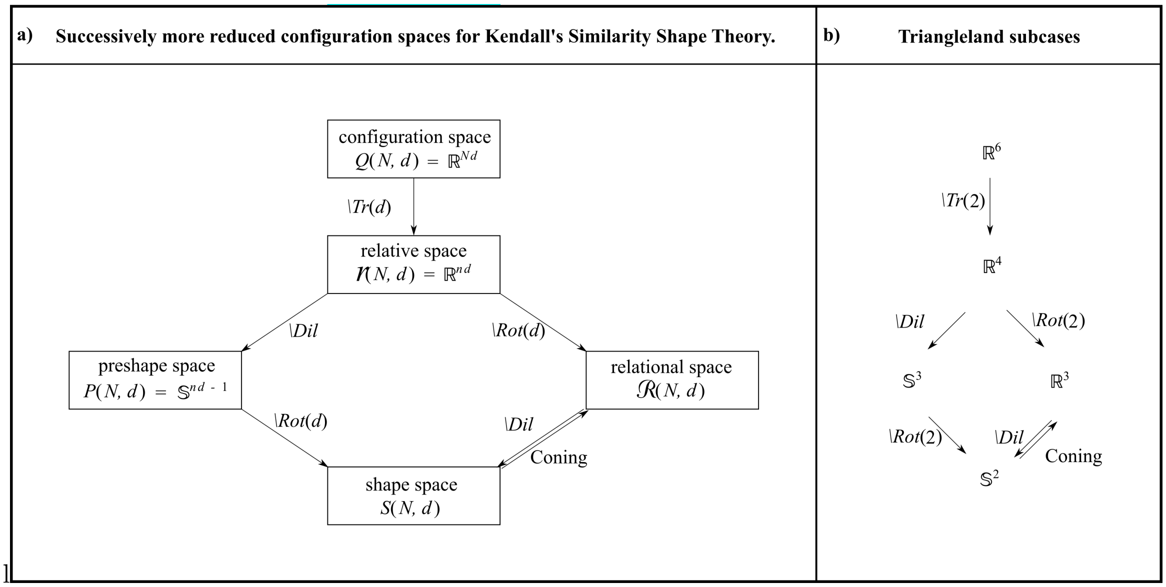

Figure 6:

Remark 3 See Fig 7 for a sketch of some features of the shape sphere of triangles and [42] for further details. See e.g. [40, 42, 46] for whichever combination of vertex-unlabelled, unoriented, and 3-d versions of triangleland.

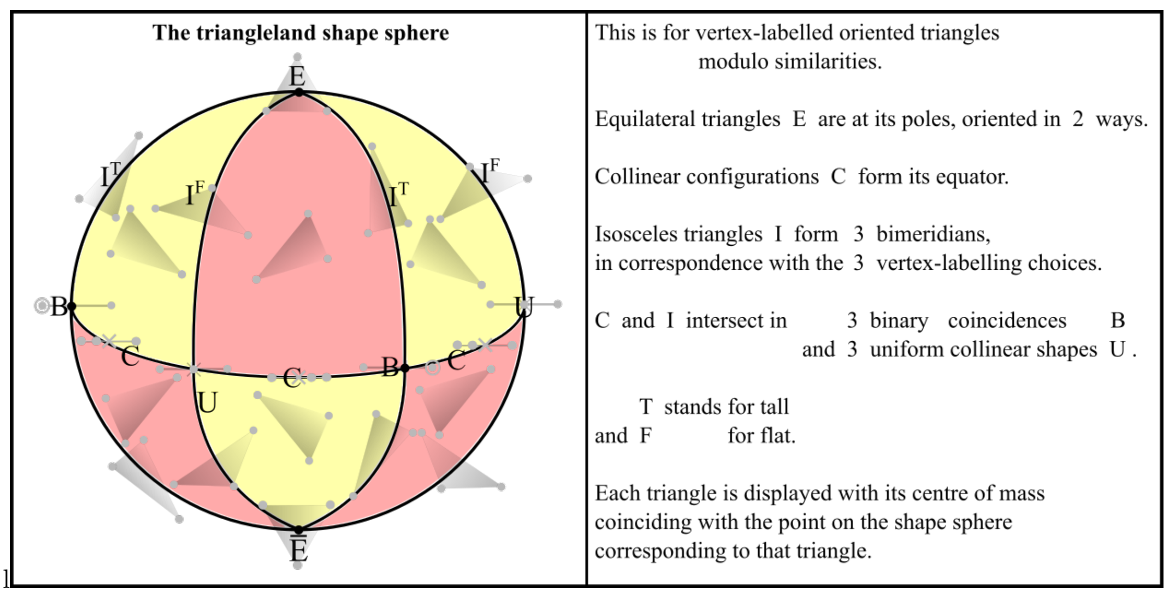

Figure 7:

C The generalized Hopf map for odd-d spheres

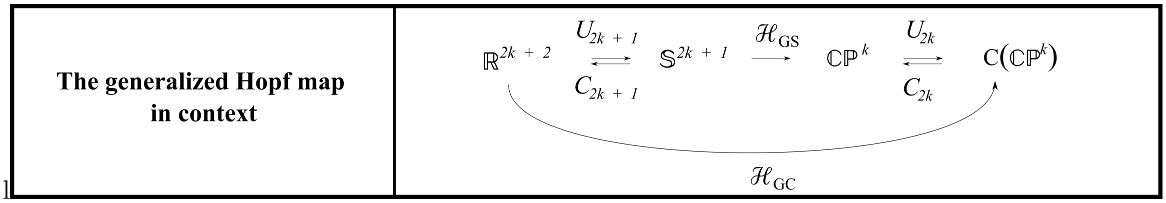

Figure 8:

Structure 1 The generalized Hopf *sphere* (now singular) map $\mathcal{H}_{\mathrm{SG}}$ from $\mathbb{S}^{2\,k\,+\,1}$ to $\mathbb{CP}^k$ is given in context in Fig 8. Our figure includes also the map unit $U_{2\,k\,+\,1}$ from the ambient $\mathbb{R}^{2\,k\,+\,2}$ for the $\mathbb{S}^{2\,k\,+\,1}$. And the map C_k forming the cone over $\mathbb{CP}^k$. The generalized Hopf map $\mathcal{H}_{\mathrm{G}\mathscr{C}}$ now has its C stand for 'cone' rather than for 'Cartesian'.

Remark 1 Observe that Appendix A's case is included by virtue of the following 'accidental relations'.

$$\mathbb{CP}^1 \;=\; \mathbb{S}^2\;. \tag{140}$$

And

$$\mathrm{C}\left(\mathbb{S}^2\right) \;=\; \mathbb{R}^3\;. \tag{141}$$

Remark 2 In considering this generalization, Appendix A's Hopf quantities are best viewed as $su(2)$ objects. Corresponding to the isometry group of $\mathbb{S}^2$ being

$$\mathrm{Isom}(\,\mathrm{Triangleland}\,) \;=\; \mathrm{Isom}\left(\mathbb{S}^2\right) \;=\; \mathrm{Isom}\left(\mathbb{CP}^1\right) \;=\; \frac{SU(2)}{\mathbb{Z}_2}\;\;[\;\;=\;\;SO(3)\;]\;. \tag{142}$$

Afterword Further study [48, 53] reveals that Lie Algebra Representation Theory is key for understanding shape quantities. Sec 9's $hopf^2$ matrices all commuting with each other however encodes the following instead.

$$\prod_{i\,=\,1}^{3} U(1) \;\stackrel{manifold}{=}\; \prod_{i\,=\,1}^{3} \mathbb{S}^1 \;=\; \mathbb{T}^3\;.$$

Which fails to capture the noncommutativity of $su(2)$. Because of this, in writing this v2 six years later, I declare that Sec 9's matrices are of limited interest.

That $\boldsymbol{H}$ and $\boldsymbol{J}$ commute with each other remains an interesting point. This encodes a symmetry of *mass-weighted sides-medians space*. Which does have a

$$\prod_{i\,=\,1}^{2} U(1) \;\stackrel{manifold}{=}\; \prod_{i\,=\,1}^{2} \mathbb{S}^1 \;=\; \mathbb{T}^2$$

acting thereupon. This is the symmetry underlying that the Jacobi versions of Heron's formula and the medians-Heron's formula are identical in mathematical form. How special mass-weighted sides-medians space is in having these features is investigated in [50].

Remark 3 Remark 2 furthermore generalizes to [35]

$$\mathrm{Isom}(\,N\text{-a-gonland}\,) \;=\; \mathrm{Isom}\left(\mathbb{CP}^{N\,-\,2}\right) \;=\; \frac{SU(\,N\;-\;1\,)}{\mathbb{Z}_{N\,-\,1}}\;. \tag{143}$$

Remark 4 The analogous Hopf quantities are now a set of

$$(\,k\;+\;1\,)^2\;-\;1\;=\;k\,(\,k\;+\;2\,)$$

objects. Which are built from $k\,+\,1$ $\mathbb{R}^2$--vectors. That span the $\mathbb{R}^{2\,k\,+\,2}$ ambient space of the $\mathbb{S}^{2\,k\,+\,1}$. See [35] for further details of these quantities for $k\;=\;2$.

D Shape-Theoretic realization of the generalized Hopf map, and generalized Kendall Theorem

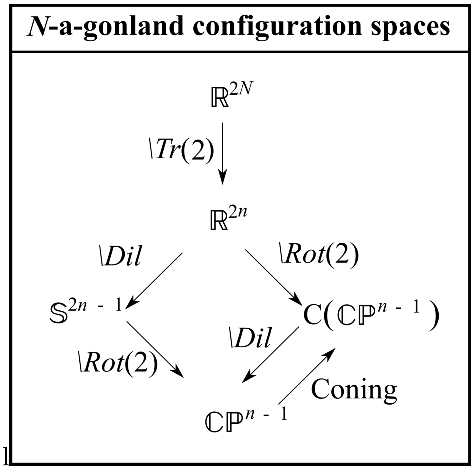

Figure 9:

Structure 1 The N-a-gonland specialization of Fig 6.a) is given in Fig 9. The $k+1=n$ $\mathbb{R}^2$--vectors are now $\underline{\rho}_a$, $a=1$ to n .

Theorem 10 (Generalized Kendall Theorem) Take N-a-gons in the Euclidean plane with vertices distinctly labelled and mirror images held to be distinct. The shape space of of these N-a-gons is topologically $\mathbb{CP}^{N-2}$. And is furthermore equipped with the standard Fubini–Study metric in the sense of Riemannian Metric Geometry.

Naming Remark 1 A conceptual name for Kendall's Little Theorem is '*Triangleland Sphere Theorem*'. Whereas a conceptual name for the generalized Kendall Theorem is ' *N-a-gonland* $\mathbb{CP}^{N-2}$ *Theorem*'.

Remark 1 The Geometrical [13, 16, 25], Mechanics-reduction [29, 33] and Hopf map [42] proofs of this carry through. It remains to be seen whether the general N-a-gon case admits an area-expression reformulation proof along the lines of the current Article. For now, our answer to this is in the negative [48].

Remark 2 The above $k=2$ example's Hopf quantities are furthermore interpreted as shape quantities in [35], in the $N=4$ case of quadrilaterals.

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