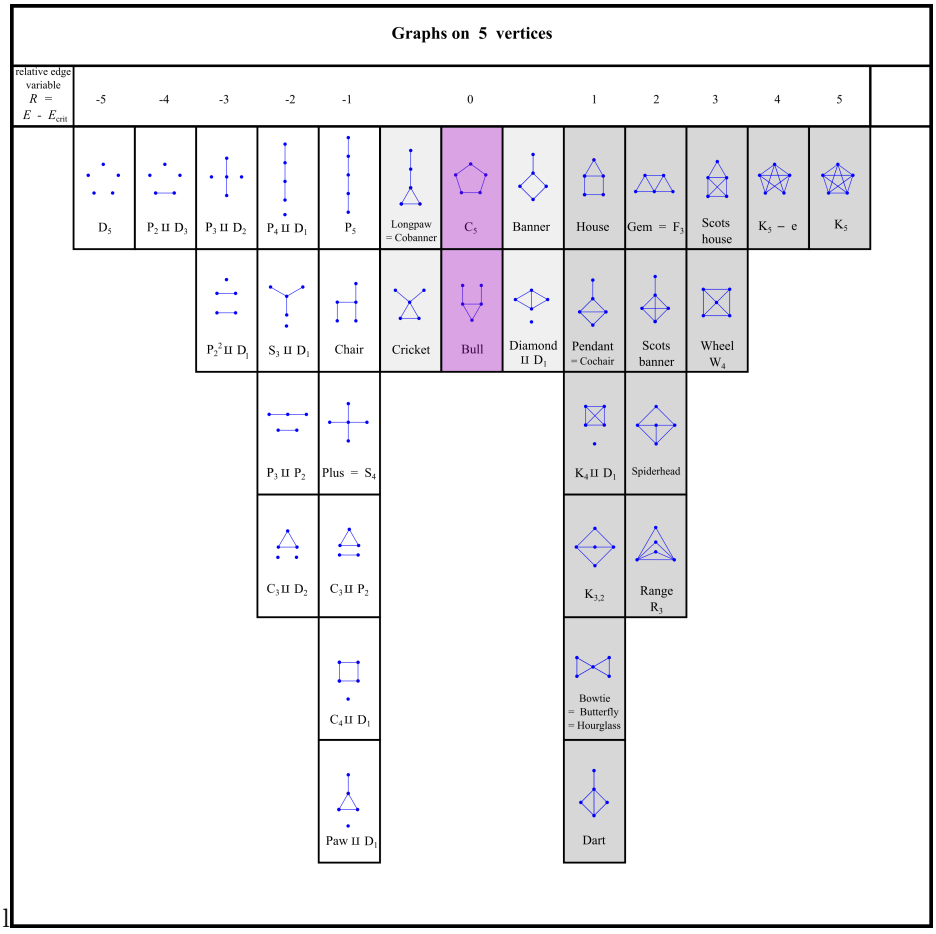
APPLIED COMBINATORICS

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Widely-Applicable Mathematics Series. Volume 0

First Edition



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This is Volume 0 of the Widely-Applicable Mathematics Series, also referred to in these books as Series 0 and the Pre-Topology Series.

Volume 0 belongs more specifically to Subseries A, whose Evolving Name is, for the present Volume, A. Improving Understanding of Everything with a pinch of Combinatorics.

Other titles in this Series

E. Anderson, *Applied Linear Algebra*, *Widely-Applicable Mathematics. A. Improving Understanding of Everything with a pinch of Combinatorics* **1**, forthcoming 2023.

E. Anderson, *Flat Geometry*, *Widely-Applicable Mathematics. A. Improving Understanding of Everything by dual-wielding Combinatorics and Linear Algebra.* **3**, forthcoming 2023.

The Reader can already see what we mean by Series A having an Evolving Name!

Back cover description

Material covered: Combinatorics essentially means systematic counting. We present some Combinatorics topics that are both simple and widely applicable, by which they are very much worth knowing, understanding and applying. These include *permutations*, *combinations*, other *selections*, *partitions*, *generating series*, and *pigeonhole* and *inclusion–exclusion principles*. *Group-theory-based enumerations*, and especially *graphs*, *posets*, and *lattices*. Applications covered include to Computer Science, Chemistry, Geometry and to the Feynman diagrams of Quantum Physics. And toward starting to gain a basic understanding of some topics in Topology: notions of connectivity, of traversability, of cover, and of separation, topological spaces, and a brief introduction to knots. This is a Mathematics book, and yet we start from scratch as regards Readers interested in applications who are not (primarily) Mathematicians. A total of almost 1000 exercises, worked proofs and worked examples are included.

Some of our innovations: Useful relative difference variables and ratio variables are systematically introduced, and applied toward understanding not only the graphs but also the spaces of graphs. And not only the partitions but also the spaces of partitions, and so on. These developments are applying successful 'Shape Theory' or 'relational' ideas – usually found in Geometry and in high-end approaches to Physics – now to basic Combinatorics as well. In particular, the theory of each type of mathematical object involves not only the structures of those objects but also the structure of the *arena*: the space formed by those objects. *This is one of the ways in which Topology enters the theory of every other STEM subject*, often accompanied by one of Order Theory – posets and lattices – or Geometry. These parts of the current Volume are not only presented accessibly for beginners, but are also innovative enough to source new research in Mathematics and in many other areas of STEM. Throughout, Conceptual Thinking, including finding Truer Names for all the main objects and results encountered, is included and encouraged.

Our format: This Volume has been made freely available in electronic form in support of the new Generation Z's considerable desire to become self-sufficient. Two affordably-priced series of Books are growing from the current Volume. Since these will on many occasions refer to the current Volume, its free availability can effectively be viewed as part of this affordable pricing.

These Book Series

Motivation: The Tech Sector is now starting to make widespread use of Topology above and beyond the past several decades' theorization in terms of Graph Theory and Networks. More generally, it is our sincere belief that basic Topology has started to be exhaustively applied to the theory and foundations of *all other* STEM subjects, and that this will dominate development of theoretical understanding for at least the next 30 years. This will be around as influential and sweeping as the widespread application of Linear Algebra in the 1950s through to the 1970s, and of Differential Geometry in the 1990s and 2000s. It is consequently also our sincere belief that for advanced-undergraduate and masters courses to remain relevant, the content of each Theory course should be rooted in Topological concepts. Generation Z people are moreover self-reliant, and want to understand and use what actually works, regardless of whether it is 'traditionally part of' your subject of study. These Book Series are written in particular with you in mind.

Our two-series format: Many introductory university-level courses are currently divided between Pre-calculus and Calculus. Analysis is rigorous Calculus, and Topological Spaces are one of the main abstractions of Analysis, so Topology is two steps forward from Calculus. As such, we divide our books into the Pre-Topological *Widely-Applicable Mathematics Series* and the *Widely-Applicable Topology Series*. In the current Pre-Topology Series 0, the main novelty is in systematic use of basic Combinatorics in the theory of all other STEM subjects. This Pre-Topology Series also gives space and thought to what subsequently turn out to be Topologically-relevant definitions and examples.

Our main intended audience: Since Topology is two steps forward from Calculus the main intended audience for our Pre-Topological *Widely-Applicable Mathematics Series* are those who have completed the second or third year of full-time undergraduate study (or equivalent) in a STEM subject. Some of you will be studying, or self-improving, in Pure Mathematics, others in Applied Mathematics, and yet others in one of Computer Science, Engineering, Physics, Chemistry, or other Natural Sciences. For the Applied Topology Series, the main intended readership are those with a degree in a STEM subject who wish for further understanding. As regards prerequisites for this Pre-Topological *Widely Applicable Mathematics Series*, we assume that you have done first- or second-year level Calculus and Linear Algebra, and that you have seen at least one way of doing Flat Geometry.

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Chapter 0

Introductions and Prefaces

0.1 Contents outline 1: what is Combinatorics?

Much of traditional [69] Combinatorics involves *enumeration*: [67, 54, 63, 42, 55, 72]: systematic and efficient means of counting. Counting the same objects in two conceptually different ways is moreover capable of producing interesting results and proofs: the *Combinatorial style of proof*.

♠ ◇ ♣

This Volume introduces various parts of Combinatorics that are both simple and very widely applicable. By which, these are very much worth knowing, understanding and systematically applying to whatever else one is studying.

a) *Pigeonhole principles* [15, 67, 76] by which e.g. if there are $p + 1$ pigeons in p pigeonholes, then at least one pigeonhole must contain more than one pigeon (Chapter 11). While these are very simple to state, they are surprisingly useful in proving things, and also lead to a magnificent theory of the extent to which a large set can manifest structure.

b) *The Principle of Inclusion and Exclusion (PIE)* [16, 67, 63, 49], about how to count up multiple intersecting sets using an alternating-sign sum (Chapter 12).

c) *Permutations* [1, 110, 49, 58], in the sense of a particular type of map that rearranges finite sets' elements. We have various reasons to start to present these in Chapter 6, though we subsequently often revisit them.

♠ ◇ ♣

d) Chapter 13 poses an array of *selection problems* [49, 60, 53, 55]: in how many ways can a balls be drawn from a bag of b balls under various modelling assumptions. Such as whether the balls are labelled, whether or not they are replaced, or whether the picks are ordered. This can also be viewed as a theory of sampling from a Probability and Statistics point of view. Some of the most commonly encountered modelling assumptions for this are solved by the *combinations* [2, 41, 42] n-choose-r, which are also the coefficients in the binomial expansion, and n-perm-r: a first return to permutations.

e) *Partitions* [67, 50, 61] can be viewed as the study of in how many ways a natural number can be split into a sum of natural numbers. They can furthermore be subjected to a number of further modelling assumptions, such as 'into odd parts' or 'into at most k parts'. Various types of partition moreover provide answers to some of the selection problems, further motivating them (Chapter 15).

f) We precede our account of partitions with Chapter 14 introductory account of *generating series* techniques [22, 55, 76, 49], since these are very useful in studying partitions. We also apply generating series to solving *recurrence relations* alias *difference equations*, thus arriving at e.g. the well-known *Fibonacci* [46, 55] and *Catalan* [49, 71] numbers. And to the following further solutions of selection problems.

g) Chapter 16 introduces the final *Combinatorial special functions* [49, 32, 76] needed to cover our main range of selection problems, which are often referred to as *Stirling numbers* of the second and third kinds. Both of these arise in both plain and summed form, one of which is often termed *exponential numbers* (often alias '*Bell numbers*') [41, 42, 65].

♠ ◇ ♣

h) Some parts of elementary Group Theory [110, 105, 96] are enumeratively useful. For instance, *Lagrange's Theorem*, *orbits*, *stabilizers*, *orbit-counting fomulae* and, indeed, *Pólya's Enumeration Theorems* [63, 42].

i) Some parts of Combinatorics are more structured, with for instance *Graph Theory* [190, 183, 164, 176, 196, 178] considering not just a set of points here called vertices but an accompanying set of edges between vertices as well.

j) *Order Theory* [49, 224] places furthermore an ordering relation on some such graphs. In the current Volume, we restrict attention to partial orderings, which equip sets to be *posets* (short for partially-ordered sets). *Lattices* are then a particularly useful kind of furtherly-structured such, describing for instance subsets, divisors and logic.

This Volume dedicates two Parts – III and IV – to basic Group Theory and its Combinatorial applications. Two large Parts – V and VI – to graph modelling assumptions and Graph Theory. Finally, Part VII outlines Order Theory, supported by Part VIII's many further interesting examples of posets and lattices.

♠ ◇ ♣

Part I itself concludes with the underlying notion of *countability* [27, 33] in Chapter 17. This gives a first few differences between finite sets and infinite sets, as well as between different kinds of infinite set.

♠ ◇ ♣

Applications that we cover in the current Volume include to Computer Science [36], Chemistry [391], Geometry [172], and to the Feynman diagrams [386, 387] of Quantum Physics [394, 388]. And toward starting to gain a basic understanding of some topics in Topology. This is a Mathematics book, and yet we start from scratch as regards Readers interested in applications who are not (primarily) mathematicians. So let us postpone explaining the particular significance of Topology to Secs 0.5-0.4, so as to firstly secure an outline of our Book's preliminary material.

0.2 Contents outline 0: this Volume's preliminary material

The current Volume does not assume that the Reader has done Discrete Mathematics before. Having one course's experience with this is of some value toward understanding our text, however. This could take the form of a first- or second- year Mathematics or Computer Science lecture course, or having instead some experience with the Combinatorics section of Mathematics Olympiads. Readers who have not done Discrete Mathematics before are however well looked-after here, by our providing Part I on sets, functions and number systems; this Part also introduces our Book Series' conceptual approach and notation. Additionally, Parts I and II provide a very widely useful collection of

techniques of proof that Readers who are not primarily studying Mathematics may have previously encountered relatively few of.

♠ ◇ ♣

Part I covers the following topics. In Chapter 1, we give a *naïve outline of sets* [75, 27]. While in Chapter 2 we introduce *functions between sets* including the very useful cases of injections, surjections, and bijections. *Proofs by two-way inclusion* [268, 242] are furthermore illustrated using these simple topics.

In Chapter 3, we outline product sets, binary operations and binary relations. In Chapter 4, we give a first outline of the basic *number systems* [46, 22] $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ so as to have these available as a source of simple examples. We next explain, and provide some practice with, *proof by mathematical induction* [67, 60] in Chapter 5.

We then return to functions in Chapter 6, covering homomorphisms, isomorphisms, endomorphisms, automorphisms, kernels [110, 96, 297, 227], characteristic functions and permutations [58]. Characteristic functions provide another method of proof for simple identities about sets, while the already-mentioned permutations are sets' own edition of automorphisms.

We next revisit number systems in Chapter 7, covering divisibility, Euclid's Algorithm, primes, mutual primes and modular arithmetic. And also Fermat's little Theorem, the Fermat–Euler Theorem and Euler's totient function. This Chapter also involves a first few simple *proofs by contradiction*.

Our first material that may be partly new to mathematical readers is in Chapter 8. Namely, families – sets of subsets – and collections: sets of sets of subsets! We further specialize families in Chapter 9 to *covers* [260] and *topological spaces* [268, 250, 260, 243, 264]. Smallest examples are widely used in the current Series, not least because of their propensity to provide *minimal counterexamples* [41] to many questions: a further method of proof. We conclude Part I by outlining the more commonly encountered idea of *equipping sets with further levels of mathematical structure* [227] in Chapter 10.

♠ ◇ ♣

We also do not assume that all Readers will have done Group Theory before, or, in some cases, enough of the right kinds of basic Group Theory to meet the current Volume's needs. Thus we also provide an introductory account of Group Theory with Combinatorial applications, as follows.

Part III outlines three foundational ways of conceiving of groups that turn out to be equivalent.

I) The *concrete approach to groups* – as subgroups of permutation groups – is outlined in Chapter 18.

II) The *abstract approach to groups* – as any structure obeying the Algebraic axioms of closure, associativity, identity and inverse, is next outlined in Chapter 19. The order of a group element – the smallest power of it that returns the identity – and of a subgroup – how many elements it contains – are also considered here.

III) The *symmetries approach to groups* – as realized by the symmetries that preserve some mathematical object – is finally provided in Chapter 20. Planar Geometry examples are included here, while further examples from 3-*d* Flat Geometry and Chemistry are postponed to Chapter 21. We also consider here how groups act on (possibly equipped) sets. And how introducing a further layer of mathematical structure can break a symmetry group down to one of its subgroups. Conceptual discussion and Combinatorial applications of symmetry groups form Chapter 22.

Part IV outlines the Group Theory that we need for our enumerative and more structured Combinatorial pursuits. More specifically, Chapter 23 introduces *orbits* and *stabilizers* for groups acting on (possibly equipped) sets. Orbits can be viewed as a way in which the group partitions up the equipped set. While stabilizers can be viewed as subgroups that each set element picks out one of. Chapter 24 introduces group *cosets*, and group *conjugation*; these can be viewed as ways of partitioning up a group. And also *Lagrange's Theorem* that the order of all elements and all subgroups of a group must divide the order of the group.

These are all a priori structural topics; Chapter 25 provides however a first few enumerative applications. Orbits and stabilizers give efficient counts, while Lagrange's Theorem can also place effective restrictions.

Chapter 26 introduces relational quantities for groups; see the next section for a first outline of what relational notions are; every Part of this Volume uses some of these! And then covers *normal subgroups*, the *quotient groups* that these enable, and *quotienting equipped sets by groups* more generally. Chapter 27 presents the *Group Homorphism Theorem*. While the above three Chapters are structural, in contrast, Chapter 28 gives and applies *Pólya's Enumeration Theorems*. These provide a systematic way of counting objects that manifest symmetries, by putting together some orbit-counting and permutational ideas.

♠ ◇ ♣

A total of almost 1000 exercises, worked proofs and worked examples are included throughout this Volume (everything labelled Proof is also implicitly a Worked Example by being very complete in detail and explanation).

0.3 Innovation 1: Relational variables and relational configurations

Consider the following four presentations of Pascal's triangle.

Fig 1.a) has the advantage of being rectangularly coordinatized in terms of the standard n and m arguments of the *combinations* n choose k. These are denoted by such as

$$^nC_k \text{ or especially often by } \left(\begin{array}{c} n \\ k \end{array} \right) ,$$

and take the numerical values

$$\frac{n!}{(n\,-\,k)!\,k!} .$$

This fails to capitalize on the obvious symmetry in the entries of the triangle, however. Fig 1.b) starts to tap into this by being centred; for Pascal's triangle, this is furthermore already a more widely-used presentation than Fig 1.a)'s. One rule generating the numbers in Pascal's triangle can be stated in the following form with reference to Fig 1.b). 'Each brick is labelled by the sums of the labels of the two bricks in contact with it from above.' An equivalent way of generating these numbers is to compute the coefficients in the binomial expansion of

$$(1\,+\,x)^n .$$

Combinations are also familiar from basic Discrete Mathematics and Probability and Statistics.

What is not so standard is to clearly state what linear-shift transformation variable centres the triangle. Since this is linear,

$$r \;=\; \lambda n \;+\; \mu k \;+\; \nu .$$

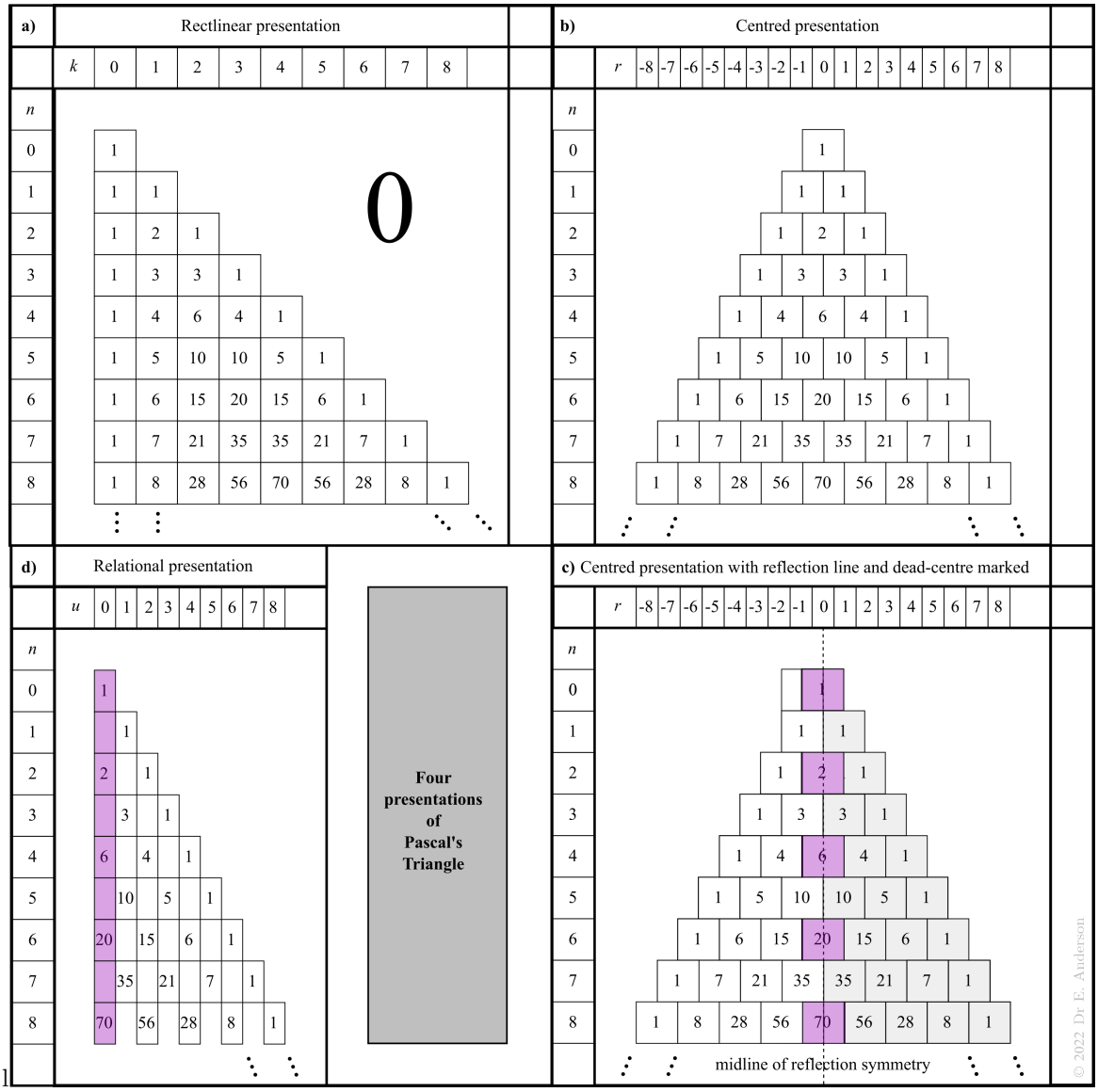


Figure 1:

The Reader can check that this is fixed to be

$$r \;:=\; 2\,k \;-\; n$$

by $(n,\, k) \;=\; (0,\, 0)$ and $(2,\, 1)$ mapping to the midline, while scaling so that $(1,\, 0)$ maps to 1.

Notice furthermore that in Fig 1.b), not all rows have an entry on the centred midline. This follows from the centred condition being

$$r \;=\; 0\;.$$

Thus

$$n \;=\; 2\,k\;.$$

But since k must be an integer, n is forced to be even. In Fig 1.c), these centred entries are highlighted in pink. So our new variable r renders manifest a symmetry in Pascal's triangle as a reflection about its midline, marked as a dashed line in Fig 1.c).

We can follow these developments up by discarding almost half of the triangle's numbers resulting in no loss of information. More precisely, one is to keep all the midline entries, but only one copy of the remaining two lobes that reflect into each other without overlap. One could for instance choose to discard Figure 1.c)'s left lobe, resulting in Fig 1.d). This constitutes a *relational* version of Pascal's triangle. Here u is the unsigned version of r,

$$u \ := \ |\, r \,| \ = \ |\, 2\, k \ - \ n \,| \ .$$

In this particular context, the relational version corresponds to Pascal's triangle getting quotiented by $C_{2\,\mathrm{ref}}$: the cyclic group of order 2 acting reflectively. The previous paragraph's midline versus other distinction of entries is then straightforwardly accounted for in terms of the corresponding orbits and stabilizers.

So far in the literature, tables of mathematical objects are usually stuck in presentations like Figure 1.a)'s. Pascal's triangle is fairly exceptional as regards having progressed to a centred presentation. Even it is seldom seen in the more compressed format of Figure 1.c). Let us for now adjourn where this development is heading until Sec 0.11, since this destination lies on the other side of the great curtain of Topology that we next open.



0.4 Preface 1: Outline of elementary Topology itself

This is the bigger picture that we are working towards. It additionally motivates the the coordinated pair of Series of Books, that emanate from the current Volume. Readers who are just interested in what Combinatorics topics this Volume covers can skip ahead to Sec 0.7. Noting however that one of the main points of the current Volume is that we carefully explain which parts of Combinatorics, and which of our examples, turn out to have further Topological significance.

So let us first outline what parts elementary-level Topology [268, 250, 258, 266, 260, 257, 243, 256, 246, 264, 261, 247, 254, 244, 274] consists of.

1) *Topology* is an abstraction of the study of *continuity*.

In contrast, *Differential Geometry* is an abstraction of the study of differentiability.

2) *Topological spaces* are ones that are well-suited to Topological ideas (see Chapter 9 for one way of formulating these and especially [268, 250, 243, 264] if interested in further details). Each topological space may or may not possess a number of interesting *topological properties*. I.e. properties preserved by Topology's most natural morphisms: continuous bijections with continuous inverses: *homeomorphisms*. Rightfully-famous such include *connectedness*, *compactness*, *second-countability* and *Hausdorffness* (a notion of *separation*).

The earliest studies of Topology [236] turned out to be [250, 260, 261] best interpreted Algebraically in two somewhat separate chunks. *Homotopy*, which loosely concerns how many kinds of loop a space supports that cannot be shrunk down to a point. For which a prototype is *winding number* (which some Readers may have already encountered in Complex Analysis). And *Homology*, which loosely concerns the extent to which a geometry, such as a surface, has holes of various dimensionalities. A prototype here is the *Euler characteristic*, or the closely-related notion of *topological genus* of a surface. Graph Theory already suffices to meaningfully encounter the Euler characteristic (Chapter 45).

Finally, some parts of Topology intimately involve active use of further equipping by differential structure, as in Differential Calculus, or more eventually Differential Geometry and its differentiable manifolds [316, 270, 332]. Perhaps the simplest most commonly encountered such part of Topology is *co-homology* [257, 322, 241, 261]: a dual and yet more differential counterpart of homology.

0.5 Preface 2: Motivating Combinatorics as a stepping stone towards understanding Topology

The particular current and future significance of Topology is that it will be systematically applied to [274, 273, 418, 426, 275, 276, 439, 278] the theory of all other STEM subjects over the next 30 years in ways that are not only profoundly changing our understanding of these subjects but also shall very largely develop how these subjects are treated at the advanced-undergraduate and masters levels. This will be at least as influential and sweepingly universal as the widespread application [292, 291, 441, 409, 315, 326, 294, 285, 286, 295, 342, 348, 344, 283, 290, 289] of Linear Algebra [280, 297, 281, 298, 292, 296, 293, 291, 441] mostly in the 1950s in the 1950s through to the 1970s. And of the Differential-Geometrization [326, 335, 334, 315, 332, 318, 328, 367, 301, 306, 312, 333, 320, 323, 368, 319, 317, 327] of all other subjects [300, 301, 341, 347, 361, 252, 369, 370, 389, 392, 396, 395, 397, 401, 403, 411, 419, 428, 427] mostly in the 1990s and 2000s.

Notice among these earlier waves of change that there always were 'early birds': authors or entire subjects who incorporated branches of mathematics that subsequently became universally applied, some decades before this universalization. So for instance, P.A.M. Dirac was an 'early bird' in realizing in the 1920s that Linear Algebra provides a useful framework for Quantum Mechanics [343, 344]. While Hilbert (in collaboration with Courant [342]) valued Linear Algebra in explaining Methods of Mathematical Physics in the 1920s he did not however apply it in giving his 1930s overview of Geometry (in collaboration with Cohn-Vossen [345]). 'Early birds' in Differential-Geometrization of other areas of STEM included Lie and Killing in the 1880s and 90s as regards differential equations and symmetries. Einstein [341] in the 1910s in his conception of the General Relativity framework that superseded Newton's approach to Gravitation. And Whitney in the 1940s [304] as regards formulating more widely-applicable generalizations of manifolds.

♠ ◇ ♣

It is thus well worth commenting on 'early birds' in applying Topology to other areas of STEM. Poincaré [236] discovered a substantial proportion of Topology in pursuit of better understanding Dynamical Systems. Dynamical Systems thereby got a large head start in Topologization, with so-called qualitative methods long coming to dominate much of the theory of the subject [400]. Furthermore, Arnol'd [361] Topologized rather than just Differential-Geometrized, Mechanics and the Principles of Dynamics.

Additionally, much of Theoretical Physics was reformulated in the 1960s and 1970s in terms of the fibre bundles version [322, 318, 328, 319] of Differential Geometry. This development covers, in particular, the Quantum Theory of Electromagnetism and of the nuclear forces (collectively referred to as *Gauge Theory*). As well as effects in the Theories of Matter, and Cosmology, as well as further refinements to General Relativity. Topological effects resulting from such reformulations were then rapidly found in the 1970s [367, 318, 328].

In talking about new and upcoming applications of Topology, we are thus *mostly not* talking about the previous two paragraph's developments. But rather about, firstly, Computer Science and Statistics catching up.

Secondly, about new waves of progress in areas that have so far only been partly Topologized. One should distinguish, for instance, between applying 1930s-era Topology [238, 239, 250, 268, 272, 267], 1960s-era Topology [243, 260, 261, 244] and modern Topology [274, 264, 261, 278].

Thirdly, one should additionally distinguish between applying Topology to the objects O of an area of STEM and applying Topology to *the space formed by* those objects, which in the current Volume we refer to as their *arena*, $\mathfrak{O}$. The notions of configuration space [347, 361] and phase space [361] in Dynamics are early-bird prototypes for what we mean by arenas. As a major motivation for the current Series and its Widely-Applicable Topology Series successor, we shall gradually provide a wealth of examples of arenas. And on how understanding the structure of these has enormous potential for revolutionizing the theory of all STEM subjects. Simple models of arenas tend to be either geometries or order structures, motivating our current Volume's drive to reach as far as Order Theory. Given some area of STEM's mathematical objects O, it is moreover always a valid and revealing question to ask which topology their arena naturally realizes.

Let us however difer further explanation about arenas to Sec 0.11's outline of the last Part of the current Volume. And briefly return to our second point above: modern Topology has started to be applied to cutting-edge Computer Science [274, 273, 418, 271, 428, 275, 276, 426, 278]: Big Data, Image Analysis, Computer Graphics, Robotics, Signalling Networks, Artificial Intelligence... These are, on the one hand, inherently significant developments and works in progress. On the other hand, it is clear from these developments that modern Topology will profoundly change the theory of many other STEM subjects. This is as opposed to for instance the small-correction or whole-universe-model-only effects upon applying 1930s to 1960s-era Topology to Theoretical Physics in the 1970s. So both applying *modern* Topology and conceiving in terms of *arenas* have enormous potential. *In this way, we provide enough new water for all the Empress' Steeds to drink*. [Enough today for the Vanguard, and for one further Corps in each of most of our twin Series' subsequent Books.]

This just being the start of an introductory Book at the Pre-Topology level, it is far from a suitable place to explain what modern Topology consists of... For the main intended Readership would first need to acquire several layers of understanding of Pre-Topological and Topological concepts first. Indeed, it is the main purpose of the current Pre-Topology Series and our subsequent Topology Series to gently introduce these layers to our Readers! The *end of* this Volume being a slightly more suitable place, a few pointers to modern Topology relevant to current research applications can be found there (Sec 65.10). Alongside some indication of how far along the current two Series one would need to work through before each of these becomes accessible.

0.6 Preface 3: our Pre-Topology Series' starts by systematically applying Combinatorics to everything else

Among basic Combinatorial topics, Graph Theory is particularly widely applicable. It also provides one of the easier routes to some of Topology. At present, it is perhaps a less widely-used route to Topology, with e.g. [258, 268, 263, 260, 243] proceeding otherwise. See however [266] for a kindred spirit, as well as smaller doses of Graph Theory in e.g. [250, 267, 261].

For Graph Theory is an 'opposite corner' of Topology to Differential Geometry (and Differential Topology [244, 245, 332, 257, 249, 321, 377]), which offer an alternative, if more mathematically-structured route into Topology. In the present Series, we elect to lead with Graph Theory, however, applying this to each other subject in turn. All the while commenting on which of the notions that arise turn out to have further Topological significance. The current Volume's two Parts V and VI on Graph Theory are the first to contain substantial amounts of novel concepts and new commentary.

When the question of what arena $\mathfrak{Q}$ is formed from one's mathematical objects O is asked of Combinatorial objects (or Discrete Mathematics objects more generally), moreover, a widespread answer is that it is a graph. More particularly, the Topological level of such arenas can be described to some extent by a graph. It can then often furthermore be *better*-described by an *arrow-graph*: a graph whose edges are equipped with arrows. And many arenas turn out to be arrow-graphs with furtherly-significant ordering structure, by which a poset or lattice description is suitable.

We thus return to a point that we have already mentioned twice before, and which is indeed the key point of the current Volume. Namely that to understand arenas, we cannot stop at Graph Theory, but must push ahead to treating at least some Order Theory: Part VII of this Volume. Only then, by the powers of the Graph Theory and Order Theory combined, do we start to systematically go after arenas, in Part VIII. This gives us several dozen concrete examples of interesting arenas, leaving us rather more able to discuss specifics about arenas in our Conclusion...

♠ ◇ ♣

To further understand the current Series' approach, observe that undergraduate subjects have hitherto often been presented piecemeal, while graduate treatments often cumulatively build towers of subjects. One exception to this is that many undergraduate courses became Linear-Algebraicized in the 1970s and 1980s. So it has become rather usual to see Linear Algebra at the base of many towers. We are however electing to place Combinatorics at the base of our tower, indeed Combinatorializing every other subject much as every other subject has been Linear-Algebraicized in previous texts.

On the one hand, this is pedagogically novel. As well as deeply profitable due to all of: Graph Theory's and Order Theory's versatility, and their entering each other subject a second time at the level of what arenas that subject sires.

On the other hand, this subject-ordering difference will shortly be compensated for. Firstly via Volume 1 being on Linear Algebra, and its two-way interaction with Combinatorics, given our 'Combinatorialize everything' motif. Secondly, via 'Combinatorialize everything' being but the zeroth floor of our Series' tower. By which, from Volume 2 onwards, everything else is to be Linear-Algebraicized as well as Combinatorialized! This is reflected by our Series' subtitle passing at this stage from 'improving understanding of everything with a pinch of Combinatorics' to 'Improving understanding of everything by dual-wielding Combinatorics and Linear Algebra'. We term this an *evolving* Series subtitle, which is very arguably the right kind for a *tower* series to have.

♠ ◇ ♣

That Differential Geometry offers other routes in to Topology will be dealt with in due course, by building up our tower to quite soon contain a dedicated Differential Geometry level. We do not head there immediately, however, for the following three reasons that motivate the three intervening Books.

• Volume 2 is in recognition of Computer Science having changed the traditional organization of STEM. As well as its algorithms, programming and mathematics packages having become key tools in studying the other areas of STEM.

• Volume 3 deals with how we need Flat Geometry before attempting Differential Geometry. This can be placed after Computer Science since *most* Computer Science makes little use of Flat Geometry (while making *copious* use of Combinatorics and Linear Algebra). With such as 'Computer Graphics, Computer Vision, Image Analysis and signalling, or navigating robots, across physical space' (*) being consequently postponed until after Flat Geometry is in play.

• Volume 5 deals with how much more (finite and discrete) Group Theory can be done, and partnered with Representation Theory, before Differential Geometry is necessary.

• Though I do still anticipate making Volume 6 on Differential Geometry available prior to Volume 5. Out of Differential Geometry being far more applicable and suggestive of Topological matters than Volume 5's topics. Only with Volume 7 – on Lie Theory, which combines Differential Geometry and Groups – shall Volume 5's contributions become inevitably necessary for the further building of our tower. Differential Geometry also affords a *wider* range of approaches to topics (*). While it and Lie Theory combined afford a new interpretation of the Principles of Dynamics and the Classical Laws of Nature [438]. By each of which, the current Series shall in due course expand to yet further Books.

But let us for now outline what basic Graph Theory and basic Order Theory consist of!



0.7 Contents outline 2: Parts V and VI on graphs and Part VII on order

Chapter 29 introduces the simplest notion of *graphs*, which consist of vertices with at most one edge between distinct vertices only. Chapter 30 introduces relational variables for graphs and then applies these in starting to consider graph arenas. Chapter 31 considers *degrees* – the number of edges emanating from each vertex, called *valencies* in Chemistry – as a start on the local structure of graphs. Chapter 32 covers some basic Combinatorics of graphs, mostly rooted in Euler's early-bird work at the very inception of Graph Theory in the 18th Century. Throughout Parts V and VI, we place particular emphasis on Topology, structure, and applications.

Chapter 33 extends our treatment to *oriented*, *di-* and *mixed arrow-graphs*: a first source of graph-modelling variants. Chapter 34 then develops new graphs from old, using *subspaces* (here subgraphs) and *sum*, *product* and *quotienting* operations. This 'Cartesian' quartet can in fact be entertained in setting up any branch of mathematics [243, 270]. Particular examples of subgraphs covered here include paths, trails, walks, cycles, circuits and tours.

Chapter 35 gives a simple account of graph *connectivity*, rooted on Whitney's early-bird work from the 1930s. Chapter 36 studies the structure of *tree graphs* in rather more detail, including rooted trees, binary trees and early-bird Computer Science applications, including some that go as far back as Cayley's work in the mid 19th century. Chapter 37 returns to more Topological matters: *independence*, *cover* and *domination* numbers. In contrast, Chapter 38 next considers notions of *distance* on graphs. This is supported by Appendix B's metric notion of distance.

Chapter 39 extends graph modelling in an initially-distinct direction, to *loop-graphs*: allowing for edges from a vertex to itself. And to *multi-graphs*: with multiple edges between some pairs of vertices. Chemical, Quantum-Theoretic Feynman diagram and Binary relation applications of these are split off into Chapter 40. Many introductory books on Graph Theory omit this paragraph's modelling variants; though without these many applications cannot be reached.



Chapters 41 and 42 consider notions of *traversability* for graphs. These cover, respectively, which graphs possess a circuit going through every edge precisely once: the so-called *Eulerian* graphs, following on from further early-bird work of his. And which possess a cycle going through every vertex precisely once: the so-called *Hamiltonian* graphs, named after another mid 19th century early bird.

Chapter 43 presents examples of *forbidden* and *forced subgraphs*. Chapter 44 considers *homeomorphs*: the core topology of graphs to which Metric information can then be attached to generate various Metrically-distinct and yet Topologically-equivalent graphs.

Chapter 45 introduces *planar graphs*, faces, and Euler's characteristic. And also *triangulations*, higher-*d simplices* beyond faces, and simplicial complexes. and to *hypergraphs*, in which edges can join more than two vertices. Chapter 46 then sketches some further Theory for planar graphs. We identify here forbidden subgraphs – quite an early-bird topic [132, 190, 183] – and consider a more systematic approach to Hamiltonianness in the case of planar graphs [159, 183, 176].

Chapter 47 provides an introduction to *colourings* of graphs, which motivated much of the development of Graph Theory. Graph colouring saw much work from the late 19th century though to its main resolution in the 1970s by Appel and Haken [165].

Part VI ends with Chapter 48, which considers rather more modern work on two quantifiers of departure from planarity: *crossing number* and *thickness* [183, 193, 194]. All of planarity, departures from planarity and colouring for graphs furthermore turn out to be rather deeply rooted in Topology. With crossings also playing a defining role for knots, this last Chapter on graphs is where we also choose to place our first simple partial outline of Knot Theory.

♠ ◇ ♣

Part VII on Order Theory begins by introducing *posets* [49, 44, 224] in Chapter 49, with simple examples including *chains* and the smallest posets beyond these. We also include a forbidden triangles condition that oriented graphs need to obey if they are to be posets. Chapter 50 then introduces *antichains*, and duality between these and chains. The topology of posets is also discussed here, and new posets from old are built as well.

In Chapter 51, we further specialize posets to *lattices* [224, 49], which are forced to be connected. Two particularly structurally significant types of lattice – *modular* and *distributive* – are the subject of Chapter 52. Each of these is given a forbidden sublattice (or forbidden poset-forming subgraph) characterization.

0.8 Innovation 0: Freedom of Conceptual Thinking, including of Truer Naming

Throughout these Series of Books, Conceptual Thinking is visibly displayed and further encouraged. This includes finding Truer Names for all the main objects and results encountered.

As a first example, I use the names *combinations triangle* or *binomial triangle* instead of Pascal's triangle. For these names tell us about the nature of this triangle in terms of even more widespread basic mathematical structures. Such Truer-Naming of mathematical objects envisages them as naturally-occurring denizens of Platonia, which are to be seen as *discovered* rather than merely invented.[1] Incidentally, this point of view very much accommodates for some of the most beautiful and natural denizens of Platonia to be independently discovered by different mathematicians and different civilizations. For instance, 'Pascal's triangle' is a structure that was already previously known and studied in China and Baghdad. It can even be argued that some of the denizens of Platonia being discovered on multiple occasions is a *testament to their naturality*.

[1] It is too early on in this Volume and Series to discuss variants of Platonia, and what Truer Names *each of those* might have...

In some cases, even multiple a priori unrelated perspectives and chains of thought lead to one and the same natural and beautiful denizen of Platonia. This was for instance Wheeler's overall 'many routes' position on the nature of General Relativity [353, 358].[2]

Multiple routes are also plainly on show in the title of Johnstone's 'Sketches of an Elephant – A Topos Theory Compendium' [262]. This title refers to one person describing an elephant as snakelike, out of only being familiar with the trunk. While another describes it as sail-like, out of only being familiar with an ear. And yet another describes it as tree-like, out of only being familiar with one of the legs...

Freedom of naming is part of Graph Theory and of Topological Spaces, in that each person working in these areas has the freedom to use their own nomenclature. *This freedom is hereby declared to be a universal freedom: a fundamental right for all intellectuals working on any topic.* In my case, I will endeavour to use this freedom so that every new name I use in these two Series of Books is a Truer Name. With each notion's previously commonly used names (if any) featuring in at least a footnote, so that Readers can make contact with other courses and previous literature.

0.9 Foreword about this Volume's presentational choices

To keep subsection, subchapter and subpart analogues of paragraph breaks mutually distinct, let us use ♠ ◇ ♣ , ♗ ♙ ♘ , and ♔ ♖ ♕ respectively.

Let us now cast the Book's principal mathematical objects in pictorial form. Each is given a distinctive shape-and-colouring scheme visualization so as to afford instant recognizability to the Reader! This includes instant recognizability of when a modelling situation, or a proof, is involving multiple actors at once! [Let us call this *manifestly-visible interdisciplinarity*.]

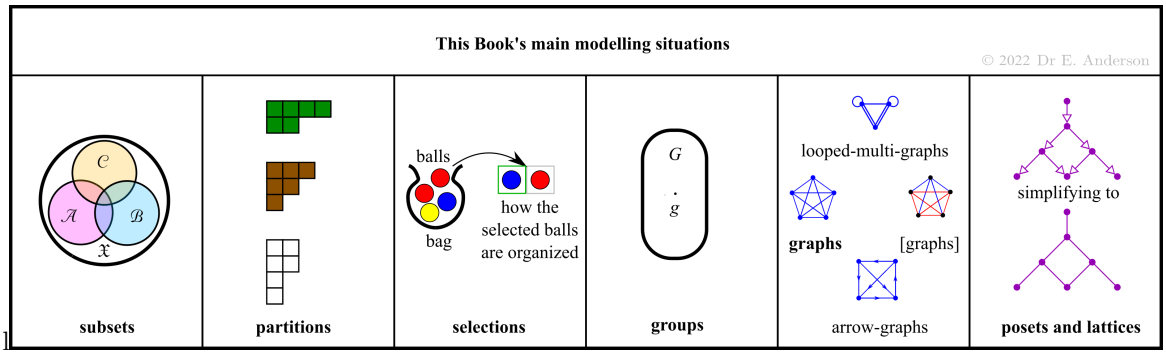


Figure 2:

One precursor of this is 'blob diagrams'. These are already quite widely used in graduate courses on and research in [270, 332] such as Differential Geometry or Topology. It is a big disservice, however, not to root the use of such on already well-understood courses on more basic material. For sets, this is quite standard (and quite comprehensively developed in [227]). While the current Pre-Topology Series further rectifies this, starting with the current Volume importing such into the most very basic theory of binary operations and of groups.

So far also an opportunity has been lost by not drawing each different type of mathematical space as a strikingly visually-distinct type of blob. So that Readers know immediately that the text is

[2] That we start this Series with Volume 0 and this Volume with Chapter 0 is itself a 'Wheelerism': that, given first principles, one is to always ask what the zeroth principles are.

evoking an open set, a manifold, a group... just by glancing at the shape-and-colour scheme of the figure on that page. Also in this way, when a result or proof involves multiple open sets, say, then all possible confusion between distinct such dissolves upon referring to the red pear and the yellow lemon within the proof's accompanying pictures. This just works better than calling them $\mathfrak{U}_4$ and $\mathfrak{U}_7$. Let alone just denoting each a priori distinct open set that a proof makes use of by $\mathfrak{U}$ or leaving them unlabelled. Given the focus of the current Volume and Series, let us add here that such depictions also very much help in not confusing objects with the arenas that those objects form.

♗ ♙ ♘

0.10 Innovation 2: study not just objects but also the arenas formed by those objects

Pascal's triangle furthermore provides us with our first arena. For it can also be viewed as $\mathfrak{C}$ombination: the space of combinations n-choose-k. [3]

The relational Pascal's triangle is furthermore a first simple example of a *more reduced arena*. Namely, the arena of [combinations]. I.e. of the quotients of combinations by reflection symmetry about the midline of their second naïve parameter: the k in n-choose-k. Quotienting by a group has the same meaning as passing to a more reduced space here. Relational spaces are reduced spaces, or at least a 'sufficiently natural' subcase of reduced spaces. The study of relational arenas thus turns out to be a substantial application of quotienting by groups.

♠ ◇ ♣

This key point – that use of such as difference and ratio variables can be naturally followed up by passing to reduced arenas – is by now quite common knowledge in the following 'early-bird' subjects. Dynamics, Foundational and Theoretical Physics and Theoretical Statistics.

Let us first further flesh out this key point, by observing that 'difference and ratio variables' (systematized in Appendix A) generalize to *relational variables*. Meaning those that are invariant under some natural action of a group G on our objects O that can be taken to be *meaningless* in our modelling. For instance, a Physicist would call such a G a physically-meaningless group, redundant group, or gauge group. While a Philosopher might call it *absolute information* and evoke Leibniz's 'identity of indiscernibles' [339] to declare this information bereft of meaningful content. Following Kendall, a Statistician might say 'shape variables' rather than 'relational variables', and a Geometer would likely agree with this naming. Possibly modulo distinguishing between an overall scale variable and 'nontrivially scale-invariant' shape variables [392, 433].

Let us next explain the context in which the above subjects became 'early birds' in acquiring such notions. As previously mentioned, configuration spaces in Mechanics are one of the earliest examples of arena. For instance, the naïve $3 \times 3 = 9$-dimensional configuration space corresponding to the 3-d 3-Body Problem dynamical system. It is then furthermore standard to pass to centre-of-mass variables, which removes from contention the 3 translations supported in 3-d. The question of further eliminating out one or both of the rotations and the overall scale from such arenas was subsequently posed. See e.g. [392, 414, 422] for more recent accounts of this.

Aside from improving technical understanding of N-body problems, this has furthermore been argued [365, 385, 414, 431] to realize various 'relational principles' along the lines of Leibniz [339] and Mach

[3] In these Series, we use mathfrak font to denote spaces, and bold mathfrak font, as here, to denote spaces of spaces or spaces of objects: collectively referred to as *arenas*. This serves to lessen the possibility of confusion between objects and spaces, and between each of these and the arenas that they form.

[340]. Thus constituting an actual mathematically-sharp resolution of the long-standing Absolute Versus Relational Motion Debate [337, 385]. And even forming part of a local resolution [438] of the classical version of the notorious Problem of Time [353, 354, 381, 379, 415, 431].

Though before we explain what this is, we should first mention several other 'early birds' that form an intervening layer of understanding. Firstly, that configuration space study was geometrized [347, 361] and, secondly that this approach was even quite promptly extended to the much more intricate case of General Relativity's configuration spaces [354, 356, 360, 390, 407, 431]. General Relativity's configurations are 3-d spatial geometries. By which General Relativity is not only Einstein's theory of Gravitation in curved geometrical 4-d spacetime form [341, 359, 368]. But also a Geometrodynamics of evolving 3-d spatial geometries [349, 350, 351, 353, 358, 368, 424].[4] Thirdly, that the intermediately-difficult case of configuration spaces for Gauge Theory has also been considered in some detail [371, 398, 399].

The Problem of Time then refers more specifically to the way that time is conceived of in General Relativity being at least apparently incompatible with how it is conceived of in Quantum Theory. This matters because it is possible for physical regimes to concurrently be Quantum-Mechanical and Generally-Relativistic, with Black Holes [359, 368] and the Early Universe [378] providing examples. The Problem of Time can furthermore be traced back to [381, 379, 431] conceptual and technical differences between background-independent theories [352, 404] such as General Relativity and background-dependent ones like ordinary Quantum Theory. By which it turns out to already be poseable at the classical level [415, 431, 438]. Where it is locally resolved by noticing that its hitherto mysterious and mutually-interfereing multiple facets [381, 379, 383, 412] crystallize into [438] a rediscovery of Lie Theory (!!) Incidentally, this also in good part explains why the current Series places our Principles of Dynamics text [447] *above* the Lie Theory floor in its growing tower.

Fourthly, an 'early bird' in Theoretical Statistics' was David Kendall, who considered the statistics of triangles in the plane [370, 389, 396]. He based this on their arena: the space of all scale-free triangles, that is of triangles modulo similarity. Which he showed to be both Geometrically and Topologically the sphere, $\mathbb{S}^2$. More generally, he found that the arena of all scale-free N-a-gons is both Geometrically and Topologically the complex-projective space $\mathbb{CP}^{N-2}$ [370, 376, 396, 416]. The Topological half of this result was in fact already known in the Dynamics and Geometry context to Smale [355].

The Author's own first original brush with arenas came in showing that [405, 414] Kendall's arena already answered the question of what Relational Mechanics' [365, 414] configuration space is (in the 2-d case). And so some of the 'early birds' allign (a third example of a multiple-routes perspective, albeit a much smaller example than General Relativity's or Topos Theory's many routes).

♠ ◇ ♣

One difficulty with relational and arenas programs so far is that relatively few people are familiar with (or have the theoretical infrastructure to become familiar with) the examples that bave hitherto been used to illustrate it in the literature. My answer to this is that, behold, *even Pascal's Triangle can be thought about in this way*! In fact, *much of basic Combinatorics can be usefully reworked in relational and arena terms*, with the current Volume being the world's first systematic exposition of this (!)

Only then in Volume 3's Flat Geometry [443] do we need to resort to the traditionally-simplest example that the space of triangles is a sphere. While Gauge Theory and General Relativity arenas are not further mentioned until our Theoretical Physics Volumes 4, 8 and our Lie-Theoretic reformulation of the Principles of Dynamics in Volume 9 [447]. Numerous global details of this [439]

[4] This is indeed part of Wheeler's aforementioned point that there are multiple different routes to General Relativity!

must however await our Widely-Applicable Topology Series. As must our treatment of the very interesting generalization [428] of Kendall's work to affine and projective shape spaces for use in such as Image Analysis and Computer Vision. For indeed a more mature understanding of Topology is needed before tackling any of these topics in any detail.

0.11 Contents outline 3: Part VIII's many examples of arenas

What our basic Combinatorics objects' arenas give is the aforementioned pile of arrow-graphs, many of which additionally carry meaningful poset or lattice structure. It thus remains for us to outline which such arenas feature in our Part VIII.

Many of our quotients involve just quotienting out the simplest nontrivial group: the only group of order 2, C_2. In each case, we need to specify how the group acts, since this exhibits more variety in our examples than what the group actually is.

♠ ◇ ♣

1) Chapter 53 presents *arenas of subsets* of a fixed set $\mathfrak{X}$, including the well-known *power set* $\mathfrak{P}(\mathfrak{X})$ in the case with labelled elements. Further identical-labels and C_2-complementation quotients of this are also entertained.

2) Chapter 53 also presents *arenas of families*, $\mathfrak{F}$amily, which admit their own separate notion of C_2 complementation.

3) Chapter 56 picks out the *arena of covers*, $\mathfrak{C}$over, as an attractor set within $\mathfrak{F}$amily The *arenas of topologies*, $\mathfrak{T}$opology, and *of partitions*, $\mathfrak{P}$artition, form more fickle patches therein.

4) *Arenas of selections*, $\mathfrak{S}$election, are considered in Chapter 54. This includes finishing with this Introduction's account of *arenas of combinations*, $\mathfrak{C}$ombination, as well as dealing with the consequences of the corresponding reflection symmetry being broken for whichever of the Stirling numbers.

5) This Chapter also includes simple cases of *recurrence arenas*, $\mathfrak{R}$ecurrence, corresponding to Combinatorial objects generated by recurrence relations.

6) The *arena of partitions*, $\mathfrak{P}$artition carries the famous Young lattice, as well as various less well-known orderings detailed in Chapter 57. These now have a meaningful C_2 action by conjugation of partitions.

♠ ◇ ♣

7) The arena of groups is too hard to consider in the current Volume. We do however consider *arenas of subgroups* and *of normal subgroups* of some specific groups in Chapter 58.

♠ ◇ ♣

8) *Graph arenas* $\mathfrak{G}$raph, including of simpler cases such as of just trees, $\mathfrak{T}$ree, are covered in the next few Chapters (59 to 63) Arrow-graph and (loop-)(multi-)graph arenas are included.

These examples support not only C_2 graph-complementation quotients but also the harder quotients by homeomorphs to form homeomorph-prime graph arenas. These are moreover examples of *incompatible quotients*: since these do not respect each other, a choice between them needs to be made in any given study. graphs.

Arrow-graphs furthermore come with a second C_2 quotient, corresponding to swapping two-way arrowed and unarrowed graphs. This and complementation are compatible quotients, so a $C_2 \times C_2$ quotient is also entertained. We also provide a somewhat simpler model arena with some features that anticipate our larger quotient, in the form of an arena of edge-coloured graphs. Colouring examples can furthermore support quotients by larger permutation groups than just C_2.

♠ ◇ ♣

9) Chapter 64 gives this Volume's last examples of *arenas: of posets*, *of lattices*, *of modular lattices* and *of distributive lattices*. These admit C_2 order-reversal quotients.

♠ ◇ ♣

We conclude in Chapter 65 with our main findings. Followed by pointers to further reading in Combinatorics, Combinatorial Group Theory, Graph Theory Order Theory and Arenas. With further applications to the Theory of many other STEM subjects.

0.12 Preface 4: which Topological conceptual types emerge in this?

This being a Pre-Topology Series that is being followed up by a Topology Series, whenever we cover a topic, we indicate which of its concepts, results and examples subsequently turn out to have Topological significance. When conceptually-similar such features recur in multiple topics, we take it as an opportunity to further discuss their conceptual type. This plants a number of seeds, which will blossom in the gardens of those Readers who continue to study, and eventually add to, the Theory of some part of STEM. Let us thus close our zeroth Chapter by listing the main families of Topological notions that the current Volume uncovers.

- *Topological spaces* themselves.
- Notions of *cover* (which enter *compactness* and many other Topological notions).
- Notions of *connectivity*.
- Notions of *traversability*,
- Notions of *separation*.
- *First* and *second countability*.
- *Homeomorphs* of graphs.
- A start on how *(non)planarity* and *colouring of graphs* turn out to be inherently Topological subjects.
- A few introductory exercises on *Knot Theory*.

Part I

Outine of sets, functions and number systems

Chapter 1

Sets

1.1 Introduction

Definition 1 A *set*

$$\mathfrak{X}$$

is a collection of objects, which are said to be elements of the set.

Remark 1 'Collection' here means a mathematically well-defined bunch of objects. Since we shall shortly be using 'collection' to mean something else, we shall use 'bunch' in the present role instead.

Notation 1 It is customary to use lower-case letters for elements and upper-case ones for sets. We furthermore have a lot of types of mathematical object to develop in this Volume and Series. So let us more specifically use lower-case italic letters for elements: x, y, x_1, x_2, x_i $i = 1$ to n, and upper case mathfrak letters for sets: $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{X}_1$, $\mathfrak{X}_2$, $\mathfrak{X}_i$ $i = 1$ to n.

Set-counting Principle The *size* (alias *cardinality*) a finite set is the number of elements in it.

Notation 2 We use $x \in \mathfrak{X}$ and $y \notin \mathfrak{X}$ to denote that the element x belongs to the set $\mathfrak{X}$, while the element y does not.

Notation 3 x_i, $i = 1$ to n will do to denote the elements of a finite set. More generally, we use x_j, $j \in \mathfrak{J}$ for $\mathfrak{J}$ some general *indexing set* for the elements of our set; this can be finite or infinite.

Remark 2 All that we are setting up here is a naïve approach to sets (compare up to p. 57 of [27]) *Axiomatic Set Theory* [38] being more advanced than the current Series.

♠ ◇ ♣

Notation 4 Let us depict sets as blobs, with finite sets' elements as marked and labelled points therein. Since this Volume and Series have quite a lot of spaces to depict, we choose the convention of depicting the default set as an ellipse. We more generally depict each plain type of space as a 2-d blob hugged within a rectangle with aspect ratio $2 : 1$, in the default 'tall' orientation.[1]

In the case of sets, blob diagrams are often called *Venn diagrams* (Figs 1.1).

The wider idea being that whenever the Reader sees an elliptic blob in this Volume and Series, then you know that it is being treated just as a set. This is as opposed to blobs of other shapes, with say and rounded-off rectangles depicting groups (Fig 2). Rectangular blobs themselves being reserved

[1] We elect to wait until non-plain versions of spaces appear before explaining what we mean by 'plain' here; see Chapter 10. See Sec 15.5 for a brief explanation of aspect ratios.

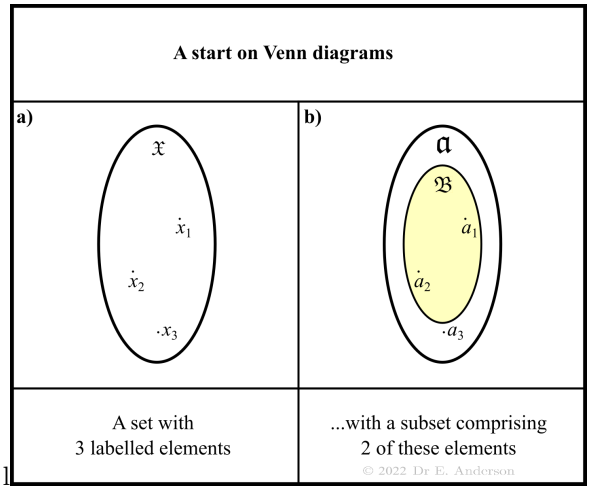


Figure 1.1:

for vector spaces (Volume 1), and square blobs for flat spaces (Volume 3). Or, when labelled as such, products of distinct spaces and powers of a single space (Chapter 3). We thus keep clear what is what, in an immediate visual way. This will help a lot as regards always understanding which space each object under consideration belongs to. This is both conceptually crucial in understanding what is going on. And often very useful in envisaging results and what constructs may be used to prove them.

1.2 Subsets

Definition 1 If all the elements of a set $\mathfrak{B}$ also belong to a set $\mathfrak{a}$, then we say that $\mathfrak{B}$ is a subset of $\mathfrak{a}$ (Fig 1.1.b).

Remark 1 We can write this as if $b \in \mathfrak{B} \Rightarrow b \in \mathfrak{a}$, then

$$\mathfrak{B} \subseteq \mathfrak{a} . \tag{1.1}$$

'Is contained in' is an alias for 'is a subset of'.

Remark 2 There are two cases to consider. If $\mathfrak{a}$ contains no further elements $a \neq \mathfrak{B}$, then we can just as well write that

$$\mathfrak{a} \subseteq \mathfrak{B} . \tag{1.2}$$

Then

$$\mathfrak{a} = \mathfrak{B} .$$

If $\mathfrak{a}$ contains at least one element $a \neq \mathfrak{B}$, however (Fig 1.1.c) , then we write

$$\mathfrak{B} \subset \mathfrak{a}$$

in place of (1.1). This reads 'is a strict subset of', 'is a proper subset of', 'is strictly contained in' or 'is properly contained in'.

Remark 3 $\mathfrak{a} \supseteq \mathfrak{B}$ is equivalent to $\mathfrak{B} \subseteq \mathfrak{a}$. This reads '$\mathfrak{a}$ is a superset of $\mathfrak{B}$'. $\mathfrak{a} \supset \mathfrak{B}$ is equivalent to $\mathfrak{B} \subset \mathfrak{a}$, pronounced '$\mathfrak{a}$ is a strict superset of $\mathfrak{B}$'.

Remark 4 That together (1.1) and (1.2) $\Rightarrow \mathfrak{B} = \mathfrak{a}$ (Fig 1.2 embodies a very significant strategy in proving that 2 different descriptions of a set are in fact referring to one and the same mathematical set. We term this *two-way inclusion*, and start to use it in Sec 1.9.

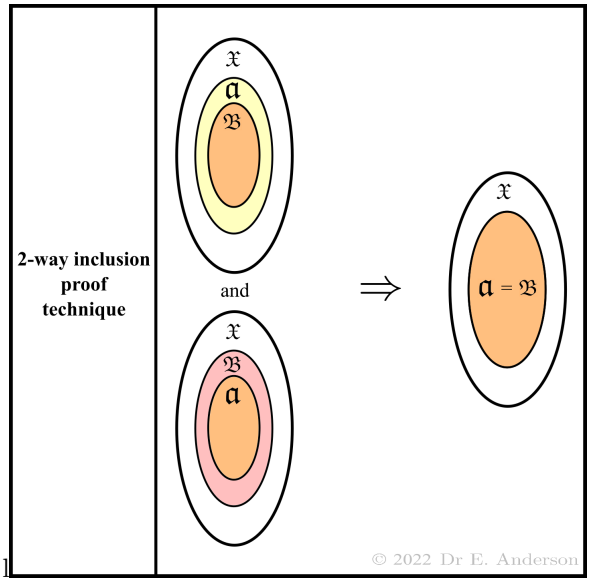


Figure 1.2:

Remark 5 We slash $\subseteq$, $\subset$, $\supseteq$, $\supset$ much as we did for $\in$, to form symbols for not being a (strict) sub- or super-set: $\nsubseteq$, $\not\subset$, $\not\supseteq$, $\not\supset$. We subsequently introduce further slashed versions of symbols to 'denote negation' without further comment.

1.3 The universe set and the empty set

Definition 1 The set with all of a given model's elements in it is called the *universe set*; we denote this by $\mathfrak{X}$.

Definition 2 The set with no elements in it is still considered to be a set: the *empty set*

$$\emptyset \;=\; \{\,\text{no elements}\,\}\;.$$

Definition 3 A subset is *nontrivial* if it is not $\emptyset$.

Remark 1 We thus arrive at a notion of *proper-and-nontrivial subsets*. I.e. those $\subset 1$ for which

$$\emptyset \;\subset\; \mathfrak{a} \;\subset\; \mathfrak{X}\;.$$

This extends to proper and nontrivial subsets of $\mathfrak{a}$, by temporarily casting $\mathfrak{a}$ in the role of $\mathfrak{X}$.

1.4 Hereditarity

Definition 1 If a set has a property that is such that all of its subsets have the same property, then we say that this property is *hereditary*.

Example 1 *Finiteness* is a hereditary property.

Counterexample 2 *Non-emptiness* is not a hereditary property.

1.5 Relative and absolute complements of sets

How about changing $\in$ to $\in$ in the definition of intersection? This gives (Fig 1.3.a) the *relative complement of* $\mathfrak{B}$ *in* $\mathfrak{a}$,

$$\mathfrak{B}\backslash\mathfrak{a} := \{\text{all } r \,|\, r \in \mathfrak{B} \text{ and } r \notin \mathfrak{a}\} .$$

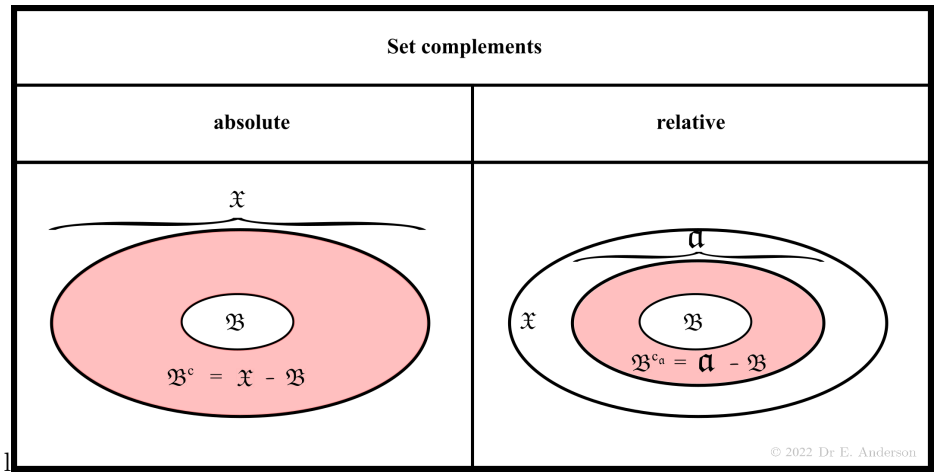


Figure 1.3:

Remark 1 := denotes 'is defined to be'. Unlike equality =, this is directional: the left-hand symbol's notion is defined to be the right-hand-side expression.

Remark 2 Passing from using i to r here is not a contentful change; it is rather just a change in name of a *dummy variable* that the definition makes use of. It is 'nice' to realign our notation from i to r to match up with the name of the new notion produced: relative complement.

Naming Remark 1 If the universe set $\mathfrak{X}$ is cast in the role of $\mathfrak{a}$, then we call the resulting complements *absolute complements*. We denote the absolute complement of subset $\mathfrak{B}$ by $\mathfrak{B}^{\mathrm{c}}$ (Fig 1.3.b). This notation often being used, let us also introduce the parallel notation $\mathfrak{B}^{\mathrm{c}\mathfrak{a}}$ for the relative complement of $\mathfrak{B}$ in $\mathfrak{a}$.



1.6 Intersections and unions

Definition 1

$$\mathfrak{I} := \mathfrak{a} \cap \mathfrak{B} := \{\text{all } i \,|\, i \in \mathfrak{a} \text{ and } i \in \mathfrak{B}\} \tag{1.3}$$

is the *intersection* of the sets $\mathfrak{a}$ and $\mathfrak{B}$ (Fig 1.4.a).

Remark 1 This covers the case in which $\mathfrak{a}$ and $\mathfrak{B}$ share at least one element, while $\subset A$ might contain elements not in $\mathfrak{B}$ and vice versa. See Fig 1.4.a).

Definition 2

$$\mathfrak{U} := \mathfrak{a} \cup \mathfrak{B} := \{\text{all } u \,|\, u \in \mathfrak{a} \text{ and/or } u \in \mathfrak{B}\} \tag{1.4}$$

is the *union* of the sets $\mathfrak{a}$ and $\mathfrak{B}$ (Fig 1.4.b).

Remark 2 Two styles of specification for sets are as follows.

1) Hitherto, we have listed the elements.

2) Another way is by specifying properties that characterize the elements of the set; for instance, (1.3) and (1.4) use specifications of this kind.

Remark 3 These are two of the most commonly encountered *binary operations*: on pairs of subsets. They are not however the only logical possibilities; let us further explore in the next two sections.

Notational Remark 1 That intersection and union are denoted by an upside-down pair of symbols is no coincidence. For these are a dual pair of operations, for which such upside-down pairs of symbols are customary. This Volume and Series will introduce many further dual pairs of concepts.

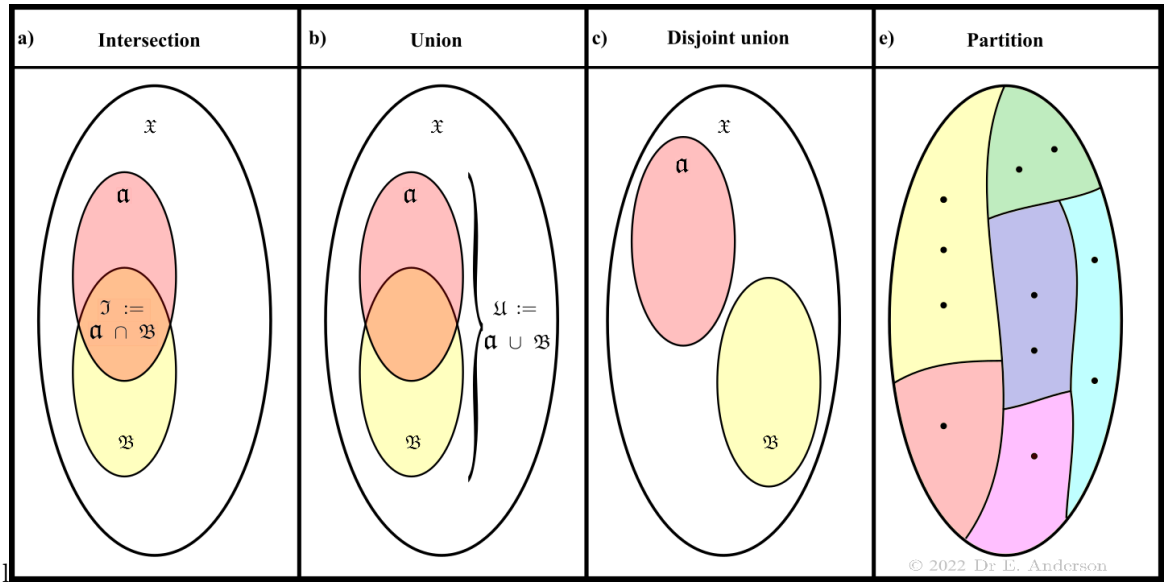


Figure 1.4:

1.7 Disjoint unions

Definition 1 Two subsets $\mathfrak{a}$, $\mathfrak{B}$ are *disjoint* if their intersection is empty,

$$\mathfrak{a} \cap \mathfrak{B} = \emptyset .$$

Unions of such are termed *disjoint unions* (Fig 1.4.c).

Notation 1 These are denoted by

$$\mathfrak{a} \amalg \mathfrak{B} .$$

Pointer 1 Cutting up a set into a disjoint union that leaves out no element is called forming a *partition* (Fig 1.4.d): the subject of Chapter 15.

Disjunctive Counting Principle In a disjoint union or partition, the count of the elements in the whole set is the sum of the counts of the elements in each of its parts. I.e. such counts are additive.

Remark 1 Counting is somewhat harder in the presence of nontrivial intersections. The next section ends by considering a first structure in this direction. A systematic counting principle for multiple nontrivially-intersecting sets is then the subject of Chapter 12.

Remark 2 A *property* that holds for the whole set if it holds for each disjoint part of the set is called *disjunctive* or *additive*. A *quantity* whose value for the whole is the sum of the values of its disjoint parts is called *additive*. Additive quantities lie on the critical boundary between often qualitatively distinct *sub-additive* and *super-additive* quantities. For instance, we shall be seeing that some notions of information content are additive, while others are sub-additive and even occasionally super-additive (literally 'more than the sum of their parts').

1.8 Further interplay between subsets conceptualized

Remark 1 Negating instead the other $\in$ in the definition of intersection does not produce a new notion, just a differently-denoted one. For

$$\{\,\text{all}\ \ r\,|\,r\ \notin\ \mathfrak{A}\ \ \text{and}\ r\ \in\ \mathfrak{B}\,\}\ =\ \{\,\text{all}\ \ r\,|\,r\ \in\ \mathfrak{B}\ \ \text{and}\ r\ \notin\ \mathfrak{A}\,\}\ =\ \mathfrak{B}\backslash\mathfrak{A}\ :$$

the relative complement of $\mathfrak{B}$ in $\mathfrak{A}$.

How about negating both at once? Then assuming that there are no more elements outside of our sets,

$$\{\,\text{all}\ \ e\,|\,e\ \notin\ \mathfrak{A}\ \ \text{and}\ \ e\ \notin\ \mathfrak{B}\,\}$$

leaves us with no elements at all: the empty set.

Remark 2 $\mathfrak{A}$ and $\mathfrak{B}$ enter the definitions of $\cap$ and $\cup$ on an equal footing: a label-exchange symmetry. This has the immediate consequence that

$$\mathfrak{B}\ \cap\ \mathfrak{A}\ =\ \{\,\text{all}\ \ i\,|\,i\ \in\ \mathfrak{B}\ \ \text{and}\ \ i\ \in\ \mathfrak{A}\,\}\ =\ \{\,\text{all}\ \ i\,|\,i\ \in\ \mathfrak{A}\ \ \text{and}\ \ i\ \in\ \mathfrak{B}\,\}\ =\ \mathfrak{A}\ \cap\ \mathfrak{B}\ .$$

And

$$\mathfrak{B}\ \cup\ \mathfrak{A}\ =\ \{\,\text{all}\ \ u\,|\,u\ \in\ \mathfrak{B}\ \ \text{and/or}\ \ u\ \in\ \mathfrak{A}\,\}\ =\ \{\,\text{all}\ \ u\,|\,u\ \in\ \mathfrak{A}\ \ \text{and/or}\ \ u\ \in\ \mathfrak{B}\,\}\ =\ \mathfrak{A}\ \cap\ \mathfrak{B}\ .$$

Remark 3 Such equal footing does not however extend to relative complement. This is clear from Fig 1.5.b), in which $\mathfrak{A}\backslash\mathfrak{B}$ has 2 elements whereas $\mathfrak{B}\backslash\mathfrak{A}$ has 3 (Fig 1.5.a-b). One question to pose, then, is whether any further 'label symmetric' version of this can be formed.

Remark 4 Let us now explore the conceptual diversity of the 'and' versus 'and/or' difference between the definitions of intersection and of union? 'And/or' looks curiously more complicated than just 'and'. This symbolic complexity is designed to avoid confusion with another type of 'or': 'or but not both'. If we use this, we have

$$\{\,\text{all}\ \ c\,|\,c\ \in\ \mathfrak{A}\ \ \text{or}\ \ c\ \in\ \mathfrak{B}\ \ \text{but not both}\ \}\ ;$$

see Fig 1.5.c).

Remark 5 This is also 'label exchange symmetric', so we do not need to mention $\mathfrak{A}$ and $\mathfrak{B}$ distinctly in defining this notion. It is also

$$(\ \mathfrak{A}\backslash\mathfrak{B}\ )\ \cup\ (\ \mathfrak{B}\backslash\mathfrak{A}\ )\ .$$

By this, it can be viewed as a *symmetrization* of relative complement, by taking the union over $\mathfrak{A}$ and $\mathfrak{B}$ label-specific objects. These observations lead to this notion's commonly used name, as follows.

Definition 5 The *symmetric difference* of $\mathfrak{A}$ and $\mathfrak{B}$ is

$$\mathfrak{A}\ -_{\mathbf{S}}\ \mathfrak{B}\ :=\ \mathfrak{S}\ :=\ \{\,\text{all}\ \ s\,|\,s\ \in\ \mathfrak{A}\ \ \text{or}\ \ s\ \in\ \mathfrak{B}\ \ \text{but not both}\,\}\ .$$

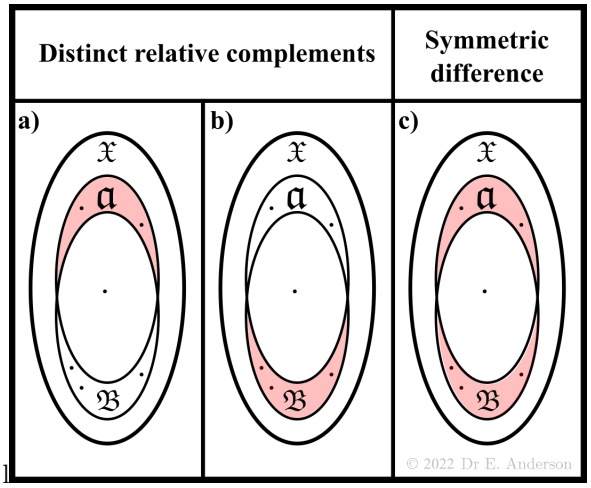


Figure 1.5:

1.9 Exercises

Exercise 1 Find i) $\mathfrak{A} \cap \mathfrak{A}$, $\mathfrak{A} \cup \mathfrak{A}$, $\mathfrak{A} \backslash \mathfrak{A}$ and $\mathfrak{A} -_{\mathbf{S}} \mathfrak{A}$.

ii) $\emptyset \cap \emptyset$, $\emptyset \cup \emptyset$, $\emptyset \backslash \emptyset$ and $\emptyset -_{\mathbf{S}} \emptyset$.

iii) $\mathfrak{A} \cap \emptyset$, $\mathfrak{A} \cup \emptyset$, $\mathfrak{A} \backslash \emptyset$ and $\mathfrak{A} -_{\mathbf{S}} \emptyset$.

Exercise 2 If you are familiar with logic gates, which three of these correspond to $\cap$, $\cup$ and $-_{\mathbf{S}}$? Why are the other four logic gates not represented at this stage as sources of variety?

Exercise 3 What happens if we use one or two $\notin$ in the definition of $\cup$?

Exercise 4 i) Show that $\mathfrak{A} \cap \mathfrak{B} \subseteq \mathfrak{A}$, $\mathfrak{A} \cap \mathfrak{B} \subseteq \mathfrak{B}$, $\mathfrak{A} \subseteq \mathfrak{A} \cup \mathfrak{B}$, $\mathfrak{B} \subseteq \mathfrak{A} \cup \mathfrak{B}$ and $\mathfrak{A} \cap \mathfrak{B} \subseteq \mathfrak{A} \cup \mathfrak{B}$.

ii) When is $\mathfrak{A} -_{\mathbf{S}} \mathfrak{B} \subseteq \mathfrak{A}$?

Worked Example 1 Show that

$$(\mathfrak{A} \cup \mathfrak{B}) \cap \mathfrak{C} = (\mathfrak{A} \cap \mathfrak{C}) \cup (\mathfrak{B} \cap \mathfrak{C}) .$$

Answer 1 Probe with an arbitrary element $x \in \mathfrak{A} \cup \mathfrak{B}) \cap \mathfrak{C}$. Then

$$x \in \mathfrak{C} \tag{1.5}$$

and

$$x \in \mathfrak{A} \cup \mathfrak{B} .$$

By the second of these, at least one of

$$x \in \mathfrak{A}$$

and

$$x \in \mathfrak{B}$$

must hold. Since 1.5 also applies, at least one of

$$x \in \mathfrak{A} \cap \mathfrak{C}$$

and

$$x \in \mathfrak{B} \cap \mathfrak{C}$$

must hold. So

$$x \in (\mathfrak{A} \cap \mathfrak{C}) \cup (\mathfrak{B} \cap \mathfrak{C}) .$$

Thus

$$(\mathfrak{A} \cup \mathfrak{B}) \cap \mathfrak{C} \subseteq (\mathfrak{A} \cap \mathfrak{C}) \cup (\mathfrak{B} \cap \mathfrak{C}) . \tag{1.6}$$

Next probe instead with an arbitrary element

$$y \in (\mathfrak{A} \cap \mathfrak{C}) \cup (\mathfrak{B} \cap \mathfrak{C}) .$$

Then at least one of

$$y \in \mathfrak{A} \cap \mathfrak{C} \tag{1.7}$$

and

$$y \in \mathfrak{B} \cap \mathfrak{C} \tag{1.8}$$

must hold. But (1.7) means that $y \in \mathfrak{A}$ and $y \in \mathfrak{C}$. Also (1.8) means that $y \in \mathfrak{B}$ and $y \in \mathfrak{C}$. Thus $y \in C$ for sure, while $y \in$ at least one of $\mathfrak{A}$, $\mathfrak{B}$ as well, and thus

$$y \in \mathfrak{A} \cup \mathfrak{B}$$

is assured. So

$$y \in (\mathfrak{A} \cup \mathfrak{B}) \cap \mathfrak{C} .$$

Thus

$$(\mathfrak{A} \cap \mathfrak{C}) \cup (\mathfrak{B} \cap \mathfrak{C}) \subseteq (\mathfrak{A} \cup \mathfrak{B}) \cap \mathfrak{C} . \tag{1.9}$$

Finally, together, (1.6) and (1.9) give our desired equality.

Exercise 5 Show that

$$(\mathfrak{A} \cap \mathfrak{B}) \cup \mathfrak{C} = (\mathfrak{A} \cup \mathfrak{C}) \cap (\mathfrak{B} \cup \mathfrak{C}) .$$

Exercise 6 Show that

i)

$$(\mathfrak{A} \cup \mathfrak{B}) \cup \mathfrak{C} = \mathfrak{A} \cup (\mathfrak{B} \cup \mathfrak{C}) .$$

ii)

$$(\mathfrak{A} \cap \mathfrak{B}) \cap \mathfrak{C} = \mathfrak{A} \cap (\mathfrak{B} \cap \mathfrak{C}) .$$

Draw Venn diagrams for each of the previous four equalities.

Exercise 7 Hypothesize about what $\mathfrak{A} -_{\mathbf{S}} \mathfrak{B} -_{\mathbf{S}} \mathfrak{C}$ should consist of ($-_{\mathbf{S}}$ is also not to depend on bracketing).

Exercise 8 Show that

$$\mathfrak{A} -_{\mathbf{S}} \mathfrak{B} = (\mathfrak{A} \cup \mathfrak{B}) \backslash (\mathfrak{A} \cap \mathfrak{B}) .$$

1.10 n-ary versions

Remark 1 Another way of generalizing the definitions of $\cap$, $\cup$ and $-_{\mathbf{S}}$ is to make them not binary but *n-ary*. I.e. depending on not 2 but n inputs, here subsets. I.e.

$$\mathfrak{I} := \{ \text{ all } i \,|\, i \in \mathfrak{a}_1, ..., i \in \mathfrak{a}_n \}$$

for a *finite intersection*.

$$\mathfrak{I} := \{ \text{ all } i \,|\, i \in \mathfrak{a}_j, ..., j \in \mathfrak{J} \}$$

for an *arbitrary intersection*.

$$\mathfrak{U} := \{ \text{ all } u \,|\, u \in \mathfrak{a}_1, ..., u \in \mathfrak{a}_n \}$$

for a *finite union*.

$$\mathfrak{U} := \{ \text{ all } u \,|\, u \in \mathfrak{a}_j, ..., j \in \mathfrak{J} \}$$

for an *arbitrary intersection*.

We will justify that these do not need brackets in the finite case in Exercise 5.2, this Chapter being the one affording ready extension to n-item proofs.

Remark 2 This arbitrary to finite distinction in unions and intersections turns out to be significant in Topology ([268] and Sec 9.5).

Chapter 2

Functions

2.1 Introduction

Definition 1 Let $\mathfrak{X}$ and $\mathfrak{Y}$ be sets. A *function* or *map*

$$f \ : \ \mathfrak{X} \longrightarrow \mathfrak{Y}$$

('from $\mathfrak{X}$ to $\mathfrak{Y}$') is a rule assigning to each $x \in \mathfrak{X}$ a unique element in $\mathfrak{Y}$.

Remark 1 $\mathfrak{X}$ here plays the role of 'source space', which is termed *domain*, $\mathrm{dom}\, f$. $\mathfrak{Y}$ here plays the role of 'target space', alias *co-domain*, $\mathrm{co\text{-}dom}\, f$. Some authors use *range* as a synonym for co-domain. Others consider range to be the set of all $f(x)$, which may be a subset of, rather than necessarily equal to, $\mathrm{co\text{-}dom}\, f = \mathfrak{Y}$.

Notation 1

$$f \ : \ \mathfrak{X} \longrightarrow \mathfrak{Y}$$

$$x \ \mapsto \ y \ = \ f(x)$$

is the arrow notation for specifying what the rule is.

Notation 2 See Fig 2.1 for diagrammatic representation of functions as arrows between blobs.

2.2 Simple examples of functions

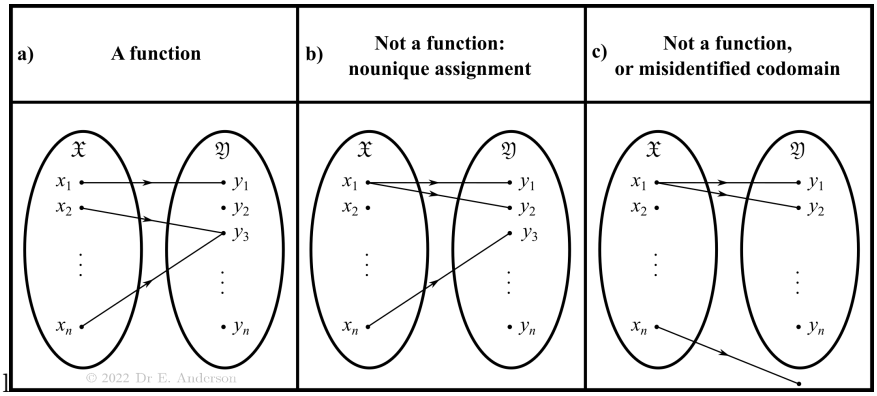


Figure 2.1:

Example 1 Fig 2.1.a) depicts a function.

Example 2 In contrast, Fig 2.1.b) is not a function, for both b_2 and b_3 are assigned to one and the same element of $\mathfrak{A}$: a_1.

Example 3 Also Fig 2.1.c) is either not a function or is the result of misidentifying what the co-domain ought to be.

Notation 3 The *graph* of a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is depicted in Fig 2.2.a). Fig 2.2.b) illustrates another non-example of function.

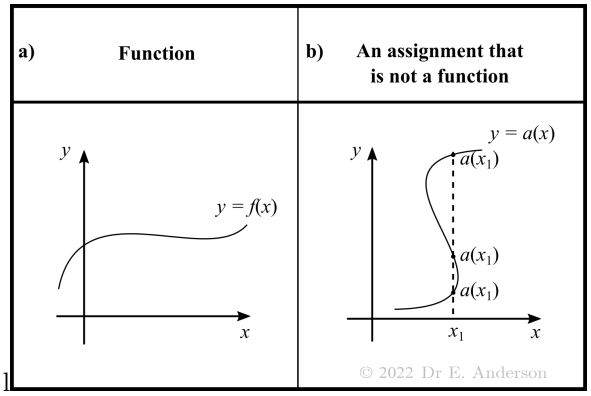


Figure 2.2:

Exercise 1$^-$ Which of the following are functions?

i) $y = \sqrt{|x|}$ for $x = 0$ to 1.

ii) $y = \sqrt{|x|}$ for $x = -1$ to 1.

iii) $x^2 + y^2 = 1$.

iv) $y = 3x^3 - 9x^2 + 6x + 1$.

v) $r = \sin(2\theta)$ in polar coordinates, with θ running from 0 included to 2π excluded.

2.3 Specific examples of functions

Example 1 The *identity function*

$$\mathfrak{X} \longrightarrow \mathfrak{X}$$
$$i : x \mapsto x$$

maps each element to itself (Fig 2.3.a).

Example 2 The *constant function*

$$\mathfrak{X} \longrightarrow \mathfrak{Y}$$
$$c : x \mapsto c$$

all to the same unique element $c \in \mathfrak{Y}$ (Fig 2.3.b).

Example 3 The *inclusion function* of $\mathfrak{X}$ in $\mathfrak{Y}$ for $\mathfrak{X} \subseteq \mathfrak{Y}$,

$$\mathfrak{X} \longrightarrow \mathfrak{Y}$$

$$\iota_{\mathfrak{X}} : x \mapsto x \text{ for } x \in \mathfrak{X}$$

(Fig 2.3.c). This is not $i_{\mathfrak{X}}$ since $\mathfrak{Y} \supset \mathfrak{X}$ is possible. It is standardly given the special arrow notation indicated on the Figure.

Remark 1 If

$$\mathfrak{X} = \mathfrak{Y}\ ,\ \ \rho_{\mathfrak{W}} = i_{\mathfrak{W}} = i_{\mathfrak{X}} = \iota_{\mathfrak{X}}\ .$$

Example 4 Let $\mathfrak{D}$ be the domain of a function f and $\mathfrak{a} \subseteq \mathfrak{D}$. Then the *restriction* of f to $\mathfrak{a}$ is (Fig 2.3.d-f)

$$f|_{\mathfrak{a}}(x) := \{\, f(x) \,|\, x \in \mathfrak{a} \text{ while being defined elsewhere} \,\}\ . \tag{2.1}$$

Example 5 Suppose that f is a function with domain $\mathfrak{a}$, and F is a function with domain $\mathfrak{B}$ and with $\mathfrak{B} \supseteq \mathfrak{a}$. Then F *extends* f provided that (Figs 2.3.g-h)

$$F = f|_{\mathfrak{a}}\ .$$

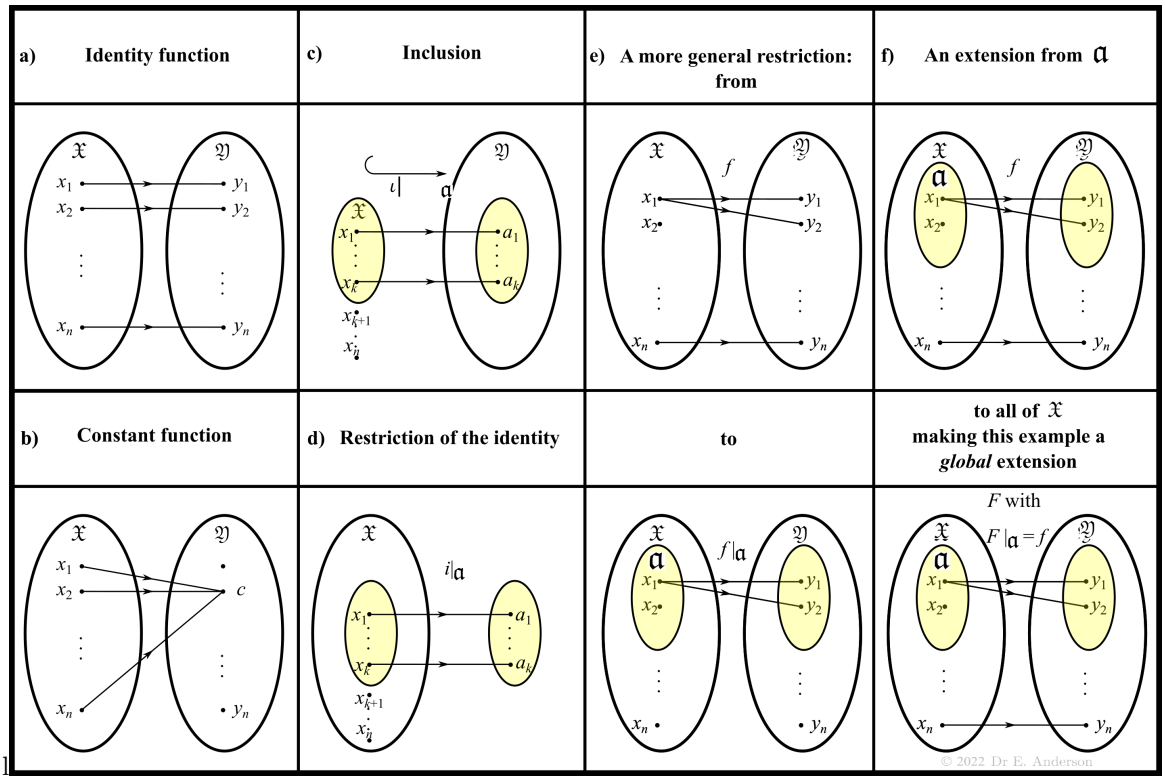


Figure 2.3:

Exercise 2 Show that for a given function f and a given subset $\mathfrak{a}$ of its domain, restrictions to $\mathfrak{a}$ are unique but extensions from $\mathfrak{a}$ to a given subset $\mathfrak{B} \supset \mathfrak{a}$ are not necessarily.

2.4 Injections, surjections and bijections

Definition 1 A function j is *injective* alias *1 to 1* if

$$f(x) = f(y) \ \Rightarrow\ x = y\ .$$

Definition 2 A function s is *surjective* alias *onto* if

$$\text{every } y \in \mathfrak{Y} \text{ is in the image of at least one } x \in \mathfrak{X} .$$

Definition 3 A function β is *bijective* if it is both injective and surjective.

Remark 1 The injectivity condition means that each element $y \in \mathfrak{Y}$ has at most one pre-image (compare with the surjectivity condition).

2.5 Examples of -jections

Example 1 Fig 2.4.a) depicts a function that is injective but not surjective. Fig 2.4.b) is surjective but not injective. Fig 2.4.c) is both surjective and injective and thus bijective. Fig 2.4.d) is a function that is neither surjective nor injective.

Exercise 3 Which of Figs 2.4.e)-h) are surjective? Injective? Bijective? Neither? What about all identity maps? All inclusions? The constant map?

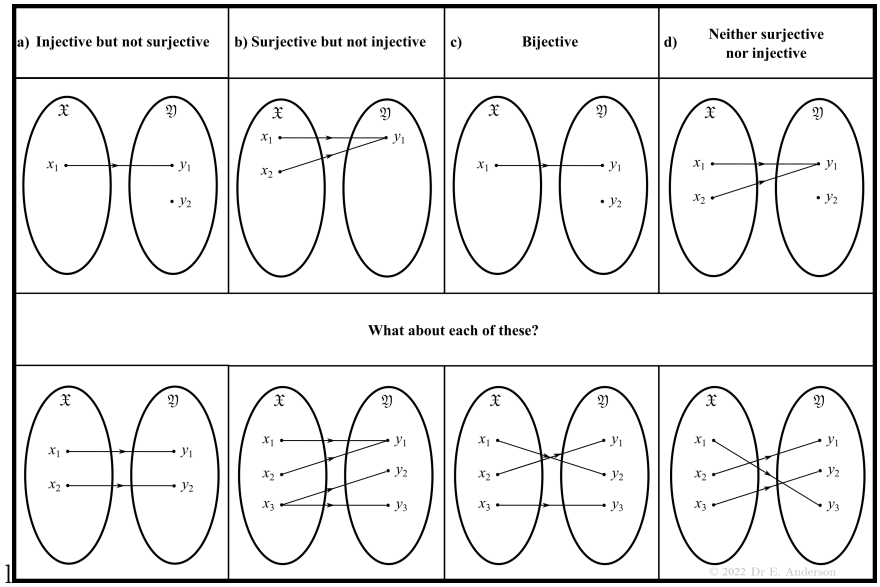


Figure 2.4:

2.6 Composing functions

Definition 1 Let

$$\begin{aligned} f \ : \ \mathfrak{X} &\longrightarrow \mathfrak{Y} \\ x &\mapsto f(x) \end{aligned}$$

and

$$\begin{aligned} g \ : \ \mathfrak{Y} &\longrightarrow \mathfrak{Z} \\ y &\mapsto g(y) \end{aligned}$$

Also suppose that

$$\operatorname{dom} g \ \subseteq \ \text{co-dom}\, f \ . \tag{2.2}$$

Then the *composite function*

$$f \circ g : \ \mathfrak{X} \longrightarrow \mathfrak{Z}$$

is well-defined, and given by

$$x \ \mapsto \ g(f(x)) \ .$$

Remark 1 (2.2) is a compositional compatibility condition. This always holds for f and g bijective (Fig 2.5.a). Or at least if f is surjective and g is injective (Fig 2.5.b). If compositional compatibility does not hold, we can restrict a more local notion of composition to proceed via

$$\operatorname{dom} g \ \cap \ \text{co-dom}\, f$$

(Fig 2.5.c). Or some subset thereof (Fig 2.5.d).

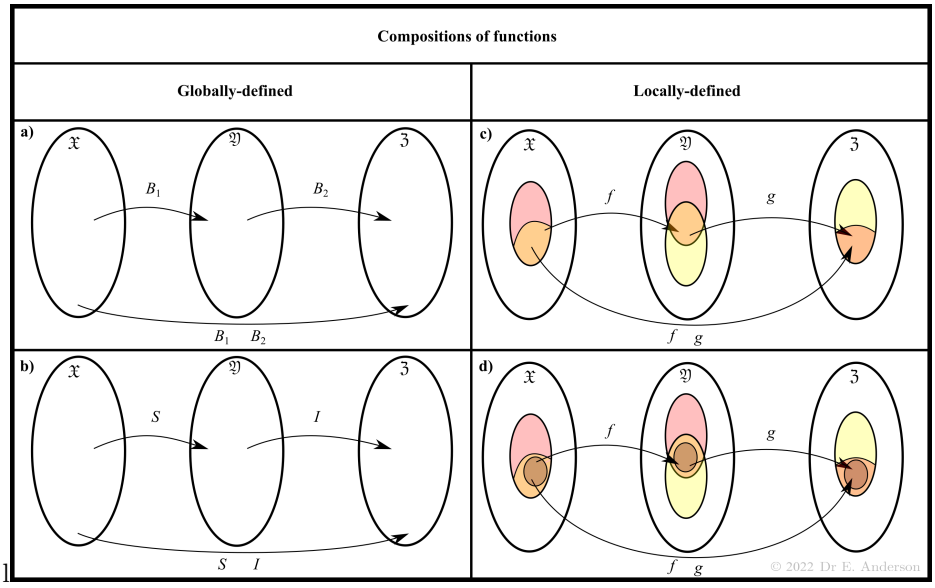


Figure 2.5:

Remark 2 Observe that the $\circ$ and bracketing notation list the composed functions use reverse orders of appearance of the functions.

Exercise 4 Show that

$$f \circ (g \circ h) \ = \ (f \circ g) \circ h \ .$$

Also draw a blob diagram for this.

Exercise 5 Let us term a property of maps *compositional* if the composition of two maps which have the property has itself the property. Show that the following compose.

a) Inclusions, restrictions, and extensions.

b) Injections, surjections and bijections.

2.7 Inverse functions

Definition 1 Let $B : \mathfrak{X} \longrightarrow \mathfrak{Y}$ be a bijection. Then the *inverse function* B^{-1} is a function such that the following hold (Fig 2.6.a).

$$B \circ B^{-1} = i_{\mathfrak{X}}$$

and

$$B^{-1} \circ B = i_{\mathfrak{Y}} .$$

Remark 1 For S surjective, the first of these conditions holds – a right-inverse exists – but not necessarily the second of these conditions (Fig 2.6.b). For I injective, the second of these conditions holds – a left-inverse exists – but not necessarily the first of these conditions (Fig 2.6.c).

Examples 1-3 The identity function i is its own inverse. The inverse function of $\ln x$ is $\exp x$ and vice versa. All of these inverses are globally valid, in the sense of being available for any x in the codomain of the function.

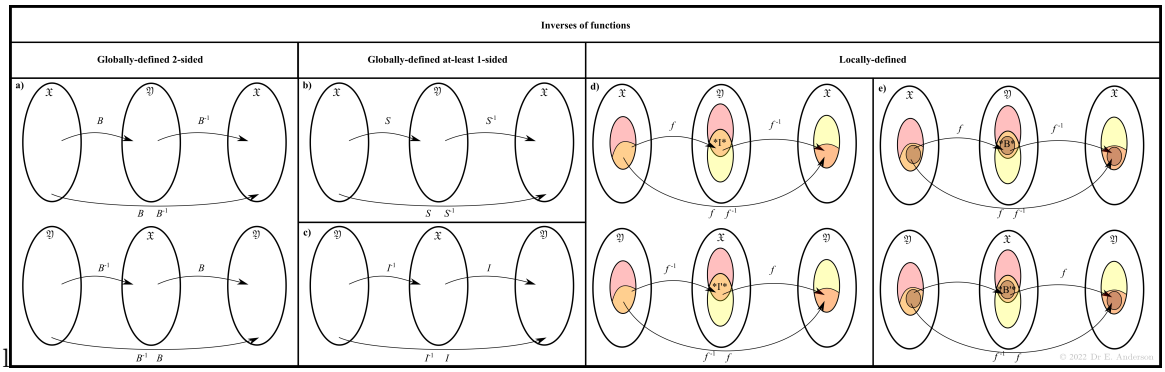


Figure 2.6:

Exercise 6 a) Find a concrete example of a function for which an inverse at most exists for just a piece of the codomain of the function [along the general lines of Fig 2.6.d) or e)].

b) Show that when an inverse exists everywhere, then it must be unique.

c) (Inversion is involutive) Show that if the inverses in question exist everywhere, then the inverse of the inverse function is the original function. What happens if the inverses are only well-defined for part of the range of each of f and f^{-1}?

d) Show that the identity map i is not unique in being its own inverse.

2.8 Images and pre-images of functions

Remark 1 For a given element $x = \mathfrak{X}$, the point or points it maps to, $f(x)$, can be viewed as its *image*. Image extends furthermore to subsets $\mathfrak{B} \subseteq \mathfrak{X}$, for which we denote them as $\text{image}_f(\mathfrak{B})$ or $f(\mathfrak{B})$. This includes for the whole of $\mathfrak{X}$, for which we recover

$$\text{co-dom}\, f = f(\mathfrak{X}) .$$

Remark 2 Suppose that we are given an element

$$b = f(a) \in f(\mathfrak{X}) \subseteq \mathfrak{Y} .$$

Then the point a can be viewed as the *pre-image* alias *inverse image* of b. Pre-image extends furthermore to subsets

$$\mathfrak{B} \subseteq f(\mathfrak{X}) \subseteq \mathfrak{Y} :$$
$$f^{\leftarrow}(\mathfrak{B}) := \{ x \in \mathfrak{X} \,|\, f(x) \in \mathfrak{B} \} .$$

And to the whole of $f(\mathfrak{X})$, for which

$$f^{\leftarrow}(\, f(\mathfrak{X})\,) := \{ x \in \mathfrak{X} \,|\, f(x) \in f(\mathfrak{X}) \} = \operatorname{dom} f .$$

Notational Remark 1 While pre-image is usually denoted by f^{-1}, this notation is shared with inverse function, which pre-image is *not*. This hitherto more common notation having caused many a misunderstanding, we do not use it here. If forced to use f^{-1} for pre-image in one of your courses, the Reader is advised to both figure out, and state, whether each working containing a f^{-1} symbol is about a pre-image or an inverse function.

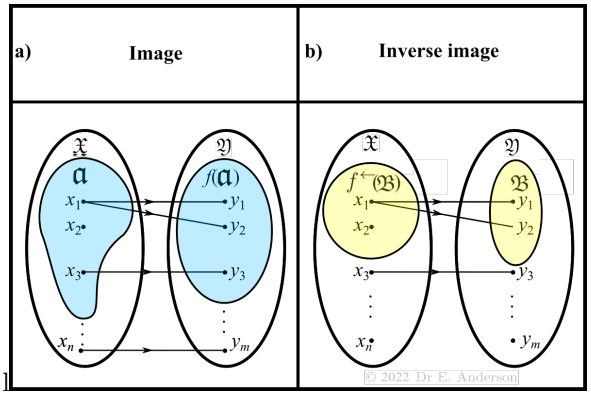


Figure 2.7:

Worked Example 1 Let $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$. For $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathfrak{X}$, show that

$$f(\mathfrak{a}_1 \cup \mathfrak{a}_2) = f(\mathfrak{a}_1) \cup f(\mathfrak{a}_2) .$$

Answer 1 Suppose that $y \in \mathfrak{a}_1 \cup \mathfrak{a}_2$. Then $y = f(x)$ for some $x \in \mathfrak{a}_1 \cup \mathfrak{a}_2$. So $x \in$ at least one of $\mathfrak{a}_1$ and $\mathfrak{a}_2$. Consequently, $y = f(x) \in$ at least one of $f(\mathfrak{a}_1)$ and $f(\mathfrak{a}_2)$. Thus, by the definition of union, $y \in f(\mathfrak{a}_1) \cup f(\mathfrak{a}_2)$. So, since y is arbitrary,

$$f(\mathfrak{a}_1 \cup \mathfrak{a}_2) \subseteq f(\mathfrak{a}_1) \cup f(\mathfrak{a}_2) .$$

This argument is furthermore reversible, so two-way inclusion and thus equality is obtained. □

Exercise 7 a) Show that however

$$f(\mathfrak{a}_1 \cap \mathfrak{a}_2) = f(\mathfrak{a}_1) \cap f(\mathfrak{a}_2)$$

cannot always hold by a minimal counterexample (a map between 2-element sets will do).

b) Yet prove that

$$f(\mathfrak{a}_1 \cap \mathfrak{a}_2) \subseteq f(\mathfrak{a}_1) \cap f(\mathfrak{a}_2)$$

does always holds.

Exercise 8 Show furthermore that for $\mathfrak{B}_1, \mathfrak{B}_2 \subseteq \mathfrak{Y}$, both

$$f^{\leftarrow}(\mathfrak{B}_1 \cup \mathfrak{B}_2) \subseteq f^{\leftarrow}(\mathfrak{B}_1) \cup f^{\leftarrow}(\mathfrak{B}_2)$$

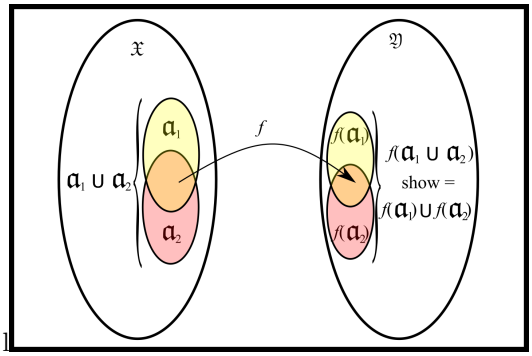


Figure 2.8:

and

$$f^{\leftarrow}(\mathfrak{B}_1 \cap \mathfrak{B}_2) \subseteq f^{\leftarrow}(\mathfrak{B}_1) \cap f^{\leftarrow}(\mathfrak{B}_2)$$

hold.

[Pre-images are better-behaved: a fact subsequently exploited in Topology.]

Exercise 9 Prove that

$$f^{\leftarrow}(\mathfrak{B}_1 \backslash \mathfrak{B}_2) \subseteq f^{\leftarrow}(\mathfrak{B}_1) \backslash f^{\leftarrow}(\mathfrak{B}_2) \ .$$

2.9 Discussion about -jections

Naming Remark 1 Injective, surjective and bijective come from the Latin 'iacere' via the French 'jeter', both meaning 'to throw'. The alternative older name 'onto' is reconciled with 'surjective', by 'surjective' being loosely translateable to 'throw onto', from 'sur' meaning 'upon', which roughly also translates to 'onto'. This means that *all of* $\mathfrak{Y}$ is realized by f, so surjections *are a means of constructing the entirety of a desired target space*. 'Injective' means 'thrown into'. Once an element x_i is thrown into $\mathfrak{Y}$, it is no longer available in $\mathfrak{X}$ to throw into $\mathfrak{Y}$ again, so it only lands in one place: a single element of $\mathfrak{Y}$. This means that *injective maps can be undone*, by sending each realized element of $\mathfrak{Y}$ back to a unique element of $\mathfrak{X}$.

Exercise 10^{-} Reconcile this with some of what was said in Sec 2.6.

Bijective is then simple to define: both of these types of throwing concurrently apply. This moreover amounts to a particularly significant accumulation of useful properties.

2.10 Further practice

Exercise 11 Characterize the local compositions and local inverses in Figs 2.5 and 2.6 as bijections formed by restricting the original functions.

Lemma 1 For

$$I : \mathfrak{X} \longrightarrow \mathfrak{Y}$$

an injective function, $\exists$ a surjective function

$$S : \mathfrak{Y} \longrightarrow \mathfrak{X}$$

and vice versa.

Proof ($\Rightarrow$) For I an injection, restricting the codomain to the image

$$I \ : \ \mathfrak{X} \ \longrightarrow \ I(\mathfrak{X})$$

we can view this as a bijection. Thus $\exists$ an inverse function I^{-1} on $I(\mathfrak{X})$. Extend this to a function E with

$$E(y) \ = \ (\text{some fixed } \ x \ \in \ \mathfrak{X})$$

for every

$$y \ \in \ \mathfrak{Y} \ - \ I(\mathfrak{X}) \ .$$

Finally observe that

$$e \ \circ \ I \ = \ \iota_{\mathfrak{X}} \ .$$

which suffices for E to be a surjection.

($\Leftarrow$) is left to the Reader as **Exercise 12**. □

Worked Example 2 Let $f \ : \ \mathfrak{X} \ \longrightarrow \ \mathfrak{Y}$ be a function and $\mathfrak{B} \ \subseteq \ \mathfrak{Y}$ Prove that

$$f \, f^{\leftarrow}(\mathfrak{B}) \ \subseteq \ \mathfrak{B} \ , \tag{2.3}$$

with equality iff f is surjective.

Answer 2 Consider some

$$y \ \in \ f \, f^{\leftarrow}(\mathfrak{B}) \ .$$

This can be rewritten as

$$y \ = \ f(x)$$

for some

$$x \ \in \ f^{\leftarrow}(\mathfrak{B}) \ .$$

Thereby,

$$y \ = \ f(x) \ \in \ \mathfrak{B} \ .$$

So we have established that $y \ \in$ both $\mathfrak{B}$ and $f(\mathfrak{X})$. So by the definition of intersection,

$$y \ \in \ \mathfrak{B} \ \cap \ f(\mathfrak{X}) \ .$$

y being arbitrary, this has established that

$$f \, f^{\leftarrow}(\mathfrak{B}) \ \subseteq \ \mathfrak{B} \ \cap \ f(\mathfrak{X}) \ . \tag{2.4}$$

Next consider some

$$y' \ \in \ \mathfrak{B} \ \cap \ f(\mathfrak{X}) \ .$$

By the definition of intersection, this means that both

$$y' \ \in \ \mathfrak{B}$$

and

$$y' \ \in \ f(\mathfrak{X}) \ .$$

By the second of these,

$$y' \ = \ f(x)$$

for some $x \ \in \ \mathfrak{X}$. This rearranges to

$$x \ = \ f^{\leftarrow}(y) \ \in \ f^{\leftarrow}(\mathfrak{B}) \ .$$

And so

$$y \;=\; f(x) \;=\; f\,f^{\leftarrow}(y) \;\in\; f\,f^{\leftarrow}(\mathfrak{B})\;.$$

y' being arbitrary, this has established that

$$f\,f^{\leftarrow}(\mathfrak{B}) \;=\; \mathfrak{B} \;\cap\; f(\mathfrak{X})\;. \tag{2.5}$$

Together, (2.7) and (2.5) yield

$$f\,f^{\leftarrow}(\mathfrak{B}) \;=\; \mathfrak{B} \;\cap\; f(\mathfrak{X}) \tag{2.6}$$

[2-way inclusion!]

When f is surjective, furthermore

$$f(\mathfrak{X}) \;=\; \mathfrak{Y}\;.$$

Then using $\mathfrak{B} \;\subseteq\; \mathfrak{Y}$,

$$\mathfrak{B} \;\cap\; f(\mathfrak{X}) \;=\; \mathfrak{B} \;\cap\; \mathfrak{Y} \;=\; \mathfrak{B}\;.$$

Thus (2.6) collapses to

$$f\,f^{\leftarrow}(\mathfrak{B}) \;\subseteq\; \mathfrak{B}\;, \tag{2.7}$$

If f is not surjective, then in general

$$\mathfrak{B} \;\cap\; f(\mathfrak{X}) \;\subseteq\; \mathfrak{B}$$

holds. This means that (2.3) holds. While the presence of at least one point in at least one $\mathfrak{B}$ that is not mapped onto blocks the reverse inclusion from occurring.

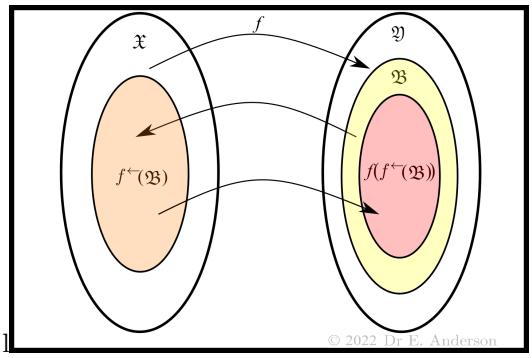


Figure 2.9:

Exercise 13^{+} Prove that

$$\mathfrak{a} \;\subseteq\; f^{\leftarrow}\,f(\mathfrak{a})$$

with equality iff f is injective.

2.11 Pointers

Pointer 1 Injective and surjective are each already very significant notions. Finding injections and surjections is common in studying Algebra. So does quite a lot of Topology [260, 261], and quite a lot of (especially Topologized) Differential Geometry [246, 240].

Pointer 2 Furthermore, every branch of Mathematics reserves a *central* place for bijections. For now, we have only introduced unstructured sets, whose bijections are known as *permutations*. See Secs 6.5-6.12 for more about these, and Sec 10.3 for bijections in some more structured branches of Mathematics.

Pointer 3: The Bijective Counting Principle Suppose that we have two finite sets $\mathfrak{X}$ and $\mathfrak{Y}$. Then if we can establish a bijection between them, their sizes – i.e. counts of elements – must coincide. This underlies much of the Combinatorial style of proof.

Pointer 4 Given Exercise 2's slackness, more stringent extensions, such as continuous extensions or differentiable extensions, are required in some applications. Though in Volume 1 we shall show that these can themselves also be nonunique.]

More specifically, a significant part in [250, 260] Topology is played by continuous extensions, and Theorems guaranteeing them, such as *Tietze's Extension Theorems*.

And a significant part in [246, 332] Differential Geometry and Differential Topology is played by differentiable extensions, and Theorems guaranteeing them, such as *Whitney's Embedding Theorems*.

Chapter 3

Binary -products, -operations, and -relations

3.1 Product sets

Definition 1 Let $\mathfrak{X}$ and $\mathfrak{Y}$ be sets. Then the *(Cartesian) product set*

$$\mathfrak{X} \times \mathfrak{Y}$$

consists of all pairs (x_i, y_j) for $x_i \in \mathfrak{X}$ and $y_j \in \mathfrak{Y}$.

Notation 1 A standard depiction for binary product sets is Fig 3.1.a)'s maps from a rectangle for unequal factors: the more general *heterogeneous binary product*. And b)'s maps from a square for equal factors; the more specialized *homogeneous binary product* alias *squared power* of the space.

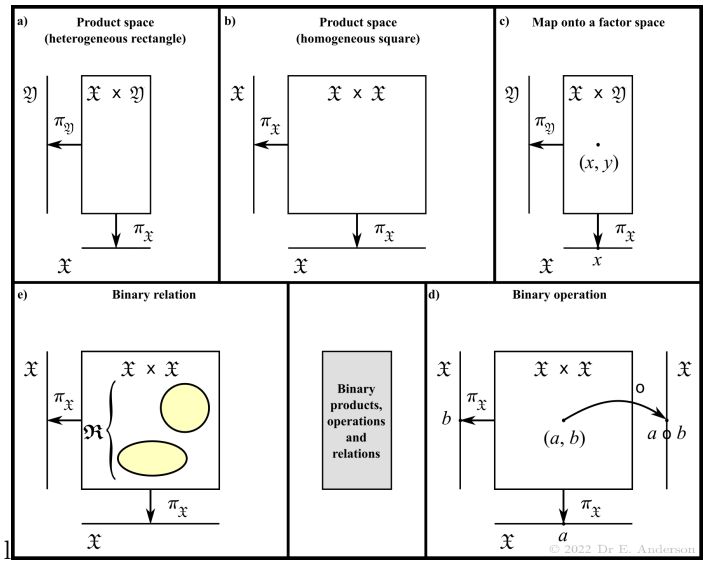


Figure 3.1:

Exercise 1 Show that the maps sending a product space down to each of its individual factors (posed in Fig 3.1.c) are surjections. We shall revisit these maps in further detail in Volume 1.

Notation 2 n-ary such products of sets can also be defined. We denote these by

$$\prod_{i=1}^{n} \mathfrak{X}_i \ .$$

While using

$$\mathfrak{X}^n \ := \ \prod_{i=1}^{n} \mathfrak{X} \ .$$

when all the factor spaces are the same: the nth power of $\mathfrak{X}$. We would draw n-ary powers as cubes for $n = 3$ or as (idealized drawings of) hypercubes for $n > 3$. And heterogeneous n-ary products as (idealized drawings of hyper)boxes for $n \geq 3$.

Product-set Counting Principle For a finite number of products of finite sets, the product set's size is the product of the sizes of each factor set.

Naming-and-Notation Remark 1 That Notation 2 for products and disjunction use an upside-down pair of symbols is no coincidence. For disjunction happens to be a type of co-product, where the prefix 'co' indicates duality. This duality generalizes to many other more structured branches of mathematics' product and sum operations. While not every mathematical name starting with 'co-' indicates duality, we reserve this to mean dual and thus set about renaming all the others. The reason that we do this is that we take Topology to be central, and certain notions of duality play a very significant part in Algebraic Topology. C.f. Sec 0.4 having already mentioned the dual pair of homology and co-homology; domain and co-domain is our only other use so far.

A. Binary operations

3.2 Introduction to binary operations

Definition 1 A *binary operation* on a set is a map

$$\beta \ : \ \mathfrak{X} \times \mathfrak{X} \longrightarrow \mathfrak{X} \ .$$

Remark 1 I.e. from the (homogeneous binary product) = (squared power) of a space to the space itself. This means that its input is two elements a, b from our set $\mathfrak{X}$ while its output is a single element in our set (Fig 3.1.d). We write any such particular or general-formula specifications as follows:

$$\beta \ : \ (a, b) \mapsto a \circ b \ .$$

Simple Examples For sets, we have already mentioned that union, intersection and symmetric difference are binary operations. And we have talked about the simplest n-ary generalization, from concatenating multiple uses of the first two of these binary operations.

See Sec 4.2 for further very well-known examples.

3.3 Adjectives for binary operations

Definition 1 A binary operation $\circ$ on $\mathfrak{X}$ is *commutative* if

$$a \circ b = b \circ a \ \forall \ a, b \in \mathfrak{X} \ .$$

Definition 2 A binary operation is *associative* if

$$a \circ (b \circ c) = (a \circ b) \circ c \ \forall \ a, b, c \in \mathfrak{X} \ .$$

Example 1 We have already shown that union and intersection are commutative.

Definition 3 Given two such operations $\circ$ and $\square$, $\circ$ is *distributive over* $\square$ if

$$a \circ (b \square c) = (a \square b) \circ (a \square c) \ \forall \ a, b, c \in \mathfrak{X} .$$

Exercise 2$^-$ Pick out where Chapter 1 has already established that unions and intersections are each commutative, and associative as well as distributive over each other. And where Chapter 2 established that composition of functions is associative; is this commutative?

Exercise 3 Draw out a blob diagram for disjunctive union of two subsets.

B. Binary relations

3.4 Introduction to binary relations

Definition 1 A *binary relation* $\mathcal{R}$ on a set $\mathfrak{X}$ is a property that each pair of elements of x_1 and x_2 may or may not possess.

Structure 1 This picks out some subset $\mathfrak{R} \subseteq \mathfrak{X} \times \mathfrak{X}$ (Fig 3.1.d). I.e. a subset of the in general heterogeneous product space. We write $a \ \mathcal{R} \ b$ if $(a, b) \in \mathfrak{R}$. And $a \not\mathcal{R} \ b$ if $(a, b) \notin \mathfrak{R}$.

Example 1 Equality is a binary relation.

Example 2 $\leq$, $\geq$, $<$, $>$, $\mathfrak{S}$ubset, $\supset$, $\subseteq$ and $\supseteq$ are binary relations.

Example 3 $:=$ is a binary relation.

Example 4 So are congruence and similarity, each in the sense of between triangles in the flat plane.

Example 5 $\Rightarrow$ is a binary relation.

Example 6 $\Leftrightarrow$ is a binary relation.

Subsequent Chapters add many further examples.

3.5 Adjectives for binary relations. i. For single elements

Structure 1 Some useful and commonly-encountered properties posed of binary relations $\mathcal{R}$ are as follows.

Definition 1 $\mathcal{R}$ is *reflexive* if

$$x \ \mathcal{R} \ x \ \forall \ x \in \mathfrak{X} .$$

Irreflexive if at least one x fails to obey this condition. Or *a-reflexive* if no x obeys it.

Remark 1 When $\mathfrak{X} = \mathbb{R}$, reflexiveness has the following geometrical interpretation. All of the line

$$\{ y = x \} \subset \mathfrak{X} \times \mathfrak{X} \tag{3.1}$$

must lie in the relation's defining subset $\mathfrak{R}$ (Fig 3.3.a). For instance, this is the case for $=$, for which this line is the entirety of $\mathfrak{R}$ This observation furthermore extends to whatever form is taken by the corresponding 'subleading' diagonal in $\mathfrak{X} \times \mathfrak{X}$.

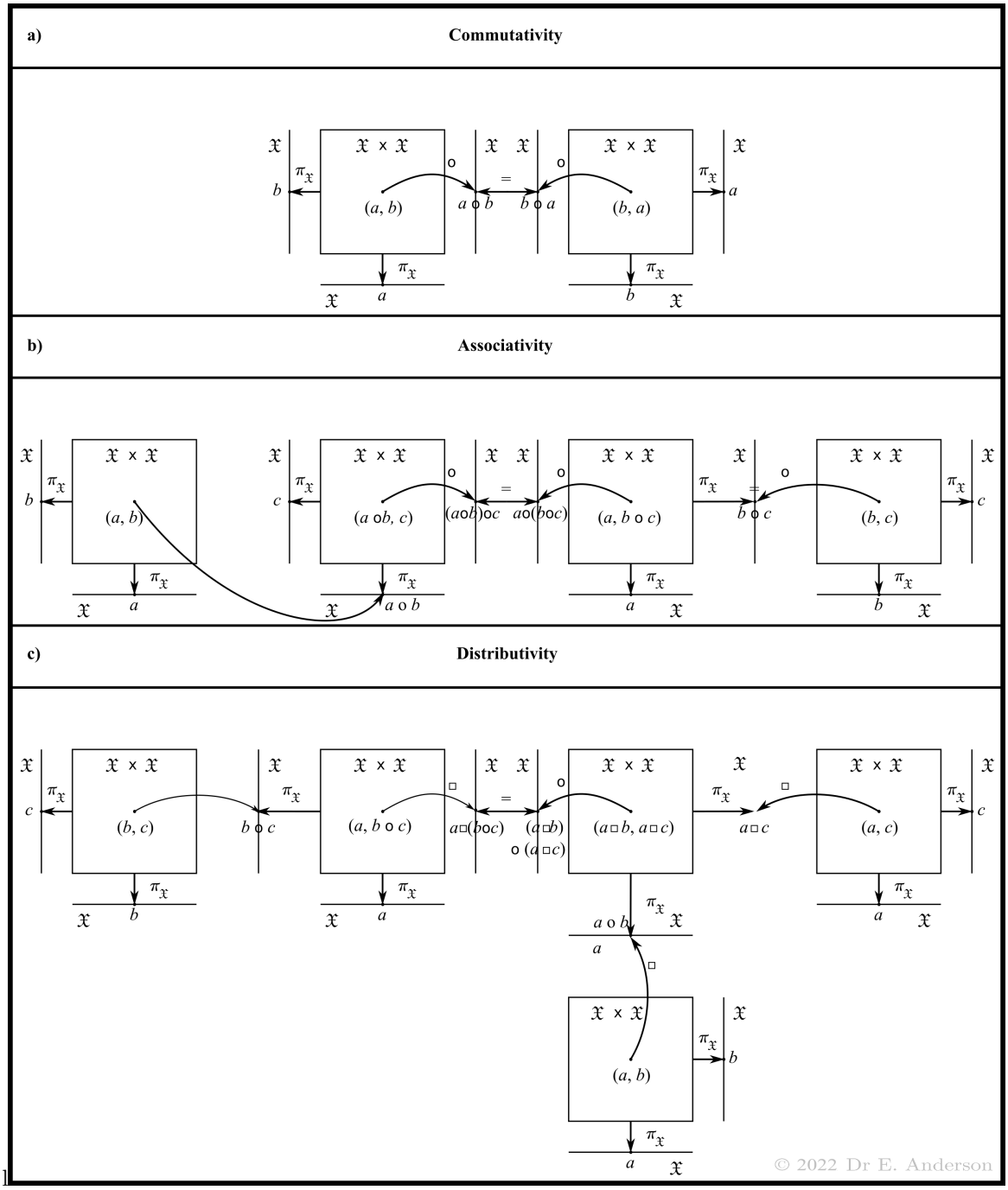


Figure 3.2:

3.6 ii. For pairs of elements

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Definition 1 $\mathcal{R}$ is *symmetric* if

$$x \mathcal{R} y \;\Rightarrow\; y \mathcal{R} x \;\forall\; x, y \in \mathfrak{X} . \tag{3.2}$$

Unsymmetric if this fails for at least one x, y pair.

A-symmetric if

$$x \mathcal{R} y \;\; \Rightarrow \;\; y \not\mathcal{R} x \; .$$

Or *antisymmetric* if

$$x \mathcal{R} y \; , \;\; y \mathcal{R} x \;\; \Rightarrow \;\; x = y \;\; \forall \;\; x, y \in \mathfrak{X} \; . \tag{3.3}$$

Remark 1 When $\mathfrak{X} = \mathbb{R}$, then symmetricness is manifested as reflection symmetry about the line (3.1); see Fig 3.3.b). Antisymmetry covers at most half of 'the product square $-$ its subleading diagonal' $y = x$, while leaving $y = x$ itself unconstrained (Fig 3.3.c).

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Definition 2 $\mathcal{R}$ is *total* if

$$\text{one or both of } \;\; x \mathcal{R} y \;\; \text{ or } \;\; y \mathcal{R} x \;\; \text{ holds } \;\; \forall \, x, y, \in \mathfrak{X} \; . \tag{3.4}$$

I.e. all pairs are related.

Naming Remark 1 'Total' is the more traditional name for this property, which lives on in the *total ordering* application that we eventually make much use of. A more modern name for this property is *strongly connected*. It makes sense to introduce what *connected* means in the context of a binary relation prior to discussing this renaming.

Definition 3 A binary relation is *connected* if (3.4) holds for all *distinct pairs* of elements x, y.

Remark 2 On the one hand, strong connectedness covers at least half of the product square, including all of its subleading diagonal. On the other hand, connectedness covers at least half of 'the product square - its subleading diagonal', while leaving the subleading diagonal itself unconstrained.

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Exercise 4 Show that a relation is a-symmetric $\Leftrightarrow$ it is both antisymmmetric and a-reflexive. Give also two examples of asymmetric relations.

Exercise 5$^-$ Show that a relation is strongly connected $\Leftrightarrow$ it is both connected and reflexive.

3.7 iii. For triples of elements

Definition 1 $\mathcal{R}$ is *transitive* if

$$x \mathcal{R} y \; , \;\; y \mathcal{R} z \;\; \Rightarrow \;\; x \mathcal{R} z \;\; \forall \, x, y, z \in \mathfrak{X} \; . \tag{3.5}$$

It is *intransitive* if at least one triple of elements does not obey transitivity. It is *a-transitive* if no triple does.

Remark 1 Geometrically, transitivity means that if points $(a, b), (b, c) \in \mathfrak{R}$, then the point 'completing the zig-zag' $(a, c) \in \mathfrak{R}$ as well (Fig 3.3.g).

Exercise 6$^-$ Motivate transitivity by showing that both equality $=$ and two-way implication $\Leftrightarrow$ obey this. Does one-way implication, $\Rightarrow$?

Exercise 7 Geometrically interpret a-reflexive, irreflexive, a-symmetric, unsymmetric, a-transitive and intransitive as regions $\mathfrak{R} \subseteq \mathfrak{X} \times \mathfrak{X}$.

Exercise 8 Show that

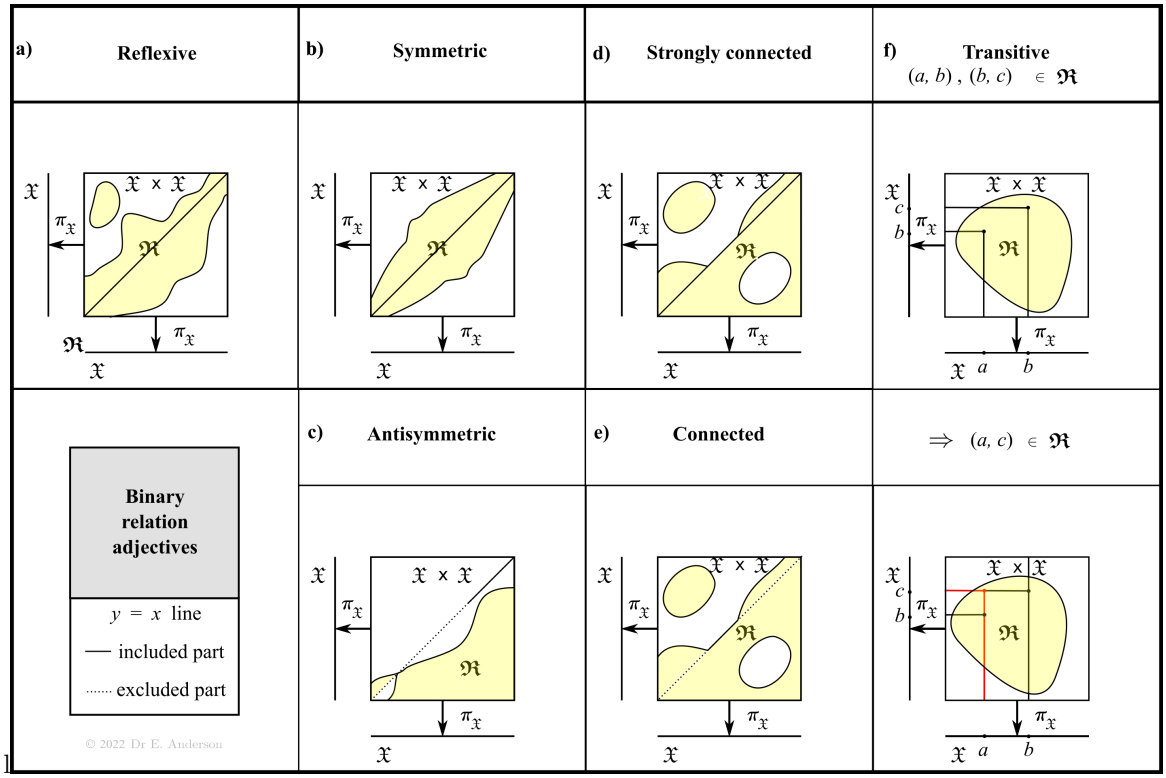


Figure 3.3:

i) a-transitive $\Rightarrow$ a-reflexive.

ii) For $\mathcal{R}$ transitive, then a-symmetric $\Leftrightarrow$ a-reflexive.

3.8 iv. Discussion

Exercise 9 The *empty binary relation* $\mathcal{E}$ is the relation that is false for any pair of elements. The *universal binary relation* $\mathcal{U}$ is the relation that is true for any pair of elements. The *identity binary relation* $\mathcal{I}$ is that which is true iff its input pair of elements are equal to each other.

a$^-$) What is $\mathfrak{R}$ for each of these?

b) Which of the above binary-relation adjectives apply to which of these three examples?

Remark 1 In the reflexive, symmetric and transitive families, the 'odd one out' among the ten adjectives presented is antisymmetry. The other three concepts in each family are of the form

$$\text{full property}\ , \quad \text{partial property and} \quad \text{total devoidness of the property}\ . \tag{3.6}$$

The reason to include giving special distinction to equality in antisymmetry stems from its being a property of $\leq$, $\subseteq$ and a large family of orders more generally. Equality also provides the difference between connected and strongly connected, for the same reason of modelling inequalities, inclusions and orders more generally.

Exercise 10 Motivate antisymmetry and strong connectivity by showing that $\leq$ has both of these properties, while $\subseteq$ has the first but not the second. Which of these properties do their strict counterparts $<$ and $\subset$ possess?

Naming Remark 1 The English 'reflexive' is essentially unchanged in spelling and meaning from its Latin precursor. Reflexive pronouns have the distinction of referring to *their own* subject rather than to any other. One meaning of 'reflecting' is *self*-contemplating. So these usages also carry the single-input connotation of the reflexive relation.

Naming Remark 2 The word 'symmetric' comes from the Greek $\sigma\upsilon\nu\mu\varepsilon\tau\rho\upsilon\nu$, meaning 'with measure'. For now, this Volume has considered a 2-input version of this, in which exchanging the order of the inputs does not affect the output. Our previous use – symmetric difference – is in fact describing a binary operation, not a binary relation. And yet the current paragraph's second sentence applies just as well to this: symmetry is a more general notion than just the current Chapter's binary-relation application! As this Volume and Series progress, further notions of symmetry will frequently appear; these will include cases in which the exchange of n inputs does not affect the output. For symmetry is also more general than its binary application.

Naming Remark 3 'Transitive' is also essentially unchanged in spelling and meaning from its Latin precursor. It means 'going across', with 'transit', 'transiting' and 'transition' sharing a similar meaning. 'Trans' means more generally 'across' or 'on the other side of', to 'cis' meaning 'on the same side as'. While some words starting with the 'trans' prefix have no, or uncommon, 'cis' prefix opposites, such as 'transfer' – to carry across– others occur in such pairs, e.g. 'cisalpine' and 'transalpine'. We will be needing 'cis' and 'trans' terminology for some of this Volume's Chemical structural formula applications. 'Transistor' – a key component in Electronics and Computer Hardware – is itself a 'portmanteau word', its front half deriving from 'transfer' and its back half from 'resistor': a more basic Electronics component.

Naming Remark 4 'A-reflexive' has hitherto mostly been called '*fully irreflexive*'. 'A-transitive' has hitherto mostly been called '*antitransitive*'. Renaming these with a-prefixes moves toward being able to work out, rather than having to remember or look up, terminology. The general form in use here is

$$(\text{a-property}) \;=\; (\text{fully devoid of that property})\ .$$

When the property starts with a vowel, however, we adhere to the lingustic convention of using the prefix 'an-'.

Notice furthermore that this aligns with the hitherto-standard definition of asymmetry. Also observe that this eliminates how the 'anti' in 'antitransitive' has a different meaning from that in 'antisymmetric'.

Exercise 11 Extend each of antisymmetric, connected and strongly connected to a triple of concepts of the form (3.6). Name them. Can you identify a particularly topologically-meaningful notion among them? [If not, try again once you have reached Sec 35.1.]

3.9 Equivalence relations

These are a key composite of binary relation adjectives, as follows.

Definition 1 $\mathcal{R}$ is an *equivalence relation*, which we denote by $\sim$, iff it is reflexive, symmetric and transitive.

Exercise 12 a) Further motivate equivalence relations by showing that $=$ and $\Leftrightarrow$ are equivalence relations.

b) Which of Examples 2 to 5 of Sec 3.4 are equivalence relations?

3.10 Pointers

Pointer 1 Chapters 4, 7, 15, 19, 23, and 44 contain further examples of equivalence relations.

Pointer 2 Chapter 4 introduces two further key composites of the above adjectives: *partial orders* and *total orders*. These are central to this Volume's account of Order Theory.

Pointer 3 In the current Chapter's context, 'connected' means mutually-connected element pairs (Fig 3.3.e), while 'strongly-connected' is stronger by including each element being self-connected as well (Fig 3.3.f). The conceptual type of 'connected' in this context means 'can be reached from', which is, firstly, a type of *path-connectedness* upon envisaging $\mathcal{R}$ as an arrow. But also a type of *adjacency*, since each element can be reached from each other not only via some concatenation of such arrows into a path but also solely by use of a single such arrow. We shall be discussing a wider variety of notions of connectivity and of adjacency in Part V.

Chapter 4

Number systems and order structure outlined

4.1 Number systems

Structures 1-4 We denote the set of all natural numbers by $\mathbb{N}$,

That of all integers by $\mathbb{Z}$.

The set of all rational numbers by $\mathbb{Q}$, that of all real numbers by $\mathbb{R}$, and that of complex numbers by $\mathbb{C}$.

4.2 The four main operations of arithmetic

The four arithmetic operations – addition $+$, subtraction $-$, multiplication $\times$ and division div also denoted $|$ – are further examples of binary operations.

Exercise 1 Which of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ are these well-defined on?

Exercise 2 Which of these are commutative? Which are associative? Which pairs of these are distributive over each other?



4.3 Introduction to ordering

This Volume's two further key composites of adjectives for binary relations $\mathcal{R}$ are as follows.

Definition 1 $\mathcal{R}$ is a *partial ordering*, which we denote by $\preceq$, if it is reflexive, antisymmetric and transitive.

Definition 2 $\mathcal{R}$ is a *total ordering* if it is both a partial ordering and total.

Naming Remark 1 A set equipped with a partial ordering is termed a *poset*. Another word for a total ordering is *chain*.

Example 1 The first four number systems above all furthermore support the notion of greater than $>$, or equivalently of less than $<$. This equivalence is embodied by $a > b$ meaning the same as $b < a$, for a, b elements within any one of our four number systems. It is then straightforward to define greater-than-or-equals $\geq$, or equivalently less-than-or-equals $\leq$. Having such a relation means that our four number systems have a *total ordering*.

Exercise 3$^-$ Check this, by building on the previous Chapter's results.

Counterexample 1 Total-ordering is not a property shared by all number systems; for instance the set of all complex numbers $\mathbb{C}$ is not totally-ordered.

Exercise 4 Geometrically interpret the partial-ordering and total-ordering combinations of binary relation adjectives, in terms of the region $\mathfrak{R} \subseteq \mathfrak{X} \times \mathfrak{X}$.

♠ ◇ ♣

Structure 1 One can also conceive of the possibility of smallest and greatest elements within a subset $\mathfrak{A}$ of our ordered number system. For instance, m is the smallest element in $\mathfrak{A}$ if $m \leq a \; \forall \, a \in \mathfrak{A}$.

4.4 Further conceptual distinctions between number systems

Remark 1 Let us next consider some of the ways in which the above four number systems differ. The integers enlarge the natural numbers in two ways. The first of these makes sense by itself: inclusion of the zero number, giving the non-negative natural numbers $\mathbb{N}_0$. The second of these is the inclusion of the negative numbers.

Remark 2 One distinct qualitative feature of $\mathbb{Z}$ is that it contains no smallest number, by containing arbitrarily large-negative integers. In contrast, 1 and 0 are the smallest numbers in $\mathbb{N}$ and $\mathbb{N}_0$ respectively. None of these three contain a largest number, by containing arbitrarily large integers. $\mathbb{Z}$ is thereby infinite, to $\mathbb{N}$ and $\mathbb{N}_0$ being *semi-infinite*. I.e. infinite in only one of their two directions: here the positive direction.

Remark 3 The rationals $\mathbb{Q}$ enlarge the integers $\mathbb{Z}$ by including nontrivial fractions as well. The reals $\mathbb{R}$ enlarge the rationals $\mathbb{Q}$ by including irrational numbers as well. $\sqrt{2}$, π and e are examples of irrational numbers. $\mathbb{Q}_0$ and $\mathbb{R}_0$ also make sense: the non-negative rationals and reals respectively. We denote the *positive rationals* and *positive reals* by $\mathbb{Q}_+$ and $\mathbb{R}_+$ respectively. All the reals between a and b inclusive for $a < b$ constitute the *closed real interval* $[\, a, \, b \,]$. If a and b themselves are excluded instead, one is considering the *open real interval* $(\, a, \, b \,)$.

Remark 4 $\mathbb{R}_0$, $\mathbb{Q}_0$ and $[\, a, \, b \,]$ contain smallest elements: 0, 0 and a respectively. Of these three, $[\, a, \, b \,]$ alone contains a largest element: b. Unlike $\mathbb{N}$, $\mathbb{R}_+$, $\mathbb{Q}_+$ and $(\, a, \, b \,)$ do not however contain smallest elements. No matter how close one gets to 0, 0, a respectively therein, one can always find another member that is closer.

Remark 5 Finally, we use a star subscript to indicate puncturing out zero, e.g. the punctured real line $\mathbb{R}_*$, or the punctured complex plane $\mathbb{C}_*$.

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Remark 6 One way of distinguishing $\mathbb{N}$ from $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ that turns out to be quite useful is as follows.

Definition 1 An ordered number system is *well-ordered* if any nonempty subset of it contains a smallest element.

Well-Ordering Axiom $\mathbb{N}$ is well-ordered.

Remark 7 In contrast, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ are not.[1] $\mathbb{Q}$ and $\mathbb{R}$ are not, by Remark 4.

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Remark 8 Courses on Abstract Algebra show that there are substantial algebraic differences between the above number systems. In another subsequent Volume [449], we shall also eventually classify these number systems, and the above-mentioned subsets, on Topological grounds. One salient difference between $\mathbb{R}$ and $\mathbb{Q}$ is that $\mathbb{R}$ supports standard Analysis, rigorously justifying Calculus on $\mathbb{R}$, while $\mathbb{Q}$ does not. This explains why $\mathbb{R}$ but not $\mathbb{Q}$ are ubiquitous in 'continuum mathematics'. On the other hand, Number Theory pursues $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$, alongside further number systems that generalize the Algebraic properties of these.

4.5 Further Exercises on orders

Exercise 5 Show that $\leq$ places a total ordering on $\{1, 2, 3\}$ and on $\mathbb{N}$. Does this place a total ordering on $\mathbb{R}$? On $\mathbb{R}^2$?

Exercise 6 The *discrete order* on a set is $x \preceq y$ iff $x = y$. Is this a total order? A partial order?

Exercise 7 Consider the Cartesian product $\mathfrak{X} \times \mathfrak{Y}$ of two ordered sets $\mathfrak{X}$ and $\mathfrak{Y}$. Thereupon, the *Cartesian product order*, alias *component-wise order*, is defined as follows.

$(x_1, y_1) \leq (x_2, y_2) \Leftrightarrow x_1 \leq x_2$ using the order on $\mathfrak{X}$ and $y_1 \leq y_2$ using the order on $\mathfrak{Y}$.

Show that, by this construct, $\mathfrak{X}$ and $\mathfrak{Y}$ being totally-ordered fails to ensure that $\mathfrak{X} \times \mathfrak{Y}$ is.

Exercise 8 8 The *lexicographic order* alias *library order* or *dictionary order* is defined as follows. In a library, books have e.g. a topic number and then various sub-topic numbers, then an author's initial and then another number. In a dictionary, it is easier to describe: a precedes b, aa precedes ab and so on, so 'aardvark' precedes 'abacus'.

a) Place a lexicographic ordering on $\mathbb{R}^2$ by referring to its components.

b) Show that the lexicographic ordering on a binary product of totally-ordered sets is itself total.

[Aside from arranging actual words into dictionary order, lexicographic ordering can be used to sort and compare strings of data. This is by e.g. ranking 101 above 011 by using the leftmost binary digit to rank first, then the next leftmost, and so on. If we were given 101 and 11 to compare, we would first render these two binary strings of equal length by inserting a 0 in front of the 11. We know from the previous Exercise that Cartesian product order would not in general succeed in providing a total ordering, by which the lexicographic order is often preferable.]

c) Nonetheless, give an example in which the product and lexicographic orders coincide.

d) Finally extend both the Cartesian product order and the lexicographic order to finite products.

Exercise 9 Show that $\subseteq$ places a partial ordering on the set of subsets of a fixed finite set. Give a minimal counterexample to this being a total ordering.

Exercise 10 Show that any subset of a totally-ordered set can be taken to inherit Given two disjoint totally-ordered sets, how might one totally-order their union?

[1] The star subscript notation $_*$ used here means excluding zero.

Chapter 5

Proof by induction

5.1 Introduction

Principle of Mathematical Induction [5, 8] We formulate our proposition under investigation as wishing to prove a statement P_n for n any integer above a certain value. Suppose that we can prove the following.

Base step: that P_a holds for some $a \in \mathbb{Z}$.

Induction step: that whenever $k > a$, P_k holds, then P_{k+1} holds.

Then it follows that P_n holds $\forall\ n \in \mathbb{Z}$ with $n \geq a$.

Remark 1 In basic examples, $a = 1$ is common (see for instance Examples 1 and 2). This is not however always possible; c.f. Example 3. Identifying the smallest valid case has nontrivial content. This should not be confused with just shifting the indexing of P_n so that the range of values being investigated is always standardized to $\mathbb{N}$.

Remark 2 To start off with, the induction step is not established. Then its first statement is a hypothesis, which becomes established if both the base step and the induction step are proven.

5.2 Summation Examples

Worked Example 1 Show that a)

$$S_1(N) := \sum_{n=1}^{N} n = \frac{N(N+1)}{2} .$$

b)

$$S_2(N) := \sum_{n=1}^{N} n^2 = \frac{N(N+1)(2N+1)}{6} .$$

Answer 1 a) Base case: for $N = 1$,

$$\frac{1 \times (1+1)}{2} = \frac{2}{2} = 1 = S_1(1) .$$

Induction hypothesis: Suppose that our result holds for some $N = P$.

Induction step: For $N = P + 1$,

$$S_1(P+1) = \sum_{n=0}^{P+1} n = \left(\sum_{n=0}^{P} n\right) + (P+1) = S_1(P) + P + 1$$

$$= \frac{P(P+1)}{2} + P + 1 = \frac{(P+1)}{2}(P+2) = \frac{(P+1)((P+1)+1)}{2} .$$

Here step 1 is the $N = P + 1$ version of the definition. Step 2 splits off the last term of this summation. Step 3 identifies our new summation with the $N = P$ version of the definition. To which the Induction Hypothesis applies: step 4. Step 5 factorizes the preceding, and step 6 finally casts our expression manifestly as the $P + 1$ version of our result.

b) Base case: for $N = 1$,

$$\frac{1 \times (1+1) \times (2 \times 1 + 1)}{6} = \frac{2 \times 3}{6} = 1 = S_2(1) .$$

Induction hypothesis: Suppose that our result holds for some $N = P$.

Induction step: For $N = P + 1$,

$$S_2(P+1) = \sum_{n=0}^{P+1} n^2 = \left(\sum_{n=0}^{P} n^2\right) + (P+1)^2 = S_2(P) + (P+1)^2 =$$

$$\frac{P(P+1)(2P+1)}{6} + (P+1)^2 = \frac{(P+1)}{6}(P(2P+1) + 6(P+1)) =$$

$$\frac{(P+1)}{6}(2P^2 + P + 6P + 6) = \frac{(P+1)}{6}(P+2)(2P+3) =$$

$$\frac{(P+1)((P+1)+1)(2(P+1)+1)}{6} .$$

The first to fourth, and last two, steps here are as per a) verbatim. The new insertion consists of step 5 expanding the last factor, which step 6 next groups.

Worked Example 2 How might one arrive at such 'candidate formulae' in the first place? For a), we could observe that adding the numbers from 1 to n involves adding n pairs

$$1 + n , \; 2 + (n-1) , \; \ldots \; n + (n - (n-1)) ,$$

all of which add up to $n + 1$. But we have 'double-counted' – included every number precisely twice – so the answer is

$$\frac{n(n-1)}{2} .$$

For a), we could instead observe that the first few values are as follows.

$$1 , \; 1 + 2 = 3 , \; 3 + 3 = 6 , \; 6 + 4 = 10 , \; \ldots$$

Including 0 up front (which makes no difference to the sum!) we can then write these on a first row, with the differences between them on a second row returning the natural numbers. And then observing that the differences between these take the constant value, 1. So, overall, we have

$$0\,1\,3\,6\,10\ldots$$

$$1\,2\,3\,4\ldots$$

$$1\,1\,1\ldots$$

This 'trapezium of differences' taking 3 rows to attain constancy leads to us making the following 3-parameter ansatz.

$$S_1(n) \;=\; a\,n^2 \;+\; b\,n \;+\; c\;.$$

Using the smallest values of n will be most convenient in fixing a, b and c. Setting $n \;=\; 0$, we immediately get that

$$c \;=\; 0\;.$$

This leaves us needing to solve $n \;=\; 1$'s

$$a \;+\; b \;=\; 1\;. \tag{5.1}$$

Simultaneously with $n \;=\; 2$'s

$$4\,a \;+\; 2\,b \;=\; 3\;. \tag{5.2}$$

(5.2) $-\;2\;\times$ (5.1) then gives

$$2\,a \;=\; 1 \;\;\Rightarrow\;\; a \;=\; \frac{1}{2}\;.$$

Thus (5.1) yields

$$2\,b \;=\; 2 \;-\; 1 \;=\; 1 \;\;\Rightarrow\;\; b \;=\; \frac{1}{2}\;.$$

So our ansatz has been fixed to be

$$S_1(n) \;=\; \frac{1}{2}\,n^2 \;+\; \frac{1}{2}\,n \;=\; \frac{n\,(\,n \;+\; 1\,)}{2}$$

by elementary factorization.

Remark 1 The $S_1(n)$ are alias T_n: the so-called *triangular numbers* (Fig 5.1's blue triangle). Our first method above – a double-count and thus combinatorial, and more specifically the 'schoolboy Gauss' method – then also geometrically amounts to the following. Stick a second 'pink' copy of our triangle back-to-front on top of the corresponding first 'blue copy' to form a rectangle. Whose area is clearly obtained by multipliciation, but is precisely twice too big to be our final answer, due to our having involved a second copy of our triangle. Thus finally divide by 2.

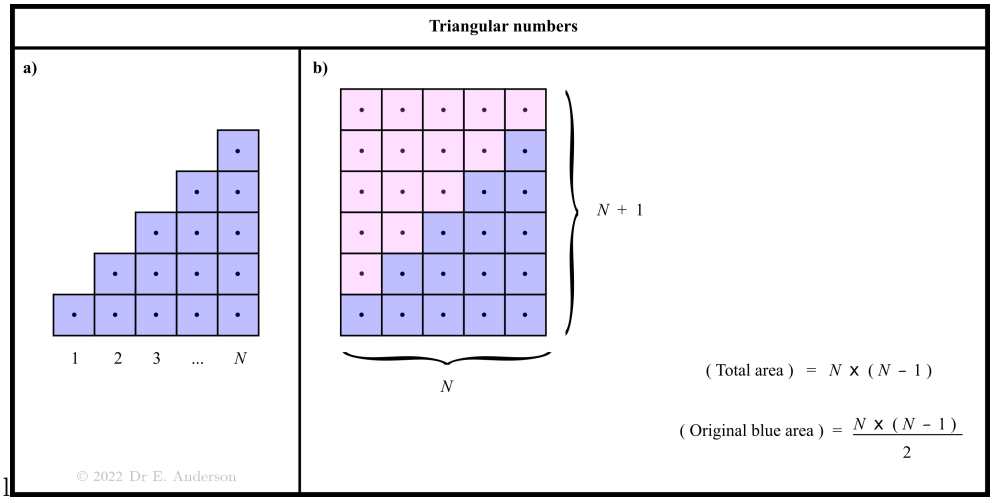


Figure 5.1:

Remark 2 In contrast, our second method above is Linear-Algebraic. While slower than the first method for our particular problem, it has the substantial benefit of generalizing to whichever other positive power.

Exercise 1 a) Derive Worked Example 1.b)'s formula by use of a 'trapezium of differences'.

b) Also propose formulae for the sums of the first $n \in \mathbb{N}$ objects of each of the following kinds, and prove by induction that they hold.

i) Cubic numbers.

ii) Quartic numbers.

iii) Odd-square numbers.

iv) Terms in the geometric progression $c\,x^{n-1}$?

5.3 Subsets Examples

Worked Example 3 Prove that no finite product of unions requires any brackets.

This holds trivially for $n = 2$.

Suppose that this holds for p-term products. Then

$$\left(\bigcup_{i=1}^{p} \mathfrak{U}_i \right) \cup \mathfrak{U}_{p+1} = \mathfrak{U}_1 \cup \left(\bigcup_{i=2}^{p} \mathfrak{U}_i \right) \cup \mathfrak{U}_{p+1} =$$

$$\mathfrak{U}_1 \cup \left(\bigcup_{i=2}^{p} \mathfrak{U}_i \cup \mathfrak{U}_{p+1} \right) = \mathfrak{U}_1 \cup \left(\bigcup_{i=2}^{p+1} \right) .$$

This is by use of the induction hypothesis twice.

Worked Example 4 Prove that

$$\left(\bigcup_{j=1}^{n} \mathfrak{a}_j \right)^{c} = \bigcap_{j=1}^{n} \mathfrak{a}_j{}^{c} .$$

[This is one of De Morgan's laws for a finite number of subsets.]

Exercise 2 a) Prove by induction that no finite product of intersections requires any bracketing.

b) Formulate and prove by induction, the other De Morgan's law in the case of a finite number of subsets.

Exercise 3 Formulate, and prove by induction, the counterparts of Worked Example 1, Exercise 1 and Exercise 2 of Chapter 2 for a finite number of subsets.

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Exercise 4$^+$ Prove each statement in Exercises 1 to 3 without use of induction. [You may need to return to some parts of this when you have read more of this Volume.]

5.4 Double induction

Principle of Double Mathematical Induction This is formed by straightforward concatenation of the usual Induction Principle [67, 60].

Suppose that we wish to prove a statement $P_{m,\,n}$, where m and n are any integers above certain values. Suppose that we can prove the following.

Base step: that $P_{a,\,b}$ holds for a and b fixed $\in \mathbb{Z}$.

Induction step 1: that if $P_{a,\,k}$ holds for some $q > b$, then $P_{a,\,k\,+\,1}$ also holds.

Induction step 2: that if $P_{l,\,k}$ holds for some $l \geq a$ and $k > b$, then $P_{l\,+\,1,k}$ also holds.

Then it follows that $P_{m,n}$ holds $\forall\, m,\, n \in \mathbb{Z}$ such that $m \geq a$ and $n \geq b$.

Exercise 5 Formulate triple induction, and k-fold induction for k finite.

5.5 Strong induction

Strong Principle of Mathematical Induction [67, 60] Suppose that we wish to prove a statement P_n for n any integer above a certain value. Suppose that we can prove the following.

Base step: that P_a holds for some $a \in \mathbb{Z}$.

Induction step: that P_p with

$$p = a\,,\ \ a\,+\,1\,,\ \ \ldots\,,\ \ k \quad \Rightarrow \quad P_{k\,+\,1}\,.$$

Then P_n is true $\forall\, n \in \mathbb{Z}$ such that $n \geq a$.

Exercise 6 Formulate strong double induction.

5.6 Pointers

Pointer 1 We next use simple induction in proving some basic formulae about divisors (Sec 7.7), the Fermat–Euler Theorem (Sec 7.12) and in handling recurrence relations (Chapter 14).

Pointer 2 We first make use of multiple induction in dealing with larger recurrence relations in Chapter 14.

Pointer 3 Our first three main uses of strong induction are in proving Euclid's Useful Lemma (Sec 7.5) the Fundamental Theorem of Arithmetic (Sec 7.7). and Euler's formula (Sec 45.3.



Chapter 6

A bit more about functions

6.1 Homomorphisms, isomorphisms, endomorphisms and automorphisms

Definition 1 A *homomorphism* is a map that preserves a given level of mathematical structure, $\boldsymbol{\sigma}$.

Remark 1 Examples of $\boldsymbol{\sigma}$ include order structure or group structure; see Chapter 10 for more about what we mean by a level of mathematical structure.

Definition 2 An *isomorphism* is a bijection that is also a homomorphism, such that its inverse map also preserves $\boldsymbol{\sigma}$.

Definition 3 An *endomorphism* is a homomorphism for which the target space coincides with the source space.

Definition 4 An *automorphism* is an isomorphism whose target space coincides with its source space.

Remark 2 We already know that when a set carries *no* $\boldsymbol{\sigma}$, its automorphisms are permutations. Let us postpone giving further examples until Chapter 10.

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Naming Remark 1 'Morphism' derives from the Greek 'morphe', meaning 'shape'. This meaning is directly used in such words as 'dimorphism' or 'anthropomorphism'. The mathematical use '-morphic to' means 'same-shaped as', taken to be established by finding a *transformation map* between the two entities being compared. The mathematical use is thereby as a word-ending for types of map.

Naming Remark 2 'Iso' and 'auto' are meant to be a matchingly-stronger pair that are elsewise analogous to 'homo' and 'endo' respectively. This extra strength corresponds to the extra mathematical requirement of having to be bijections. 'Homo' means 'the same as', while 'iso' means the stronger 'is equal to', which we use along the lines of 'is mathematically equivalent to by being in bijective correspondence with'. In the above use, 'endo' means 'inner' or 'internal', and hence, in the context of a map, 'to itself'. While 'auto's extra strength is manifested through its directly meaning 'self'.

All in all, the preceding three sections' four types of map are already well-named in standard texts.

Remark 3 If we furthermore take the stance of identifying all sets that are isomorphic to each other as constituting the same set, then isomorphisms coincide with automorphisms. Some applications

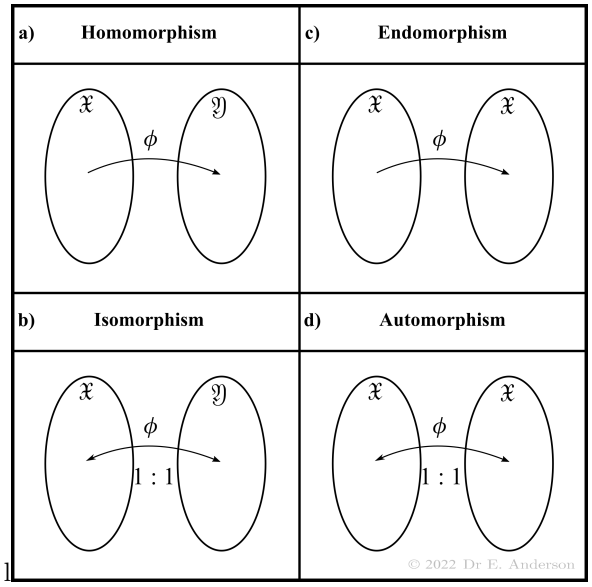


Figure 6.1:

and authors warrant this, while others do not. Not doing so allows for discussion of what turns out to be mathematically the same structure arising from different principles or in different applications. In some pairs of models, one would need to envisage asking whether and then proving that the structure of the one is isomorphic to that of the other. See e.g. Part III for Group Theory examples of this.

Nomenclature In these Series, we call a homomorphism *proper* if it is not an isomorphism. The *trivial homomorphism* sends every element in any group G_1 to the identity of another group G_1 .

6.2 Kernels and co-kernels

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be (possibly equipped) sets supporting a meaningful notion of identity element, 0. Also let $\mu \ : \ \mathfrak{X} \ \longrightarrow \ \mathfrak{Y}$ be a map.

Definition 1 The *kernel* of a map is the inverse image of the identity element,

$$\ker \mu \ := \ \{ \, x \ \in \ \mathfrak{X} \, | \, \mu(x) \ = \ 0 \, \} \ . \tag{6.1}$$

This is a subset of the source space.

Definition 2 If we consider a map ν running in the opposite direction, the name *co-kernel* is on some occasions used to make distinction.

$$\text{co-ker} \, \nu \ := \ \{ \, y \ \in \ \mathfrak{Y} \, | \, \nu(y) \ = \ 0 \, \} \ . \tag{6.2}$$

This is a subset of the previous map's source space.

Exercise 1 What possible kernels can planar rotation matrices have? What are the corresponding images?

Exercise 2^{-} A map μ is *idempotent* if applying the map a second time does not alter the outcome. This can be phrased as

$$\mu^2 \ = \ \mu \ .$$

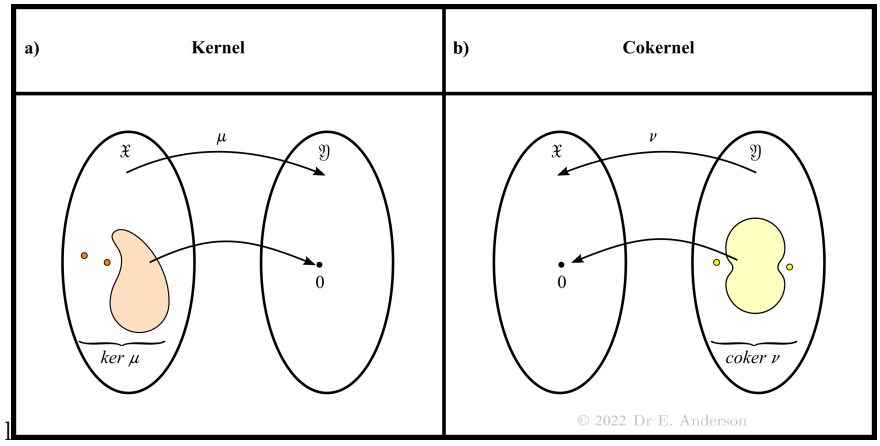


Figure 6.2:

a) Show that mapping from a product onto a factor by the map π in Fig 6.2.c) is idempotent.

b) Find a 2×2 matrix that is idempotent.

Remark 1 We postpone giving further examples until Sec 6.5 Chapter 10, Chapters 19, Chapters 19.4 and our Linear Algebra Volume 1 [441].

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Naming Remark 1 'Kernel' means the seed within a grain (compare Fig 6.2). This botanical analogy is due to Fredholm, who made it in French ('noyau'). It was subsequently popularized by Hilbert, who however translated it into German ('kern'), which is where the notation *ker* comes from.

Naming Remark 2 One major type of duality, that the *co−* in co-image refers to in particular, is that of reversing the senses of the arrows in the maps.

Pointer 1 This discussion continues in Volume 1's linear maps subcase, where the corresponding cardinalities, images and *their* cardinalities also get 'names and co-names'.

End-Remark 1 The next two Subchapters are each about a particular kind of function.

A. Characteristic functions

6.3 Introduction to characteristic functions

Definition 1 The *characteristic function* of subset $\mathfrak{a} \subseteq \mathfrak{X}$ is the function

$$\chi_{\mathfrak{a}} : \mathfrak{X} \longrightarrow \{0, 1\}$$

$$x \mapsto \begin{cases} 1 & \text{if } \{ x \in \mathfrak{a} x \notin \mathfrak{a} \\ 0 \end{cases}$$

Lemma 1 Two subsets $\mathfrak{a}, \mathfrak{B} \subseteq \mathfrak{X}$ are equal iff

$$\chi_{\mathfrak{a}} = \chi_{\mathfrak{B}} \, . \tag{6.3}$$

Proof ($\Rightarrow$) Let $\mathfrak{A} = \mathfrak{B}$. Then either $x \notin \mathfrak{A}, \mathfrak{B}$, by which

$$\chi_{\mathfrak{A}}(x) = 0 = \chi_{\mathfrak{B}}(x) .$$

Or $x \in \mathfrak{A}, \mathfrak{B}$, whence

$$\chi_{\mathfrak{A}}(x) = 1 = \chi_{\mathfrak{B}}(x) .$$

Thus exhaustively (6.3) holds.

($\Leftarrow$) Probe with an element: suppose that there is some $x \in \mathfrak{X}$ such that $x \in \mathfrak{A}$ but $\notin \mathfrak{B}$. Then

$$\chi_{\mathfrak{A}}(x) = 1 \neq 0 = \chi_{\mathfrak{B}}(x) .$$

So

$$\chi_{\mathfrak{A}} \neq \chi_{\mathfrak{B}} .$$

Finally apply the contrapositive. □

Naming Remark 1 These are alias *indicator functions* in Probability Theory. They characterize, or indicate, what the corresponding subset is, element by element. They can furthermore be viewed as answering whether $x \in \mathfrak{A}$; in this sense, a truer name is none other than *truth functions*!

Motivation 1 This is an efficient alternative approach to Chapter 1's two-way inclusion proofs as regards establishing equalities between a priori different set constructs.

6.4 Worked Example and Exercises

Worked Example 1 Prove that

$$\chi_{\mathfrak{A} \cap \mathfrak{B}} = \chi_{\mathfrak{A}} \chi_{\mathfrak{B}} .$$

Answer 1 Suppose that

$$x \in \mathfrak{A} \cap \mathfrak{B} .$$

Then

$$\chi_{\mathfrak{A} \cap \mathfrak{B}} = 1 .$$

But also by the definition of $\cap$, $x \in \mathfrak{A}, \mathfrak{B}$. Thus

$$\chi_{\mathfrak{A}} = 1 = \chi_{\mathfrak{B}} .$$

So

$$\chi_{\mathfrak{A}} \chi_{\mathfrak{B}} = 1 \times 1 = 1 = \chi_{\mathfrak{A} \cap \mathfrak{B}} .$$

Next suppose that

$$x \notin \mathfrak{A} \cap \mathfrak{B} .$$

Then

$$\chi_{\mathfrak{A} \cap \mathfrak{B}} = 0 .$$

But also by the definition of $\cap$, $x \notin$ at least one of $\mathfrak{A}$ and $\mathfrak{B}$. Thus at least one of

$$\chi_{\mathfrak{A}} = 0 \text{ and } \chi_{\mathfrak{B}} = 0$$

hold. So

$$\chi_{\mathfrak{A}} \chi_{\mathfrak{B}} = 0 = \chi_{\mathfrak{A} \cap \mathfrak{B}} .$$

Thus exhaustively, our result holds. □

Exercise 3 a) Prove that

$$\chi_{\mathfrak{a} \cup \mathfrak{B}} = \chi_{\mathfrak{a}} + \chi_{\mathfrak{B}} - \chi_{\mathfrak{a}} \chi_{\mathfrak{B}} .$$

b) Prove that

$$\chi_{\overline{\mathfrak{a}}} = 1 - \chi_{\mathfrak{a}} .$$

c) Consequently re-prove De Morgan's Laws (both for 3 subsets and for a finite number of subsets).

Exercise 4 Prove that $\cup$ and $\cap$ are a) commutative, b) associative and c) distributive over each other.

Exercise 5^{+} a) What is $\chi_{\mathfrak{a} \, - \mathbf{S} \, \mathfrak{B}}$?

b) Prove that this is commutative and associative.

c) Which combinations of $\cup$, $\cap$ and $-_{\mathbf{S}}$ that involve $-_{\mathbf{S}}$ at least once, exhibit distributivity?

B. Permutations

6.5 The automorphisms of a bare set are its permutations

Definition 1 *Permutations* [76, 55, 110, 49, 58] are bijections between sets.

Remark 1 With plain sets having no further levels of mathematical structure, bijections and isomorphisms coincide here.

Remark 2 Our first of many ways of representing permutations in this Series is by which elements map to which elements. This is usually done by giving an input row and then an output row of where each element in the first row maps to. With each element in the first row mapping to the element vertically beneath it.

6.6 Simple examples of permutations

Example 1 The 1-element set supports just the one *identity permutation*,

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

Its image is 1 and its kernel is 1 as well.

Example 2

$$\begin{pmatrix} 1\,2 \\ 1\,2 \end{pmatrix}$$

is the identity permutation on the 2-element set $\{\,1,\,2\,\}$. Its kernel is 1 while its image is the whole set $\{\,1, 2\,\}$. While

$$\begin{pmatrix} 1\,2 \\ 2\,1 \end{pmatrix}$$

is the only other – and nontrivial – permutation on the 2-element set $\{\,1,\,2\,\}$. Its kernel is 2 while its image is again the whole set $\{1, 2\}$. There are no other possibilities for permutations on the 2-element set.

Example 3 There are precisely 6 permutations on the 3-element set $\{\,1,\,2\,3\,\}$. The identity

$$\begin{pmatrix} 1\,2\,3 \\ 1\,2\,3 \end{pmatrix} .$$

The two 3-cycles

$$\begin{pmatrix} 1\,2\,3 \\ 2\,3\,1 \end{pmatrix} , \quad \begin{pmatrix} 1\,2\,3 \\ 3\,1\,2 \end{pmatrix} .$$

And the three *transpositions* – two-element switches –

$$\begin{pmatrix} 1\,2\,3 \\ 1\,3\,2 \end{pmatrix} , \quad \begin{pmatrix} 1\,2\,3 \\ 2\,1\,3 \end{pmatrix} , \quad \begin{pmatrix} 1\,2\,3 \\ 3\,2\,1 \end{pmatrix} .$$

These are cycles as well, though now using only two of the three elements.

Naming Remark 1 In everyday speech, 'transposition' means changing the order of *two or more* things. But in the mathematical context of permutations, one is to additionally remember that it means *just two*. I.e. that the mathematical use of 'transposition' refers specifically to the *minimal subcase* of the everyday use of 'transposition'.

Exercise 6^{-} Show that there are $N!$ distinct permutations on the set of N elements.

Counterexample 1 A set endomorphism that is not a permutation is exhibited in Fig 6.3.

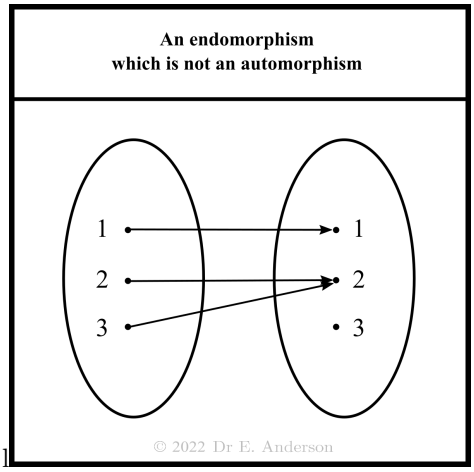


Figure 6.3:

6.7 Permutations as group elements

Exercise 7 Referring to Chapter 19 for the axioms if necessary, show that permutations form a group under composition of functions.

Exercise 7 Show that each of the following is an automorphism.

i) The composition of two automorphisms of a given space.

ii) The identity morphism.

iii) The inverse of each automorphism.

Deduce that the automorphisms of any given space also form a group.

[This gives further substance to automorphisms being a significant kind of map.]

6.8 Permutations as listings

Remark 1 The top row in our mapping notation for permutations is just 1 to n. [If the finite set elements were labelled otherwise, they could still be relabelled 1 to n.] So having a top row is somewhat wasteful. Let us thus rewrite the permutations that we have encountered so far as (1), (12), (21), (123), (231), (312), (132), (213) and (321).

Remark 2 This is now the target listing, but we can furthermore think of the permutation as just a listing [76].

Notation 1 Another notation is then $(\,\sigma(1),\,\ldots,\,\sigma(n)\,)$. This stands for the listing produced by where the permutation σ sends each element 1 to n.

Motivation for permutations Permutations play a a very significant part of basic Combinatorics (Chapter 13), with considerable applications in basic Probability and Statistics [70, 289] and both basic (Chapters 18 and 28) and not-so-basic Group Theory [59, 43, 92].

Remark 3 Further very significant examples of bijections are listed in Sec 10.3, just after introducing a way of passing from sets to many further mathematical structures.

6.9 Cycles in the study of permutations

Remark 1 Let us now pay attention to the above cycle notion to obtain a very usefully generalizable approach to permutations.

Definition 1 An m-cycle is a permutation such that a labelling can be found in which its action is as follows. It sends a_i to $a_{i\,+\,1}$ for $i \;=\; 1$ to m, a_m to a_1 and leaves every other element unchanged.

Definition 2 Two cycles are said to be *disjoint* if they have no element in common. I.e. the one has nontrivial action just on some subset $\mathfrak{A}$ of elements while the other has nontrivial action just on some other subset $\mathfrak{B}$ of elements. So $\mathfrak{A} \cap \mathfrak{B} \;=\; \emptyset$. Thus $\mathfrak{A} \amalg \mathfrak{B}$.

Lemma 2 Cycles can be cycled. I.e.

$$(\,\pi(1),\,\pi(2),\,\pi(3),\,\ldots\,,\,\pi(n)\,) \;\text{ is equivalent to }\; (\,\pi(2),\,\pi(3),\,\ldots\,,\,\pi(n),\,\pi(1)\,)\;,$$

$$(\,\pi(3),\,\ldots\,,\,\pi(n),\,\pi(1),\,\pi(2)\,)\;\ldots$$

Proof The first expression here involves sending 1 to 2, 2 to 3, ... $n \;-\; 1$ to n, and n to 1. This is clearly equivalent to sending 2 to 3, ... , $n \;-\; 1$ to n, n to 1 and 1 to 2. And so on. □

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Example 1 For any set $\mathfrak{X}$, we can think of the identity map as a permutation sending each element $x \;\in\; \mathfrak{X}$ to itself. This is composed of one 1-cycle per element. For $\mathfrak{X} \;=\; \emptyset$, there are no elements and thus zero 1-cycles, or for that matter, no cycles at all.

Example 2 The simplest nontrivial permutations are transpositions, consisting of precisely one 2-cycle, and nothing else. [Fig 6.4.a) illustrates the 2-point set's such, and Fig 6.4.b) the 3-point set's 3 such.

Naming Remark 1 The 3-point set also supports two 3-cycles: the *even* one denoted by η in Fig 6.4.c) and the *odd* one denoted there by ω.

Exercise 8 Find all the permutations on 4 objects. What are their kernels? What cycle lengths do each of these permutations possess?

Exercise 9 Show that all permutations on 5 objects can be thought of as products of disjoint cycles. Which distinct combinations of cycle lengths are involved?

♠ ◇ ♣

Lemma 3 Any permutation of a finite set can be thought of as a product of disjoint cycles.

Proof If our set $\mathfrak{X} \;=\; \emptyset$, its permutations are just the identity as realized by the empty product of no cycles at all.

Otherwise, $\mathfrak{X}$ is nonempty, so we can start by picking an element $x \;\in\; \mathfrak{X}$. Act on this with our permutation until one has returned to x. Then pick another element that has not been involved

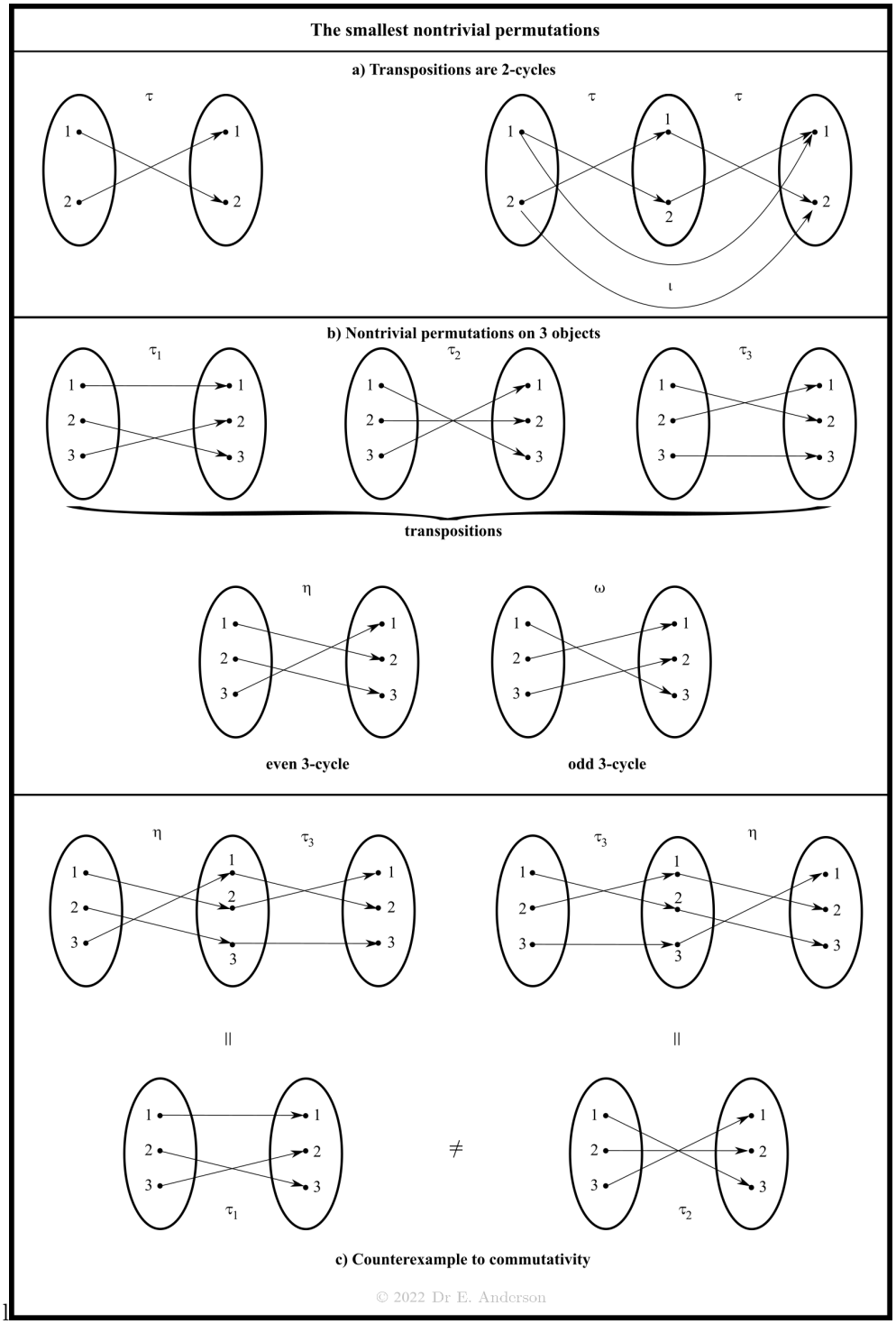


Figure 6.4:

yet. Keep on going; for a finite set, all elements get involved thus within a finite number of steps. A family of disjoint cycles representing our arbitrary permutation of a finite set has thus been found. □

Remark 2 In Sec 27.4 we show furthermore that it makes no difference in which order we perform

disjoint cycles. We thus arrive at the following notion.

6.10 Cycle type

Definition 1 The *cycle type*

$$[\, a_1,\, a_2,\, \ldots,\, a_n \,] \tag{6.4}$$

of a permutation consists of its associated cycle lengths a_i, including the multiplicities with which they occur [49, 58]. I.e. how many copies of each given length occur concurrently in our permutation. Which can be built in long-hand by listing multiple equal entries, with each such batch grouped together. Or shorthand as

$$[\, a_1{}^1,\, a_2^{m_2},\, \ldots,\, a_j^{m_l} \,] \tag{6.5}$$

for a_j with $j \;=\; 1$ to l the distinct cycle lengths with correponding multiplicities m_j.

Remark 1 Lemma 2 and the supporting well-definedness provided by Lemma 1 of Sec 27.4 mean that cycle type *classifies* permutations of a finite set. This is the second of many ways in which we will be representing permutations.

Remark 2 Furthermore, from a structural point of view, cycle type is a particular example of partition (partitions are the subject of Chapter 15).

Notation We do not usually bother listing 1-cycles, so all elements ‘ not mentioned in the cycle presentation of a permutation realize further 1-cycles. When we think of cycle type in terms of permutations, however, we need to remember to add one single-occupancy box per usually unlisted 1-cycle.

Exercise 10 What cycle types are possible for the permutations on 6 objects?

♠ ◇ ♣

Lemma 4 Every permutation can be represented as product of transpositions.

Proof That every permutation can be expressed as a product of disjoint cycles simplifies what we need to prove. Rearrange

$$(\, a_1\, a_2 \ldots a_n \,) \;=\; (\, a_1\, a_2 \,)\,(\, a_2\, a_3 \,) \ldots (\, a_{n-1}\, a_n \,)\ .$$

This is licit since the LHS cycles are not disjoint. □

Remark 3 The Reader should convince themself that this is not a unique representation.

Corollary 1 The number of transpositions representing a given permutation is of fixed parity.

6.11 Derangements

Definition 1 *Derangements* are permutations with no 1-cycles in their cycle type.

Remark 1 An alternative phrasing for the definition of derangements is as permutations with no fixed points.

Remark 2 We can furthermore characterize any permutation by its *number of* fixed points [55, 441]. The identity permutation and the derangements constitute the two extreme cases of this for each N.

Naming Remark 2 Permutation means 'to change completely'; this does not capture that some permutations have fixed points. It does capture the most general case of permutation, for which we have since developed the term 'derangement'. Which in turn means 'throw into disorder', to be interpreted as maximal disorder: no fixed points.

6.12 Pointers

Pointer 2 See Exercises 14.6 and 8 for more about derangements.

Pointer 3 See Parts III and IV for more about permutations from a Group-Theoretic point of view.

Pointer 4 The permutation groups point of view gives a further representation of permutations. Permutation cycles acquire further structural consideration in Chapter 28 (cycle-indexes). And again in Chapter 43 from the point of view of graphs of cycles formed by the maps between elements of the underlying sets. Permutation matrices are set up in our Linear Algebra Volume 1 [441].

Chapter 7

A bit more about the integers

7.1 Divisibility

Definition 1 Let $m, n \in \mathbb{Z}$. We say that m *divides* n, denoted by $m \mid n$, if the following two conditions hold.

i) $n \neq 0$

ii) $n = d\,m$ for some $d \in \mathbb{Z}$.

d is furthermore called a *factor* alias *divisor*, of n.

Definition 2 A divisor d of $n \in \mathbb{Z}$ is *proper and nontrivial* if $d \neq \pm 1, \pm n$.

Naming Remark 1 For divisors, this is often just called 'proper'. I am however aligning 'proper' to have exactly the same meaning in every branch of Mathematics, and so say 'proper and nontrivial' in this very commonly-encountered case.

Exercise 1$^-$ a) Show that divisibility is pairwise $\mathbb{Z}$-linear-combination extendable. I.e. that whenever $m \mid n_1$ and $m \mid n_2$, then

$$m \;\Bigg|\; \sum_{i=1}^{2} \lambda_i\, n_i \quad \text{for any} \quad \lambda_i \in \mathbb{Z}$$

as well.

b) Mathematically phrase and prove that divisibility is arbitrary-finite $\mathbb{Z}$-linear-combination extendable.

Exercise 2 Show that if $m \mid a\,b$, then at least one of $m \mid a$ and $m \mid b$ must hold.

Exercise 3 Show that division equips $\mathbb{N}$ to be a poset but not an equivalence relation. What happens if we work on $\mathbb{Z}_*$ instead?

7.2 The Division Algorithm

Theorem 1 (Division Algorithm) For $a, b \in \mathbb{N}$, $\exists\ !\ q, r \in \mathbb{N}$ such that

$$n = m\,q + r\ , \quad r \in [\,0, m\,)\ . \tag{7.1}$$

Remark 1 r is the *remainder*, whereas q is the *integer part of the quotient*. In the event of r being zero, q then meets the definition of factor or divisor, which we previously denoted by f.

Proof $\exists$. Take the largest possible q for which $m\,q \;\leq\; n$. Then set $r \;=\; n \;-\; m\,q$.

! . Suppose that there are distinct q', $r' \;\in\; \mathbb{N}$ for which

$$n \;=\; m\,q' \;+\; r' \tag{7.2}$$

and $r' \;\in\; [\,0,\, m\,)$. Swapping primed and unprimed versions if needs be, we can assume that $r' \;\geq\; r$.

(7.2) $-$ (7.1) followed by factorization gives

$$0 \;=\; n \;-\; n \;=\; m\,q' \;+\; r' \;-\; (\,m\,q \;+\; r\,) \;=\; m\,(\,q' \;-\; q\,) \;+\; (\,r' \;-\; r\,)\;.$$

Thus

$$m\,(\,q' \;-\; q\,) \;=\; (\,r \;-\; r'\,)\;.$$

I.e.

$$m \,|\, (\,r \;-\; r'\,)\;.$$

However, since

$$r' \;-\; r \;\leq\; r' \;<\; m\;,$$

$$r' \;=\; r$$

is forced. Thus

$$m\,(\,q' \;-\; q\,) \;=\; 0\;.$$

Since $m \;>\; 0$,

$$q' \;=\; q$$

is thus also forced. □

7.3 Highest common factor and least common multiple

Definition 1 Let $a, b \;\in\; \mathbb{N}$. Then $h \;\in\; \mathbb{N}$ is the *highest common factor (h.c.f.)*, denoted by

$$hcf(a,\, b)\;,$$

if the following hold.

i) (commonness)

$$h \,|\, a \;\text{ and }\; b\;.$$

ii) (highest)

$$\text{If some other } \; h' \;\in\; \mathbb{N} \;\text{ obeys } \; h_1 \,|\, a \;\text{ and }\; b\;, \;\text{ then } \; h_1 \,|\, h\;.$$

Definition 2 Let $a,\, b \;\in\; \mathbb{N}$. Then $l \;\in\; \mathbb{N}$ is the *least common multiple (l.c.m.)*, denoted

$$lcm(a,\, b)\;,$$

if the following hold.

i) (commonness)

$$a \;\text{ and }\; b \,|\, l \;.$$

ii) (least)

$$\text{If some other } \; l' \;\in\; \mathbb{N} \;\text{ obeys } \; a \;\text{ and }\; b \,|\, l'\;, \;\text{ then } \; l \,|\, l'\;.$$

Notation 1 (,) and [,] are widely used for $hcf(\ ,\ )$ and $lcm(\ ,\ )$. This is however a Series geared toward Topology, so intervals are ubiquitous therein, and vector components will soon be frequently evoked as well (and so on). So we use the entirely unconfuseable, if slightly longer notation above. *Greatest common divisor* is furthermore a common alias for highest common factor, so $gcd(a,\, b)$ would also be an entirely acceptable notation.

Worked Example 1 Given any pair of numbers $a,\, b \in \mathbb{N}$, show that $hcf(a,\, b)$ is !.

Answer 1 Suppose that both h and h' are $hcf(a,\, b)$. Then both $h' \mid h$ and $h \mid h'$, forcing $h' = h$. □

Let $n \in \mathbb{N}$ Suppose that we write this as

$$n = m\,q + r$$

for some given $m \in \mathbb{N}$ Where the *quotient* q and *remainder* r are to be $\in \mathbb{N}$ as well, and with $r \in [\,0,\, q\,)$.

Exercise 4 Given any pair of numbers $a,\, b \in \mathbb{N}$, show that $lcm(a,\, b)$ is !.

7.4 Prime and mutually-prime numbers

Definition 1 $p \in \mathbb{N}$ is *prime* if $p \neq 1$ and p possesses no proper-and-nontrivial divisors. Those $1 \neq c \in \mathbb{N}$ that are not prime are called *composite*.

Definition 2 $a,\, b \in \mathbb{Z}_*$ are said to be *mutually prime* if the following holds.

$$hcf(\,|a|,\, |b|\,) = 1\ .$$

Naming Remark 2 *Coprime* and *relatively prime* are common aliases. We do not use 'coprime' since this starts with 'co' while not being a dual notion to prime. It is, rather, a mutual extension of primality: from no proper-and-nontrivial divisors of a single number rendering it prime to no such being shared between two numbers rendering them mutually prime. We also reserve 'relative' for relative-difference notions: a simple type of relational notion as per Sec 0.3.

Worked Example 2 Suppose that

$$a \,|\, p\,q \tag{7.3}$$

and

$$hcf(a\,,p) = 1\ ,$$

where $a,\, p,\, q \in \mathbb{Z}_*$. Deduce that $a \,|\, q$.

Answer 2 Set

$$a\,x + p\,y = 1\ . \tag{7.4}$$

for some $x,\, y \in \mathbb{Z}$ and apply the Euclidean Algorithm. Thus

$$q\,a\,x + q\,p\,y = q\,(\,a\,x + p\,y\,) = q\,1 = q\ . \tag{7.5}$$

Here the first step is a factorization, the second step is by (7.4) and the third step is just the multiplicative unit property of 1.

Now a manifestly obeys

$$a \,|\, q\,a\,x\ . \tag{7.6}$$

While also

$$a \,|\, qpy \ .$$

follows from (7.3).

Together, (7.6) and (7.4) imply, by Exercise 1.a) that

$$a \,|\, (\, q\,a\,x \ + \ q\,p\,y\,) \ .$$

But this can be viewed as

$$a \,|\, (\,\text{LHS of (7.5)}\,) \ .$$

So we can deduce that

$$a \,|\, (\,\text{RHS of (7.5)}\,)$$

as well. But

$$(\,\text{RHS of (7.5)}\,) \ = \ q \ ,$$

and so we are done.

7.5 The Euclidean Algorithm

Useful Lemma 1 (Euclid) Let a and $b \ \in \ \mathbb{N}$ with

$$a \ = \ b\,q \ + \ r \tag{7.7}$$

for quotient q and remainder r. Then

$$hcf(a,\, b) \ = \ hcf(r,\, b) \ .$$

Proof By definition,

$$hcf(a,\, b) \,|\, a \ \text{ and } b \ .$$

So, by Exercise 1.a),

$$hcf(a,\, b) \,|\, (\, a \ - \ b\,q\,) \ .$$

Thus by (7.7)

$$hcf(a,\, b) \,|\, r \ .$$

So

$$hcf(a,\, b) \,|\, hcf(r,\, b) \ . \tag{7.8}$$

But also by definition

$$hcf(r,\, b) \,|\, r \ \text{ and } b \ .$$

So, by another use of Exercise 1.a),

$$hcf(r,\, b) \,|\, (\, b\,q \ + \ r\,) \ .$$

Thus by (7.7) again,

$$hcf(r,\, b) \,|\, a \ .$$

So

$$hcf(r,\, b) \,|\, hcf(a,\, b) \ . \tag{7.9}$$

Together, (7.8, 7.9) give our desired equality (two-way inclusion!) □

The Euclidean Algorithm consists of applying this recursively:

$$a \ = \ b\,q_1 \ + \ r_1 \ \ 0 \ < \ r_1 \ < \ b \ ,$$

$$b \ = \ r_1\,q_2 \ + \ r_2 \ \ 0 \ < \ r_2 \ < \ r_1 \ ,$$

$$r_1 \;=\; r_2\, q_3 \;+\; r_3 \;\; 0 \;<\; r_3 \;<\; r_2\;, \;\; ...$$

I.e.

$$r_{i\,+\,1} \;=\; r_{i\,-\,1} \;-\; q_{i\,+\,1}\, r_i \tag{7.10}$$

with initial values

$$r_{-1} \;=\; a\;,$$

$$r_0 \;=\; b\;.$$

Until $r_N \;=\; 0$ is hit in the $(\,i \;=\; N\,)$th step, which is to be the terminal step.

The Euclidean Algorithm's objective is moreover r_n for $n \;:=\; N \;-\; 1$. For by recursive use of the above Useful Lemma, this returns an old friend of ours.

$$r_n \;=\; hcf\,(r_n,\, r_{n\,-\,1}) \;=\; hcf\,(r_{n\,-\,1},\, r_{n\,-\,2}) \;=\; ... \;=\; hcf\,(r_{-1},\, r_0) \;=\; hcf\,(a,\, b)\;.$$

It thus provides *a systematic method for computing the h.c.f.*

Exercise 5$^-$ a) Check that the Euclidean Algorithm works for 9 and 12, the h.c.f. of which you should certainly be able to spot without...

b) Next find $hcf(5304,\, 23205)$, which might otherwise be a somewhat harder problem for you to answer...

Naming Remark 3 The above are the *'h.c.f. passes to the remainder' Lemma.* And the *'efficient h.c.f.-finding' Algorithm.* Further comments on *how* efficient it is are however difered to Sec 7.7 and Volume 2.

♠ ◇ ♣

Worked Example 3 Let us next extend Euclid's Algorithm to have two new columns s_i and t_i such that the following holds. s_i and t_i are both to obey the same recurrence as r_i: Eq. (7.10). But now starting from initial values

$$s_{-1} \;=\; 1 \;=\; t_0\;.$$

and

$$t_{-1} \;=\; 0 \;=\; s_0\;.$$

a) Show that these s_i and t_i obey

$$r_i \;=\; a\, s_i \;+\; b\, t_i \tag{7.11}$$

for each $i \;\geq\; -\,1$.

b) Deduce that the h.c.f. can be written as a pairwise $\mathbb{Z}$-linear combination of its two input natural numbers.

Answer 3 a) Base step: a) holds for each of $i \;=\; -\,1$ and 0 by the above choice of initial values.

Strong induction hypothesis: suppose that a) holds for all $i \;\leq\; k$.

Induction step:

$$r_{k\,+\,1} \;=\; r_{k\,-\,1} \;-\; q_{k\,-\,1}\, r_k \;=\; a\, s_{k\,-\,1} \;+\; b\, t_{k\,-\,1} \;-\; q_{k\,-\,1}\,(\,a\, s_k \;+\; b\, t_k\,)$$

$$=\; a\,(\,s_{k\,-\,1} \;-\; q_{k\,-\,1}\, s_k\,) \;+\; b\,(\,t_{k\,-\,1} \;-\; q_{k\,-\,1}\, t_k\,) \;=\; a\, s_{k\,+\,1} \;+\; b\, t_{k\,+\,1}\;.$$

The first move is (7.10). The second is by our strong induction hypothesis. The third is from expanding and then distinctly factorizing. Finally, the fourth is by s_i and t_i being governed by the same recurrence as r_i: (7.10).

b) We know that the Euclidean Algorithm terminates for some $i = N$ when the zero-remainder condition

$$r_N = 0$$

is hit. Furthermore, we know that the preceding row $n = N - 1$ attains the actual objective of finding the h.c.f.,

$$r_n = hcf(a, b) .$$

So setting $i = n$ in a),

$$hcf(a, b) = s_n a + t_n b . \tag{7.12}$$

Since s_n and t_n are certainly $\in \mathbb{Z}$ by construction, we are done.

Naming Remark 4 Working with r_i, q_i, s_i and t_i columns is known as the *extended Euclidean Algorithm*. While (7.12) is a special case of a more general identity of Bézout [11] pertaining to Abstract Algebra. This special case having already been known to Méziriac [6], it could be referred to as the Méziriac–Bézout identity. A truer name for it is provided by what b) above poses: 'h.c.f. $=$ $\mathbb{Z}$-linear combination of its input numbers' identity.

Exercise 6 a) Show furthermore that the above-defined s_i and t_i obey

$$s_j t_i - s_i t_j = (-1)^i$$

for each $i \geq 0$ and $j := i - 1$.

b) Deduce that s_i and t_i are mutually prime for each $i \geq 0$.

7.6 Solving linear equations in integers

Problem 1 Solve

$$A x + B y = C \text{ for } A, B \in \mathbb{Z}_* \text{ and } C \in \mathbb{Z} \tag{7.13}$$

for all possible $x, y \in \mathbb{Z}$.

Proposition 1 Case A) If

$$hcf(A, B) \not| C ,$$

then $\nexists$ solutions in $\mathbb{Z}$ to (7.13).

Case B) But if

$$hcf(A, B) \,|\, C ,$$

then $\exists$ an infinity of solutions in $\mathbb{Z}$. Namely,

$$x = x_0 + \frac{B}{hcf(A, B)} , \tag{7.14}$$

$$y = y_0 - \frac{A}{hcf(A, B)} . \tag{7.15}$$

Where (x_0, y_0) is any one particular solution.
Proof Case A) Suppose that

$$hcf(A, B) \,|\, A, B .$$

Then by Exercise 1.a),

$$hcf(A, B) \,|\, (A x + B y) .$$

So by (7.13),

$$hcf(A, B) \,|\, C .$$

Then apply the contrapositive.

Case B) By transforming to $-x$ and $-y$ variables if needs be, we can take $A,\, B\, >\, 0$. Also pass to the ratio variables

$$a_k \;:=\; \frac{A_i}{hcf(A,\, B)}\;,$$

for $a_k \;=\; a,\, b,\, c$ with, correspondingly, $A_k \;=\; A,\, B,\, C$. So

$$a\,x \;+\; b\,y \;=\; c\;. \tag{7.16}$$

is equivalent to our original problem (7.13), but now with

$$hcf(a,\, b) \;=\; 1$$

guaranteed.

Then the Euclidean Algorithm ensures there are $x_0,\, y_0 \;\in\; \mathbb{Z}$ such that

$$a\,x_0 \;+\; b\,y_0 \;=\; c\;.$$

Next, suppose that $x_1,\, y_1 \;\in\; \mathbb{Z}$ is another solution. Then

$$a\,x_1 \;+\; b\,y_1 \;=\; c \;=\; a\,x_0 \;+\; b\,y_0\;.$$

So factorizing the first-and-third forms,

$$a\,(\,x_1 \;-\; x_0\,) \;=\; b\,(\,x_1 \;-\; x_0\,)\;.$$

$$b\,|\,(\,x_1 \;-\; x_0\,) \;\overset{\text{mutual primality}}{\Rightarrow}\; x_1 \;-\; x_0 \;=\; i\,b \;\text{ for }\; i \;\in\; \mathbb{Z}\;.$$

Similarly,

$$y_1 \;-\; y_0 \;=\; i\,a\;.$$

Back-substituting to the original variables, this means that (7.14) and (7.15) are forced. □

Exercise 7 a) Where possible, solve in integers:

i)

$$235\,x \;+\; 238\,y \;=\; 1187\;.$$

ii)

$$34\,x \;+\; 51\,y \;=\; 666\;.$$

iii)

$$12\,x \;+\; 16\,y \;=\; 44 \;\text{ in }\; \mathbb{N}_0\;.$$

b) What happens in Problem 1 if one or both of a and b are zero?

c) Solve
i)

$$12\,x \;+\; 16\,y \;+\; z \;=\; 50 \;\text{ in }\; \mathbb{N}_0\;.$$

ii)$^+$

$$12\,x \;+\; 16\,y \;+\; z \;=\; 50 \;\text{ in }\; \mathbb{Z}\;.$$

Pointer 1 Requiring this for

$$\sum_{i\,=\,1}^{k} a_i\,x_i \;=\; c$$

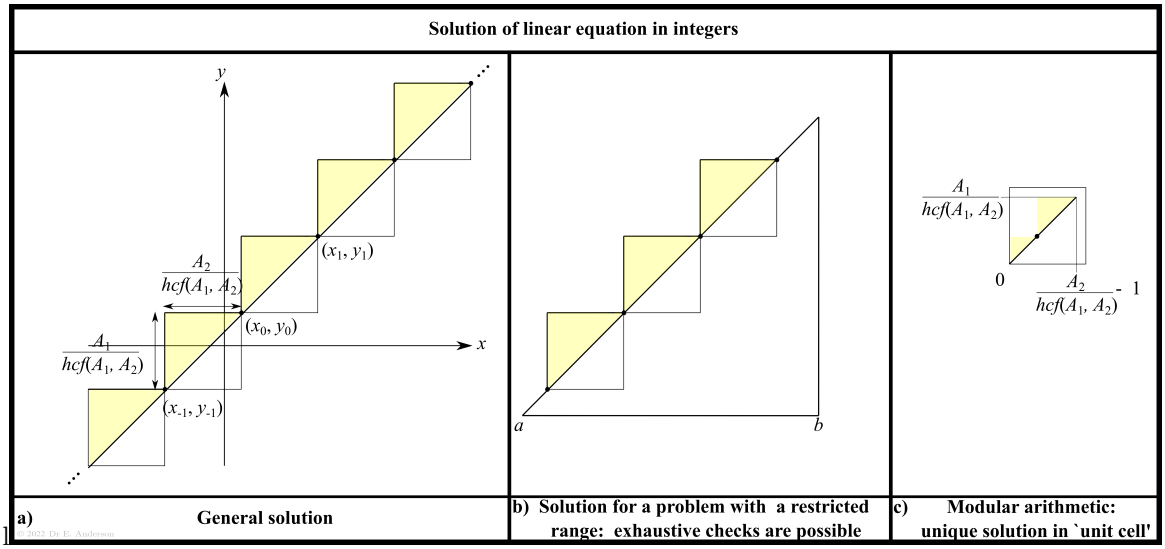


Figure 7.1:

turns out to be rather standard throughout STEM. So figuring out how to handle c.ii) above is rather crucial.



7.7 Prime decompositions

Examples 0 2 is the smallest prime. Thus all subsequent primes must be odd. 3, 5 and 7 manage to be prime. The next odd number is $9 = 3^2$, which is thus not a prime. Discount all subsequent multiples of 3 as well. 11, 13, 17, 19 and 23 manage to be prime. The next odd number is $25 = 5^2$: not prime. So we should be discounting all multiples of 5 as well (and so on).

Proposition 2 There is no such thing as the largest prime number.

Proof For suppose that there was. Then multiply every single prime together and add 1. Then none of these primes divide our constructed number. So this itself meets the definition of prime. Contradiction! □



Remark 1 Any $n \in \mathbb{N}$ can be written as a prime decomposition: a product whose factors are powers of primes.

Theorem 2 (Fundamental Theorem of Arithmetic) The prime decomposition of a given $n \in \mathbb{N}$ is unique (discounting different orderings of the same product).

Proof Let us first write out the Theorem as

$$n = \prod_{j=1}^{J} {p_j}^{m_j} \ . \tag{7.17}$$

Where the p_j are distinct primes and the corresponding powers m_j are known as *multiplicities*.

$\exists$. Given $n \in \mathbb{N}$, set up a strong induction on the total number N of proper-and-nontrivial factors it has.

Base step: If n does not subdivide into any such, n is itself prime, and so n is its own prime decomposition. This guarantees that our result holds for $N = 0$.

Induction hypothesis: Suppose that our result holds for all $N \leq$ some K.

Induction step: Then for $N = K + 1$, we can split it into two factors $n_1, n_2 \in \mathbb{N}$. Since n_1 has to be a proper-and-nontrivial factor, its own number J_1 of such factors is ≥ 1, by which $J_2 \leq N$. And vice versa, so our induction hypothesis applies to each of n_1 and n_2. Concatenating these two products then gives a further such product for our n, by which it also has a prime decomposition.

! m. Suppose that some n admits a second prime decomposition,

$$n = \prod_{i=1}^{I} {p'_j}^{m_j} \ . \tag{7.18}$$

Then

$$p_1 \mid n \overset{(7.18)}{\Rightarrow} p_1 \left| {p'_1}^{m_1} \prod_{i=2}^{I} {p'_j}^{m_j} \right. \ .$$

So either

$$p_1 = p'_1$$

or

$$p_1 \left| \prod_{i=2}^{I} {p'_j}^{m_j} \right.$$

Proceed recursively for $\leq I$ steps to get $p_1 =$ some p'_k. Then cancel this common pair of factors.

Proceed recursively (at a second level of recursion) for $\leq J$ steps to likewise pair off every p_j. This leaves us with two identical decompositions, so no distinct second decomposition is possible, and our uniqueness claim is established. □

Exercise 8 Find the prime decompositions of 48, 7776, and 392863.

Remark 2 As a matter of efficiency, to find the prime decomposition of some number a, we check whether it divides the smallest prime, 2, keeping on dividing by 2 until we reach an odd number. We then divide by successive powers of 3, and so on. We stop moreover when the prime p_t ('t' for 'terminal') we are dividing by exceeds the square root of the hitherto undecomposed factor of a. I.e.

$$a = u \prod_i p_i^{\alpha_i} \ , \ \ p \not| u \ \text{ for any } \ p \ \text{ prime } \ < p_t \ \text{ minimal such that } p_t > \sqrt{u}$$

$$\Rightarrow u \ \text{ is itself the last prime factor of } \ a \ .$$

For beyond this point, there is no space for the number to be the product of two primes (whether or not distinct) that both exceed this square root. This method is known as the *sieve of Eratosthenes.*

Exercise 9 To start off with in thus searching for prime decompositions, we further benefit from simple tests for divisibility by 2, 3, 5, and 11. Numbers divisible by 2 must be even, and numbers

divisible by 5 must end in 0 or 5. Those divisible by 3 have digits whose sum is a multiple of 3. Those divisible by 11 have

$$\sum (\quad \text{odd digits} \quad) \;=\; \sum (\quad \text{even digits} \quad) \;.$$

Ensure that you can prove these well-known rules, as well as the slightly more effective one for divisibility by $9 \;=\; 3^2$ (digits adding up to a multiple of 9). Thus the actual dividing performed in finding prime factorizations would at most start with dividing by 7, 13...[1]

Exercise 10 Prove that

$$|a|\,|b| \;=\; hcf(a,\,b)\;lcm(a,\,b)\;.$$

Naming Remark 5 Fundamental Theorems are invariably well-named (at least in basic subjects). So Readers should take particular note of what each of these say and what each can be used for. 'Arithmetic' itself derives from the Greek '$\alpha\rho\iota\theta\mu o\varsigma$', meaning 'number'. In its modern usage, it is usually taken to mean a basic level of manipulation of numbers.



7.8 Modular arithmetic. i. Single congruences

Definition 1 Let $a,\, b \;\in\; \mathbb{Z}$ and $k \;\in\; \mathbb{N}$. Then we say that $a,\, b$ are *congruent modulo* k if

$$k \,|\, (a \;-\; b)\;.$$

It is standard to denote this by

$$a \;\cong\; b \pmod{k}\;.$$

Remark 1 This is another way of phrasing that

$$a \;=\; q\,k \;+\; b \;\text{ for some }\; q \;\in\; \mathbb{Z}\;.$$

Exercise 11 Prove that $\cong$ is an equivalence relation.

Exercise 12 Prove that, given some fixed $k \;\in\; \mathbb{N}$, each integer
$j \;\cong\;$ (precisely one of 0 to $k \;-\; 1$), namely its *residue* (mod k).



Worked Example 4 The preceding Exercise leaves *modular arithmetic tables* well-defined. Work out binary-operation tables for addition $\pmod 1$ through to $\pmod 5$, and for multiplication likewise.

Answer 4 For addition, see the top row of Fig 7.2. Within this, each row of numbers comprises a permutation. For multiplication, see the bottom row. With the non-prime number 4 first illustrating how these times tables can exhibit more interesting patterns.

Exercise 13 What are the multiplication tables mod 6, 8, 9 and 12?

Exercise 14 Observe that each of these tables has reflection symmetry about its leading diagonal.

a) Prove that this must hold for all modular arithmetic addition and multiplication tables.

[1] All of the above is with reference to base 10.

Modular arithmetic sum and times tables

a) (mod 1)

+	0
0	0

×	0
0	0

b) (mod 2)

+	0	1
0	0	1
1	1	0

×	0	1
0	0	0
1	0	1

c) (mod 3)

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

d) (mod 4)

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

×	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

e) (mod 5)

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

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Figure 7.2:

b) Give a change of variables from the naïve inputs to linear-shift relative variables (as explained in the Introduction) that vertically centre this axis of symmetry.

c) Write out the corresponding triangular arrays for the 5 cases in Fig 7.2, and for the 4 further cases in Exercise 13.

d) Quotient out the redundant information, ending up with triangles slightly larger than half of the previous ones.

e) Give a formula for the area ratio of the general answer to d) over the general answer to c).

Worked Example 5 When do solutions of

$$n\,x \;\cong\; m \pmod{k}$$

exist, and are they unique?

Answer 5 By Remark 1, Proposition 1 and Fig 7.1.c), congruences can either have one solution or none. Depending on whether or not

$$hcf(n,\,k)\;|\,m \quad .$$

Exercise 15 a) Solve $5\,x \;=\; 3 \pmod 7$.

b) Why does $3\,x \;=\; 2 \pmod 6$ require a bit more care to solve?

c) Show that $24 \;\cong\; -\,1 \pmod 5$.

Worked Example 6 Prove *Wilson's Theorem*: that for p prime,

$$(\,p \;-\; 1\,)!\;\cong\; -\,1 \pmod p\ .$$

Answer 6

$$hcf(a, p) \;=\; 1$$

for each $a \;\in\; [\,0, p\,)$. So by Worked Example 5, $\exists\;!$ solution to

$$a\,x \;\cong\; 1 \pmod{p}\;.$$

Trick 1: Set $x \;=\; a$ therein to obtain

$$a^2 \;-\; 1 \;\cong\; 0 \pmod{p}\;.$$

So by the difference of two squares,

$$p\,|\,(a \;-\; 1\,)(a \;+\; 1\,)\;.$$

Thus by Exercise 2, at least one of the following must hold.

$$p\,|\,(a \;-\; 1\,)\;\;,\;\;p\,|\,(a \;+\; 1\,)\;\;.$$

Trick 2: For odd primes, the above gives a grouping of the residues $\in [\,2, p \;-\; 1\,)$ into inverse pairs. I.e. with the product of each pair being 1, all of whose elements are distinct. Thus

$$(\,p \;-\; 2\,)! \;\cong\; (\text{pair 1}) \;\times\; (\text{pair 2})\,... \;=\; 1^{(p\,-\,1)/2} \;\times\; 1 \;=\; 1\;.$$

The first step here is just a permutation ordering out these pairs.

So finally multiplying both sides by $(\,p \;-\; 1\,)$,

$$(\,p \;-\; 1)! \;=\; (\,p \;-\; 1\,)(\,p \;-\; 2)! \;\cong\; (\,p \;-\; 1\,) \;\times\; 1 \;=\; (\,p \;-\; 1\,) \;\cong\; -1 \pmod{p}.$$

This leaves $p \;=\; 2$. But here our identity just reads

$$(\,2 \;-\; 1\,)! \;=\; 1! \;=\; 1 \;\cong\; -1 \pmod{2}\;.\;\square$$

♠ ◇ ♣

Naming Remark 6 Aside from Wilson not being the first to state this Theorem, it was Lagrange who gave the earliest known proof of it. Since for integers the Gamma function is just the factorial of the number smaller by 1, Wilson's Theorem can be rephrased as

$$\Gamma(p) \;\cong\; -1 \pmod{p}\;.$$

By which, a truer name for it is 'Gamma prime-modular-negunarity Theorem'.

Notation 1 For later use, let $\mathfrak{Unit}_a$ denote the set of units $\pmod{a}$.

Exercise 16 Extend the triangle of combinations (i.e. Pascal's triangle) to at least 12 floors. What pattern do you observe upon passing to the $\pmod 2$ version of this triangle? [This is the Book's fifth presentation of Pascal's Triangle.]

7.9 ii. Simultaneous congruences

Problem 2 What if multiple congruence hold at once?

Theorem 3 (Chinese Remainder Theorem) i) Suppose that

$$x \;=\; a \pmod{k} \tag{7.19}$$

and

$$x \;=\; b \pmod{l}\;. \tag{7.20}$$

Then if k and l are mutually prime, a solution x to the system (7.19, 7.20) exists and is unique $(\mathrm{mod}\ k\,l\,)$.

ii) Suppose that

$$x \;=\; a_j \pmod{k_j}$$

for $j \;=\; 1$ to J and $a_j,\, k_j \;\in\; \mathbb{Z}$, with

$$hcf\,(a_i,\, a_j) \;=\; 1 \;\text{ for }\; i \;\neq\; j \;:\quad \text{pairwise mutually prime}\;.$$

Then a solution for x exists, which is furthermore unique

$$\left(\mathrm{mod} \prod_{j\,=\,1}^{J} k_i\right)\;.$$

Proof of i) $\exists$. We can set

$$i\,k \;+\; j\,l \;=\; 1 \;\text{ for }\; i,\, j \;\in\; \mathbb{Z}\;. \tag{7.21}$$

This is by the Euclidean Algorithm, given that mutual primeness means that

$$hcf(\,k,\, l\,) \;=\; 1\;.$$

Next observe that

$$y \;=\; b\,i\,k \;+\; a\,j\,l \tag{7.22}$$

obeys both our congruences. This trick uses the 'back to front' product combination, in the sense that the original congruences pair a with k and b with l, rather. This works because, firstly proceeding $(\mathrm{mod}\ k)$,

$$y \;=\; a\,j\,l \;=\; a\,1 \;=\; a\;.$$

The first step here uses (7.22), the second (7.21) and the third the unit property. Secondly, the same sequence of steps $(\mathrm{mod}\ l)$ yields

$$y \;=\; b\,i\,k \;=\; b\,1 \;=\; b\;.$$

! Suppose that y' also solves. Then

$$y \;\cong\; y' \pmod{k}$$

and

$$y \;\cong\; y' \pmod{l}\;.$$

These can be rephrased as

$$k,\, l\,|\,(\,y \;-\; y'\,)\;.$$

Since k and l are mutually prime, this means that

$$k\,l\,|\,(\,y \;-\; y'\,)\;.$$

Uniqueness is thereby established. □

Naming Remark 7 'Remainder Theorem' is a useful conceptual type. To be more specific, among all Remainder Theorems, this Section's is the *Simultaneous Linear Congruences Remainder Theorem*. For which a mnemonic is *SiLiCon Remainder Theorem*. Then reading the first two letters of this syllabe-acronym forwards and the last two letters backwards spells out *Sino*: a prefix meaning Chinese (!)

Exercise 17 a) Solve the simultaneous congruences

$$3\,x \;\cong\; 2 \pmod{13}\;,\quad 5\,x \;\cong\; 4 \pmod{36}\;.$$

b) Prove ii) above.

c) Solve the following system of congruences.

$$2\,x \;\cong\; 1 \pmod{7}\;,\quad 3\,x \;\cong\; 2 \pmod{13}\;,\quad 5\,x \;\cong\; 4 \pmod{36}\;,\quad 7\,x \;\cong\; 8 \pmod{121}\;.$$



7.10 Fermat's little Theorem

Theorem 4 (Fermat's little Theorem) Let p be prime and $a \in \mathbb{Z}$ such that $p \not| a$. Then

$$a^{p\,-\,1} \;\cong\; 1 \pmod{p}\;.$$

Remark 1 In other words, given p prime and a p-free, the $(p\,-\,1)$th power of a $\in \mathfrak{Unit}_p$.

In yet other words, this is for p prime alongside some other integer a that is mutually prime with it. Which, by the definition of mutually prime, means that

$$hcf(p,\,a) \;=\; 1\;.$$

Naming Remark 8 This is a *prime modular unarity Theorem*. We are not far enough into the current Volume to attempt a more specific truer name.

Remark 2 We leave proofs of this Theorem to Exercise 23 and 13.3.

7.11 Euler's totient function

Definition 1 For $n \in \mathbb{N}$, *Euler's totient function*

$$\varphi(n) \;:=\; (\,e \in \mathbb{N} : 0 \,<\, e \,\leq\, n\;,\;\; hcf(e,\,n) \;=\; 1\,)\;.$$

In other words, it is the counting function whose output is the number of mutually-prime residues $(\mod n\,+\,1)$. Or, equivalently, of units $(\mod n\,+\,1)$.

Example 0

$$\varphi(1) \;=\; 1\;.$$

Example 1

$$\varphi(p) \;=\; p\,-\,1\;.$$

Exercise 18$^-$ a) Find $\varphi(4)$, $\varphi(8)$, $\varphi(16)$ and $\varphi(32)$.

b) Find $\varphi(6)$, $\varphi(10)$, $\varphi(15)$, $\varphi(24)$ and $\varphi(30)$.

Naming Remark 9 'Totient' is the 'quantum tantum' partner-word of 'quotient'. Where 'quantum' is used to ask how many, and 'tantum' to answer how many, Latin and consequently many Romance languages having a number of such pairs of words. In this vein, the quotient q of a number a by a number b concerns how many copies of b fit inside a. To the totient φ of a number a specifying the number of ways that this is mutually-prime $(\mod n\,+\,1)$. For all that this is not a watertight correspondence. The word 'totient' was coined by Sylvester [17], while the symbol φ for it is previously due to Gauss (who thus had no reason to use such as t or τ for it).

♠ ◇ ♣

Worked Example 7 A function f is *multiplicative* if

$$f(a\,b) \;=\; f(a)\,f(b)\;.$$

Prove that φ is multiplicative.

Answer 7 Let us use the number of units conception. Also set

$$A \;:=\; a \;+\; 1$$

and

$$B \;:=\; b \;+\; 1\;.$$

Let x be a unit $\pmod A$, and y be a unit $\pmod B$ Then the simultanenous linear congruences

$$z \;\cong\; x \pmod A\;,$$

$$z \;\cong\; y \pmod B$$

are guaranteed by the SiLiCon $=$ Sino- $=$ Chinese Remainder Theorem to be solvable by a unique z that is a unit $\pmod{A\,B}$. This furthermore exhaustively gives all the possible units $\pmod{A\,B}$. □

Exercise 19 In fact, Euler originally defined φ slightly differently: with $<\; n$ in place of $\leq\; n$. Show that the only resulting difference is that $\varphi_{\text{original}}(1) \;=\; 0$. And yet that this is enough to spoil multiplicativity.

[On the one hand, multiplicativity is a significant enough property to merit this change to the modern conception. On the other hand, this is a small enough modification that the resulting totient remains attributed to Euler.]

Exercise 20 Deduce from the above worked Example that

$$\varphi(n) \;=\; n \prod_{p\,|\,n} \left(1 \;-\; \frac{1}{p}\right) \;:$$

the product over all prime divisors of n.

Exercise 21 a)$^-$ Show that

$$\varphi(1) \;+\; \varphi(p) \;=\; 1\;,$$

$$\varphi(1) \;+\; \varphi(2) \;+\; \varphi(5) \;+\; \varphi(10) \;=\; 10 \;\text{ and}$$

$$\varphi(1) \;+\; 2\,\varphi(2) \;+\; \varphi(4) \;=\; 4\;.$$

b) Propose (and check!) analogues of these formulae for $\varphi(30)$ and $\varphi(32)$.

Exercise 22 What other multiplicative functions can you think of?

7.12 Fermat–Euler Theorem

Theorem 5 (Fermat–Euler Theorem) Suppose that $j \in \mathbb{Z}$ and $n \in \mathbb{N}$ with j and n mutually prime. Then

$$j^{\varphi(n)} \cong 1 \pmod{n} .$$

In other words, given any $n \in \mathbb{N}$, its totient power of any mutual prime is $1 \pmod{n}$.

Naming Remark 10 Thus a truer name would be 'Modular unarity of mutually-prime totient powers Theorem'. The acronyms FET and FlT for the Fermat–Euler and Fermat's little Theorems are also quite widespread.

Proof . The totient function picks out t of the m possible remainders supported by working $\pmod{n}$

$$\{ r_i , \ i = 1 \text{ to } t \} .$$

Then since both j and r_i are mutually prime with n, so is $j\, r_i$.

We need to check furthermore that there are no repeats among the $j\, r_i$. But suppose that

$$j\, r_a \cong j\, r_b \pmod{n}$$

for some $a, b \in [1, t]$. This can be rewritten as

$$n \mid j\, (r_a - r_b)$$

But by Worked Example 2 and the definition of mutually prime,

$$n \mid (r_a - r_b)$$

is then forced. And this can be rewritten as just

$$r_a \cong r_b \pmod{n} .$$

The $j\, r_i$ are thus in bijective correspondence with the r_i. I.e. multiplying by j just permutes the remainders.

Then use that the permutation of the order of a finite product's factors does not affect the value of the product that they form. So in our particular case, writing this backwards gives the first of the following working's steps.

$$\prod_{i=1}^{t} r_i \cong \prod_{i=1}^{t} j\, r_i \cong j^t \prod_{i=1}^{t} r_i .$$

The last step is just pulling out the product-index-i-independent factors. But by definition $t = \varphi(m)$, so cancelling off the identical products gives the Theorem.

The cancellation in question being licit since a simple induction gives that

$$hcf\left(n, \prod_{i=1}^{t} r_i \right) = 1 . \ \square$$

Remark 1 Since counting up the objects in the labelled order and in the permuted order give distinct expressions which must be equal, the above can be envisaged as the Book's first use of the Combinatorial style of proof.

Exercise 23 a) Quickly prove FlT as a Corollary of FET, and Wilson's Theorem in turn as a Corollary of FlT.

b) Prove FlT from first principles by developing a simpler version of the above FET proof's permutation of remainders.

Exercise 24 FET permits evaluating

$$a^k \pmod{p}$$

for large powers k, provided that

$$p \not| a \ .$$

Evaluate $168^{400} \pmod{401}$ and $293^{1202} \pmod{401}$. Also see what happens if you try to check these answers by brute force using your calculator, or even your computer!

7.13 Epilogue on Naming, Number Theory, and Cryptography

End-Remark 1 A truer name for FlT is consequently *prime-FET*. With conversely FET showing how to think about FlT in order to generalize from its 'little' prime p and p-free pairs context to the larger context of mutually-prime but elsewise unrestricted pairs. So 'prime-only' is a truer name for the 'little' in FlT. Enabling this generalization is the first of many reasons why studying Euler's totient function is a worthwhile investment.

End-Remark 2 Wilson's Theorem has the added virtue of being provable within multiple distinct branches of Mathematics. As such, we shall be seeing it again in Group Theory [444] and in Multilinear Algebra [441]. Some such approaches further generalize to FlT and FET.

Pointer 2 The converse of FlT is false. I.e. there exist composite $n \in \mathbb{N}$ such that the FlT statement

$$a^{n-1} \cong 1 (\text{mod } n)$$

holds. Thus, on the one hand, FlT cannot be used to diagnose primes. On the other hand, the numbers arising thus – called *pseudoprimes* – are rather interesting in their own right [26, 35].

Exercise 25 Show that

$$2^{340} \cong 1 \pmod{341} \ ,$$

and yet that 341 is not prime. This establishes, by minimal counterexample, that the converse of FlT is false.

Pointer 3 The adjective 'little' was however introduced to distinguish Fermat's little Theorem from Fermat's Last Theorem, which moreover took several more centuries to prove [45]. It is furthermore an example of a Theorem that is very hard to prove while nonetheless being very simple to *pose*.

" $a^n + b^n = c^n$ is not solvable in integers for any natural number $n > 2$. "

Pointer 4 This is a notorious example of how, while solving linear equations in integers is straightforward along the lines in this Chapter, solving cubic-or-higher equations in integers can be tough, impossible, or with no method yet known. The topic of solving polynomial equations in integers is often referred to as *Diophantine equations* [22]. At least some parts of the quadratic case are of intermediate difficulty, thus making it into some introductory textbooks [57] and Olympiad problems [64]

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Pointer 5 It is easy for somebody to multiply large primes together. But much harder for anyone else who does not know which primes were used to factorize such products back into large primes. In Volume 2 on Computer Science, we shall estimate how many steps a number of algorithms require

as functions of their task's size, including these two. The above discrepancy in steps being large renders numbers with large prime factors a common ingredient in Cryptography.

That FET also plays a sizeable part in Cryptography can then also be gleaned from Exercise 24's also very large discrepancy between this and brute-force computation for large powers $\pmod{m}$.

Exercise 26 To balance out the Exercises set, let us end by inviting the Reader to try to find a reasonably efficient way of decomposing 58545167 into its prime factors. While noting that this is many orders of magnitude smaller than what public-key Cryptography would use.



Chapter 8

Sets of (sets of) subsets!

8.1 Sets of subsets

Definition 1 For a given set $\mathfrak{X}$ with fully distinguishably labelled elements, let us denote the corresponding arena – the *space of subsets* – by

$$\mathfrak{LS}\text{ubset}(\mathfrak{X}) \ .$$

As a set, this consists of all the possible subsets of our incipient labelled set, $\mathfrak{LX}$.

Definition 2 For a given set $\mathfrak{X}$ with fully indistinguishable elements, let us denote its arena of subsets by

$$\mathfrak{S}\text{ubset}(\mathfrak{X}) \ .$$

Remark 1 In the current Volume, we only consider arenas of subsets for finite sets. These being uniquely characterized up to isomorphism by their cardinality $N \in \mathbb{N}$ we can rewrite (8.1) as

$$\mathfrak{S}\text{ubset}(N) \ . \tag{8.1}$$

Lemma 1 For a finite set with fully distinguishably labelled elements,

$$|\, \mathfrak{LS}\text{ubset}(N)\,| \ = \ 2^N \ .$$

Proof In forming each subset, can independently include or exclude each of the set's elements. This gives a total count of

$$(1 \ + \ 1)^N \ = \ 2^N \ . \ \square$$

Naming Remark 1 This count gives $\mathfrak{LS}\text{ubset}(N)$ a new alias – *power set* – with corresponding notation

$$\mathfrak{P}(N) \ .$$

The notation

$$\mathfrak{P}(\mathfrak{X}) \ = \ 2^{\mathfrak{X}}$$

is furthermore in common use even if $\mathfrak{X}$ is not finite.

Remark 2 If a finite set's elements are indistinguishable, then

$$|\, \mathfrak{S}\text{ubset}(N)\,| \ = \ N \ + \ 1 \ .$$

For there are now $N \ + \ 1$ possible subsets, covering all cardinalities from 0 to N.

Remark 3 Throughout these Series, we use square brackets for up to order N dependency, e.g. $\mathfrak{S}\text{ubset}[N]$. Arenas depending on this can furthermore be viewed as cumulative approximands to the corresponding arena for all values of N. Which we denote with no dependency at all, e.g. here $\mathfrak{S}\text{ubset}$.

8.2 The smallest examples

Example 0

$$\mathfrak{P}(0) \;=\; \emptyset \;=\; \mathfrak{S}\text{ubset}(0)\ .$$

Example 1

$$\mathfrak{P}(1) \;=\; \{\,\emptyset,\, \varsigma\text{ingleton}\,\} \;=\; \mathfrak{S}\text{ubset}(1)\ ,$$

where ςingleton denotes a subset consisting of a single element.

Example 2

$$\mathfrak{P}(2) \;=\; \{\,\emptyset,\, \varsigma\text{ingleton}_1,\, \varsigma\text{ingleton}_2,\, \mathfrak{X}\,\} \;\neq\; \{\,\emptyset,\, \varsigma\text{ingleton},\, \mathfrak{X}\,\} \;=\; \mathfrak{S}\text{ubset}(2)\ .$$

Remark 1 See Fig 8.1 for depictions of the indistinguishable-elements subcase on 0 to 4 elements.

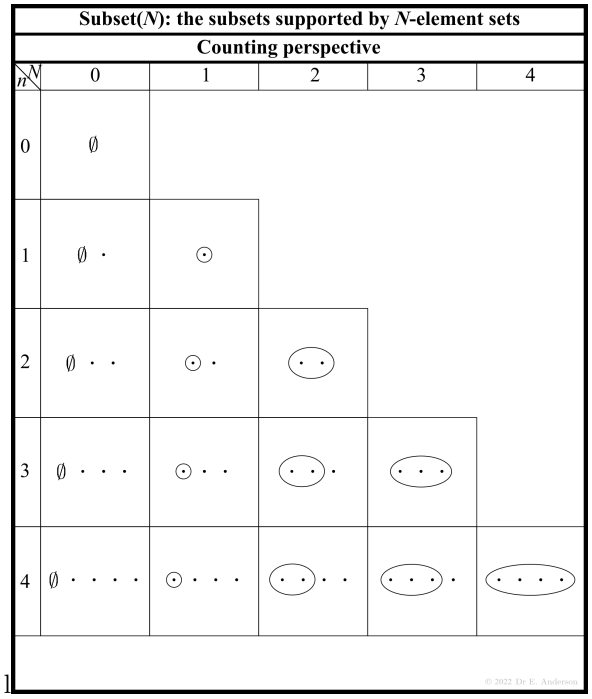


Figure 8.1:

8.3 Complementation for power sets

Remark 1 A result in Chapter 1 can readily be arranged to give the following *complementation formula*.

$$|\,\mathfrak{a}\,| \;+\; |\,\mathfrak{a}^{\mathrm{c}}\,| \;=\; |\,\mathfrak{X}\,|\ . \tag{8.2}$$

This can be useful for finite universe-sets.

Definition 1 A [subset] is a subset modulo complementation. I.e. its number of included elements is equal to its number of excluded elements.

Remark 2 This can only happen if the total number of points N is even. For it requires

$$M_{\mathrm{crit}} \;:=\; \frac{N}{2}$$

to be realized as both the number of included elements M and the number of excluded elements $N - M$.

Definition 2 A subset $\mathfrak{A}$ is *homocomplementary* if

$$\mathfrak{A} \cong \mathfrak{A}^c .$$

Remark 3 For this to occur,

$$N \text{ even} \quad \text{and} \quad M = M_{\text{crit}} = \frac{N}{2}$$

constitute a joint set of both necessary and sufficient conditions.

8.4 Relational variables for subset size

We here consider a universe set $\mathfrak{X}$ of finite size $|\mathfrak{X}| = N$.

Definition 1 The *linearly-shifted number of subsets* relative variable is

$$R = M - M_{\text{crit}} = M - \frac{N}{2} . \tag{8.3}$$

Definition 2 Its *unsigned* counterpart is

$$U = |M - M_{\text{crit}}| = \left| M - \frac{N}{2} \right| . \tag{8.4}$$

Structure 3 The *subset relative size* is

$$(y|z) := \frac{|\mathfrak{Y}|}{|\mathfrak{Z}|} .$$

Structure 4 The *subset absolute size* is

$$y := \frac{|\mathfrak{Y}|}{|\mathfrak{X}|} .$$

In the finite universe set case, we can write this as

$$m := \frac{M}{N} .$$

Example 1 For a map $\phi : \mathfrak{A} \longrightarrow \mathfrak{B}$,

$$\frac{|\ker(\phi)|}{|\text{image}\phi)|}$$

is a quite commonly encountered example of subset cardinality ratio.

♠ ◇ ♣

Structure 5 The *subset number ratio* is

$$\rho = \frac{M - M_{\text{crit}}}{N} = \frac{M}{N} - \frac{1}{2} = m - \frac{1}{2} .$$

This runs from -1/2 to 1/2.

Structure 6 The *unsigned subset number ratio* is

$$\sigma \; = \; 2 \, \frac{\left| \, M \; - \; \frac{N}{2} \, \right|}{N} \; = \; \left| \, 2 \, m \; - \; 1 \right| \; .$$

This runs from 0 to 1.

Remark 1 Using R as centring variable on the incipient arena of subsets gives Fig 8.2.a) The cyan shading here picks out the homocomplementary cases.

Remark 2 Keeping say the white lobe while suppressing the darker grey lobe then gives Fig 8.2.b) for the subsets modulo complementation.

Remark 3 Letting

$$y^{\mathrm{c}} \; := \; \frac{|\, \mathfrak{Y}^{\mathrm{c}} \,|}{|\, \mathfrak{X} \,|} \; ,$$

it follows that

$$y^{\mathrm{c}} \; = \; \frac{|\, \mathfrak{X} \,| \; - \; |\, \mathfrak{Y} \,|}{|\, \mathfrak{X} \,|} \; = \; 1 \; - \; \frac{|\, \mathfrak{Y} \,|}{|\, \mathfrak{X} \,|} \; = \; 1 \; - \; y \; .$$

Thus

$$y \; + \; y^{\mathrm{c}} \; = \; 1 \; .$$

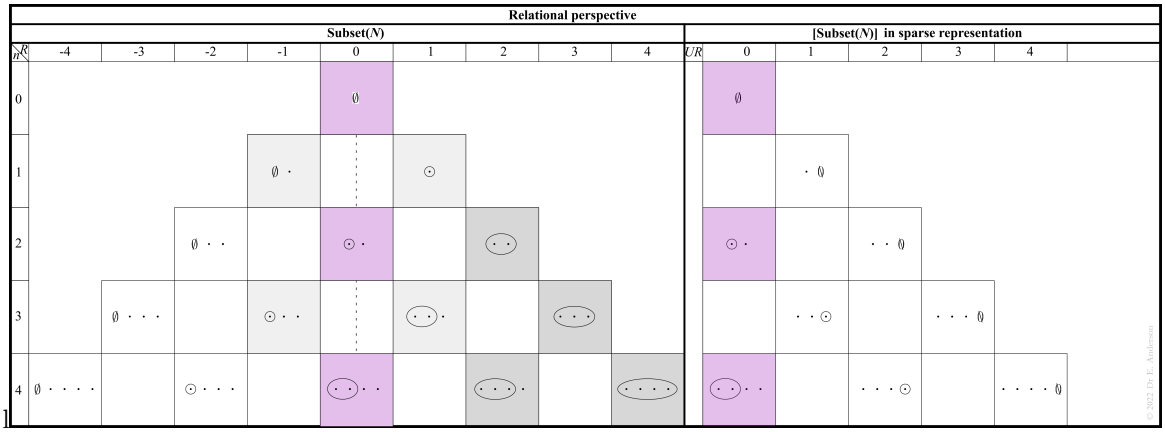


Figure 8.2:

8.5 Further relational quantities for subsets

Structure 1 The *symmetric difference* is a somewhat more sophisticated relative set notion. From this, we can form the count

$$|\, \mathfrak{a}_1 \, -_{\mathbf{S}} \; \mathfrak{a}_2 \,| \; = \; |\, \mathfrak{a}_1 \,| \; + \; |\, \mathfrak{a}_2 \,| \; - \; 2 \, |\, \mathfrak{a}_1 \; \cap \; \mathfrak{a}_2 \,| \; .$$

This is not only a relative count but also symmetric under interchanging labels 1 and 2.

Worked Example 1 In forming ratios out of this, we would like this symmetry to be preserved, by picking denominators that also respect it.

Answer 1 One such possibility is the absolute

$$\frac{|\mathfrak{a}_1 -_{\mathfrak{S}} \mathfrak{a}_2|}{|\mathfrak{X}|} .$$

Another is

$$\frac{|\mathfrak{a}_1 -_{\mathfrak{S}} \mathfrak{a}_2|}{|\mathfrak{a}_1 \cup \mathfrak{a}_2|} = \frac{|\mathfrak{a}_1| + |\mathfrak{a}_2| - 2|\mathfrak{a}_1 \cap \mathfrak{a}_2|}{|\mathfrak{a}_1| + |\mathfrak{a}_2| - |\mathfrak{a}_1 \cap \mathfrak{a}_2|} .$$

This does not however succeed in being relational, since the denominator's numbers of positive and negative terms do not coincide. To achieve this, we need to compare distinct symmetric differences. The simplest such possibility is

$$\frac{|\mathfrak{a}_1 -_{\mathfrak{S}} \mathfrak{a}_2|}{|\mathfrak{a}_3 -_{\mathfrak{S}} \mathfrak{a}_4|} .$$

Further ways of attaining this are given in Sec 12.4.

8.6 Viewing some set operations as binary operations on the power set

Remark 1 When dealing with union, intersection, difference, and symmetric difference 'for sets' we are actually dealing with binary operations *on the power set*. I.e. maps

$$\circ : \mathfrak{P}(\mathfrak{X}) \times \mathfrak{P}(\mathfrak{X}) \longrightarrow \mathfrak{P}(\mathfrak{X}) .$$

For the input here is a pair of subsets, i.e. a pair of elements of the power set, while the output is a single element thereof.

8.7 Further discussion about labels

Case 0) We could have fully-indistinguishable labels. This is equivalent to having no labels. For unlabelled elements can still be taken to be all labelled with the indistinguishable *unlabel*. $\mathfrak{LS}$pace is the labelled version of a habitually-unlabelled arena $\mathfrak{S}$pace. Conversely, $\mathfrak{US}$pace is the unlabelled version of a habitually-labelled arena $\mathfrak{S}$pace.

Case 1) We could have fully-distinguishable labels. In the finite case, these could be taken to be 1 to L. We then denote our fully-distinguishable finite set by $[1, \ldots, L]$.

Case M) In between these two extremes lie a number of possible cases of partly-distinguishable labels. Fig 8.3 gives some examples.

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Remark 1 The above examples of labels are *meaningful*. Each of the numbers 1, 2 and 3 have distinct properties. Each of blue, red and yellow balls reflect different wavelengths of light. Each of carbon, hydrogen and oxygen have distinct chemical and physical properties, rooted in particular in distinct electron structures and nuclear masses. Each of hydrogen and deuterium have more subtly different properties, electron structure now being shared but a distinction in nuclear mass remaining. Each of carbon-12 and carbon-14 are somewhat more distinct, since aside from having different nuclear masses, carbon-14 is also unstable to radioactive decay on a humanly-appreciable timescale.

Other examples of labels can however be *meaningless*.

Definition 1 A label-value is *meaningless* if its values have arbitrary names that can be interchanged at will.

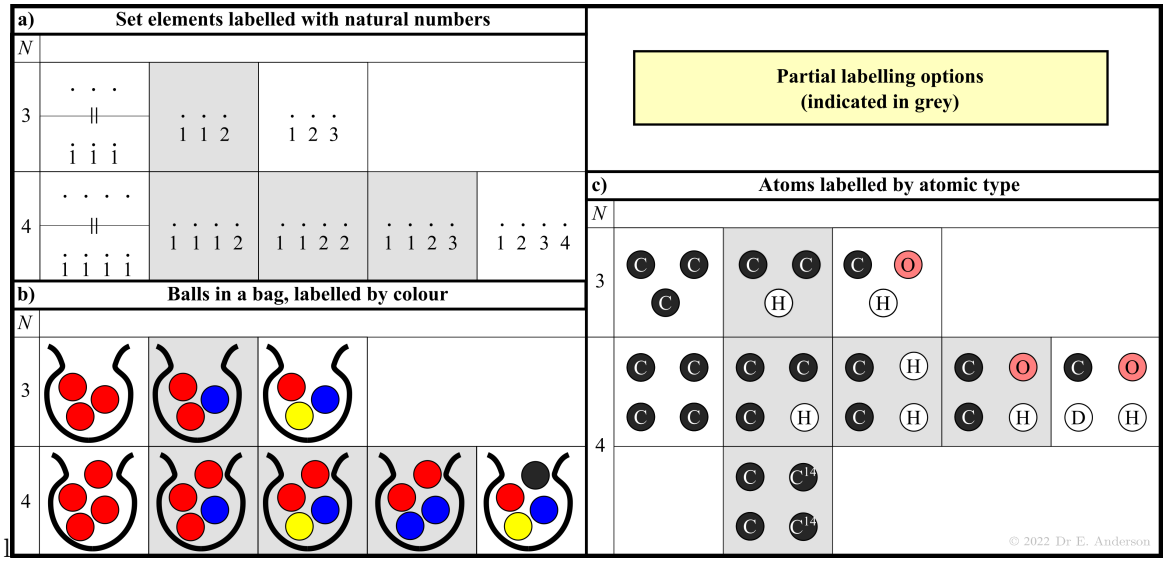


Figure 8.3:

Example 1 Model N objects by points. If two of them are related, join them by a blue edge. If two of them are not related, join them by a red edge. If the relation has no intrinsic meaning, however, the blue and red colour assignments could just as well be reversed. We make use of this Example in Parts V and VI.

Example 2 There is no way in which spin-'up' in QM has any properties of upness.

Example 3 Nor is there anything actually red about the 'red'-charge type label in Quantum Chromodynamics: the modern theory of the strong nuclear forces [387].

Remark 2 Meaninglessness can itself also be partial. See Fig 8.4 for examples.

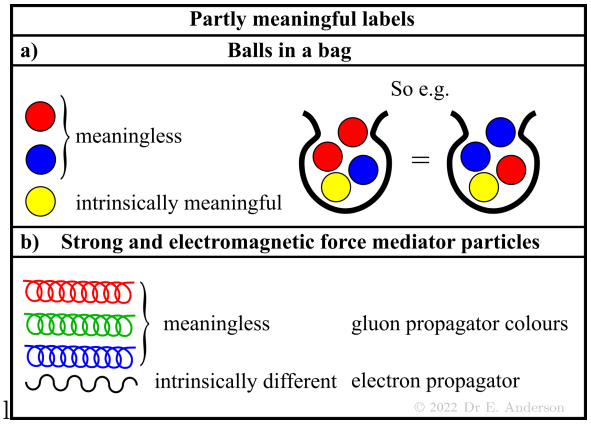


Figure 8.4:

End-Remark 1 We shall see in Sec 22.1 how to classify the partly-indistinguishable and partly-meaningless labelling cases for a given underlying unlabelled set. The current Section moreover considerably multiplies the number of arenas to explore.



8.8 Sets of sets of subsets

Naming Remark 2 Having considered an instance of sets of sets in the previous three Sections, we now turn our gaze to all such instances. In such ventures, we would like to avoid repetitiveness and confuseability. As such, we elect to use *families* to mean sets whose elements are (sub)sets. And *collections* for sets of sets of sets' ternary level of generality. In particular, arenas of families are themselves collections. These include both the arena of all families

$$\mathfrak{F}\text{amily}(\mathfrak{X})$$

on a fixed finite set, and numerous further examples of much use in Topology. The Reader should be warned that conventions for these names do vary between authors. We shall not presently consider any quaternary or higher versions.

Structure 1 Families of subsets of a given labelled set are the subsets of the power set itself. Thus the arena of families is

$$\mathfrak{LF}\text{amily}(\mathfrak{X}) \;=\; \mathfrak{P}(\mathfrak{P}(\mathfrak{X}))\ .$$

Lemma 2 i)

$$|\,\mathfrak{LF}\text{amily}(N)\,| \;=\; 2^{2^N}\ .$$

Proof Twice iterate the counting argument for the power set. □

Example 0

$$\mathfrak{LF}\text{amily}(0) \;=\; \{\emptyset, \{\emptyset\}\}\ .$$

Notice the difference between the two elements here. The first is the *empty family*, while the second is the nonempty family consisting of the empty set.

Example 1 For the 1-element set,

$$\mathfrak{F}\text{amily}(1) \;=\; \{\,\emptyset,\, \{\emptyset\},\, \{\mathfrak{X}\},\, \{\,\emptyset, \mathfrak{X}\,\}\,\} \;=\; \{\,\emptyset,\, \{\emptyset\},\, \{\varsigma\text{ingleton}\},\, \{\,\emptyset, \varsigma\text{ingleton}\,\}\,\}\ .$$

We depict these as per Fig 8.5.

Example 2 See row 3 of this figure for the diversity of families based on the 2-element set. In part a) of this Figure, we present the families of subsets of sets with 0 to 2 objects using family size and subset number variables.

8.9 Family complementation

Remark 1 Complementation is exhibited by families upon switching the $\in$ and $\notin$ operations.

Definition 1 A [*family*] is a family modulo complementation.

Definition 2 A family of subsets is *homocomplementary* if it takes the same form as its complement.[1]

[1] Again, the literature has hitherto mostly used 'self-complementary'.

Remark 2 A necessary but not sufficient condition for this is for the number of subsets n involved to be

$$n \;=\; n_{\text{crit}} \;=\; \frac{N_{\max}}{2} \;=\; \frac{N}{2}\;.$$

So if family-homocomplementarity is to be present, N is forced to be even.

Remark 3 In the case of families, homocomplementarity is furthermore not realized. While for unlabelled sets the lack of structure means that counting alone suffices to describe the objects, families have more freedom to be structured. Then there being just the one subset with no elements, $\emptyset$, is enough to end any chance of homocomplementarity. I.e. if a family contains the universe set, then its family-complement cannot.

Example 1 See Figs 8.5.b) and 8.6 for [families] on up to 3 elements.

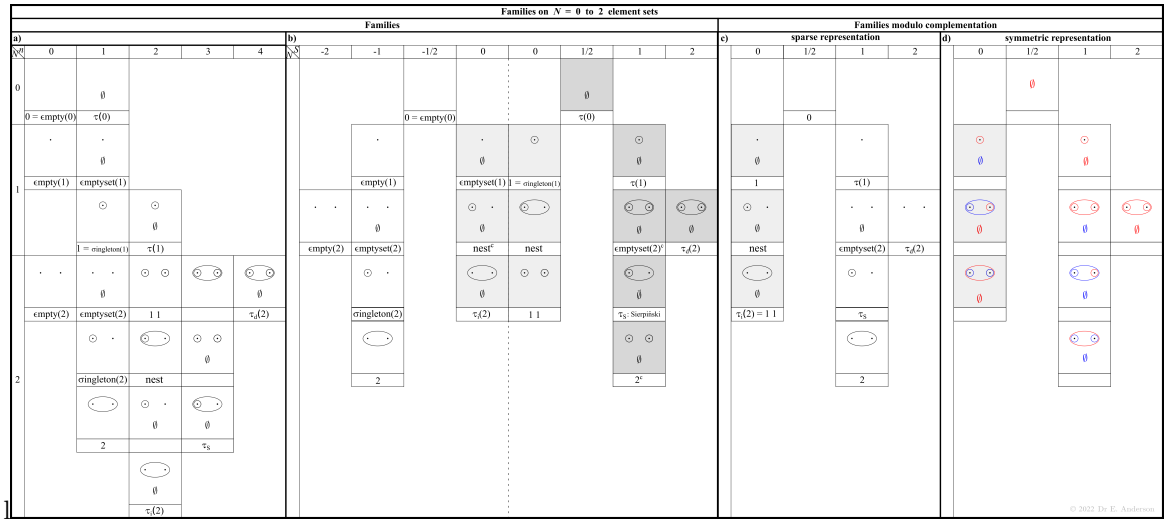


Figure 8.5:

Exercise 1 Find all the familes of subsets of the 3-element set.

Exercise 2 Find all the [families] of subsets of the N-element sets for $N \;=\; 0$ to 3.

[For subsequent use, the answers to Exercises 1 and 2 can be found in Figs 8.5 and Fig 8.6.]

8.10 Relational variables for families

Remark 1 It turns out to be convenient to centre about the n_{crit} reflection line using the linearly-shifted relative variable

$$R \;:=\; n \;-\; n_{\text{crit}} \;=\; n \;-\; \frac{N}{2}\;. \tag{8.5}$$

The corresponding unsigned version is

$$U \;:=\; |\,n \;-\; n_{\text{crit}}\,| \;=\; \left|n \;-\; \frac{N}{2}\right|\;. \tag{8.6}$$

Definition 2 The *subset density* is

$$\rho \;:=\; \frac{n}{n_{\max}}\;.$$

The *critical subset density* is

$$\rho_{\text{crit}} \ := \ \frac{n}{n_{\text{crit}}} \ .$$

Remark 2 Ratio versions of the above relative variables are as follows. Firstly,

$$\frac{n \ - \ n_{\text{crit}}}{n_{\text{crit}}} \ = \ \rho_{\text{crit}} \ - \ 1 \ .$$

Secondly,

$$\frac{|\, n \ - \ n_{\text{crit}} \,|}{n_{\text{crit}}} \ = \ |\, 1 \ - \ \rho_{\text{crit}} \,| \ .$$

Fig 8.5.b) reorganizes this using relative subset number.

Remark 3 In modelling for which the inclusion–exclusion distinction is meaningless, then inclusion–exclusion symmetry applies. And so one is to pass to the arena quotiented by this. The configurations for this are given in Fig 8.5.c) in the sparse presentation or Fig 8.5.d) in the belonging-symmetric presentation.

Naming Remark 1 $\mathcal{T}_{\text{nest}}$ exhibits *nesting*: each smaller subset lies inside the next largest. 1 1 denotes the smallest nontrivial partition in the sense of being into more than one piece. The τ are explained in Secs 9.5 and 9.6.

8.11 Families based on the 3-element set

These are tabulated in Fig 8.6.

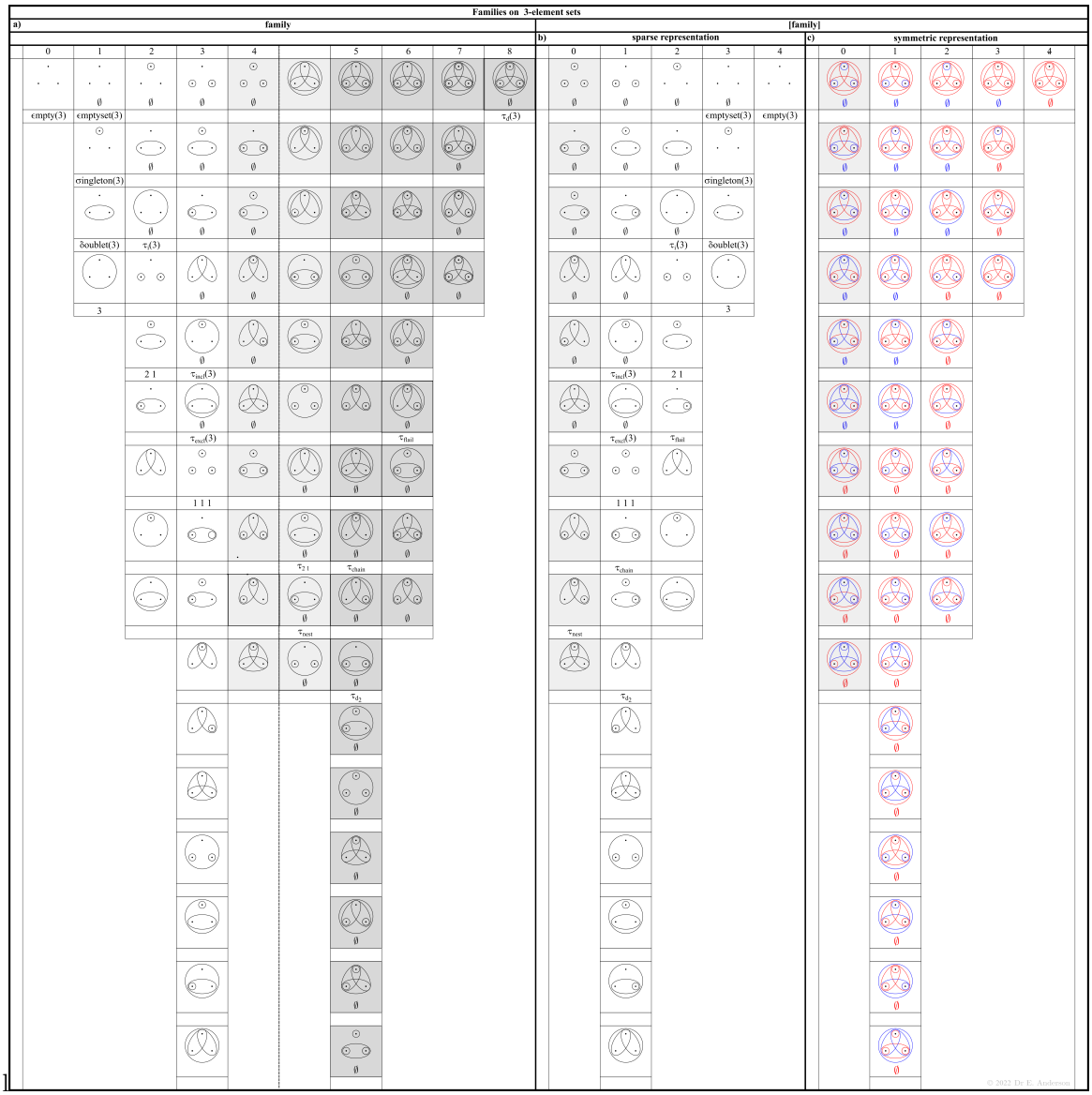


Figure 8.6:

Naming Remark 2 δoublet is the 'include two elements counterpart of singleton's 'include one element'. 3, 2 1, 1 1 1 are further partitions (a notation explained and improved upon in Sec 15.1). The new τ are also explained in Sec 9.6.

Chapter 9

Covers and topologies



9.1 Covers

Definition 1 A *cover* is a family – a set of subsets – whose union is the whole underlying set. I.e.

$$\{ \mathfrak{U}_i \subseteq \mathfrak{X} \}_{i \in \mathfrak{I}}$$

such that the *covering condition*

$$\bigcup_{i \in \mathfrak{I}} \mathfrak{U}_i = \mathfrak{X} \tag{9.1}$$

holds.

Definition 2 The collection comprising of all possible covers on a given set $\mathfrak{X}$ is the *arena of covers*,

$$\mathfrak{C}\text{over}(\mathfrak{X}) \ .$$

9.2 Smallest examples of covers

Examples 0-2 The covers on the sets with 0 to 2 elements are shaded orange in Fig 9.1.

Definition 1 The *indiscrete cover* consists of just the universe set $\mathfrak{X}$.

Definition 2 The *discrete cover* consists of all the singletons supported by $\mathfrak{X}$.

Remark 1 These coincide for $N = 0, 1$ only.

Remark 2 $N = 2$ furthermore supports covers which are neither of these (Fig 9.1).

Remark 3 The complement of a cover is not in general a cover. The minimal example for this is the 1-element set's singleton, whose complement is the empty set. The smallest counterexample consists of the 2-element set's pair of singletons, whose complement is the 2-element universe alongside the empty set.

Exercise 1 Find all the covers of the 3-element set.

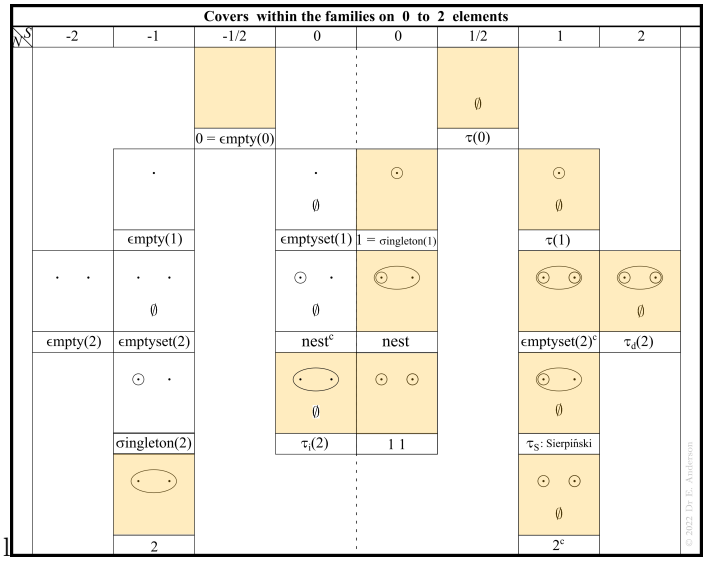


Figure 9.1:

[For subsequent use, the answer is provided overleaf, in Fig 9.2.]

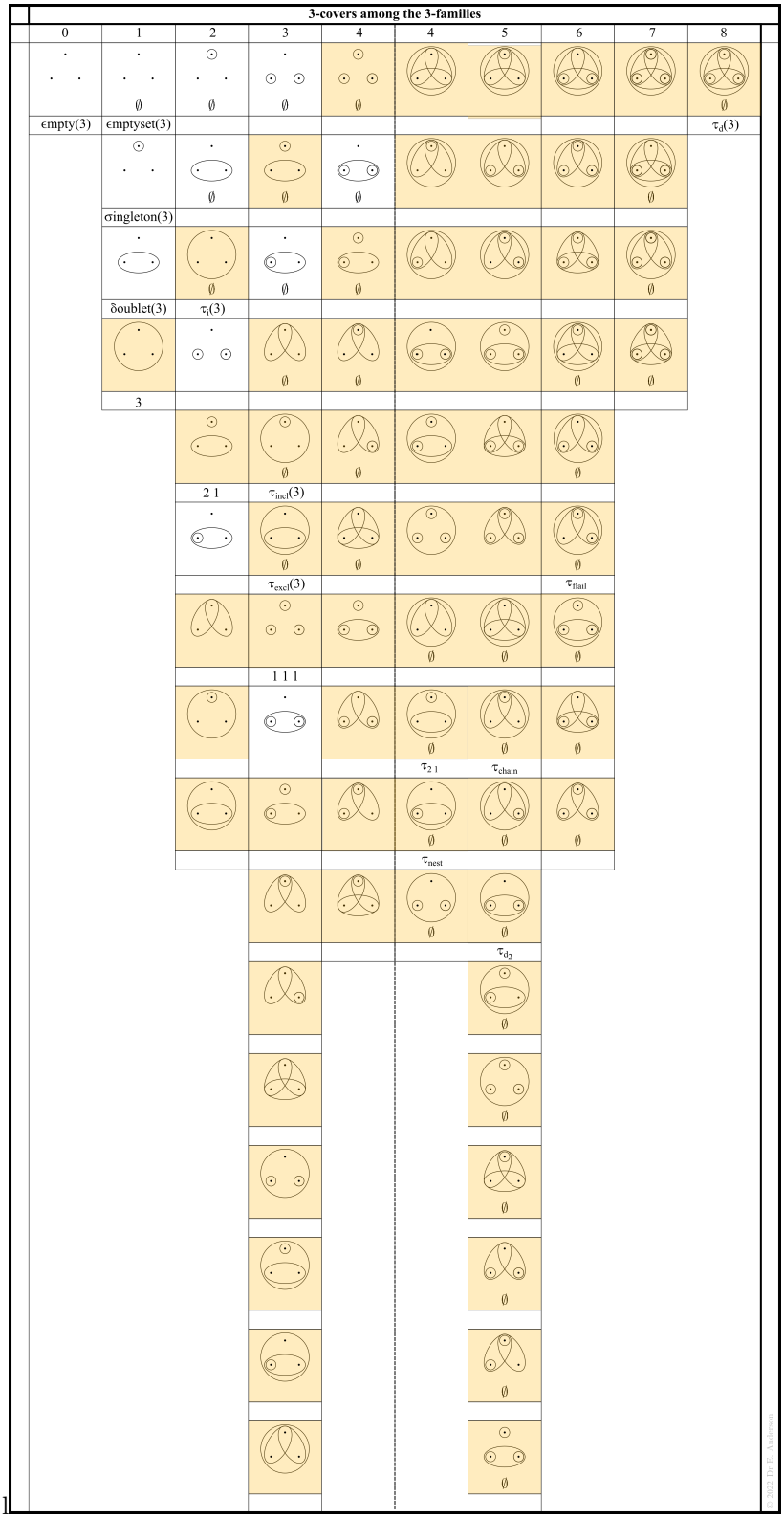

3-covers among the 3-families
0
1
2
3
4
4
5
6
7
8
∅
ϵmpty(3)
ϵmptyset(3)
$\tau_d(3)$
σingleton(3)
δoublet(3)
$\tau_1(3)$
3
2 1
$\tau_{incl}(3)$
$\tau_{excl}(3)$
τ_{flail}
1 1 1
$\tau_{2\,1}$
τ_{chain}
τ_{nest}
τ_{d_2}

Figure 9.2:

9.3 Subcovers and refinements

Definition 1 A *subcover* $\mathfrak{S}$ of a given cover $\mathfrak{C}$ is a subfamily thereof which is itself an open cover.

Definition 2 An open cover

$$\mathfrak{R} \ = \ \{\mathfrak{P}_k\}_{k \in \mathfrak{K}}$$

is a *refinement* of a given cover

$$\mathfrak{C} \ = \ \{\mathfrak{U}_i\}_{i \in \mathfrak{I}}$$

if the following holds.

$$\forall \ \mathfrak{V}_k \ \exists \ \mathfrak{U}_i \ \text{ such that } \ \mathfrak{V}_k \ \subset \ \mathfrak{U}_i \ .$$

Exercise 2 Show that subcover $\Rightarrow$ refinement but not vice versa.

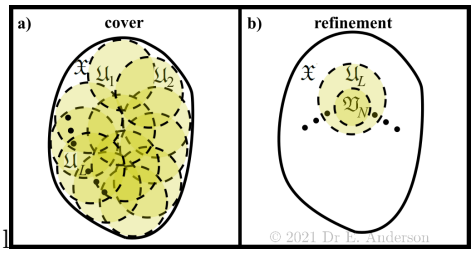


Figure 9.3:

9.4 Cover numbers

This is more relevant for covers of a given type, since if not the whole universe set is always available as a single-subset cover.

Definition 1 The *cover number* is the number of subsets in a cover's family.

Remark 1 For a finite set, this can be normalized by $|\mathfrak{X}| \ = \ N$, or $|\mathfrak{P}(\mathfrak{X})| \ = \ 2^{|\mathfrak{X}|}$. Such can more readily be used to compare between finite sets of different cardinalities.

Pointer 1 There are more subtle notions of cover number for the metric spaces that are introduced in Appendix B. This notion of cover number bounds the sizes of the subsets involved. In this regard, Readers who are already familiar with Analysis on metric spaces can look up *Lebesgue number* ($\mathbb{R}_+$-valued), and the associated useful Lemma usually named after Lebesgue, in [260]



9.5 Topologies

Definition 1 Topologies (in the sense of topological spaces) can be taken to be families τ obeying the following three axioms.

i) The universe set and the empty set belong to this family:

$$\mathfrak{X}, \emptyset \ \in \ \tau \ .$$

ii) Whichever finite intersections of subsets in the family also belong to the family.

$$\mathfrak{a}_i \in \tau \quad \Rightarrow \quad \bigcap_{i \in \mathfrak{I}} \mathfrak{a}_i \in \tau \ .$$

iii) Whichever arbitrary union of subsets in the family also belongs to the family.

$$\mathfrak{a}_\iota \in \tau \quad \Rightarrow \quad \bigcup_{\iota \in \mathfrak{I}} \mathfrak{a}_\iota \in \tau \ .$$

Naming Remark 3 These subsets are called *open sets*; these carry further Analysis connotations [268, 449]. The above-mentioned 'open intervals' are a simple subcase of open sets.

Remark 1 By 'can be taken to be', we mean that there are other equivalent conceptualizations of topological space axioms [260, 449], the above being the open-sets such.

Remark 2 For finite sets, the number of distinct subsets is also finite and thus there is no difference between finite and arbitrary unions. The topological space axioms for unions and intersections are thus 'symmetrized' in this special case (which the rest of this Volume concentrates on).

9.6 Examples of topologies

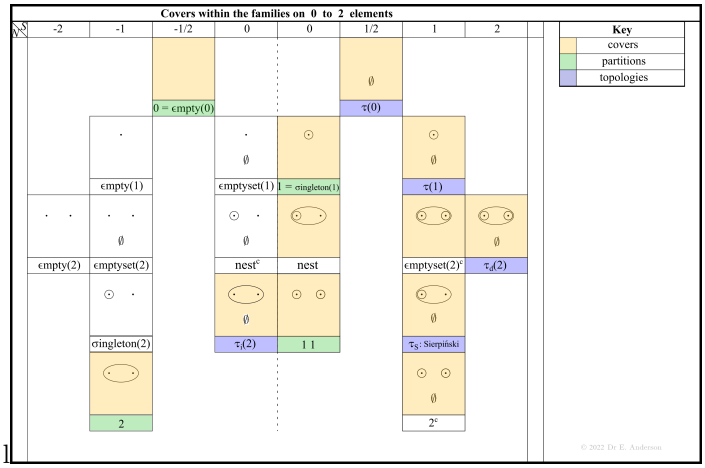


Figure 9.4:

Examples 0-2 The smallest examples are marked in blue in Figs 9.4 and 9.5. In particular, there is only one topology on each of the 0-element and 1-element sets.

Example 3 The *indiscrete topology* $\mathcal{T}_{\mathrm{i}}$ consists of just $\mathfrak{X}$ and $\emptyset$.

Example 4 The *discrete topology* $\mathcal{T}_{\mathrm{d}}$ consists of the entire set of subsets $\mathfrak{S}\mathrm{ubset}(\mathfrak{X})$.

Remark 1 We first observe a distinction between the discrete and indiscrete topologies for $N = 2$. This also has the first topology in excess of this: the Sierpiński topology, $\mathcal{T}_{\mathrm{S}}$ (Fig 9.4).

Exercise 3 [Long] a) Show that everything marked in blue on Figs 9.4 and 9.5 are topologies.

b) Prove that the above are all the topologies on $N \leq 3$ unlabelled-element sets.

c)$^+$ Find all the topologies on the 4-element set.

c) Prove that being a topology is not in general preserved by family complementation.

Naming Remark 4 Among $N = 3$'s middling topologies, the name $\mathcal{T}_{2\,|\,1}$ is self-explanatory. Modulo this referring to the $\mathfrak{X}, \emptyset$ copy's shadow of the actual partition family (marked in green on Figs 9.4 and 9.5). $\mathcal{T}_{\text{incl}}$ and $\mathcal{T}_{\text{incl}}$ are widespreadly well-named as the *included-point* and *excluded point* topologies. Every $N \geq 3$ has one of each of these. The name 'nest' is also clear, and somewhat widely used. $\mathcal{T}_{\text{d}_2}(3)$ is named for its discrete 2-topology subfamily. 'chain' and 'flail' are personal names; flail has one extra piece at one end of its chain subfamily.

Exercise 4 a)$^-$ Which further names do the above induce on the Sierpinski topology?

b) How many topologies on 4-elements does the above incipient naming system leave unamed?

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Example 5 On $\mathbb{R}$, the *standard topology* arises from letting the open intervals (a, b) [for $a < b$] determine the open sets.

Example 6 One way of extending Example 5's approach to $\mathbb{R}^n$ (including to $\mathbb{R}^2$ viewed a fortiotri as $\mathbb{C}$) is as follows. Use *open balls*

$$\mathbb{B}(\underline{x}, r) := \{ \underline{y} \in \mathbb{R}^n \,|\, ||\underline{y} - \underline{x}|| < r \} \tag{9.2}$$

to determine the open sets. Each such is centred about the point $\underline{x}$ and is of radius up to but not including r. It is straightforward to see that the open intervals are indeed recovered from this in the $n = 1$ case.

Example 7 Suppose that we have a list of subsets that are to be interpreted as open sets. Next adjoin any unions that these produce which are not already on our list (and $\mathfrak{X}$, $\emptyset$ if necessary). Then we have a topology provided that we can Families all finite intersections of our enlarged list are also on this enlarged list.

Example 8 If we additionally permit adjoining intersections to our list when these are not already on our list, then we are guaranteed to generate a topology from our list.

Remark 2 In more advanced texts [268, 260], Example 7's idea can be streamlined into 'generating a topology from a base' and Example 8's into 'generating a topology from a sub-base'. In Part V, we use Example 6's approach for graphs. In Part VII, Example 5's approach will turn out to suffice for posets, since the intersections in question are always guaranteed to be on the type of initial list used there.

Exercise 5 What topology on the 3-element set with elements labelled a, b, c is generated by the subsets $\{a\}$, $\{a, b\}$ and $\{b, c\}$? What is the smallest possible base for the discrete topology on 3 points, $\mathcal{T}_{\text{d}}(3)$? [A *base* is a family of subsets of a topological space such that every open subset in the topology can be expressed as the union of some of the base subsets.]

Remark 3 On the one hand, we now have a reasonable idea of what 'letting the open intervals (a, b) for $a < b$ determine the open sets' signifies.

On the other hand, we point out that there can be multiple ways of arriving at the same topology, which the current Volume does not attempt to explore. More generally, almost all topologies considered in the current Volume are 'discrete' (and indeed finite). In contradistinction to 'continuum' topologies (of which 5 and 6 above are Examples). Further diversity of, and intuition for, continuum

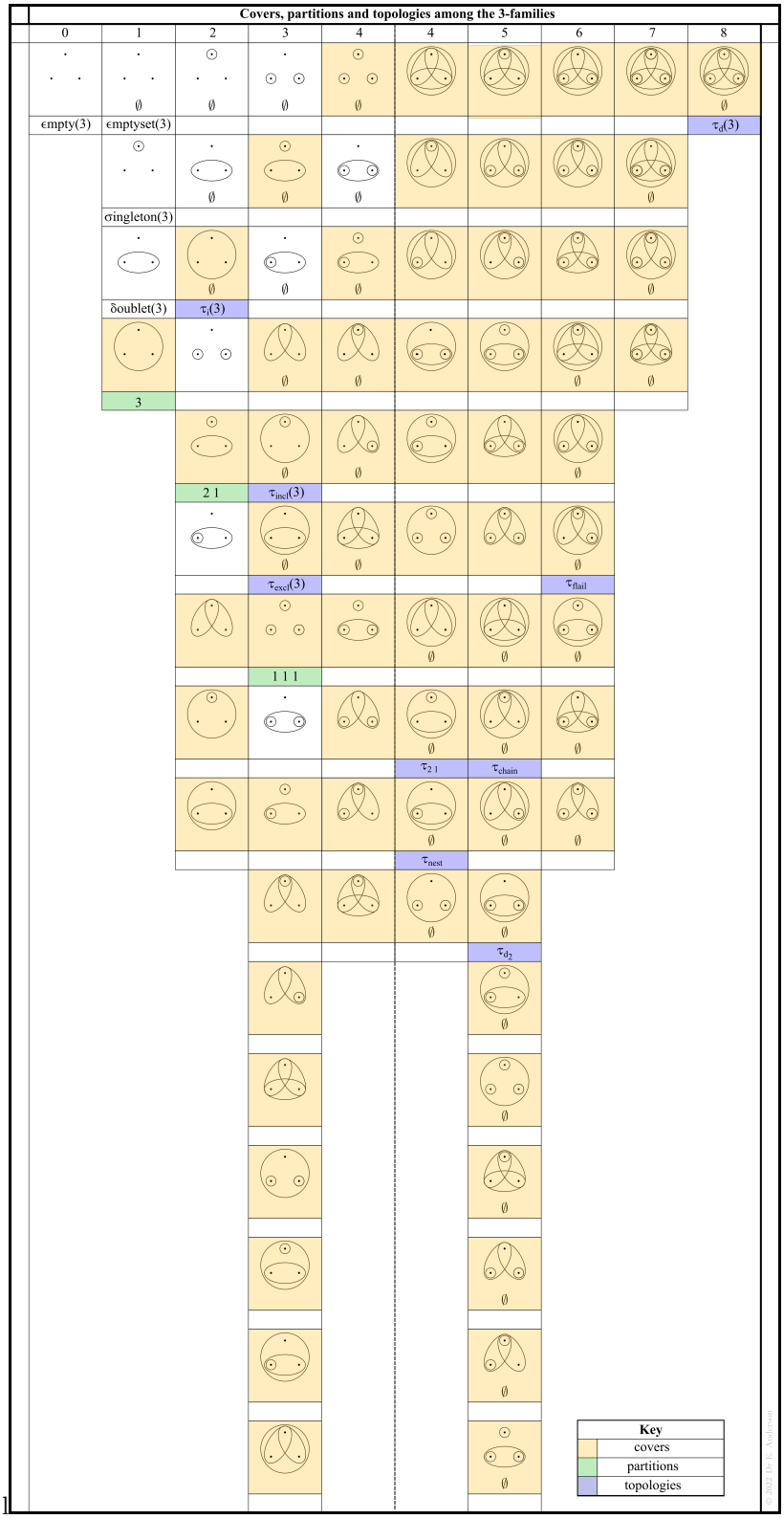

Covers, partitions and topologies among the 3-families
0
1
2
3
4
4
5
6
7
8
∅
ϵmpty(3)
ϵmptyset(3)
τ∅(3)
σingleton(3)
δoublet(3)
τ1(3)
3
2 1
τinel(3)
τexcl(3)
τflail
1 1 1
τ2 1
τchain
τnest
τd2
Key
covers
partitions
topologies

Figure 9.5:

topologies shall rather only start in Volume 6 of the current Series [445], and shall only mature in Volume 0 of the next Series [449].

9.7 Neighbourhoods

Definition 1 Let x be a point in a topological space $\mathfrak{X} = \langle \mathfrak{S}, \tau \rangle$. Then a subset $\mathfrak{N}_x$ is said to be a *neighbourhood* of x if the following holds.

$$x \in \mathfrak{N}_x \subseteq \mathfrak{U} \subseteq \mathfrak{X} \text{ for } \mathfrak{U} \text{ open in } \tau. \tag{9.3}$$

I.e. firstly, the point must belong to this neighbourhood. Secondly, this neighbourhood must be a subset of some subset $\mathfrak{U}$ that is open in the sense of belonging to the topology τ

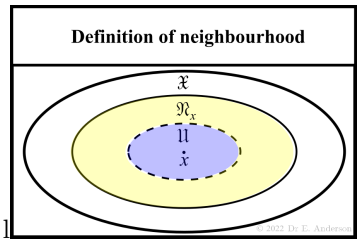


Figure 9.6:

Naming Remark 5 Some authors require $\mathfrak{N}_x$ itself to be open, so the Reader should always check conventions in each text consulted that makes mention of neighbourhoods. In these Series, we however just term such *open neighbourhoods*.

Naming Remark 6 Indeed, on occasion we also talk about *closed neighbourhoods*. Closed carrying distinct Analysis connotations – including all of its limit points – to open, with closed intervals being an aforementioned example. Open and closed turn out moreover to be well-named as opposite notions that form a further dual pair.

Naming Remark 7 We also occasionally mention *punctured neighbourhoods*. In which, for technical reasons the incipient point is itself deleted from the neighbourhood in a manner perhaps most familiar from the punctured version of the complex plane, $\mathbb{C}_*$.

9.8 Motivation for topological spaces

Motivation 1 Topological spaces provide the most widely useful abstraction of Analysis. With such as notions of connectedness generalizing the Intermediate Value Theorem [402] and compactness generalizing the Heine–Borel [260] and Bolzano–Weierstrass Theorems [402].

Motivation 2 Topological spaces provide a useful conceptual analysis for each of graphs and posets. We cover this in Chapters 41 35, 37, 45, 47, and 50.

Naming Remark 8 Topology means the study – λογος – of τοπος: Greek for 'place' or 'location'. This name was coined by Listing, somewhat before he wrote the world's first book on the subject [235] in 1847. Topology had previously been considered under the names of *Geometria Situs* – coined in its posing by Leibniz [233] in the 17th century, whose first significant result came in work by Euler [234] in the 18th century – and *Analysis Situs*. These names mean, respectively the Geometry, or Analysis, of 'position'; the English 'site, and 'situated at' are closely related to the Latin 'situs'. Poincaré continued to use the name *Analysis Situs* in his subject-forming work [236]. Talking about place has some connotations of further generality and abstraction than talking about position. Not

referring to Topology as a type of Geometry or Analysis emphasizes furthermore that it is a subject in its own right.

Motivation 3 Topology proper is the study of continuous functions. Poincaré [236] developed Topology so as to improve humanity's understanding of Dynamics. This led to qualitative analysis of Dynamical Systems and of stability [384, 400]. Topological spaces were themselves put forward by Hausdorff [237] as the natural home for Topological notions.

Motivation 4 Topology also already provides a much more unified understanding of a large portion of Complex Analysis [263].

5) *Differential Geometry* [326, 335, 332, 317, 327] can be viewed correspondingly as the study of differentiable functions, which are a useful subcase of the continuous functions; this benefits enormously from tame topologial properties.

Structure 6 Topology is useful in understanding the large-scale structure of an observed space with far too many details to model with precision. Such as large data sets. The Internet. Signalling, or moving robots, across urban landscapes. Or indeed, the physical Universe that we live in.

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Motivations 6-9) Topology is useful for understanding partial differential equations (PDEs) on spaces which support them. And in Functional Analysis [373] underlying each of this, the theory of integrability (though this uses a further abstraction of Analysis called Measure theory [417]) and in advanced formulations of Quantum Physics.

Motivation 10 Topology is essential in considering arenas.

9.9 Motivation for covers

Motivation 1 Chapter 15's widely useful partitions can be viewed as a more structured version of covers: the disjoint-union rather than just plain union version.

Motivation 2 Various simple notions of cover appear in Graph Theory (see Chapters 37 and 48), and in Order Theory (see Chapters 49 and 50).

Motivation 3 More notably, in Topology, covers and subcovers enter the definition of the majorly significant notion of *compactness* [268, 260, 270]. Specifically, a space is *compact* if every cover admits a finite subcover (Fig 9.7.a).

Motivation 4 Covers and refinements enter the definition of another rather significant topological property: paracompactness [260, 270, 243]. Specifically, a space $\mathfrak{X}$ is *paracompact* if every open cover admits a locally-finite refinement.

'Locally-finite' here means that each point $x \in \mathfrak{X}$ possesses an open neighbourhood $\mathfrak{N}_x$ such that the following holds. Only finitely many of the $\mathfrak{R}_k$ in our refinement $\mathfrak{R}$ overlap with $\mathfrak{N}_x$, i.e. have

$$\mathfrak{N}_x \cap \mathfrak{R}_k \neq \emptyset$$

(Fig 9.7.b).

Remark 1 Compactness and paracompactness are mostly useful in continuum(-like') rather than 'discrete' models. So we shall largely not mention them again until Volume 6 [445]. Nor are they wielded for their enormous capacity for proving results until Volume 0 of Series 1 [449].

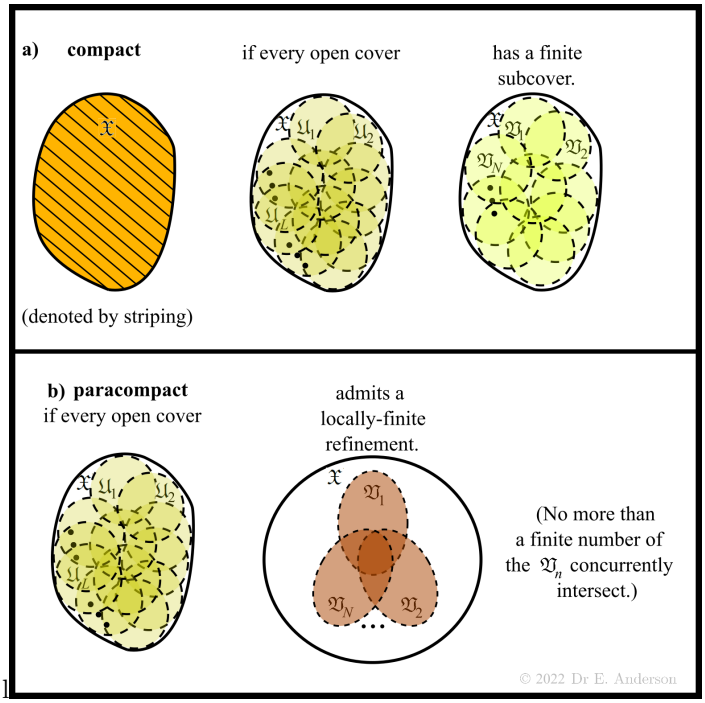


Figure 9.7:

Motivation 5 Topology also makes ample use of *covering spaces* [260, 270] and *universal covers* [270], which are indeed cover concepts in the current Chapter's sense.

Chapter 10

Equipped sets

10.1 *Equip* and *Forget*

Notation 1 Let $\langle\, \mathfrak{X}, \boldsymbol{\sigma} \,\rangle$ be a set $\mathfrak{X}$ as further equipped with levels of mathematical structure, $\boldsymbol{\sigma}$.

Definition 1 *Equip* is the map equipping the bare set $\mathfrak{X}$ with the levels of structure $\boldsymbol{\sigma}$. The reverse map is called a *forgetful map*, $Forget$, which involves forgetting these levels of structure (Fig 10.1.a).

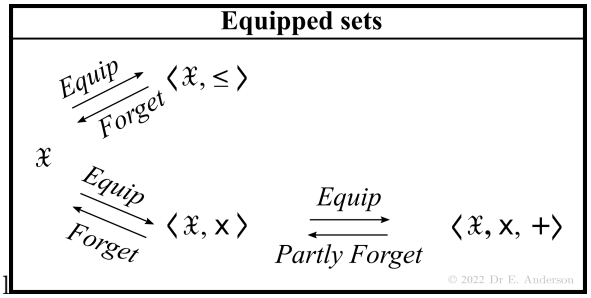


Figure 10.1:

Remark 1 When levels of structure are sufficiently independent of each other, partial structuralization is possible in stages (Fig 10.1). In reverse, we can then partially forget as well.

10.2 Examples of levels of structure

For instance, we could equip our set with each of the following.

Example 0 Labelling structure. Labels, whether fully or partly distinguishable, and whether meaningful or meaningless, are a level of structure. We shall see in Part III that the diversity of labelling is group-valued.

Example 1 A *binary operation*, which might or might not upgrade our set to a *group* (Parts III and IV).

Example 2 Two binary operations, which might furthermore satisfy the field axioms or the vector space axioms (Volume 1).

Example 3 Some of our set's elements could be inter-related by edges, turning our set into a graph (Parts V and VI).

Example 4 We could give our set an ordering relation, turning it into such as a chain or a poset (Part VII). We could then furthermore equip it with two binary operations that might or might not upgrade our poset to a lattice.

Example 5 We could give our set a topological level of structure. I.e. a distinguished family of subsets obeying Sec 9.5's axioms; see [268, 250] and Volume 0 of Series 1 for further details. Among these, those carrying the metric level of structure outlined in Appendix B are an interesting and frequently-encountered subcase.

Example 6 In some cases, we could follow up giving our set a topological level of structure by giving it a geometrical level of structure as well (Books 3 and 5).

Example 7 Labels can themselves be more structured. For instance, these could themselves be groups [99] . This however lies outside of the scope of the current Volume.

10.3 Further examples of homo-, iso-, endo- and auto- morphisms

Example 0 Labelling a set's elements is capable of changing what symmetries it has, i.e. what its automorphism group is. This will be among the examples of symmetry breaking considered in Part III, as well as of counting distinguishable objects in Part IV.

Example 1 Group isomorphisms and homomorphisms are the subject of Secs 19.7-19.8 and Chapter 27 Groups' own automorphism groups turn out to be a more complicated matter, which we postpone until Volume 5.

Example 2 Vector spaces' morphisms are *linear maps*, which may be more familiar to the Reader when represented by *matrices*. All of homomorphisms, isomorphisms and endomorphisms for these are covered in detail in Volume 1.

Example 3 Graph homomorphisms and isomorphisms are covered in Secs 29.6 and 33.3 while graph automorphisms are considered in Exercises 29.5-6, 30.6, 33.8, 40.8.b), and 44.9.

Example 4 Homomorphisms and isomorphisms are considered for chains and posets in Sec 49.2 and lattices in Sec 51.6.

Example 5 Homeomorphisms are the main kind of isomorphism considered in Topology: continuous bijections with continuous inverses. See [246, 261] for introductions to other types of isomorphism of interest in Topology.

Example 6 In order to talk about morphisms in Geometry, we need to specify first more precisely which kind of Geometry we are talking about. In Euclidean Geometry in flat space isomorphisms are quite often taken to be *isometries*, though for some purposes they can furthermore be taken to be *similarities*. See Secs 19.6 and 21.4 for what each of these are. In much more general Differential Geometry, the isomorphisms are diffeomorphisms: differentiable bijections with differentiable inverses.[1]

[1] Some applications require particular strengths of differentiability, such as doubly-continuously-differentiable or infinitely-differentiable, alias smooth; see Books 1 and 5.

10.4 How much of Mathematics does Equipped Sets cover?

Remark 1 Equipped sets is a Foundational System of Mathematics.

Remark 2 It is not a unique such.

Alternative 1 and Pointer 1 Some texts use Category Theory [227] instead. Here, one uses (object, morphism) pairs, called *categories*. Envisaging morphisms as arrows between objects, the duality underlying the prefix 'co-' then reverses these arrows. We will say a bit more about categories in Sec 22.5 after having taken a look at the morphism-centric approach to Geometry in Part III.

Pointer 2 Sec 2.3's restriction maps provide the infrastructure for the very widely applicable *sheaf* construct [336].

Alternative 2 and Pointer 3 *Topos Theory* [225] offers a more radical if also considerably more mathematically-advanced alternative. We briefly comment on this in Sec 52.12 and in the Conclusion.

Remark 3 Equipped Sets is however the Foundational System of Mathematics used in the current Pre-Topology Series. Within Topology itself, Algebraic Topology in particular has considerable need of Category Theory.



10.5 Equivalence relations partition sets into equivalence classes, which form the corresponding quotient sets

Structure 1 Suppose that we are given a set $\mathfrak{X}$ on whose elements an equivalence relation $\mathcal{R}$ is defined. Then $\mathfrak{X}$ is partitioned by the equivalence classes of $\mathcal{R}$.

Structure 2 The space whose elements are these equivalence classes is called the *quotient set* (Fig 10.2.a).

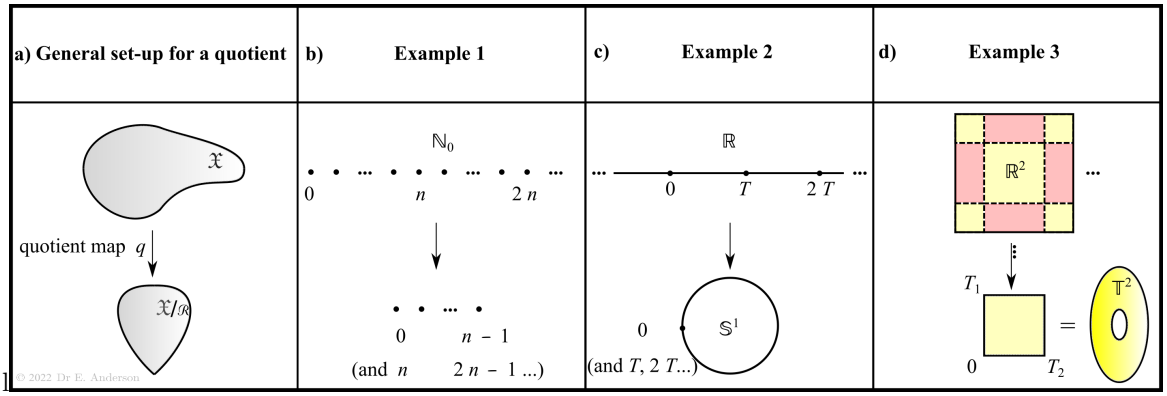


Figure 10.2:

Worked Example 1 $a \cong b \pmod{n}$ acting on $\mathbb{N}_0$ has n equivalence classes $[0]$ to $[n-1]$: the remainders (alias residues) $\pmod{n}$. The quotient space that these form (Fig 10.2.b) is the interval of integers $[0, n-1]$.

Worked Example 2 Suppose that the real numbers are periodically identified with period T. Then the equivalence classes of $\mathbb{R}$ are the values of the real interval $[0, T)$. Which is itself most conveniently represented, as a quotient space, by the circle $\mathbb{S}^1$ (Fig 10.2.c). This entails identifying the end-points of the interval $[0, T]$. By which $[0, T)$ is itself represented 1 to 1.

Worked Example 3 Suppose that the real plane $\mathbb{R}^2$ is periodically identified in two perpendicular directions with periods T_1 and T_2. Then the equivalence classes of $\mathbb{R}$ are the values in the product of real intervals $[0, T_1) \times [0, T_2)$. The identification in question is that marked in Fig 10.2.d). This is most conveniently represented by a torus, formed by gluing along the pairs of identified arrows in the direction that these indicate.

Exercise 1$^-$ Suppose that we identify $\mathbb{N}^2$ by identifying the first factor's numbers $\pmod{m}$ and the second's $\pmod{n}$. What are the equivalence classes, and what quotient space do they form?

Exercise 2 How else can opposite pairs of edges on the rectangle carry arrows? Which surfaces result from identifying each of these arrowed rectangles?

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Remark 1 On the one hand, Examples 1 and 3 involve partitions with no further structure. On the other hand, Examples 2 and 4 conventionally involve further topological and geometrical levels of structure. Indeed, applications of partitions often involve more structure than just the partitioning itself.

Remark 2 This is furthermore to be distinguished from more structured notions of *quotient itself*, of which Chapter 26.5's quotient group is this Volume's leading example.

Remark 3 Whenever we are given a property of a space, asking whether this property is quotientive – i.e. preserved by quotients – is always interesting.

Exercise 3 What are the equivalence classes in each of the above four examples of quotients?

10.6 The Cartesian combination of operations for obtaining new spaces from old

Hereditarity – that if a space has a property then all of its subspaces have the same property, – is an interesting question to pose of any type of space and any property thereof. So are the additivity, productivity and quotientivity properties. The study of these four questions (occasionally more) is thus a part of the study of any kind of mathematical objects. From Part V of Volume 0 onward, we always ask these four questions side by side under the title of 'new spaces from old' [243, 260, 270].

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10.7 Arenize

Remark 1 The previous two sections could be described as 'Pre-Category Theory'. The arenization map *Arenize* is rather tougher to study than this, however. It looks to be much less predictable, and is so far largely unexplored, let alone understood.

Structure 1, with Examples 1 to 4 As a map, *Arenize* sends each object (in general nontrivially-structured) to a single point in the arena of such objects. Fig 10.3 gives a Wheeler-type [353] depiction for the first step of the Arenization program, for what few cases the current Volume has presented so far. The context in which Wheeler put forward this depiction was rather more structured: the configuration space of General Relativity! Yet it is a universally-applicable idea. And thus does well to be presented for simpler examples first. Subfigure b)'s for subsets of a fixed finite set, and Subfigure a)'s for the arena of combinations $\mathfrak{C}$ombination interpretation of Pascal's triangle being the current Volume's simplest two examples.

Aside 1 Contrast the following two *notions of belonging*. On the one hand, $\in$ takes us from the objects to the space that the objects live in: an arenization move. On the other hand, 'is a subset of' $\subseteq$ is in general a binary relation on subsets, i.e. on elements of the power set: internal to the arena. However, its alternative use as 'is a subset of a given universe' is an arenization move. This involves passing from treating subsets as objects to considering the space of all such possible objects within a given universe.

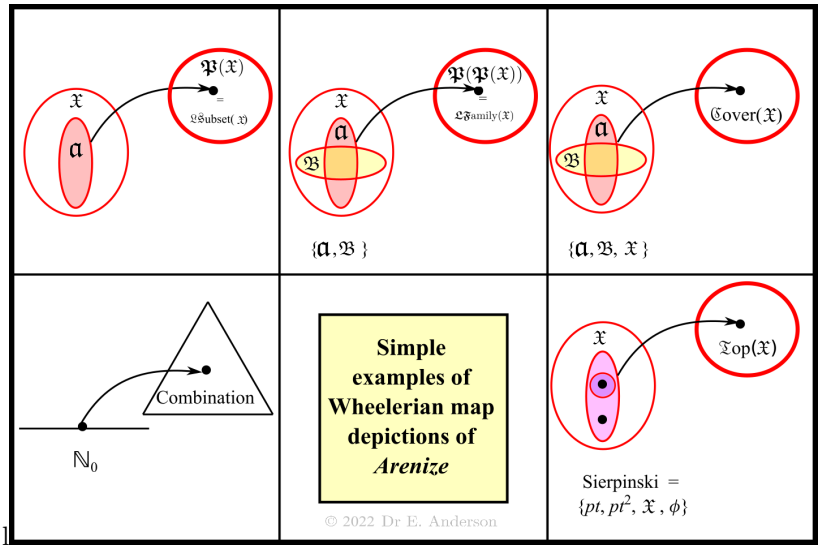


Figure 10.3:

Structure 2 The second step is to ask, given any specific example of objects or structures, what mathematical space the corresponding arena is.

Examples 1 to 4 revisited For Fig 10.3's simple examples, Part VIII's first subpart works this out, with posets, lattices and their underlying graph skeletons by and large sufficing. The main observation to make from performing this second step is as follows. The arena of some specific objects or structures *has no a priori reason* to have the same levels of mathematical structure as its constituent objects or structures do. As a further simple example, the arena of triangles up to similarity turns out to be a sphere [370, 396, 434]. In the current Series, we work this out in Volume 3 [443]; it also doubles as a 3-Body Problem Dynamics configuration space [392, 414, 422].

More generally, simple objects or structures' arenas tend to be either order structures (Parts VII and VIII) or geometries (Books 3 and 5). Given some area of STEM's objects or structures, however, it is moreover always a valid and revealing question to ask which *topology* their arena naturally realizes.

Open Question 1 It is nonetheless an interesting question to ask which mathematical structures are fixed points of the arenization map *Arenize*. Meaning that their arena is a space with the same levels of mathematical structure as the objects themselves. One reason that this is interesting is that, at least from the study of simple examples so far, such would appear to be rare. In the current Volume, we shall Families chains and posets can be interpreted as fixed points of *Arenize*. While the traditional example of fixed point of *Arenize* is that Gromov [324] showed that the arena of metric spaces is itself a metric space; see Appendix B.

Open Questions 2 and 3 It might however be that fixed points of *Arenize* are non-generic. *Arenize* could even be a *wild map* in the sense of generically not exhibiting any patterns for which level of mathematical structure map to which others. It is far too early to determine whether these hypotheses hold.

10.8 Quotient arenas

Structure 1 Many of the most interesting arenas that have been studied so far happen to be quotients. It turns out to be particularly useful to view these as quotients by a group. Which in general requires us to develop Parts III and IV on groups first. We then return to discussion of quotient arenas in Part VIII, Parts V and VI's graphs and Part VII's posets and lattices also being necessary to handle the specifics of our simple examples.

Remark 1 And yet quite a few of our examples involve quotienting out just the simplest nontrivial group – the unique 2-element group, C_2. So we can for now give the following simple examples of quotient arenas.

Example 1 Quotienting the arena of combinations $\mathfrak{C}$ombination by reflection about the midline of the number of chosen objects, k (Fig 1.d).

Example 2 Quotienting the arena of subsets $\mathfrak{S}$ubset by subset complementation [Fig 8.2.b) in the unlabelled case].

Example 3 Quotienting the arena of families $\mathfrak{F}$amily by family complementation [Figs 8.5.b) and 8.6.a) in the unlabelled case).

Remark 2 Both of these operations cease to make sense for covers partitions, or topologies, however. For they do not preserve the definition of cover, partition or topology. By which applying them can kick us out of the cover or topology subarena of $\mathfrak{F}$amily. For family complementation, this is manifest from the asymmetry in Figs 26.12 and 26.13's heat maps.

Exercise 4 Show that taking the set-complements of the subsets in a) a cover does not necessarily produce a cover, and in b) a topology does not necessarily produce a topology.

Part II

Combinatorial counts, and countability

Chapter 11

Pigeonhole principles

11.1 Introduction

One occasionally encounters problems that are immediately solvable by the following innocent-looking idea.

Proposition 1 (Pigeonhole Principle) i) Suppose that $p + 1$ objects are placed in $p \in \mathbb{N}$ pigeonholes. Then at least one pigeonhole must contain more than one object.

ii) Suppose that we put

$$o = p\,q + 1$$

objects in p pigeonholes, for $p, q \in \mathbb{N}$. Then at least one pigeonhole must contain more than q objects.

Proof ii) Suppose that each pigeonhole contains $\leq q$ objects. So the totality of pigeonholes contains $\leq p\,q$ objects. Contradiction!

i) then follows as the $q = 1$ subcase. □

Remark 1 Some of the problems solved by this are (somewhat) contrived. Yet the Pigeonhole Problem has various fundamental applications: Ex 2, Pointers 2 to 6 ...

Naming Remark 1 Dirichlet called it a 'drawer principle'. Pigeonholes refer to large arrays of compartments, usually arranged on a wall, for receiving or sorting mail. Since this is becoming an increasingly unfamiliar practice, some depictions of it have started to involve actual dove-cotes with birds in them. The boxes and balls, standardly used in discussion of selection or sampling, will also do. Though Pigeonhole Principles are a specific family of fast counting principles, that does well not to be confused with further intricacies of selection problems (Chapters 13 and 16).

11.2 Exercises

Exercise 1 For a set of objects a, show that

$$a_{\min} \leq \langle\, a \,\rangle \leq a_{\max}\ .$$

Exercise 2 A party host wishes to further build up a community by ensuring that either at least 3 guests already all know each other or least 3 guests do not already know each other in any combination. What is the minimum number of guests that they should invite to their parties?

[This is the start of the magnificent theory of structure alluded to in Sec 0.1.]

11.3 Floor and ceiling functions

Definition 1 [39] The *floor function*

$$\lfloor\ \rfloor\ :\ \mathbb{R} \Longrightarrow \mathbb{Z}$$
$$x \Longrightarrow \left(\text{the greatest integer}\quad \leq\ x\right) .$$

Definition 2 [39] The *ceiling function*

$$\lceil\ \rceil\ :\ \mathbb{R} \Longrightarrow \mathbb{Z}$$
$$x \Longrightarrow \left(\text{the least integer}\quad \geq\ x\right) .$$

Remark 1 The *integer part* of $x\ \in\ \mathbb{R}$ is $\lfloor x \rfloor$ for $x\ <\ 0$ but $\lceil\ \rceil$ for $x\ >\ 0$. Either of these suffice to include the final possibility of $x\ =\ 0$, as a subcase of

$$\lfloor n \rfloor\ =\ n\ =\ \lceil n \rceil\ \forall\ n\ \in\ \mathbb{N} .$$

Exercise 3^{-} Sketch $\lfloor\ \rfloor$, $\lceil\ \rceil$, and 'the integer part of' on the interval $[\,-3,\,3\,]$. Also show that all of these functions are idempotent.

Remark 2 Observe the *alternation* in these definitions: greatest $\leq$ in $\lfloor\ \rfloor$ to least $\geq$ in $\lceil\ \rceil$. This pair of notions are clearly opposite (and in fact dual).

Exercise 4 Encode the remaining *fractional part* of $x\ \in\ \mathbb{R}$ by defining a further function. Sketch this function over the interval $[\,-3,\,3\,]$.

Naming-and-Notational Remark 2 Legendre [14] was the first known user, under the name of '*entire part of* function (a Truer Name being '*integer part of* function). The modern notation $\lfloor\ \rfloor$ and $\lceil\ \rceil$ is due to Computer Scientist Kenneth Iverson [28]. In between these times, floor and ceiling were generally not given separate notations, with Gauss's $[x]$ being a notation for 'integer part of' itself. $floor(x)$ and $ceil(x)$ are also acceptable. Though Iverson's notation uniquely pairs the two, while this might not be obvious for *floor* and *ceil* in a text containing a lot of 'word function notation'.

Exercise 5 Phrase the Pigeonhole Principle for $o,\ p,\ q\ \in\ \mathbb{R}_{+}$ using firstly the floor function, and secondly the ceiling function.

11.4 Pointers

Pointer 1 Various subsequent Chapters make use of the floor and ceiling functions as well, on occasion hidden in answers to Exercises.

Pointer 2 Basic Number Theory uses [51] Dirichlet's Pigeonhole Principle.

Exercise 6 Reformulate the Pigeonhole Principle in terms of injections' domain and co-domain's relative sizes.

Pointer 3 Exercise 2 is often presented as a first toy model of Ramsey Theory. Here partitioning a large structured space into two parts guarantees that one of the parts remains structured. The Reader might consequently like to try the corresponding problem for 4 in place of 3, and for one 4 and one 3 (**Exercise 7**$^{+}$).

Pointer 4 Pigeonhole principles are useful in studying pattern-avoiding permutations [76, 58] (another more advanced Combinatorics topic).

Pointer 5 Volume 3 [443] includes various examples of Geometrical Pigeonhole Principle.

Chapter 12

Principle of Inclusion and Exclusion (PIE)

12.1 Introduction

This [67, 49, 16, 42, 29] is a counting principle. It generalizes Sec 1.2's

$$|\mathfrak{A} \cup \mathfrak{B}| = |\mathfrak{A}| + |\mathfrak{B}| - |\mathfrak{A} \cap \mathfrak{B}|$$

to larger numbers of subsets.
The answer to Ex 7.ii) of Sec 1.2 is the first case of thereof. This takes the form

$$|\mathfrak{A} \cup \mathfrak{B} \cup \mathfrak{C}| = |\mathfrak{A}| + |\mathfrak{B}| + |\mathfrak{C}| - |\mathfrak{A} \cap \mathfrak{B}| - |\mathfrak{B} \cap \mathfrak{C}| - |\mathfrak{C} \cap \mathfrak{A}| + |\mathfrak{A} \cap \mathfrak{B} \cap \mathfrak{C}| ;$$

observe the sign alterntion.

Theorem 1 (PIE) Let $\mathfrak{A}_i \in \mathfrak{P}(\mathfrak{X})$ for $i = 1$ to n,

$$\mathfrak{J} \subseteq [1, n]$$

and

$$\mathfrak{A}_\mathfrak{J} = \bigcap_{i \in \mathfrak{J}} \mathfrak{A}_i = \{ x \in \mathfrak{X} \,|\, x \in \mathfrak{A}_i \ \forall \ i \in \mathfrak{J} \} .$$

Then

i)

$$\left| \bigcap_{i=1}^{n} \bar{\mathfrak{a}}_i \right| = \sum_{\mathfrak{J} \subseteq [1,n]} (-1)^{|\mathfrak{J}|} |\mathfrak{A}_\mathfrak{J}| : \tag{12.1}$$

the *sieve form* of PIE.

ii)

$$\left| \bigcup_{i=1}^{n} \bar{\mathfrak{a}}_i \right| = \sum_{\emptyset \neq \mathfrak{J} \subseteq [1,n]} (-1)^{|\mathfrak{J}|-1} |\mathfrak{A}_\mathfrak{J}| : \tag{12.2}$$

the *union in terms of intersections form* of PIE.

Proof i) and ii) are equivalent by

$$\left| \bigcup_{i=1}^{n} \mathfrak{A}_i \right| = |\mathfrak{X}| - \left| \overline{\bigcup_{i=1}^{n} \mathfrak{A}_i} \right| = |\mathfrak{X}| - \left| \bigcap_{i=1}^{n} \overline{\mathfrak{A}}_i \right| .$$

Step 2 here uses De Morgan's Law.

The sieve version is given by the following 2-case exhaustion.

Case 1. For $x \neq$ any $\mathfrak{A}_i$, each side of gives 1.

Case 2. If not,

$$\mathfrak{Z}_i \;=\; \{\, i \;\in\; [1,\, n\,] \,|\, x \in \mathfrak{A}_i \,\} \;>\; 0$$

for at least some $\mathfrak{A}_i$. Then the RHS receives a contribution

$$\sum_{\mathfrak{W} \subseteq \mathfrak{Z}} (-1)^{|\,\mathfrak{W}\,|} \;=\; \sum_{k \leq |\,\mathfrak{K}_i\,|} \left(\begin{array}{c} |\,\mathfrak{K}_i\,| \\ k \end{array} \right) (-1)^k \;=\; (\,1 \;-\; 1\,)^{|\,\mathfrak{K}_i\,|} \;=\; 0\;.$$

Furthermore, the LHS clearly remains 0.

Naming Remark 1 The inclusion and exclusion in the name is referring to the effect of the alternating signs in the formula.

12.2 Exercises

Exercise 1 Construct an alternative proof that makes use of characteristic functions.

Exercise 2 ('Successive layers of a sieve') How many numbers $\leq$ 3000 are not divisible by all of 3, 5, 7 and 11?

Worked Example 1 (application to Euler's totient function). Show that for

$$P \;:=\; \prod_{r\,=\,1}^{R} {p_r}^{k_r}\;,$$

Euler's totient function returns

$$\varphi(P) \;=\; P \prod_{r\,=\,1}^{R} \left(\, 1 \;-\; \frac{1}{p_r} \right)\;.$$

Answer 1 Also let

$$\mathfrak{D}_i \;=\; \#\,(\,n \;\in\; \mathbb{N} \,|\, n \;<\; P\;,\quad p_i \,|\, n\,)\;.$$

Then

$$\varphi(P) \;=\; \left|\; \bigcap_{i\,=\,1}^{n} \bar{\mathfrak{D}}_i \right|\;.$$

Observe that for $\mathfrak{R} \;\subseteq\; [\,1,\, R\,]$,

$$|\,\mathfrak{D}_r\,| \;=\; \frac{P}{\prod_{r \in \mathfrak{R}} p_i}\;.$$

Feeding this into the PIE formula,

$$\varphi(P) \;=\; P \;-\; \sum_{r\,=\,1}^{R} \frac{P}{p_r} \;+\; \sum_{r\,=\,1}^{R} \sum_{s\,<\,r} \frac{P}{p_r\, p_s} \;-\; \ldots \;+\; (-1)^R \frac{P}{\prod_{r\,=\,1}^{R}\, p_r} \;=\; P \prod_{r\,=\,1}^{R} \left(\, 1 \;-\; \frac{1}{p_r} \right)\;.$$

Exercise 3 Deduce a second proof that φ is multiplicative for mutually-prime numbers.

12.3 Application to Hall's Pairs-Matching Theorem

Definition 1 Suppose that we are given a family of sets

$$\{ \mathfrak{X}_i \}_{i \in \mathfrak{I}}$$

Where the finite index i takes the values in

$$\mathfrak{I} \ = \ \{ 1, 2, ..., n \} \ .$$

Pick one element x_i from each $\mathfrak{X}_i$ as its representative. Suppose that these are distinct,

$$x_i \ \neq \ x_j \ \text{ for } \ i \ \neq \ j \ .$$

Then these are said to form a *system of distinct representatives (S.D.R).*

Naming Remark 2 This is alias a *transversal* or *cross-section* in bijection with the indexing set.[1] These names have a somewhat more geometrical and topological flavour, while S.D.R. is a truer name from the current combinatorial perspective.

Definition 2 Let

$$\mathfrak{X}(\mathfrak{K}) \ := \quad \cup \ _{k \ \in \ \mathfrak{K}} \mathfrak{X}_k$$

for any

$$\mathfrak{K} \ \subseteq \ \mathfrak{I} \ . \tag{12.3}$$

Remark 1 Immediately, if $\{ \mathfrak{X}_i \}_{i \in \mathfrak{I}}$ possesses a S.D.R., then

$$| \mathfrak{X}(\mathfrak{K}) | \ \geq \ | \mathfrak{K} | \ . \tag{12.4}$$

What is less obvious is that the converse must also hold.

Theorem 2 (Hall's Pairs-Matching) (12.3) possesses a S.D.R. iff (12.4) holds for every (12.3).

Proof

Assume that our condition holds. Remove elements from our sets until we reach a minimal family

$$\{ \mathfrak{M}_l \}_{l \in \mathfrak{L}}$$

for which removing any further element would invalidate our condition.

Claim: All the $\mathfrak{M}_l$ are singletons, and thus themselves constitute an S.D.R.

For suppose not. Then w.l.o.g. $\mathfrak{M}_l$ contains 2 distinct elements m, m'. Minimality then requires that $\exists \ \mathfrak{Z}_j$, $j \ = \ 1, 2$ such that

$$| \mathfrak{Z}_j | \ \geq \ | \mathfrak{Y}_j | \ =: \ \left| \left(\bigcup_{z \ \in \ \mathfrak{Z}_j} \mathfrak{M}_z \right) \ \cup \ (\mathfrak{M}_l \ - \ \{ x_j \}) \right| \ . \tag{12.5}$$

1) Summing these over $j \ = \ 1, 2$ then yields the following.

$$\sum_{j \ = \ 1}^{2} | \mathfrak{Z}_j | \ \geq \ \sum_{j \ = \ 1}^{2} | \mathfrak{Y}_j | \ = \ | \mathfrak{Y}_1 \ \cup \ \mathfrak{Y}_2 | \ + \ | \mathfrak{Y}_1 \ \cap \ \mathfrak{Y}_2 | \ .$$

[1] Some authors occasionally use these terms without the bijective requirement; we shall not do so.

The last expression here is by PIE.

But also, given the definition nested in (12.5), firstly,

$$\mathfrak{Y}_1 \cup \mathfrak{Y}_2 = \left(\bigcup_{z_1 \in \mathfrak{Z}_1} \mathfrak{M}_{z_1} \right) \cup (\mathfrak{M}_l - \{x_1\}) \cup \left(\bigcup_{z_1 \in \mathfrak{Z}_2} \mathfrak{M}_{z_2} \right) \cup (\mathfrak{M}_l - \{x_j\}) = \left(\bigcup_{w \in \mathfrak{Z}_1 \cup \mathfrak{Z}_2} \mathfrak{M}_{z_1} \right) \cup \mathfrak{M}_l \ .$$

Secondly,

$$\mathfrak{Y}_1 \cap \mathfrak{Y}_2 = \left(\bigcup_{z_1 \in \mathfrak{Z}_1} \mathfrak{M}_{z_1} \right) \cup (\mathfrak{M}_l - \{x_1\}) \cap \left(\bigcup_{z_1 \in \mathfrak{Z}_2} \mathfrak{M}_{z_2} \right) \cup (\mathfrak{M}_l - \{x_j\}) = \bigcup_{w \in \mathfrak{Z}_1 \cap \mathfrak{Z}_2} \mathfrak{M}_w \ ,$$

where we have used distributivity.

So the corresponding cardinalities obey, by use of (12.4), firstly,

$$|\mathfrak{Y}_1 \cup \mathfrak{Y}_2| \geq |\mathfrak{Z}_1 \cup \mathfrak{Z}_2| + 1 \ .$$

Secondly,

$$|\mathfrak{Y}_1 \cap \mathfrak{Y}_2| \geq |\mathfrak{Z}_1 \cap \mathfrak{Z}_2| \ .$$

2) So, adding,

$$|\mathfrak{Y}_1 \cup \mathfrak{Y}_2| + |\mathfrak{Y}_1 \cap \mathfrak{Y}_2| \geq |\mathfrak{Z}_1 \cup \mathfrak{Z}_2| + 1|\mathfrak{Z}_1 \cap \mathfrak{Z}_2| > |\mathfrak{Z}_1 \cup \mathfrak{Z}_2| + |\mathfrak{Z}_1 \cap \mathfrak{Z}_2| = \sum_{j=1}^{2} |\mathfrak{Z}_j| \ .$$

The second step here is a *throwaway inequality*, in which a positive term, here 1, is thrown away in order to obtain a strict inequality. While the final step once again evokes PIE.

But 1) and 2) form a contradiction! Thus our claim is established, proving the Theorem. □

Remark 2 We postpone discussing the name of this Theorem until Sec 37.6.

12.4 Weighted PIE, and Pointers

Remark 1 One can write down weighted counterparts of the PIE expressions that are relative differences in the stronger sense of containing equal numbers of positive and negative terms.

Pointer 1 One of the more usual contexts in which this arises is Probability Theory, where non-trivial weights encode non-uniform discrete probabilities of events. Then for instance, the *Bonferroni inequalities* of Probability Theory can be thought of along PIE lines [55].

Remark 2 The particular case that the current Volume should mention, however, is the following *symmetric PIE* that arises from the relationally-significant symmetric difference operation. The three-subset version of this is

$$|\mathfrak{A} -_{\mathbf{S}} \mathfrak{B} -_{\mathbf{S}} \mathfrak{C}| = |\mathfrak{A}| + |\mathfrak{B}| + |\mathfrak{C}| - 2|\mathfrak{A} \cap \mathfrak{B}| - 2|\mathfrak{B} \cap \mathfrak{C}| - 2|\mathfrak{C} \cap \mathfrak{A}| + 3|\mathfrak{A} \cap \mathfrak{B} \cap \mathfrak{C}| \ .$$

This generalizes to

$$\left| \bigcap_{i\,=\,1}^{n} \bar{\mathfrak{A}}_i \right| \;=\; \sum_{\mathfrak{J} \subseteq [\,1\,,\,n\,]} n\,(-1)^{|\,\mathfrak{J}\,|}\;|\,\mathfrak{A}_{\mathfrak{J}}\,|\;\;:$$

the sieve form of 'symmetric PIE'. This answers how to form more general relational subset expressions than by single use of symmetric difference. As well as giving a way to form an n-ary version of the symmetric difference operation.

Example 1 The simplest nontrivial relational expression thus arrived at is

$$\frac{|\,\mathfrak{E} \mathbin{-\!\!\!\mathbf{S}} \mathfrak{F}\,|}{|\,\mathfrak{A} \mathbin{-\!\!\!\mathbf{S}} \mathfrak{B} \mathbin{-\!\!\!\mathbf{S}} \mathfrak{C}\,|} \;=$$

$$\frac{|\,\mathfrak{E}\,| \;+\; |\,\mathfrak{F}\,| \;-\; |\,\mathfrak{E} \cap \mathfrak{F}\,|}{|\,\mathfrak{A}\,| \;+\; |\,\mathfrak{B}\,| \;+\; |\,\mathfrak{C}\,| \;-\; 2\,|\,\mathfrak{A} \cap \mathfrak{B}\,| \;-\; 2\,|\,\mathfrak{B} \cap \mathfrak{C}\,| \;-\; 2\,|\,\mathfrak{C} \cap \mathfrak{A}\,| \;+\; 3\,|\,\mathfrak{A} \cap \mathfrak{B} \cap \mathfrak{C}\,|}\;.$$

Structure 2 The general such is

$$\frac{\left| \mathbin{-\!\!\!\mathbf{S}}_{i\,=\,1}^{n} \mathfrak{B}_i \right|}{\left| \mathbin{-\!\!\!\mathbf{S}}_{j\,=\,1}^{m} \mathfrak{A}_j \right|}\;. \tag{12.6}$$

Pointer 2 We encounter PIE again in Exercise 14.8 on the large-n asymptotic proportion of permutations that are derangements. And in proving Dirac's 42.3 and Ore's 42.4 Theorems about overcrowding conditions for Hamiltonianness in graphs.

Chapter 13

Selections

13.1 Combinations and permutations

Remark 1 We can choose k 'balls' from a 'bag' containing n 'balls' [1] in the following numbers of ways, each corresponding to widely-applicable modelling assumptions.

Case 1 Suppose that our 'balls' are distinguishable, our choices are ordered 'in a row of boxes', and the 'balls' are not replaced in the 'bag'. Then there are n ways of choosing the first object, $n - 1$ for the second, and so on, down to $n - k + 1$ for the kth object. Thus

$$(\, n \text{ choose } k \text{ distinguishable, ordered, no replacements}\,) = n\,(n-1)\,...\,(n-k+1)\,. \quad (13.1)$$

Case 2 Suppose that our 'balls' are distinguishable, our choices are not ordered, and the 'balls' are not replaced in the 'bag'. Now we can start as above, but then we need to divide by the number of possible orders. This is the number of transformations that leave the selected number k of 'balls' invariant. Since the 'balls' are distinguishable, these transformations are the permutations of n 'balls', of which there are $k!$. Thus

$$(\, n \text{ choose } k \text{ distinguishable, unordered, no replacements}\,) = \frac{n\,(n-1)\,...\,(n-k+1)}{k!}\,. \quad (13.2)$$

It helps to tidy up at this stage, prior to considering further modelling assumptions. Because the second case involves obtaining less structured data – an unordered set rather than an ordered one – it is regarded as more primary. It is thus on occasion referred to simply as n choose k, denoted by ${}^nC_k$. Here the C additionally stands for *combinations*,

$$\left(\begin{array}{c} n \\ k \end{array} \right)\,.$$

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Remark 2 The first case is referred to as n perm k, denoted by ${}^nP_k$. Here the P stands for *permutations*. In this context, this really means *k-permutations*: permutations of a subset of size k of a set of size n.

Remark 3 If our second case is taken to be primary, however, one obtains the first case from it by multiplication by the number of permutations. I.e. [49]

$${}^nP_k = {}^nC_k \times (\text{number of permutations}) = {}^nC_k \times k!$$

From a computational point of view, however, our first case is simpler.

[1] This useful conceptualization features in e.g. [49].

Remark 4 We can furthermore espy our first case to be a truncation of the factorial $n!$. Namely,

$$ {}^{n}P_{k} \;=\; \frac{n!}{(\,n \;-\; k\,)!} \;. \tag{13.3} $$

Thus also

$$ \left(\begin{array}{c} n \\ k \end{array} \right) \;=\; {}^{n}C_{k} \;=\; \frac{n!}{(\,n \;-\; k\,)!} \;\times\; \frac{1}{k!} \;=\; \frac{n!}{(\,n \;-\; k\,)!\,k!} \;=\; \frac{n!}{k!\,(\,n \;-\; k\,)!} \;. $$

The last computational form here is probably rather familiar to the Reader.

Lemma 1

$$ \left(\begin{array}{c} a \;+\; b \\ a \end{array} \right) \;=\; \left(\begin{array}{c} a \;+\; b \\ b \end{array} \right) \;. $$

Exercise 1^{-}. Prove this both Combinatorially, and Algebraically.

♠ ◇ ♣

Remark 5 A further reason to be interested in n choose k is its occurrence as the coefficients in the *Binomial Theorem*. I.e.

$$ (\,x \;+\; y\,)^{n} \;=\; \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) x^{k}\, y^{n \;-\; k} \;. \tag{13.4} $$

Exercise 2 Give Combinatorial proofs and Algebraic proofs of the following.

i)

$$ \left(\begin{array}{c} n \\ k \end{array} \right) \;=\; \left(\begin{array}{c} n \\ n \;-\; k \end{array} \right) \;. \tag{13.5} $$

ii) *Pascal's identity*

$$ \left(\begin{array}{c} n \\ k \end{array} \right) \;=\; \left(\begin{array}{c} n \;-\; 1 \\ k \end{array} \right) \;+\; \left(\begin{array}{c} n \;-\; 1 \\ k \;-\; 1 \end{array} \right) \;. \tag{13.6} $$

Remark 6 The usual method of building up the binomial or combinations triangle (also commonly alias Pascal's triangle), described in the Introduction as 'each brick is labelled by the sums of the labels of the two bricks in contact with it from above' is encoded by ii). This permits each successive row to be built from the previous one, probably most conveniently starting from the second row. The Introduction's reflection symmetry in the binomial triangle is encoded by i). It thereby persists for the whole triangle rather than just for truncations thereof such as that exhibited in Fig 1. Indeed, ii) acts so as to preserve i) row by row.

Exercise 3 Show that for p prime,

$$ (\,x \;+\; y\,)^{p} \;\cong\; x^{p} \;+\; y^{p} \pmod{p} \;. $$

Deduce a second proof of Fermat's little Theorem.

13.2 Falling factorials

Remark 1 If we pursue computational primality, it makes some sense to give (13.3)'s factorial ratio a shorthand. The *computational form of* this gives it the name *falling factorial*, or, with more precision, *n to the k falling* [39, 47], for which Donald Knuth provided the notation

$$ n^{\underline{k}} \;. $$

Remark 2 When expanded out, $n^{\underline{k}}$'s coefficients are well-known as the *Stirling numbers of the first kind*, $s(n,\, k)$. We shall return to these in Secs 16.2 and 16.1.

Notation 1 Let us start to build up our own notation here, by calling the corresponding problem $ORI(n,\, k)$. This stands for n choose k ordered picks, with replacements, for indistinguishable objects. If any of these modelling conditions are in the negative, place complement bars thereupon. When talking about both a modelling condition and its complement, we place its letter in square brackets, [ ]. We call this 'regardless of', so e.g. $[O]$ stands for 'regardless of ordering'.

Remark 3 In this notation, the traditional notation is re-represented as follows.

$$
\begin{aligned}
{}^{n}P_{k} &= \bar{O}R\bar{I}(n,\, k)\ , \\
{}^{n}C_{k} &= \bar{O}\bar{R}\bar{I}(n,\, k)\ .
\end{aligned}
$$

13.3 Multi-combinations, multi-permutations and rising factorials

Definition 1 A *multi-set* is a set in which repeated elements are allowed.

Case 3 ('multi-combinations') Suppose that our 'balls' are distinguishable, our picks are unordered, and our 'balls' are replaced in the 'bag'. This can also be envisaged as the number of binary strings of length $(\, n \ + \ k \ - \ 1\,)$ that contain k zeros and $(\, n \ - \ 1\,)$ ones.

But the latter count is just a particular case of the n choose k problem. Thus

$$
\overline{O}R\overline{I}(q,p) \ = \ \left(\begin{array}{c} q \ + \ p \ - \ 1 \\ p \end{array} \right) \ = \ \frac{(\, q \ + \ p \ - \ 1\,)!}{p!\,(\, q \ - \ 1\,)!}\ .
$$

Remark 1 This is a further example of Combinatorial proof.

Remark 2 Since there are many other situations in which multi-combinations arise, however, introducing them is not wasteful. For instance, multi-combinations are the coefficients in the multi-nomial expansion [49].

Case 4 Suppose that our 'balls' are distinguishable, our choices are ordered, and our 'balls' are replaced in the 'bag'. Then we need to multiply the preceding by the number of permutations of our chosen objects, $k!$. This yields

$$
\overline{O}R\overline{I}(n,\, k) \ = \ \frac{(\, n \ + \ k \ - \ 1\,)!}{k!\,(\, k \ - \ 1\,)!} \ \times \ k! \ = \ \frac{(\, n \ + \ k \ - \ 1\,)!}{(\, n \ - \ 1\,)!}\ .
$$

Naming Remark 1 Given the preceding names, it makes sense for us to call these '*multi-k-permutations*'.

Naming Remark 2 The *computational form of* this gives it the name *rising factorial*, or, with more precision, *n to the k rising* [39, 47], for which Knuth provided the notation

$$
n^{\overline{k}}\ .
$$

Observe that this discussion and that of falling factorials usefully form an opposite pair, both in all Naming and in all notation used!

13.4 Further cases

Case 5 Suppose that our 'balls' are indistinguishable. Then whether or not we make replacements, and whether or not we order the 'balls' we choose makes no difference. There is only one outcome.

$$
[OR]I(n,\, k) \ = \ 1\ .
$$

Remark 1 One common response to this is to drop 'distinguishable' from the modelling assumptions. If we followed this trend, we would have just used OR for our notation. The reason that we did not – and put the I last in ORI – is covered in Sec 22.1.

13.5 The twelvefold way

The twelvefold way[2] extends our account as follows.

a) The boxes are now permitted to be indistinguishable instead of being assumed to be distinguishable.

b) The *footballer card collectors*' version of selection is now permitted. Here, one keeps on sampling until one has seen each distinct object in the sample at least once. [In building up such a collection, one would usually furthermore aim to keep one copy of each so as to end up with a complete collection.]

c) The twelvefold way does not however include the orientation choice.

This leaves us with 3 parameters RII independently taking 3, 2 and 2 values, which multiplies out to the nominal 12 cases.

13.6 The selection function interpretation

Remark 1 From the point of view of a selection function

$$f \ : \ \text{Bag's set of balls} \to \text{Boxs' set of balls} \ ,$$

the following three cases [49] have particular Combinatorial significance.

i) Unrestricted f.

ii) Injective f.

iii) Surjective f.

Exercise 4 Show that this is an alternative interpretation of the twelvefold way's 3-valued parameter.

13.7 The twentyfold way (a priori 32-fold way)

Bogart's twentyfold way [60] permits a fourth selection mode value. Namely, precise equality, corresponding to the bijections: intersection of injection and surjection. The ordering ambiguity is also permitted here. This gives $4 \times 2 \times 2 \times 2 = 32$ cases, though he culls this down to 20 by removal of some blocks of triviality. This is covered by ORII notation.



[2] This was presented in lectures given by Gian-Carlo Rota, while its name was coined by J. Spencer [49]. See e.g. [60, 49] for reviews.

13.8 Batches and blocks

Remark 1 One way of extending the above 12- and 32-folds is permitting the complements of $\leq$, $\geq$ and $=$. I.e. $>$, $<$ and $\neq$. These do not however have independent solutions.

Remark 2 A second way involves introducing the following notions [62] and then using them to build various layers of problems.

Definition 1 A *batch* is a particular number of balls that a box is to contain. A *block* is a nonempty batch. A *batch decomposition* is a partition of the number of picks k into batches k_i, $i = 1$ to d for one of the admissible values of d ($\leq k$).

Remark 3 These are subject to the linear constraint

$$\sum_{i=1}^{d} k_i = k \ .$$

The ≥ 1 and ≤ 1 cases then generalize to $\geq k$ and $\leq k$ in the batch decomposition context. In so doing, the bounding $= 1$ case loses its extra mathematical significance of being a bijection, thus removing the motivation to list it as a separate case. The case of arbitrary *block* number can also be added to the modelling assumptions. This turns Bogart's factor of 4 into a factor of 5. So the number of problems goes up to $5 \times 8 = 40$. Of which Proctor's 30-fold way is a 5×6 restriction that removes trivial cases.

Remark 4 Proctor's next move is to individually fix each batch size k_i. And then to generalize the $\geq$, $\leq$ and arbitrary-picked-items-number cases to this context. This turns the factor of 5 into a factor of 6. So $6 \times 8 = 48$ problems are supported. Among which, Proctor picks a particular 42-fold.

13.9 Yet larger n-fold ways

Remark 1 Two further subtleties are required to complete useful modelling's scope: meaningless labels and that all of distinguishability, meaninglessness, and batch-size fixing are in general partial.

For now, out of not yet having covered groups or posets, we count each instance of the first two of these as the addition of a single partial case to each corresponding multiplicative factor. This yields the set of n-fold ways in Fig 13.1. We return to partial cases once armed with groups in Secs 26.8 and 22.2.



Larger n-fold ways			
$81 = 3^4 =$	(totally, partially, or not distinguishable ball labelling)		
	x (" box ")		
	x (" meaningful ball ")		
	x (" box ")		
Traditional count of cases			**Including partial-distinguishability, -meaninglessness and -fixing**
Rota	12	x 81 =	972
Pre-Bogart	32		1620
Bogart	20		2592
Pre-Proctor 1	40		3240
Proctor 1	30		2430
Pre-Proctor 2	48		3888
Proctor 2	42		3402
partially-fixed k_i Pre-Proctor 2	56		4536

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Figure 13.1:

13.10 Stirling's approximation

Proposition 1

$$n! \sim \sqrt{2\pi n}\, n^n \exp(-n) \text{ as } n \longrightarrow \infty . \tag{13.7}$$

Naming Remark 3 A truer name for this is thus the *large-n asymptotic formula for n*!. To be more specific, it is the more accurate form of the leading-order term therein. A less accurate version – without exact knowledge of the constant of proportionality involved – was already previously known to De Moivre.

Remark 1 Since proving this involves Calculus, we postpone doing so until Volume 1 [441].

Remark 2 This is a first result in *Asymptotic Combinatorics* [65]. We could then for instance also seek corresponding asymptotic results for the Combinatorial functions arising in Chapters 15 and 16. This involves the opposite extreme to the current Series' main approach of seeing to what extent the smallest instances of mathematical objects provide minimal couterexamples. Neither of these approaches address, however, the usually harder matter of middle-sized objects.

Worked Example 1 For all of n, m and $n - m$ large,

$$\left(\begin{array}{c} n \\ m \end{array} \right) \sim \frac{n!}{m!\,(n-m)!}$$

$$= \frac{\sqrt{2\pi n}\, n^n \exp(-n)}{\sqrt{2\pi m}\, m^m \exp(-m)\, \sqrt{2\pi (n-m)}\, (n-m)^{n-m} \exp(-(n-m))}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{n^{n+\frac{1}{2}}}{m^{m+\frac{1}{2}} (n-m)^{n-m+\frac{1}{2}}} .$$

One way in which our above modelling assumptions could hold is the hierarchy $n \gg m \gg 1$, by which $n - m \simeq n$. In this case, we can furthermore write

$$\left(\begin{array}{c} n \\ m \end{array} \right) \sim \frac{1}{\sqrt{2\pi}} \frac{n^{n+\frac{1}{2}}}{m^{m+\frac{1}{2}} n^{n-m+\frac{1}{2}} \left(1 - \frac{m}{n}\right)^{n-m+\frac{1}{2}}} .$$

$$= \frac{1}{\sqrt{2\,\pi}} \frac{n^m}{m^{m\,+\,\frac{1}{2}} \left(1 \,-\, \frac{m}{n}\right)^{n\,-\,m\,+\,\frac{1}{2}}} \simeq \left(\frac{n}{m}\right)^m \ .$$

In the last step here, we binomially expand in m/n, keeping just the leading term while discarding the constant of proportionality.

In progressing through this Volume, the Reader shall eventually encounter another way of satisfying this Worked Example's modelling assumptions.

Exercise 5 a) What is the corresponding asymptotic formula if n becomes large while m maintains a fixed small value?

b) What is the corresponding analysis for ${}^nP_m$ in each of these two modelling situations?

Chapter 14

Generating series

14.1 Difference equations, alias recurrence relations. I. First-order linear.

Example 1 Let A_n be a sequence obeying the recurrence relation

$$A_n \;=\; A_{n\,-\,1}\;.$$

What particular form must the A_n then take?

For this particular example, it is straightforward to see that any constant sequence solves this recurrence relation.

When such recurrence relations are linear and with constant coefficients a more systematic approach is to consider the *power-law ansatz*

$$A_n \;=\; a^n\;.$$

Then

$$a^n \;=\; a^{n\,-\,1} \;\;\Rightarrow\;\; a^{n\,-\,1}(\,a \;-\; 1\,) \;=\; 0 \;\;\Rightarrow\;\; a \;=\; 1\;.$$

Thus

$$A_n \;=\; 1$$

solves. This recurrence relation is also linear, so (as we shall detail further in Volume 1) linear combinations of solutions are themselves solutions. So

$$A_n \;=\; k \;\times\; 1 \;=\; k$$

also solves.

Example 2 Suppose that

$$B_n \;=\; 2\,B_{n\,-\,1}\;.$$

Then the power-law ansatz returns

$$a^{n\,-\,1}(\,a \;-\; 2\,) \;=\; 0 \;\;\Rightarrow\;\; a \;=\; 2\;.$$

Thus

$$B_n \;=\; k\,2^n$$

solves.

Definition 1 A *recurrence relation* between the terms A_n in a sequence is an equation of the following form.

$$A_n \; = \; A_n(\, A_{n \, - \, 1}, \, ... \, , A_{n \, - \, m} \,) \; .$$

This is alias *difference equation*.

Remark 1 Difference equations are a discrete counterpart of differential equations. The current Chapter's all depend on just one variable n, due to which they are in more detail analogous to ordinary differential equation (ODEs). One can however espy that Exercise 13.2.ii)'s identity can be interpreted as a difference equation with 2 dependent variables. This is now analogous to a partial differential equation (PDE) in 2 dependent variables. It is but an easily-solvable example, though Chapter 16 proceeds to consider various less trivial such.

Remark 2 First-order linear difference equations are of the form

$$A_n \; = \; f(n) \, A_{n \, - \, 1} \; . \tag{14.1}$$

These need one initial condition to fix the solution (removing the freedom posed by multiplying by the constant k).

Exercise 1 To illustrate such with non-constant coefficients, consider

$$F_n \; = \; n \, F_{n \, - \, 1} \; .$$

Prove by induction that this with initial condition $F_0 \; = \; 1$ is solved by the factorial function $F_n \; := \; n!$.

14.2 II. Second-order linear

Remark 1 We are rather more interested in second-order and/or quadratic recurrence relations in this Volume. Second-order linear recurrence relations are of the form

$$A_{n \, + \, 1} \; = \; f(n) \, A_n \; + \; g(n) \, A_{n \, - \, 1} \; .$$

Worked Example 1 Another simple and well-known example of recurrence relation is as follows. Starting with 1 and 1, form a sequence of numbers in which each is the sum of the previous two. This corresponds to the second-order linear constant-coefficients difference equation

$$U_{n \, + \, 1} \; = \; U_n \; + \; U_{n \, - \, 1} \; , \tag{14.2}$$

together with the two initial conditions

$$U_1 \; = \; 1 \; = \; U_2 \; .$$

[Second-order linear difference equations admit two solutions, linear combinations of which are also solutions.]

Our equation being linear and with constant-coefficients, the power-law ansatz (14.1) applies. This gives

$$a^{n \, + \, 1} \; = \; a^n \; + \; a^{n \, - \, 1} \;\; \Rightarrow \;\; a^{n \, - \, 1} \left(a^2 \; - \; a \; - \; 1 \right) \; = \; 0 \; .$$

$$\Rightarrow \;\; a \; = \; a_{\pm} \; := \; \frac{1 \; \pm \; \sqrt{5}}{2} \; .$$

Thus

$$U_n \; = \; k \left(\frac{1 \; - \; \sqrt{5}}{2} \right)^n \; + \; l \left(\frac{1 \; - \; \sqrt{5}}{2} \right)^n \; . \tag{14.3}$$

On the one hand, setting $n = 1$,

$$1 = k\left(\frac{1 - \sqrt{5}}{2}\right) + l\left(\frac{1 - \sqrt{5}}{2}\right) . \tag{14.4}$$

On the other hand, viewing U_0 as 0 so that the recurrence relation extends to give

$$U_2 = U_1 + 0 = U_1 ,$$

$$0 = k + l .$$

(14.4) then yields

$$k = \frac{1}{\sqrt{5}} .$$

Thus

$$U_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1 + \sqrt{5}}{2}\right) - \left(\frac{1 - \sqrt{5}}{2}\right)\right) . \tag{14.5}$$

Naming Remark 1 These are widely known as the *Fibonacci numbers*[1] A truer name for them which we use in the current text is *EUSOLDE numbers*: the everywhere-unary second-order linear difference-equation numbers. 'Everywhere-unary' is here taken to include, firstly, the equation having constant coefficients, which are furthermore ones. Secondly, that both initial conditions are ones. 'Both' here being the right amount of data for an equation of this kind, as we shall explain in Volume 1 [441] under the name of *well-determinedness*.

Remark 2 Let us first check that these are indeed integers.

P_n: F_n is an integer for all n up to some k.

$$\left(1 + \sqrt{5}\right)^n - \left(1 - \sqrt{5}\right)^n = 2^n\sqrt{5}\,m \text{ for } m \in \mathbb{N} .$$

Straightforwardly, this holds for $n = 1$ with $m = 1$. Induction Hypothesis: suppose that P_n holds for some $n = k$. Then for $n = k + 1$,

$$\begin{aligned}\left(1 + \sqrt{5}\right)^{n+1} - \left(1 - \sqrt{5}\right)^{n+1} &= \left(\left(1 + \sqrt{5}\right)^n - \left(1 - \sqrt{5}\right)^n\right)\left(\left(1 + \sqrt{5}\right) - \left(1 - \sqrt{5}\right)\right) \\ &\quad - \left(1 + \sqrt{5}\right)^n\left(1 - \sqrt{5}\right) + \left(1 - \sqrt{5}\right)^n\left(1 + \sqrt{5}\right) \\ &= 2^n m_n\sqrt{5} \times 2 - \left(\left(1 + \sqrt{5}\right)^{n-1} - \left(1 - \sqrt{5}\right)^{n-1}\right)\left(1 + \sqrt{5}\right)\left(1 - \sqrt{5}\right) \\ &= 2^n m_n\sqrt{5} \times 2 + 2^{n-1}m_{n-1}\sqrt{5} \times 4\end{aligned}$$

by two uses of the induction hypothesis.

Thus we have reached the form

$$2^{n+1}\sqrt{5}\,m_{n+1}$$

for

$$m_{n+1} = m_n + m_{n-1} .$$

So P_n holds for $n = k + 1$ as well, by strong induction.

Remark 5 The first few EUSOLDE numbers are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144...

Exercise 2 The *Lucas numbers* are the $\tilde{U}_0 = 2$, $\tilde{U}_1 = 1$ initial data counterpart of the above. Give a closed form expression for the nth Lucas number.

[1] This refers to a name for the 13th Century Mathematician Leonardo of Pisa. These numbers – beautiful, succinctly poseable and realized in Nature – were already studied in India over 1 millennium previously [77].

14.3 III. Quadratic

Structure 2 (Inhomogeneous) quadratic recurrence relations in the current Volume and Series are of the form

$$A_{n\,+\,1} \;=\; \sum_{i\,=\,1}^{n} \sum_{j\,=\,1}^{m} f_{ij}(n)\, A_i\, A_j \;.$$

Remark 1 For a concrete example of quadratic recurrence relation and its solution, see Sec 36.12.

Exercise 3 This is not the most general way in which a quantity can be inhomogeneously-quadratically related to other quantities. For we could include a sum of linear terms and a constant term. Isolate A_{n+1} if:

a) the only place that this is present is among the quadratic terms.

b) If this is present as a quadratic term and a linear term.

c) If it solely features as a linear factor in overall-quadratic product terms.

14.4 Generating series as a further method for solving difference equations

Definition 1 [60, 42, 41, 22, 49, 76] The *(ordinary) generating series* corresponding to a sequence A_n is

$$G(z) \;:=\; \sum_{n \in \mathbb{N}} A_n\, z^n \;.$$

Remark 1 Encoding all the information in the sequence in this way is useful, simplifying some workings.

Naming Remark 2 Generating series are often called *generating functions*. This is in many ways a misnomer, however, so we do not use this name in the current Series. The truer name *generating series* is fortunately already in widespread use in the Algebraic Combinatorics literature.

Pointer 1 Some conceptual diversity in place of 'ordinary' is outlined in Sec 14.6.

Example 1 Set

$$U(z) \;=\; \sum_{n\,\in\,\mathbb{N}} U_n\, z^n \;.$$

Then

$$U(z)\,(\,1 \;-\; z \;-\; z^2\,) \;=\; U_0 \;+\; (\,U_1 \;-\; U_0\,)\,z \;=\; z$$

by our recurrence relation (14.2), and then the initial conditions. Thus the particular generating series for the EUSOLDE numbers is

$$U(z) \;=\; \frac{z}{1 \;-\; z \;-\; z^2} \;.$$

We can then use partial fractions to solve for what the EUSOLDE numbers are, as follows.

$$U(z) \;=\; \frac{z}{1 \;-\; a_-} z \;-\; a_+ \;=\; \frac{1}{a_+ \;-\; a_-} \left(\frac{1}{1 \;-\; a_+} \;-\; \frac{1}{1 \;-\; a_-} \right).$$

Then expanding by 2 uses of the Binomial Theorem and equating coefficients of z^n yields

$$U_n \;=\; \frac{1}{\sqrt{5}} \left({a_+}^2 \;-\; {a_-}^2 \right) \;.$$

Exercise 4 Repeat this analysis for the Lucas numbers introduced in Exercise 2.

Remark 2 The generating series method for difference equations has the substantial advantage of being applicable to solving nonlinear difference equations as well (see e.g. Sec 36.12).

Exercise 5 The Lucas numbers can be viewed as a type of MUSOLDE: *mostly unary*. What are the other MUSOLDE problems that derivate by having a single 2? Solve each of these as well!

Exercise 6 Show that the derangements on n objects satisfy the following recurrence relations.

a)
$$D_n \;=\; n\,D_{n\,-\,1} \;+\; (-1)^n \;.$$

b)
$$D_n \;=\; (\,n \;-\; 1\,)\,(\,D_{n\,-\,1} \;+\; D_{n\,-\,2}\,) \;.$$

c) These are to be treated as separate difference equations rather than as a simultaneous pair. Classify these as types of recurrence relation and show that they indeed yield the same series for the derangements.

14.5 Manipulating generating series (and power series more generally)

1) Two generating series are equal iff all their coefficients are equal:

$$\sum_{n\,\in\,\mathbb{N}} a_n\,z^n \;=\; \sum_{n\,\in\,\mathbb{N}} b_n\,z^n \;\text{ iff }\; a_n \;=\; b_n \;\;\forall\;\; n \;\in\; \mathbb{N} \;.$$

2) Addition of such is straightforward:

$$\sum_{n\,\in\,\mathbb{N}} a_n\,z^n \;+\; \sum_{n\,\in\,\mathbb{N}} b_n\,z^n \;=\; \sum_{n\,\in\,\mathbb{N}} (\,a_n \;+\; b_n\,)\,z^n \;. \tag{14.6}$$

3) So is scalar multiplication:

$$\lambda \sum_{n\,\in\,\mathbb{N}} a_n\,z^n \;=\; \sum_{n\,\in\,\mathbb{N}} (\,\lambda\,a_n\,)\,z^n \;. \tag{14.7}$$

4) Products have the complication that multiple terms contribute to each power. This is encoded by *convolution sums*

$$c_n \;=\; \sum_{k=0}^{n} a_n\,b_{n\,-\,k} \;=:\; a \;*_n\; b \;, \tag{14.8}$$

as follows.

$$\sum_{n\,\in\,\mathbb{N}} a_n\,z^n \;\cdot\; \sum_{n\,\in\,\mathbb{N}} b_n\,z^n \;=\; \sum_{n\,\in\,\mathbb{N}} c_n\,z^n \;.$$

Exercise 7 a) For
$$G(z) \;=\; \sum_{n\,\in\,\mathbb{N}} a_n\,z^n$$
a generating series, show that
$$G(z)^2 \;=\; \sum_{n\,\in\,\mathbb{N}} a \;*_n\; a \;.$$

b) Show that the *Zhu–Vandermonde identity*,

$$(\,p \;+\; q\,)^{\underline{r}} \;=\; \sum_{s=0}^{r} \left(\begin{array}{c} p \\ s \end{array}\right) x^{\underline{s}}\,q^{\underline{p\,-\,s}} \;,$$

can be reformulated as follows.

$$\binom{p+q}{r} = \sum_{s=0}^{r} \binom{p}{s} \binom{q}{r-s} . \tag{14.9}$$

This second form, due to Cauchy, gives us a truer name for our identity: *convolution of combinations*. Also prove this identity both Combinatorially and Algebraically.

c) Deduce that

$$(1+z)^m \times (1+z)^n = (1+z)^{m+n} . \tag{14.10}$$

14.6 Further types of generating series

Pointer 2 An *exponential generating series* is an expansion of the form

$$E(z) = \sum_{n \in \mathbb{N}_0} e(n) \frac{z^n}{n!} .$$

Remark 1 In the case with all the $e_n = 1$, we recover the MacLaurin series for the exponential function itself. See Sec 16.12 for further examples.

Application 1 Generating series can also be used to encode series of continuous functions. See our Volume 1 on Linear Algebra [441] for some widely-useful examples.

Application 2 Probability and Statistics involves *mgfs*: *moment generating functions*; see e.g. [289, 55].

Pointer 3 Dirichlet series are of the form

$$D(z) = \sum_{n=1}^{\infty} \frac{d_n}{n^z} .$$

These are particularly useful in Number Theory [22]. For instance, the simplest Dirichlet series is the celebrated *Riemann zeta function*, for which all the $d_n = 1$.

Exercise 8 a)$^-$ Show that

$$\sum_{n \in \mathbb{N}_0} \frac{(-1)^n}{n!} = \frac{1}{e} .$$

b)$^+$ Re-envisage the number of derangements D_n as the cardinality of an n-fold intersection. Use PIE to establish that the fraction of permutations that are derangements tends to $\frac{1}{e}$ for n large.

Pointer 4 Generating series in several variables are also straightforward to conceive of [49, 441].



Chapter 15

Partitions

15.1 Introduction

Definition 1 For $N \in \mathbb{N}$, a *partition*[1] $\mathrm{p}(N)$ of N is a split of N into various $N_i \in \mathbb{N}$ such that

$$\sum_i N_i = N \, .$$

Definition 1 The corresponding *partition number*

$$p(N) := \#(\text{ ways of writing } N \text{ as the sum of some } N_i \in \mathbb{N}) \, .$$

Example 0 We exceptionally also permit 0 to be partitioned into a single copy of 0. This gives the '*unpartition*'

$$\mathfrak{U} = 0 := \emptyset \, .$$

Thus $p(0) = 1$. $\mathfrak{U}$ is exceptional in not just being a partition, but furthermore lacking any kind of structure. For instance, it is also the empty set $\emptyset$, the unpoint, the ungraph, and the unposet. In our notation, we mark this exceptional lack of structure by using a mathfrak capital letter $\mathfrak{U}$, truer name 'un' or 'unstructured'. All our other partitions' letter-based notation involves straight lower-case font.

Notation 1 These font conventions serves to distinguish partitions from subsets, for which we continue to use mathfrak letters. From families, for which we use the italic font. From graphs (Part V, including point clouds), for which we use straight font with leading capital letters. And from groups (Parts III and IV), for which we use italic font with leading capital letters. These being the three principal structures in the current Volume. [Our convention for posets (Part VII) being to lead with their underlying graph's name, followed by hyphening on a lower-case descriptor of how the poset is bent.]

Example 1 One object can be partitioned into a single piece, which we denote by 1. Being a special case, we also call this unit. There being no other way of partitioning one object, $p(1) = 1$.

Example 2 2 can be partitioned into a single piece of size 2. Or into $1 + 1$ (using additive notation), another notation being $1 \,|\, 1$. Meaning 1 in one 'compartment', 'bin', 'urn', 'piece' (or more confuseably 'box') and 1 in another such. Or just 1 1, the space serving to disambiguate this from the number (or larger single-piece partition) 11. Or the rapidly more efficient 1^2, if there is no

[1] See e.g. [49, 76] for introductory material, and [20, 22, 30, 31, 43, 50, 61] for detailed reviews.

chance of confusing these with the power-index notation (here one-squared = one).[2] Since there are no further ways[3] of partitioning two objects, $p(2) = 2$.

Example 3 3 can be partitioned in any of the following ways. 3, $2\,1$ or 1^3. Thus $p(3) = 3$.

♠ ◇ ♣

Definition 3 We refer to the partition of N into just the one piece N as the *indiscrete partition*, $\mathrm{p_d}(N)$. The partition of N into N pieces 1^N is the *discrete partition*, $\mathrm{p_i}(N)$.

Remark 1 This parallels the notation already given for families, covers, and topologies.

Remark 2 $N = 2$ is minimal for discrete–indiscrete distinction. This is since $\mathrm{p_i}(0)$ and $\mathrm{p_d}(0)$ are just further aliases for $0 = \mathfrak{U} = \emptyset$. And $\mathrm{p_i}(0)$ and $\mathrm{p_d}(0)$ are just further aliases for for $1 =$ unit. In contrast,

$$\mathrm{p_i}(2) = 2 \neq 1^2 = \mathrm{p_d}(2) \; .$$

Remark 3 $N = 3$ is minimal to have a partition other than the discrete and indiscrete ones: $2\,1$. This is also minimal as regards having a partition into parts which all have distinct occupation numbers.

Example 4 4 can be partitioned in the following ways. 4, $3\,1$, 2^2, $2\,1^2$ and 1^4. Thus $p(4) = 5$. 4 objects is minimal to have two distinct partitions into the same number of pieces: $3\,1$ and 2^2. 4 objects is also minimal to support there being two distinct partitions equal in their largest occupation number: 2^2 and $2\,1^2$.

♠ ◇ ♣

Example 5 5 can be partitioned as follows. 5, $4\,1$, $3\,2$, $3\,1^2$, $2^2\,1$, $2\,1^3$ and 1^5. Thus $p(5) = 7$. To systematically find all of these, we search lexicographically in our list is by smallest number of pieces first, and then by largest piece first (and present our findings ordered likewise).

Example 6 6 can be partitioned as follows. 6, $5\,1$, $4\,2$, 3^2, $4\,1^2$, $3\,2\,1$, 2^3, $3\,1^3$, $2^2\,1^2$, $2\,1^4$ and 1^6. Thus $p(6) = 11$.

Example 7 7 can be partitioned as follows. 7, $6\,1$, $5\,2$, $4\,3$, $5\,1^2$, $4\,2\,1$, $3^2\,1$, $3\,2^2$, $4\,1^3$, $3\,2\,1^2$, $2^3\,1$, $3\,1^4$, $2^2\,1^3$, $2\,1^5$ and 1^7. Thus $p(7) = 15$.

Remark 4 Pieces of partitions of the same size are indistinguishable, since say no 1 involved is any different from any other 1.

Remark 5 Partitions are covers but covers are not necessarily partitions. Partitions are marked among the covers by shading their labels in green in Fig 26.12 and 26.13's heat maps. As regards a specific characterization, partitions are the disjoint union specialization of covers:

$$\coprod_i \mathfrak{W}_i = \mathfrak{X} \; . \tag{15.1}$$

This can be unpacked into the cover condition (9.1)

$$\bigcup_i \mathfrak{W}_i = \mathfrak{X}$$

alongside the pairwise-disjointness condition

$$\mathfrak{W}_i \cap \mathfrak{W}_j = \emptyset \;\; \forall \;\; i \neq j \; .$$

[2] In reading about Mathematics, take care not to confuse instances of this notation for partitions with the notation for prime decompositions!

[3] We take $\mathfrak{U}$ to be the only case in which 0 can be an admissible *part* N_i, alias *occupation number for one of the compartments*, of a partition.

15.2 Partition box diagrams, alias Young diagrams

♠ ◇ ♣

Depiction 1 *Young diagrams* [19, 48, 50] are a useful depiction for small partitions. Consider the partition $N_1, \ldots, N_k$ with

$$N_1 \geq N_2 \geq \ldots \geq N_k \ . \tag{15.2}$$

Place N_1 squares into a first row. N_2 into a second row, aligned on the left with the first row, and so on.

Naming Remark 1 A truer name is *partition box diagram*. Provided that it is understood that the boxes in question are objects being partitioned, not bins that objects are being put into. If we use dots instead of boxes, we have instead a *Ferrers diagram* for the partition. Since decorating partition diagrams turns out to be significant, and boxes are more convenient to decorate, we pick the box option.

Examples 0 to 7 See Fig 15.2 for the $N = 0$ to 7 cases of partition box diagrams.

♠ ◇ ♣

Remark 1 Each Young diagram is hugged by a rectangle that it in general only partly occupies. By (15.2), this rectangle is of width

$$W = N_1 \ .$$

It is also of height

$$H = k \ :$$

the number of pieces that the partition involves.[4]

Naming Remark 2 On the one hand, Young diagrams lend themselves to according vivid names for small or simply-structured partitions. The unpartition $\mathfrak{U}$ corresponds to no boxes, so 'unbox' is another alias. Single-part partitions are clearly already a family indexed by N, for which N is itself already an insightful name. Let us use v for $2\,1$. hockey for $3\,1$, on account of being hockey-stick-shaped. And square for 2^2.

Let us then use e-hockey for $(\,e\,+\,2\,)\,1$, the e referring to *excess handle length*. And l-square for l^l, with side-length l. So 1-square is just another name for 1, and 2-square $=$ square (the most common nontrivial square in simple applications). The notation l-square itself thus starts to be needed for 3-square $= 3^3$. Let us also use e-v for $(\,e\,+\,2\,)\,1^{e\,+\,1}$; the e here now referring to *excess stroke length*.

The next shapes to appear are (Fig 15.2) driver and icehockey (for the thickest-headed golf-club and for its longer blade respectively). These start off handle-length families. Also grid – the rectangular counterpart of square – starting a 2-parameter family grid(W, H). And triangle $= 3\,2\,1$, which starts a family which acquires a new member on every occasion that N hits a triangular number.[5] The next three are cleaver $= 2^3\,1$. pistol $= 4\,2\,1$. And flag $= 3\,2^2$. These mark the beginning of three further 1-parameter families (indexed by handle length, barrel length and handle length again.)

Remark 2 On the other hand, diversity among partitions is sufficiently moderate that the counting notation, or the Young diagram itself, constitutes a systematic notation for partitions. This type of good fortune will not however be repeated for graphs, posets, or groups. So while the present chapter's account of naming and notation systems exceeds what partitions themselves need, it has an eye on these other applications.

[4] [43] refers to this as *length*.
[5] [43] refers to these as *staircase* partitions.

Exercise 1 Find the partitions of 8 and 9 objects.

[The answer to this question can be read off Fig 15.3.]

15.3 Conjugate partitions

Definition 1 The partition $\bar{\mathrm{p}}$ is *conjugate to* the partition p if its Young diagram is the reflection of that of p about its leading diagonal. See Fig 15.1.a).

Notation 2 Let us also denote conjugation between two partitions p and q as the binary relation $\mathrm{p} \mathcal{P} \mathrm{q}$.

Example A

$$\mathrm{p_i} \mathcal{P} \mathrm{p_d} \ .$$

Lemma 1

$$p(N;\, k \ \text{ parts}) \ = \ p(N; \ \ \text{of largest part} \ \ k) \ . \tag{15.3}$$

Proof The Young diagrams of these form a conjugate pair. So the counts of these two kinds of object must coincide. □.

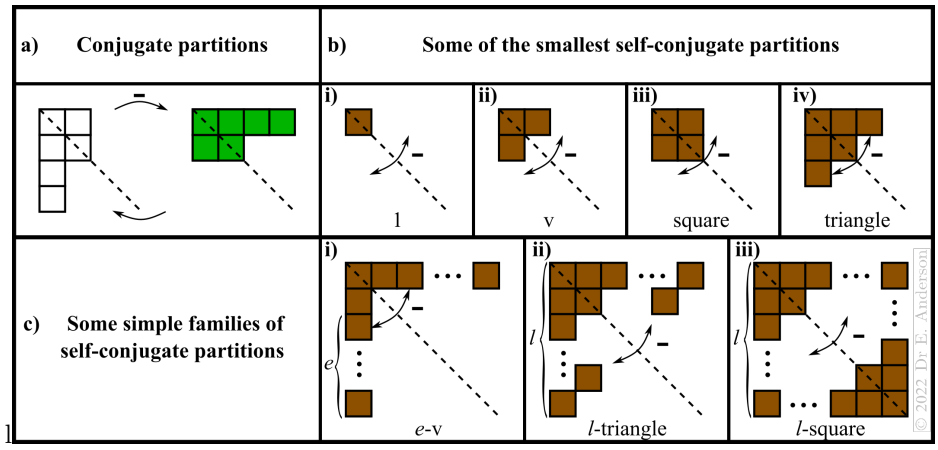


Figure 15.1:

Remark 1 This is the first of a number of insightful double counts concerning partitions.

15.4 Homoconjugate partitions

Remark 1 For each partition, conjugation always maps any given partition either to a unique partner, or to itself.

Definition 1 A partition is *homoconjugate* if it is equal to its own conjugate. Otherwise, it is *heteroconjugate*, forming part of a *conjugate pair*[6]

Notation 1 We colour homoconjugate partitions in brown.

[6]For the first of these, most of the literature has hitherto followed MacMahon [20] in using 'self-conjugate'. Yet as arenas take root more widely, need to refer to 'non-self-conjugate' partitions increases. And such as a 'homo' and 'hetero' pairing of prefixes starts to look more attractive.

Notation 2 Let $s(N)$ denote the number of homoconjugate partitions of N.

♠ ◇ ♣

Example 0 The partition $\mathfrak{U} = 0$ of 0 objects is trivially homoconjugate by involving no object to preserve under our reflection.

Example 1 The partition 1 of 1 object is homoconjugate (Fig 15.1.b.i).

Example 2 v $= 2\,1$ is also homoconjugate (Fig 15.1.b.ii).

Example 3 So is square $= 2^2$ (Fig 15.1.b.iii).

Example 4 And triangle $= 3\,2\,1$ (Fig 15.1.b.iv).

Examples 5-7 In fact, all members of the v, square and l-triangle families are homoconjugate (Fig 15.1.c). These triangles are 'right-angled isosceles', with l referring to their shortest side-length.

Exercise 2$^-$ Show that $\mathcal{P}$ is an *involution*: a map M such that $M^2 = \iota$: the identity map. And symmetric, but neither reflexive nor transitive.

Exercise 3 a) Classify the partitions on up to 9 vertices into heteroconjugate pairs and homoconjugate singletons.

b) By counting homoconjugate partitions once and (say) tall heteroconjugate partitions, obtain p(10) and p(11).

c) Which N is minimal for there to be more than one homoconjugate partition?

d) Finish working out all the homoconjugate partitions for $N \leq 15$.

e) Find the smallest N admitting 2 homoconjugate partitions that involve the same number of parts.

♠ ◇ ♣

Remark 1 Our naming stopped around half way down the partitions being introduced because the remaining partitions can be named via their conjugates. It is quite widely conventional to do this by denoting conjugation with an over-bar. It is more convenient to write $\bar{N}$ than 1^N. It is also clear that (in)discrete notation is not necessary for partitions, since these are in any case realized by the clearly and succinctly-named N and $\bar{N}$ respectively. See Fig 15.2 for this upgrade to our notation for partitions.


Young diagrams for partitions of 0 to 7 objects
0 = trivial = 1¹ = 0
1 = p(1) = unit = p₁(1) = 1
2 = p(2)
1² = p₂(2) = 2
3 = p(3)
2 1 = v
1³ = p₃(3) = 3
4 = p(4)
3 1 = hockey
2² = square = 2²
2 1² = hockey
1⁴ = p₄(4) = 4
5 = p(5)
4 1 = 2-hockey
3 2 = driver
2³ 1 = driver
3 1² = 2-v
2 1³ = 2-hockey
1⁵ = p₅(5) = 5
6 = p(6)
5 1 = 3-hockey
4 2 = 2-driver
3³ = grid(3,2)
4 1² = icehockey
3 2 1 = steps
2³ 1 = grid(3,2)
3 1³ = icehockey
2² 1³ = 5-driver
2 1⁴ = 4-hockey
1⁶ = p₆(6) = 6
7 = p(7)
6 1 = 4-hockey
5 2 = 3-driver
5 1² = 2-icehockey
4 3 = cleaver
4 2 1 = pistol
4 1³ = v
3 2–² = flag
3 2 1² = pistol
3 1⁴ = 2-icehockey
3¹ 1 = flag
2³ 1 = cleaver
2² 1³ = 2-driver
2 1⁵ = 4-hockey
1⁷ = p₇(7) = 7
N
H
W
© 2022 Dr. E. Anderson

Figure 15.2:

Remark 2 Fig 15.3 extends our repertoire to partitions of 8 and 9 objects, thus answering Exercises 1 and 3.a)

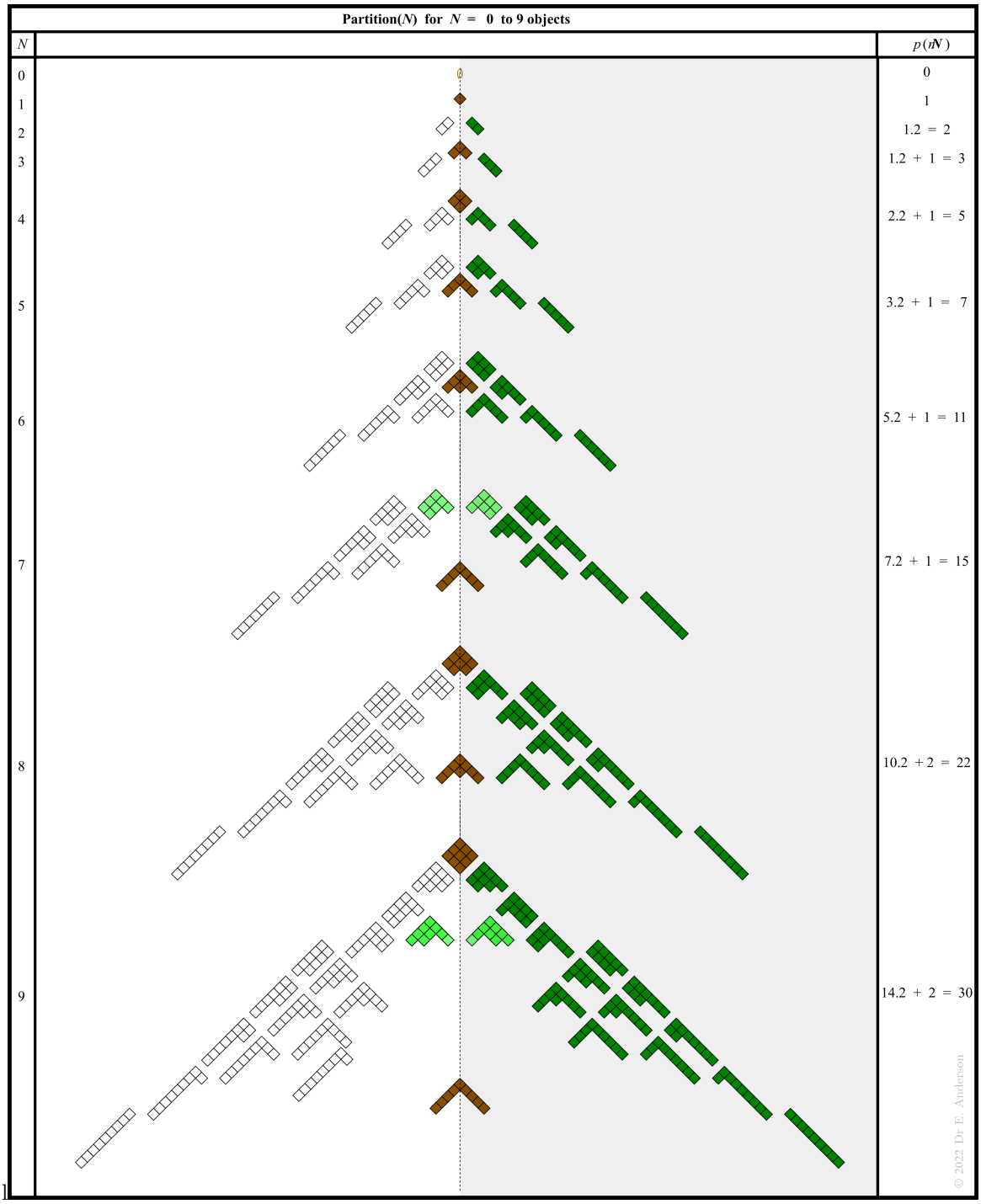


Figure 15.3:

15.5 Relational variables for partitions

Definition 1 The *(signed) anisotropy* of a partition is

$$A \ := \ W \ - \ H \ . \tag{15.4}$$

Definition 2 The *unsigned anisotropy* of a partition is

$$UA \ := \ |W \ - \ H| \ . \tag{15.5}$$

Definition 3 The *aspect ratio* of a partition is

$$\alpha \ := \ \frac{W}{H} \ . \tag{15.6}$$

Remark 1 This notion of aspect ratio can be taken to reduce to that for the rectangle hugging the Young diagram. [Aspect ratio is the standard variable for describing rectangular shapes [443].]

Remark 2 Anisotropy means departure from isotropy – being the same in all directions – presently referring to the rectangle's principal axes [441].

Remark 3 To have isotropy in this context, we need zero departure

$$0 \ = \ A \ = \ W \ - \ H \ .$$

I.e.

$$H \ = \ W \ ,$$

meaning that our rectangular shape must be square.
This is most obviously realized by the l-square family of partitions. It is however also realized by e.g. the e-v and l-triangle families, which incompletely box up their hugging square. Indeed, all homoconjugate partitions are isotropic in this sense.

Exercise 4$^-$ a) Prove this, also demonstrating that b) its converse is false, thus establishing a necessary but not sufficient criterion for homoconjugacy.

Remark 4 For $A \ > \ 0$, let us call the partition *flat*, using instead *tall* for $A \ < \ 0$. We colour these in green and white respectively. Finally, we colour isotropic and yet not homoconjugate partitions in a shade of lawn green that lies in between the preceding shade of green and white. Any lawn-green partition permits an answer to Exercise 4.b).

♠ ◇ ♣

Remark 5 For a given N, A takes values from $1 \ - \ N$ to $N \ - \ 1$. UA from 0 to $N \ - \ 1$. And α from $1/N$ to N.

Remark 6 We could normalize by dividing by say height, giving

$$a \ := \ \frac{A}{H} \ = \ \frac{W \ - \ H}{H} \ = \ \alpha \ - \ 1 \ . \tag{15.7}$$

This can be viewed as a relational variable which corrects aspect ratio by subtracting off its isotropic value,

$$a \ = \ \alpha \ - \ \alpha(\text{isotropic}) \ . \tag{15.8}$$

Standardizing the range to $[\,0,\,1\,]$ requires instead passing to

$$\alpha \ := \ \frac{N\,\alpha \ - \ 1}{N^2 \ - \ 1} \ .$$

Remark 7 (15.7) has the limitation of normalizing in a non-invariant way as regards conjugation. This may in particular not do when processing the conjugation-invariant variable UA. One way out of this would be to divide by the object number N, though using $N - 1$ turns out to be a structurally better fit.

Indeed, we shall see thoughout the current series that using $N - 1$ in place of N is natural to relative settings. In the present context of partitions, this arises as the maximum value of UA. This maximum is attained by a row of N squares which still however has a height of 1 square to subtract off.

Thus we arrive at

$$ua_1 := \frac{UA}{N - 1} = \frac{|w - h|}{N - 1} ,$$

which has the standardized range $[0, 1]$.

Remark 8 Another way out would be to divide by $\Sigma := W + H$. This is conjugation-invariant by

$$\bar{\Sigma} = \overline{W + H} = \bar{W} + \bar{H} = H + W = W + H .$$

This gives

$$ua' := \frac{UA}{\Sigma} = \frac{|W - H|}{W + H} = \frac{|\alpha - 1|}{\alpha + 1} .$$

Given Appendix A's study of this function, we can furthermore rescale this to

$$ua_{\mathrm{R}} := \frac{N + 1}{N - 1} \frac{|\alpha - 1|}{\alpha + 1}$$

so as to attain standard range $[0, 1]$.

Exercise 5$^-$ Modify the A counterpart of this to have standard range.

Remark 9 a and its close relations are indeed another example of forming a linearly-shifted relative difference quantity by subtracting off a critical value: eq. (15.8). Fig 15.3 furthermore proceeds to use a such as a centring variable.

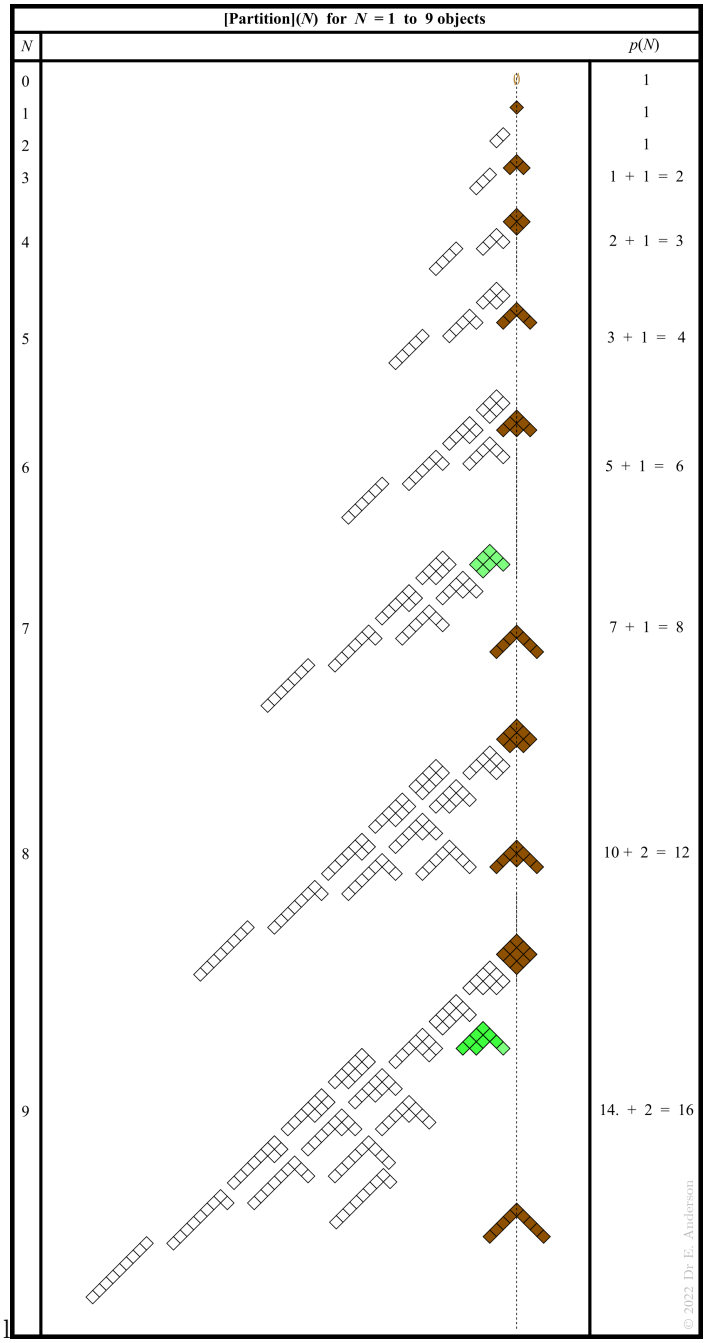


Figure 15.4:

♠ ◇ ♣

Remark 10 We have also mentioned various relative variables along the lines of

$$e \;=\; (\,\text{excess handle length}\,) \;=\; (\,\text{handle length}\,) \;-\; (\,\text{minimal handle length}\,)\;.$$

See Fig 15.5. Our study of graphs will also involve various conceptually-similar excess variables.

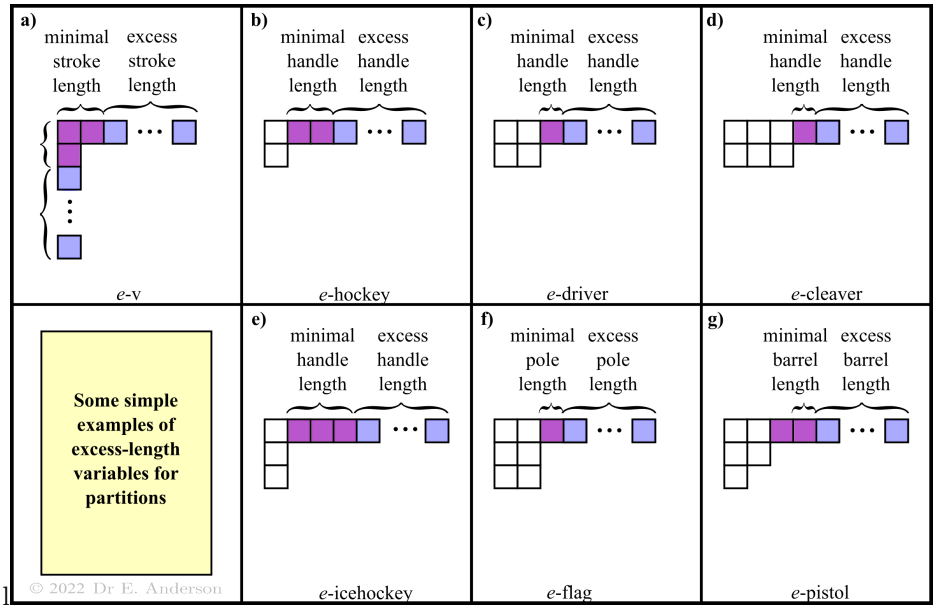


Figure 15.5:

Remark 11 Finally, in our study of arenas of partitions, we make use of the ratio variable

$$n := \frac{N}{W\,H} \quad :$$

the number of boxes per unit area of the hugging rectangle.

Exercise 6 a) Show that this is invariant under conjugation.

b) For each fixed N, which partitions maximize and minimize this variable? What are the corresponding maximal and minimal values?

c) What is the first pair of partitions that are not distinguished by N and UA? Show that n distinguishes these. What is the first pair of partitions that are not distinguished by N, UA and n?

15.6 Operations on partitions

Operation 1 'Add box', which we denote by $\Box$, involves adding a box to a given Young diagram in any licit way (so that the outcome is itself a Young diagram).

Remark 1 $\Box$ sends a given partition of N objects to various possible partitions of $N\,+\,1$ objects. A natural operation on partitions which preserves N is as follows.

Operation 2 'Slide box', which we denote by $\swarrow$, involves sliding down a box in a given Young diagram by the minimal licit amount.

Exercise 7^{-} a) What does $\Box$ v give?

b) $\Box$ driver?

c) What does $\swarrow$ give when acting upon on each homoconjugate partition for $N\;=\;1$ to 6?

Remark 2 We make use of these two operations (and ensuing orderings) in studying arenas of partitions, $\mathfrak{P}$artition, in Chapter 57.1.



15.7 Generating series for (types of) partition

Proposition 1 The generating series for partition numbers is

$$P(z) \;=\; \prod_{N \,\in\, \mathbb{N}} \frac{1}{1 \;-\; z^N} \;. \tag{15.9}$$

Proof Binomially expand each factor. All the expansions' coefficients are all 1. Thus z^N picks up a contribution of 1 whenever we select a piece that is part of a partition of N. □

Proposition 2 For partitions with no parts equal, the generating series is

$$P(\,z;\,\text{no parts equal}\,) \;=\; \prod_{N \,\in\, \mathbb{N}} (\,1 \;+\; z^N\,) \;. \tag{15.10}$$

Exercise 8$^+$ a) Prove Proposition 2.

b) Show that the generating series for partitions into pieces of size $\leq\; k$ is

$$P(\,z;\quad \leq\; k \text{ sized pieces}\,) \;=\; \prod_{N \,=\, 1}^{k} \frac{1}{1 \;-\; z^N} \;. \tag{15.11}$$

c) What is this also a partition function for?

d) What is the generating series for the partition into all parts odd? Even? $a\,(\,\mathrm{mod}\,b\,)$?

e) Prove that

$$p(\,N;\quad \text{all parts odd}\,) \;=\; p(\,N;\quad \text{all parts distinct}\,) \;. \tag{15.12}$$

[This Exercise illustrates how generating series can be very amenable to considering restricted subclasses of standard objects.

Exercise 9 Let $p(N;\,k)$ be shorthand for $p(N;\;\text{into precisely}\;\;k\;\;\text{parts}\,)$. Prove the recurrence relation

$$p(N;\,k) \;=\; p(\,N \;-\; k;\,k) \;+\; p_{m\,-\,1}(\,N \;-\; 1;\,k \;-\; 1\,) \;.$$

15.8 Reconceptualization of homoconjugate partitions as books

Proposition 3 The number of homoconjugate partitions

$$s(N) \;=\; p(\,N;\,\text{just distinct odd parts}\,) \;.$$

Proof This uses the following counting trick.

View each homoconjugate partition as an *open book* consisting of a stack of pairs of equal pages joined by a single-square 'spine' (Fig 15.6). These pairs of pages are of variable size. The first pair of pages comprises the top row and the leftmost column of the partition's Young diagram. Upon deleting these, the second pair of pages is formed by the top row and the leftmost column of the

remaining diagram, and so on. It is by homoconjugation that each page in a pair has to be the same size as its partner. Adding 1 for the corner 'spine', this forces a split into odd parts (Fig 15.6).

From how Young diagrams are by definition ordered, such books must furthermore have strictly-decreasing page size in moving from the 'cover' to the innermost pair of pages. Thus each pair of pages must be of a distinct size (Fig 15.6).

So partitions into distinct odd parts admit a 'folding' bijection converting them into homoconjugate partition 'open books'. So finite counts for each of these two types of object must coincide. □

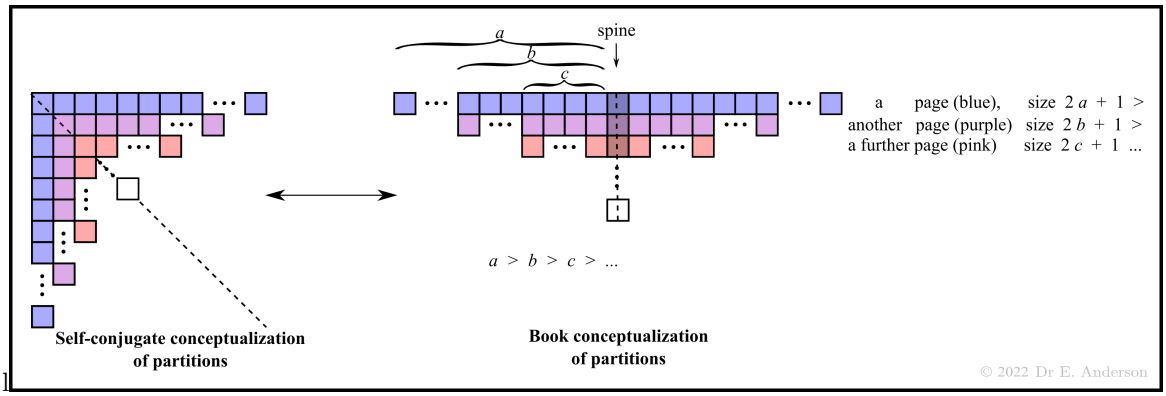


Figure 15.6:

Remark 1 We can thus use strictly-decreasing sequences of odd numbers to denote homoconjugate partitions. To avoid confusing these with the row-by-row occupation number notation for partitions, let us use round brackets and separation by commas: a fairly standard notation for sequences. In this notation, 0 is (0): the empty book (sole exception to the page size oddness requirement). 1 is (1) (Fig 15.7.a). v = 2 1 is (3) (Fig 15.7.b).

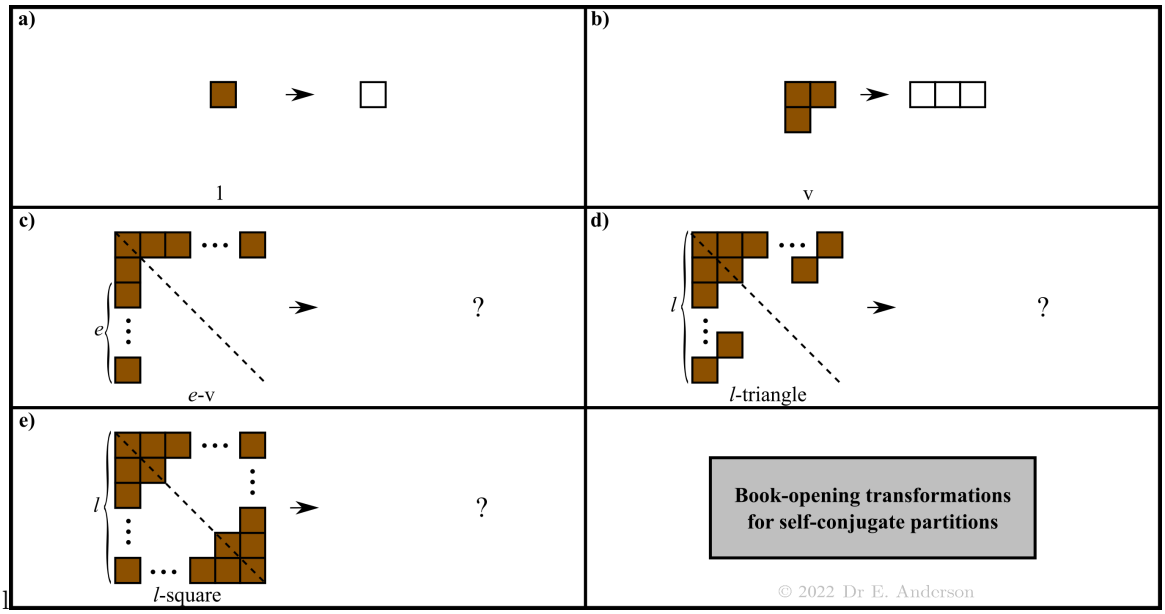


Figure 15.7:

Exercise 10 a) What are the v, triangle and square families in this notation? [I.e. finish filling in Fig 15.7.] What is the double-v family in this notation?

b) Redo Exercise 3.d) using books, now further extending your list of homoconjugate partitions to $N \leq 21$.

c) Reinterpret Exercise 3.e) in terms of books.

Remark 2 This book page sequence notation thus both supersedes, and extends, our use of families such as the four in Exercise 10.a). These are all covered by a countably-infinite parameter superfamily of books.

Remark 3 Further ancillary variables describing homoconjugate partitions then include *book page number* and *largest book page*.

15.9 Two Theorems of Euler concerning partitions

Theorem 1 [22, 50]

$$\prod_{i \in \mathbb{N}_0} (1 + z^{2i+1}) = 1 + \prod_{j \in \mathbb{N}} \frac{z^{j^2}}{\prod_{k=1}^{j} (1 - z^{2k})} . \tag{15.13}$$

Proof The LHS is the book conceptualization's form for the generating series for homoconjugate partitions.

The RHS equates to this via finding a further conceptualization for the homoconjugate partitions. Namely that they consist of a maximal square,[7] of side l, and two equal tails: a smaller partition and its conjugate. This partition is of

$$\frac{(N - l^2)}{2} ,$$

and must consist of $\leq l$ parts in order to be licit (Fig 15.8.a).

$$G(z) = \frac{1}{\prod_{q=1}^{m} (1 - z^{2q})}$$

is the generating series for partitions of N into $\leq m$ even parts. We can thus build up the generating series for partitions of

$$\frac{N - p}{2}$$

into $\leq m$ parts as follows.

$$G'(z) = \frac{z^p}{\prod_{q=1}^{m} (1 - z^{2q})} .$$

Next set

$$m = l$$

and

$$p = l^2 .$$

Whereupon the generating series is

$$G'' = \frac{z^{l^2}}{\prod_{q=1}^{l} (1 - z^{2q})} .$$

[7] Alias *Durfee square* [49, 50, 76].

Finally summing over l counts up all the homoconjugate partitions, guaranteeing equality with the RHS. □

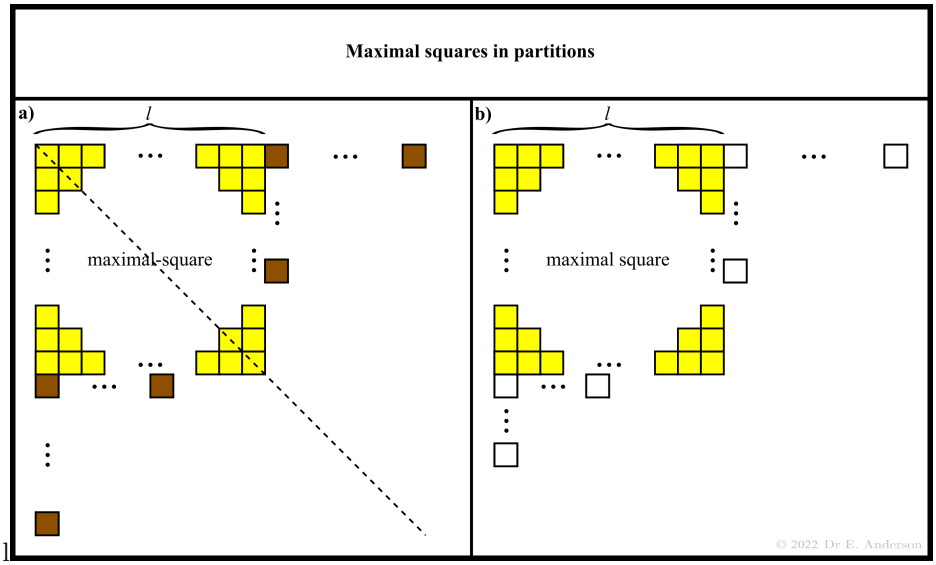


Figure 15.8:

Theorem 2 [22, 50] An alternative expression for the generating series for all the partitions is

$$P(z) \;=\; 1 \;+\; \prod_{j \,\in\, \mathbb{N}} \frac{z^{j^2}}{\prod_{k\,=\,1}^{j} (\,1 \;-\; z^k\,)^2} \;. \tag{15.14}$$

Proof Extend the maximal square decomposition to the general partition (Fig 15.8.b). Now the two tails will not be a conjugate pair (unless the overall partition is homoconjugate). We leave it as a small **Exercise 11** to encode these tails into suitable generating series expressions to give this Theorem. □



15.10 Motivation for partitions

Motivation 1 Partitions often arise in counting further objects, including as parts of larger counts with further interplay with structural restrictions. We shall see some examples of this in Chapters 28.1 and 38.

Motivation 2 Some notions of cover are a fortiori notions of partition, as for instance in Chapter 48.B.

Motivation 3-5 Aside from using Young diagrams in further manipulations of partitions, we shall encounter them again in Volume 3 [443] (in connection with the Representation Theory of[8] the permutation groups S_n [59]). And in Volume 7 [446] (in connection with the Representation Theory of Lie Algebras [59]). We here label Young diagrams' squares with numbers to form *Young tableaux* [48], this major application giving reason to use Young rather than Ferrers diagrams. Both of these

[8] See [43] for use of partition functions in the closely-related theory of the symmetric functions.

uses feed into further applications to Volume 6's Differential Geometry [445]. For instance, we shall see that [445, 446] square encodes the symmetries of the famous Riemann curvature tensor.

Chapter 16

Combinatorial special functions applied to selections

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16.1 Introduction

Remark 1 While e.g. ${}^nC_r$ and ${}^nP_r$ are just compositions of standard arithmetic and factorial elements, other Combinatorial situations can give rise to nonstandard functions. In cases in which a given (family of) such arise in multiple useful applications, such functions merit the name of *Combinatorial special functions*. These often have known and tabulated properties and inter-relations.

Remark 2 The previous chapter's partition functions are already examples of Combinatorial special functions. The two types of Stirling numbers encountered so far are further such. So are the Bell numbers (Sec 16.11), and a parallel of these, (Sec 16.13 both of which are considered below, our main objective being to exhaustively populate the 'twenty-fold way's' variety of conceptions of selections with mathematical answers.

16.2 From rising and falling factorials to Stirling numbers of the first kind, $s(n,\, k)$

Recollection 1 The coefficients in the expansion of the falling factorial $z^{\underline{k}}$ are the *Stirling numbers of the first kind*, $s(n,\, k)$. I.e.

$$z^{\underline{n}} \;=\; \sum_{k \,\in\, \mathbb{N}} s(n,\, k)\, z^k \;. \tag{16.1}$$

Exercise 1$^-$ Show that if we expand the raised factorial $z^{\overline{n}}$ instead, then the coefficients coincide with the unsigned values of the $s(n,\, k)$,

$$\left[\begin{array}{c} n \\ k \end{array} \right] \;:=\; |\, s(n,\, k)\,| \;.$$

Notational Remark 1 Denoting the unsigned $s(n,\, k)$ in this way is part of what has come to be known as *Karamata notation* for Stirling numbers. We view this as part of a slightly larger *Combinatorial bracket notation* for the basic 20-fold of selections. A previously (and still) common notation for the $\left[\begin{array}{c} n \\ k \end{array} \right]$ is $C(n,\, k)$. Some reasons to favour the Combinatorial bracket notation are covered in Exercise 8, the two Notational Remarks under this, Volume 1 [441], Exercises 15 to 18 and the Naming Remark under these.

Remark 1 The general kind of idea behind special functions is as follows. Asking such as "what are the coefficients in the series of" does not always return an already-known function as its answer. This is itself an acceptable way of defining further functions, particularly if these turn out to be repeatedly useful. Similar applies when an unknown function arises as a series with coefficients of known closed form. And again for solutions to recurrence relations or difference equations. We shall furthermore see various continuum-modelling Calculus parallels of these situations in Volume 1 [441], alongside giving a comparison of these with the current Volume's Combinatorial special functions. A further common way of obtaining special functions is as *transforms* of known functions (be this a continuum transform or a discrete one, there e.g. being Fourier transforms of both of these kinds).

16.3 Smallest examples of Stirling numbers of the first kind

Example 0 Expanding $n^{\underline{k}}$,

$$\left[\begin{array}{c} 0 \\ 0 \end{array} \right] = 1 \, , \quad \left[\begin{array}{c} n > 0 \\ 0 \end{array} \right] = 0 = \left[\begin{array}{c} 0 \\ m > 0 \end{array} \right] .$$

Example 1 $\left[\begin{array}{c} n \\ n \end{array} \right] = 1$.

Example 2 $\left[\begin{array}{c} 2 \\ 1 \end{array} \right] = 1$. Thus $n = 2$'s nontrivial entries read 1, 1, which is still symmetric.

Example 3 But then $\left[\begin{array}{c} 3 \\ 1 \end{array} \right] = 2$ and $\left[\begin{array}{c} 3 \\ 2 \end{array} \right] = 3$. So $n = 3$'s nontrivial entries are minimal for displaying asymmetry: 2, 3, 1. This asymmetry subsequently persists.

Exercise 2 a) Show that

i) $\left[\begin{array}{c} n \\ 1 \end{array} \right] = (n - 1)!$

ii) $\left[\begin{array}{c} n \\ n - 1 \end{array} \right] = \left(\begin{array}{c} n \\ 2 \end{array} \right)$.

iii)

$$\left[\begin{array}{c} n \\ n - 2 \end{array} \right] = \frac{n(n - 1)(n - 2)(3n - 1)}{24} .$$

iv)

$$\left[\begin{array}{c} n \\ n - 3 \end{array} \right] = \frac{n^2 (n - 1)^2 (n - 2)(n - 3)}{48} .$$

b) Attach signs to Examples 0 to 3 and part a) above, so as to obtain the corresponding expressions for the $s(n, m)$.

16.4 Recurrence relations

Lemma 1

$$s(n + 1, k) = s(n, k - 1) - n\, s(n, k) . \tag{16.2}$$

Proof We immediately have a recurrence relation for the falling factorials themselves:

$$z^{\underline{k+1}} = (z - k)\, z^{\underline{k}} .$$

Thus

$$\sum_{k \in \mathbb{N}_0} s(n + 1, k)\, z^k = \sum_{k \in \mathbb{N}_0} (z - n)\, s(n, k)\, z^k \overset{\text{expanding}}{=} \sum_{k \in \mathbb{N}_0} s(n, k)\, z^{k+1} - n \sum_{k \in \mathbb{N}_0} s(n, k)\, z^k .$$

The first term here can be rewritten under the index substitution

$$k' \ := \ k \ + \ 1$$

as

$$\sum_{m' \in \mathbb{N}_0} \ s(n, \, k' \ - \ 1) \, z^{k'} \ .$$

Now apply the summation dummy index change $k' \ \Rightarrow \ k$ to cast the first term into the form

$$\sum_{k \in \mathbb{N}_0} \ s(n, \, k \ - \ 1) \, z^{k} \ .$$

We can now finally just equate coefficients of z^k to obtain our recurrence. □

Remark 1 This is a linear first-order recurrence relation. It is furthermore a recurrence relation *in two variables*: now analogous to a 2-variable PDE rather than an ODE. N-variable recurrence relations are a target for N-fold induction proofs (even though these may not be the most efficient method of proof).

Exercise 3 Can you prove (16.2) using a double induction instead? Compare the length of your proof with that of the above generating series type proof.

Lemma 2

$$\left[\begin{array}{c} n \ + \ 1 \\ m \end{array} \right] \ = \ \left[\begin{array}{c} n \\ m \ - \ 1 \end{array} \right] \ + \ n \left[\begin{array}{c} n \\ m \end{array} \right] \ .$$

Exercise 4 Prove this.

16.5 More about [ ]

Structure 1 Fig 16.1.a) presents a table of these, using our smallest and series results, and then the recurrence relation to fill in the somewhat harder middle (coloured deep purple). An alternative strategy would be to keep on finding further series results for south-east diagonals sweeping from right to left. The number of simple series involved being subjective (four are marked in our Figure: the yellow and red diagonals and the green and blue columns). When two simple series intersect, in the current Series of Books, we *superpose* their colours.

Structure 2 The centring variable that we use for Stirling numbers is

$$s \ := \ 2 \, k \ - \ n \ - \ 1 \ . \tag{16.3}$$

This differs from the centring variable r that we used for the Pascal alias ( ) triangle, since now $n \ = \ 0 \ = \ k$ is not included in our centred table. Now to be on the midline,

$$0 \ = \ s \ = \ 2 \, n \ - \ k \ - \ 1$$

is required. So

$$k \ \text{ must be odd} \ .$$

Remark 1 Both the combination and Stirling centring variables can be recast in terms of the difference of the row variable that is getting centred, k, and of the critical value for centreing,

$$n_{\text{crit}} \ = \ \frac{n}{2} .$$

This even more directly aligns these examples with how we have so far treated power sets, families and partitions, and how we subsequently treat graphs and posets.

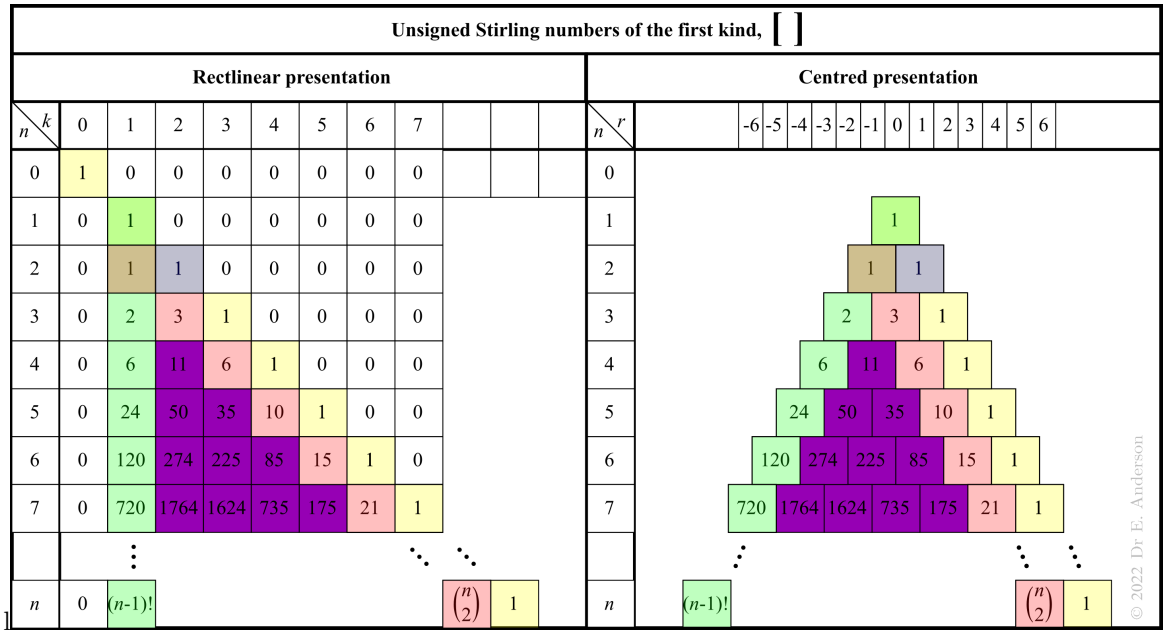


Figure 16.1:

Remark 2 The centred tables of Stirling numbers exhibit no reflection symmetry, there is no way of producing a further relational quotient version for these unlike for the ( ) This rests upon a difference between the corresponding recurrence relations. As Remark 2 of Sec 16.10 further details after we have shown that these two Remarks apply just as well to the other kinds of Stirling numbers.



16.6 Stirling numbers of the second kind, { }

Structure 1 We here select n *distinguishable* balls and place them in m indistinguishable boxes, each of which is at least singly-occupied. This returns another a priori unrelated Combinatorial special function: the *Stirling numbers of the second kind*, [60, 42, 41, 23, 32, 40, 66] which we denote by

$$\left\{ \begin{array}{c} n \\ k \end{array} \right\} .$$

16.7 Smallest examples of { }

Example 0 By convention, $\left\{ \begin{array}{c} n \\ 0 \end{array} \right\} = 0$.

Example 1 $\left\{ \begin{array}{c} n \\ 1 \end{array} \right\} = 1$ for $n \in \mathbb{N}$, since there is precisely 1 way of splitting $n \in \mathbb{N}$ distinguishable objects into a single box.

Example 2 $\left\{ \begin{array}{c} n \\ n \end{array} \right\} = 0$, since we must put one ball in each box to succeed in occupying each box.

Example 3 $\left\{ \begin{array}{c} 3 \\ 2 \end{array} \right\} = 3$,
since there are 3 ways of leaving one ball by itself: $12\,|\,3$ and cyclic permutations.

Example 4 $\left\{ \begin{array}{c} 4 \\ 2 \end{array} \right\} = 7$, since there are 4 ways of leaving one ball out: $123\,|\,4$ and cyclic permutations. Plus 3 ways of making pairs: $12\,|\,34$, $13\,|\,24$ and $14\,|\,23$. Also $\left\{ \begin{array}{c} 4 \\ 3 \end{array} \right\} = 6$, since there are $\left(\begin{array}{c} 4 \\ 2 \end{array} \right) = 6$, ways of choosing which 2 balls to pair up.

Remark 1 'Smallest examples' here really become 'smallest nontrivial central examples'. I.e. to the exclusion of various peripheral series, a first three of which are given above.

Remark 2 Our above method for $\left\{ \begin{array}{c} 4 \\ 3 \end{array} \right\}$ turns out to immediately generalize to give a fourth simple series. Namely,

$$\left\{ \begin{array}{c} n \\ n-1 \end{array} \right\} = \left(\begin{array}{c} n \\ 2 \end{array} \right) = \frac{n(n-1)}{2} .$$

Exercise 5 a) Derive a fifth simple series: that $\left\{ \begin{array}{c} n \\ 2 \end{array} \right\} = 2^{n-1} - 1$.
b) what happens if $k = 3$ instead?

16.8 Recurrence relation for { }

Lemma 3

$$\left\{ \begin{array}{c} n \\ m \end{array} \right\} = \left\{ \begin{array}{c} n-1 \\ k-1 \end{array} \right\} + k \left\{ \begin{array}{c} n-1 \\ k \end{array} \right\} . \tag{16.4}$$

Proof Let p_k be a partition of $[1, n]$ into k parts. Then the following two ways of treating the nth ball are exhaustive and mutually-exclusive, and thus add up to give all the possibilities.

Either the nth ball sits in its own private compartment. This gives 1 case per problem with 1 ball less, of which there are

$$\left\{ \begin{array}{c} n-1 \\ k-1 \end{array} \right\} .$$

Or the nth ball shares compartment with some of the other balls. In such a case, our partition takes the form of an m-partition of $[1, n-1]$ with the nth ball added to one of its parts. This addition can be performed in m ways for each of the $\left\{ \begin{array}{c} n-1 \\ k \end{array} \right\}$ partitions.

It then follows that $\left\{ \begin{array}{c} n \\ k \end{array} \right\}$ is the sum of these outcomes. I.e.

$$1 \times \left\{ \begin{array}{c} n-1 \\ k-1 \end{array} \right\} + m \times \left\{ \begin{array}{c} n-1 \\ k \end{array} \right\} . \square \tag{16.5}$$

Remark 1 This direct Combinatorial proof is again superior to setting up a double induction proof.

16.9 The { } triangle

Structure 1 We can use this recurrence relation to populate the nontrivial middle of the table of Stirling numbers. See Fig 16.2.a). We can then centre our table as per Fig 16.2.b), using the same shifted variable s as in Eq (16.3). Again, there is no reflection symmetry about the midline.

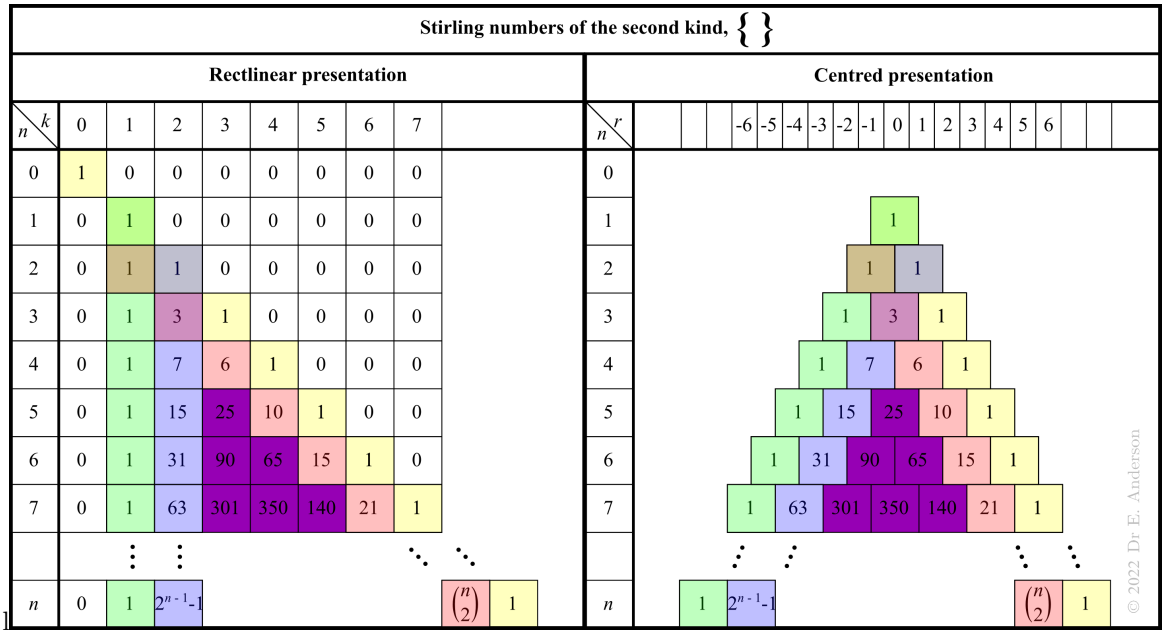


Figure 16.2:

Exercise 6 On this occasion, the part of the $\left\{ \quad \right\}$ triangle exhibited in our Figure at least has its maxima running down the midline. Show that this does not however persist, by finding the smallest n for which this centring of maximal values fails.

Exercise 7 We can also use the above recurrence relation to find formulae for the next most peripheral series.

a) Use it, results from Chapter 5 and a short induction to prove the following.

$$\left\{ \begin{matrix} n \\ n-2 \end{matrix} \right\} = \frac{n(n-1)(n-2)(3n-5)}{24} .$$

b) Also prove this using a direct Combinatorial argument.

c) Show that $\left\{ \begin{matrix} n \\ 3 \end{matrix} \right\}$ obeys the following first-order linear inhomogeneous difference equation.

$$\left\{ \begin{matrix} n \\ 3 \end{matrix} \right\} = 3 \left\{ \begin{matrix} n-1 \\ 3 \end{matrix} \right\} + 2^{n-2} - 1 .$$

Hence find an expression for $\left\{ \begin{matrix} n \\ 3 \end{matrix} \right\}$. Compare this method with the one you used in Exercise 5.b).

Exercise 8 Which $\left(\ \right)$ identities that you have encountered so far in the current Volume possess close analogues for $\left[\ \right]$ and $\left\{ \ \right\}$? Which do not?

Notational Remark 2 This partial analogy motivates using the *Combinatorial brackets notation* $\left(\ \right)$, $\left[\ \right]$, $\left\{ \ \right\}$. See in particular Knuth's account [40] for further motivational discussion about using these three symbols.

Exercise 9 Show that

$$(x+y)^{\underline{n}} = \sum_{k=0} \left(\begin{matrix} n \\ k \end{matrix} \right) x^{\underline{n-k}} x^{\underline{k}} .$$

What better-known result is this quite a close analogue of? Is there a further analogue for falling factorials?

16.10 Stirling numbers of the third kind, ⌊ ⌋

Notational Remark 3 See also [441] for why Stirling numbers of the third kind (alias *Lah numbers* [25, 65]) should be treated on an equal footing with [ ] and { }. We thereby ascribe to the notation [74] ⌊ ⌋ for these, rather than using the hitherto more common $L(n, k)$. This shape of bracket has quite plausibly been chosen due to its opening half's resemblance to the letter L. They will not be confused with floor functions when their arguments are present, for they have two to floor functions having just one. Since we occasionally present each without arguments, however, we make the further distinction that our combinatorial brackets are in bold.

Remark 1 In fact, the ⌊ ⌋ are not special functions at all, since these can be expressed as a product of basic functions:

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor = \left(\begin{matrix} n-1 \\ k-1 \end{matrix} \right) \frac{n!}{k!} .$$

Exercise 10 Show that this answers the distinct-and-ordered balls into identical boxes case of the collector's selection problem.

Exercise 11 Since we have the above simple closed formula for them, the ⌊ ⌋ are less in need of a recurrence relation. Show nonetheless that these obey

$$\left\lfloor \begin{matrix} n+1 \\ k \end{matrix} \right\rfloor = (n+k) \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor + \left\lfloor \begin{matrix} n \\ k-1 \end{matrix} \right\rfloor .$$

Structure 2 See Fig 16.3 for the first few entries of the ⌊ ⌋ triangle.

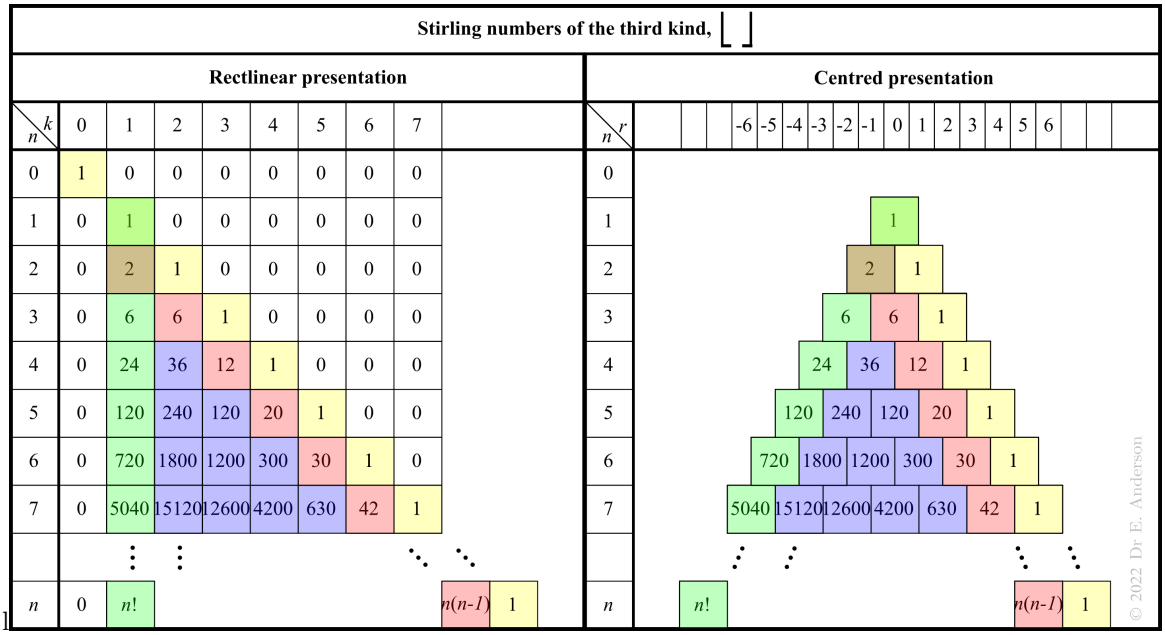


Figure 16.3:

Remark 2 Thus indeed none of these three kinds of Stirling numbers has reflection symmetry about the midline and its row maxima quickly depart from the midline as well. Thus overall, centring is less

motivated for Stirling numbers than for the triangle of combinations. In particular, the weighting in the recurrence relation – the extra k, n or $n + k$ factor in one of each relations' two summands – can readily be envisaged to be preventing a mirror symmetry from taking root.

Remark 3 See e.g. [41, 42, 76] for other introductory treatments of Stirling numbers, and e.g. [23, 32, 66] for more advanced reviews.



16.11 $\mathbb{2}_n = \sum \cdot \left\{ \right\}$, alias Bell or exponential numbers

Structure 1 Let us next consider

$$\mathbb{2}_n := \sum_{k=1}^{n} \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} . \tag{16.6}$$

Naming Remark 1 These are often elsewhere called *Bell numbers*, denoted by B_n [41, 42, 65, 66], or *exponential numbers*.

Example 1 Summing up $\left\{ \right\}$, the first few $\mathbb{2}_n$ are $1, 1, 2, 5, 15$ and 52.



Remark 1 The following recurrence relation gives each exponential number as the totality of smaller exponential numbers.

Lemma 4 [42, 41]

$$\mathbb{2}_n = \sum_{i=1}^{n} \left(\begin{matrix} n-1 \\ k \end{matrix} \right) \mathbb{2}_k = \sum_{i=1}^{n} \left(\begin{matrix} n-1 \\ k-1 \end{matrix} \right) \mathbb{2}_{n-k} .$$

Remark 2 This recurrence relation is linear, but of no fixed order: each n's has order one higher than the previous.

Exercise 12 Prove Lemma 4.

Exercise 13$^-$ Find the next two $\mathbb{2}_n$, firstly by Example 1's method and secondly by using Lemma 4. Which method is faster?

16.12 Examples of exponential generating series

Example 1 The exponential generating series for the exponential numbers $\mathbb{2}_n$ is [49]

$$E(z) := \sum_{n \in \mathbb{N}_0} \frac{\mathbb{2}_n}{n!} z^n = \exp^{\exp(z)-1} . \tag{16.7}$$

Example 1 Stirling numbers of the second kind also have a useful exponential generating series [49],

$$S_k(z) := \sum_{n \geq k} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \frac{z^n}{n!} = \frac{(e^z - 1)^k}{k!} \text{ for } k \geq 0 . \tag{16.8}$$

Pointer 1 While we are now reasonably well-served with examples of exponential generating series, we however await Volume 1's Calculus reinforcements before deriving these results.

Naming Remark 2 'Exponential numbers' is a truer name for the Bell numbers,[1] due to the three exponentialities that enter their generating series. Namely, that their exponential generating series features the exponential of the exponential function! As one sees more mathematics, however, being 'thrice exponential' may cease to be a unique feature. So on the long run, we prefer naming them by the symbol $\sum \cdot \left\{ \right\}$. This emphasizes their formulation as sums of Stirling numbers of the second kind. In formulae, we choose to denote them by the shorthand $\mathbb{2}_n$.

16.13 Motivational Exercises

Exercise 14 Show that the Bell numbers count the partitions of a labelled set.

Exercise 15 Show that the 2-input sum of second-Stirlings

$$\mathbb{2}_{n,\,k} \;=\; \sum_{i\,=\,1}^{n} \left\{ \begin{matrix} i \\ k \end{matrix} \right\}$$

answers the distinct-balls and identical-boxes selection problem. Let us denote this by $\sum \left\{ \right\}$ (now with no dot, for Multilinear-Algebraic reasons that we postpone to Volume 1).

Exercise 16 Show that (16.6) furthermore answers the Combinatorial question of the number of partitions of an n-element set 'into at most m pieces'.

Exercise 17 Show that the 2-input sum of third-Stirlings

$$\mathbb{3}_{n,\,k} \;=\; \sum_{i\,=\,1}^{n} \left\lfloor \begin{matrix} i \\ k \end{matrix} \right\rfloor \tag{16.9}$$

answers the distinct-and-ordered balls and identical boxes.

Exercise 18 a)$^-$ Define the $\mathbb{3}_n$ analogously to the $\mathbb{2}_n$. Tabulate these up to $n \;=\; 5$.

b)$^+$ Can you find a recurrence relation for the $\mathbb{3}_n$?

c) Thus (or otherwise), extend your table up to $n \;=\; 7$.

Naming Remark 3 The (16.9) remain 'officially unnamed' to date [74]. One possible interim name is 'para-Bell numbers' (possibly to be pronounced more like 'parable').

While 'exponential numbers' is a truer name for the Bell alias $\mathbb{2} \;=\; \sum \left\{ \right\}$ numbers, 'para-exponential numbers' would not do for the $\mathbb{3}_n$. This is because the $\left\{ \right\}$ to $\left\lfloor \right\rfloor$ parallel does not happen to extend to them having similar generating series. Indeed, placing value on this parallel gives further reason to Naming Remark's 2 long-term suggestion to abandon the name 'exponential numbers'. In contrast, the notation $\mathbb{2}$ and the name $\sum \left\{ \right\}$ of course immediately extend to $\mathbb{3}$ and $\sum \left\lfloor \right\rfloor$ respectively. With the notation $\sum \left\{ \right\}$ and $\sum \left\lfloor \right\rfloor$ covering the other two sums in the current section.

♠ ◇ ♣

Exercise 19$^+$ (Long) Derive everything that is left to be derived in the nontrivial 15-fold core of the 20-fold (or 32-fold) approach in Fig 16.4.

[1] According to e.g. [77] Japanese mathematicians were aware of these some centuries before Bell.

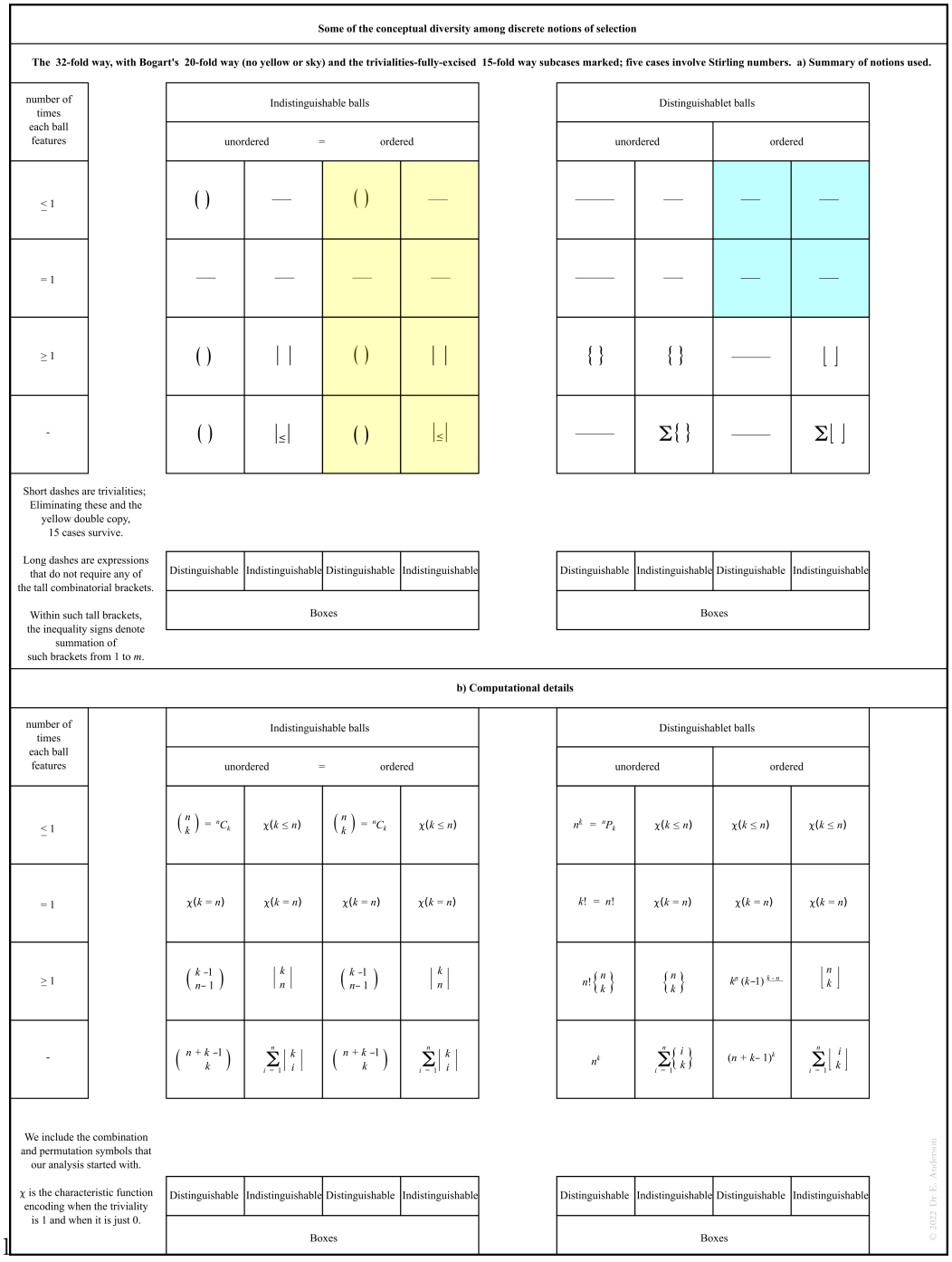


Figure 16.4:

End-Remark 1 Fig 16.4 ends Part II's account of selections. The 15-fold core includes 5 uses of Stirling numbers. And 5 uses of bona fide Combinatorial special functions: 3 of $\left\{ \right\}$ and 2 of partition functions $\left| \right|$. Our notation here for the partition functions is meant to mirror the rest of our tall Combinatorial brackets notation.

End-Remark 2 Some partial-distinguishability versions of $\left\{ \right\}$, $\lfloor \rfloor$, 2, and 3 have been put forward and studied [34, 68, 74]. Though once we are armed with some Group Theory, in Sec 22.1

we can argue that these far from exhaust all cases of possible partial distinguishability.

16.14 Further Combinatorial Exercises

Exercise 20 Show that the number of permutations with m disjoint cycles yields the Stirling numbers of the first kind for $m = 1$ to 4.

Exercise 21 How many functions, injections, surjections and bijections are supported between an m-element set and an n-element set? [Take m and n to both be finite.]

Exercise 22 What is the number of possible equivalence relations on a finite set?



Chapter 17

Countability

17.1 Introduction

Definition 1 There are precisely two cases of sets that are said to be *countable* (alias *enumerable*).

a) The finite sets.

b) The sets whose elements are in bijective correspondence with $\mathbb{N}$. These are called *infinitely-countable* or *denumerable*.

Remark 1 a) and b) are equivalently the sets which can be finitely and infinitely listed on the line respectively.

Exercise 1$^-$ Prove this equivalence.

17.2 Bijective correspondences with $\mathbb{N}$

Proposition 1 If $\mathfrak{Y} \subseteq \mathbb{N}$, then $\mathfrak{Y}$ is countable.

Proof Since the elements of $\mathfrak{Y}$ are distinct elements of $\mathbb{N}$, they can be ordered by size. We can consider a map sending the smallest element of $\mathfrak{Y}$ to 1, the second-smallest to 2 and so on. So our ordering by size can be interpreted as a unit-separation linear list on $\mathbb{N}$. $\square$

Diagnostic Lemma 1 A set $\mathfrak{X}$ is countable iff either of the following hold.

i) There is an injective map

$$j : \mathfrak{X} \longrightarrow \mathbb{N} .$$

ii) There is a surjective map

$$s : \mathbb{N} \longrightarrow \mathfrak{X} .$$

Proof i) The following two cases are exhaustive.

1) For $\mathfrak{X}$ finite, the result is clear.

2) For $\mathfrak{X}$ countably-infinite, ($\Rightarrow$) is clear.

($\Leftarrow$) Let j be such an injective function. Then $j(\mathfrak{X}) \subseteq \mathbb{N}$. So Proposition 1 applies.

ii) Lemma 1 of Chapter 2 gives that i) $\Leftrightarrow$ ii). $\square$

Proposition 2 The integers $\mathbb{Z}$ are countable.

Proof

$$\beta \; : \; \mathbb{Z} \; \longrightarrow \; \mathbb{N}$$

$$x \; \mapsto \; \begin{cases} -2\,x & \begin{cases} x \; < \; 0 \\ x \; \geq \; 0 \end{cases} \\ 2\,x \; + \; 1 \end{cases} \quad \text{is a suitable bijection .}$$

In words, this sends the negative numbers to the even numbers, and the non-negative numbers to the odd numbers. □

Exercise 2^{-} Show that the following are countable.

i) The odd natural numbers.

ii) The odd integers.

iii) The integers that are 1 (mod p) for any p prime.

Proposition 3 (Hereditarity) Any subset of a countable set is countable.

Exercise 3 Prove this.

Proposition 4 (Finite productivity) i) $\mathfrak{X}^k$ is countable for k finite if $\mathfrak{X}$ is.

ii)

$$\prod_{i \; = \; 1}^{k} \mathfrak{X}_i$$

is countable if each $\mathfrak{X}_i$ is.

Proof i) We establish an injective map

$$j \; : \; \mathbb{N} \; \longrightarrow \; \mathbb{N} \; \times \; \mathbb{N} \; .$$

$$(m,\, n) \; \mapsto \; 2^m \, 3^n \; .$$

The injectivity of this is guaranteed by the Fundamental Theorem of Arithmetic. Finally apply Diagnostic Lemma 1.i).

ii) $P_n \; := \; (\, \mathbb{N}^n \; \cong \; \mathbb{N} \,)$.

Base step: this is obviously true for $n \; = \; 1$

Inductive step: suppose that P_n is true for $n \; = \; k \; \geq \; 1$. Then

$$\mathbb{N}^{k \; + \; 1} \; = \; \mathbb{N}^k \; \times \; \mathbb{N} \; \cong \; \mathbb{N} \; \times \; \mathbb{N}$$

maps bijectively to $\mathbb{N}$ by i). Thus P_n holds for $n \; = \; k \; + \; 1$ and thus $\forall \; k \in \mathbb{N}$ by induction. □

Remark 1 The primes 2 and 3 are chosen for convenience here to be the smallest two; any other pair of distinct primes would do in this role.

Exercise 4 Prove ii) straight off – bypassing i) – by evoking the k smallest primes.

Corollary 1 The rationals $\mathbb{Q}$ are countable.

Proof Represent the rational numbers in lowest form x/y, and also w.l.o.g. with $y > 0$. Then the following injective map is well-defined.

$$\beta : \mathbb{Q} \longrightarrow \mathbb{Z} \times \mathbb{Z}$$

$$\frac{x}{y} \mapsto (x, y) :$$

in vector component notation. Furthermore, we have a bijection between $\mathbb{Z}$ and $\mathbb{N}$ by Proposition 2, And another between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$ by Proposition 4. So our target space above, $\mathbb{Z} \times \mathbb{Z}$, is countable, and thus Diagnostic Lemma 1.i) applies. □

Proposition 5 (Finite unitivity)

$$\bigcup_{i=1}^{k} \mathfrak{X}_i$$

is countable if each $\mathfrak{X}_i$ is.

Proof We again establish an injective map into $\mathbb{N}$ and then apply Diagnostic Lemma 1.i). First of all, we know that there are injections

$$j_i : \mathfrak{X}_i \longrightarrow \mathbb{N}$$

for each $i = 1$ to k. Then the following is a well-defined injection.

$$j : \mathfrak{X} \longrightarrow \mathbb{N}$$

$$x \mapsto 2^i \, 3^{j_i(x)} ,$$

for i the minimal value for which $x \in$ the corresponding $\mathfrak{X}_i$. □

17.3 Countable versus uncountable infinity

Proposition 6 The reals $\mathbb{R}$ are uncountable.

Proof Let $\{x_i\}_{i \in \mathbb{N}}$ be any countable sequence of real numbers. Express each x_i in the form

$$x_i = w_i + \frac{1}{2} b_{i1} + \frac{1}{4} b_{i2} + ...$$

Here, $w_i := \lfloor x_i \rfloor$. Also the binary 'bigits' b_{ik} are all to take values 0 or 1.

Next let

$$y := \frac{1}{2} b_1 + \frac{1}{4} b_2 + ...$$

The new binary 'bigits' here are to be 0 if $b_{jj} \neq 0$ and yet 1 if $b_{jj} = 0$. Then $y \in \mathbb{R}$ but $y \neq$ any x_i in our sequence. So $\mathbb{R}$ cannot be in bijective correspondence with $\mathbb{N}$. $\mathbb{R}$ is thus uncountable by the contrapositive statement to the definition of countability. □

Question 1 What concepts of infinity are possible?

The cardinality of $\mathbb{N}$ is denoted by $\aleph_0$, pronounced aleph-zero (aleph being the first Hebrew letter). It can then be shown that the cardinality of $\mathbb{R}$ is $2^{\aleph_0}$. *Ordinals* are enumerative numbers, in contrast to *cardinals*, which measure the sizes of sets. These two notions coincide for finite sets, by which most people are not aware that these two notions can be distinctly realized. And yet for infinite sets, vast differences between these two notions become possible. $\aleph_1$ is then used to denote the number

of ordinals on an infinite set, which is qualitatively vastly larger than the corresponding number of cardinals $\aleph$.

The *Continuum Hypothesis* is that

$$\aleph_1 \ = \ 2^{\aleph_0} \ . \tag{17.1}$$

Exercise 5 Which of the following are countable?

i)

$$\bigcap_{i \ = \ 1}^{k} \mathfrak{X}_i \ \text{ for each } \ \mathfrak{X}_i \ \text{ countable .}$$

ii)

$$\bigcup_{n \in \mathbb{N}} \mathfrak{X}_n \ \text{ for each } \ \mathfrak{X}_n \ \text{ countable } \ .$$

iii) $\mathbb{N}^{|\mathbb{N}|}$.

iv) The complex numbers $\mathbb{C}$.

v) $\mathfrak{X}^{|\mathfrak{Y}|}$ for any $\mathfrak{X}$, $\mathfrak{Y}$ countable.

vi)

$$\prod_{n \in \mathbb{N}} \mathfrak{X}_n \ \text{ for each } \ \mathfrak{X}_n \ \text{ countable .}$$

Exercise 6 The trick in Proposition 6's proof, due to Cantor [18], is well-named as a *diagonal argument* [33]. This is a *useful trick*, meaning that a Mathematics student may well encounter other uses for it, and that those that become Mathematicians will likely make many uses of such. For now, use a diagonal argument to establish an alternative proof for Exercise 5.ii)'s statement.

Remark 1 Less significantly, binary 'bigits' encapsulate the binary analogue of 'going beyond the decimal point' with *decimal digits*). Decimal digits would do, as would working in any other base. Though, like using the smallest primes elsewhere in the current Chapter, the 'binary' presentation has the slight benefit of also being *minimal*.

17.4 Further motivation

Motivation 1 *Discrete Mathematics* [75, 57, 67, 63, 55, 76, 49, 72] is the study of countable mathematical objects, spaces, games... So if we can determine that some entities that we are studying are finite, we know that some of the resources from this large field of Mathematics may be usefully tapped.

Discrete Mathematics entities that are finite can furthermore in principle be stored, retrieved and processed using a large enough computer. Finitely-countable such can be treated likewise, if using a cut-off point is acceptable.

Remark 1 We can now extend 1.1's discussion of indexing sets to discern between, firstly, countably-infinite ones, denoted by such as $n \ \in \ \mathbb{N}$. Secondly, uncountably-infinite ones, denoted by such as $x \ \in \ \mathbb{R}, \mathbb{R}_+$ or $[0, 1]$: the standard closed interval subset of $\mathbb{R}$.

Exercise 7 Show that the last two of these are indeed uncountably infinite.

Remark 2 We have already encountered one example of countably-infinite parameter family of mathematical objects, in Sec 15.8 .

Remark 3 The Discrete Mathematics regime is in many ways qualitatively distinct from *Continuum Mathematics*. Though it would be wrong to assume that a given entity is continuous or discrete; some reasons for this are as follows.

1) Different aspects of our entity could manifest both of these regimes at once. See e.g. Sec 19.6 for an example.

2) It is in principle possible to use discrete models to model continuum systems and vice versa. This is how it is possible in practice for computers to model continuum (and infinite discrete) systems. 'Finite discretizations' and 'continuum-approximation limits' are some of the denizens of this second point.

3) Such as Topology sweeps out 'conceptual possibilities that lie between' discrete and continuum modelling. For some topological spaces can exhibit features partly in between continuum and discrete regimes, while others can have unexpected wild features that do not correspond to either regime.

Thus in many ways, using such as Topological modelling supersedes 'continuum or discrete' dichotomies (which are thereby usually seen to have been *false* dichotomies).

♠ ◇ ♣

Motivation 2 (Topology) One may wish to avoid having too many things under certain circumstances. E.g. i) too many subsets or ii) too many intersections between subsets.

One option for this is to demand that at-most finite quantities of these occur. Look back to the definitions of compact and paracompact in Sec 9.9 for examples of i) and ii) respectively.

Another option is however to demand that at-most countable quantities of such occur. I.e. what happens if one demands merely *countable* subcovers or locally-*countable* refinements. On some occasions, one of a finite-or-countable pair is more advanced than the other [264], or can substantially affect what can be proven.

A further way in which numbers of open subsets can be limited in a topological space $\mathfrak{X}$ is restricting to each point having a countable neighbourhood basis. This is called *first countability*.

First recollect what a base for a topological space is from Sec 9.6. A *neighbourhood base* is then a base for a neighbourhood of a point in a space, rather than for the whole space.

Removing the 'to each point' restriction and corresponding mention of 'neighbourhood', one arrives at the stronger notion of *second countability*.

First countability roughly corresponds to sequences sufficing to probe for most topological properties. This is one of the main two conditions to be able to use fairly standard Analysis. Second countability is a very common (if not quite universal) requirement so as to have a manifold. Manifolds are moreover extremely widely-used structures in Topology, Differential Geometry, and applications from General Relativity and the Principles of Dynamics to Computer Graphics and Robotics.

Countable and complement-countable (hitherto more often termed co-countable) are also two further types of useful and commonly-encountered topologies on point sets. There are also corresponding finite and complement-finite topologies (hitherto more widely termed co-finite).

In such ways, Topology accumulates quite a number of concepts that are at least in part notions of countability.

Exercise 8 What bases are supported on sets with 0 to 3 elements?

Exercise 9 Show that the finite, co-finite, countable and co-countable topologies are indeed topologies.

Motivation 3 Our discussion of posets in Part VII makes repeated use of the distinction between countably-infinite and uncountably-infinite objects.

Motivation 4 There are multiple ways of defining ordinals in (various kinds of) Axiomatic Set Theory, and further ways of obtaining yet higher alephs as well [46].

Part III

Three Pillars of Group Theory

Chapter 18

Pillar 1: Concrete groups

18.1 Permutations form groups

Notation 1 Let $\mathfrak{X}$ be a set. The bijective maps

$$B \ : \ \mathfrak{X} \ \longrightarrow \ \mathfrak{X}$$

are then of interest, as is the arena that these form,

$$\mathfrak{Bij}(\mathfrak{X}) \ .$$

But for finite $\mathfrak{X}$, we have already mathematically identified the B to be permutations σ. In this case, we can re-denote $\mathfrak{Bij}(\mathfrak{X})$ as

$$S(\mathfrak{X})$$

in acknowledgment of this mathematical identification.

Remark 1 In the finite case, bijections enforce that $\mathfrak{X}_1 \ \longrightarrow \ \mathfrak{X}_2$ are such that

$$|\,\mathfrak{X}_1\,| \ = \ |\,\mathfrak{X}_2\,| \ .$$

For unlabelled and elsewise unstructured sets, this narrows attention down to $\mathfrak{X} \ \longrightarrow \ \mathfrak{X}$ alone.

Remark 2 In the *concrete perspective*, we are to build up what turn out to be the group axioms from properties of bijections (and thus of injections and of surjections). $S(\mathfrak{X})$ is thereby not just a set of permutations of a finite number of objects. It is furthermore equipped as a *a group* by the operation of map composition,

$$\langle \, S(\mathfrak{X}) \ , \ \ \circ \rangle \ .$$

Notational Remark 1 This Volume chooses to use the notation σ and $S(\mathfrak{X})$ because an alias for permutation is *shuffle*; using S to denote permutation groups is very standard.

Theorem 1 Permutations obey the group axioms. Specifically,

i) *Closure*: for

$$\sigma_1, \, \sigma_2 \ \in \ S(\mathfrak{X}) \ ,$$

$$\sigma_1 \circ \sigma_2 \ = \ \sigma_1 \ (\, \sigma_2(_) \,) \ \in \ S(\mathfrak{X}) \tag{18.1}$$

also (with $\sigma_1 \, \sigma_2$ being a common shorthand for this).

ii) *Associativity*: for

$$\sigma_1, \, \sigma_2, \, \sigma_3 \ \in \ S(\mathfrak{X}) \ ,$$

$$\sigma_1 \ (\, \sigma_2 \, \sigma_3 \,) \ = \ (\, \sigma_1 \, \sigma_2 \,) \ \sigma_3 \ . \tag{18.2}$$

iii) The *identity map* $\iota \ \in \ S(\mathfrak{X})$ obeys

$$\iota \, \sigma \ = \ \sigma \ = \ \sigma \, \iota \ . \tag{18.3}$$

iv) *Inverses*: for any $\sigma \in S(\mathfrak{X})$, there is a $\sigma^{-1} \in S(\mathfrak{X})$ such that

$$\sigma\,\sigma^{-1} = \iota = \sigma^{-1}\,\sigma\ . \tag{18.4}$$

Proof This was set as Ex 6.7. □

18.2 Concrete groups as subgroups of permutation groups

Remark 1 The concrete approach to groups proceeds via the notion of (concrete) subgroup.

Definition 1 A *concrete subgroup* is a subset $K \subseteq S(\mathfrak{X})$ obeying the following properties.

i)

$$\forall\ \sigma_1, \sigma_2 \in K\ ,\ \ \sigma_1\,\sigma_2 \in K\ .$$

ii)

$$\iota \in K\ .$$

iii)

$$\forall\ \sigma \in K\ ,\ \ \sigma^{-1} \in K\ .$$

Exercise 1$^-$ a) Why do we not need to check associativity?

b) Show that ii) can be phrased alternatively as $K \neq \emptyset$.

c) Show that subgroups of $S(\mathfrak{X})$ always include *id* and G itself.

Definition 2 A finite *concrete group* is any subgroup of some finite permutation group $S(\mathfrak{X})$.

Advanced Remark 1 Finiteness is indispensable for all of $S(\mathfrak{X})$'s objects to meet simple intuitions.

Naming-and-Notation Remark 2 We denote that K is a concrete subgroup of $S(\mathfrak{X})$ by

$$K \leq S(\mathfrak{X})\ .$$

If $K \neq S(\mathfrak{X})$ itself, we write

$$K < S(\mathfrak{X})$$

and term K a *proper subgroup* of $\mathfrak{X}$. If $K \neq id$ as well, we call K a *nontrivial subgroup* of $S(\mathfrak{X})$:

$$id < K < S(\mathfrak{X})\ .$$

Other authors may vary as regards this terminology; our terminology matches the one we have used so far for subsets and for divisors.

18.3 Commutative Groups

Definition 1 A (concrete) group is *non-commutative* alias *non-abelian* if

$$\sigma_1\,\sigma_2 \neq \sigma_2\,\sigma_1 \ \text{ for some } \ \sigma_2, \sigma_1 \in K\ . \tag{18.5}$$

Remark 1 Groups are not in general commutative. $id = S(\{1\})$ and $S(\{1, 2\})$ clearly are. Whereas the following shows that $S(\{1, 2, 3\})$ is not.

Worked Example 1 Show that the permutation group on three objects is not commutative.

Answer 1 From Chapter 6, we can piece together that the six permutations in this case are the identity ι. Three transpositions τ_i $i = 1$ to 3, each leaving the corresponding point i fixed. And two three-cycle permutations: the even η and the odd ω. Fig 6.4 then shows that

$$\eta\,\tau_3 = \tau_1 \neq \tau_2 = \tau_3\,\eta\ .$$

Exercise 2$^-$ Work out all the other products of pairs of permutations on three objects.

Chapter 19

Pillar 2: Abstract groups

19.1 Introduction

In this approach, an abstract group is defined as one obeying the group axioms now allowing for the composition to be 'some operation' $*$. This is as opposed to being functional composition of permutations: the particular operation involved in the concrete case.

Definition 1 $\langle G\ ,\ * \rangle$ is an *abstract group* if G is a set and the following hold.

i) *Closure*: for

$$g_1, g_2 \in G\ ,$$

$$g_1 * g_2 \in G$$

as well (with $g_1\, g_2$ being a common shorthand for this).

ii) *Associativity*: for

$$g_1, g_2, g_3 \in G\ ,$$

then

$$g_1 * (g_2 * g_3) = (g_1\ g_2) * g_3\ .$$

iii) *Identity*: G contains an element e such that

$$e * g = g = g * e \ \forall\ g \in G\ . \tag{19.1}$$

iv) *Inverses*: for each $g \in G$, there is a $g^{-1} \in G$ such that

$$g * g^{-1} = e = g^{-1} * g\ . \tag{19.2}$$

Remark 1 Concrete group $\Rightarrow$ abstract group, by setting $* = \circ$ and $e = \iota$. We establish that the opposite is true in Sec 19.9.

Remark 2 Let us henceforth suppress the symbol $*$ other than in workings with more than one such. Thus writing $g_1\, g_2$ for $g_1 * g_2$.

Example 1 $\langle \mathbb{R}\ ,\ + \rangle$ is a group. The identity here is 0, while inverses are negative numbers.

Example 2 $\langle \mathbb{R}_*\ ,\ \cdot \rangle$ is a group. The identity here is 1, while inverses are reciprocals. Not removing 0 would leave it with no inverse defined.

Exercise 1 Derive Fig 19.1's table of (counter)examples to groups.

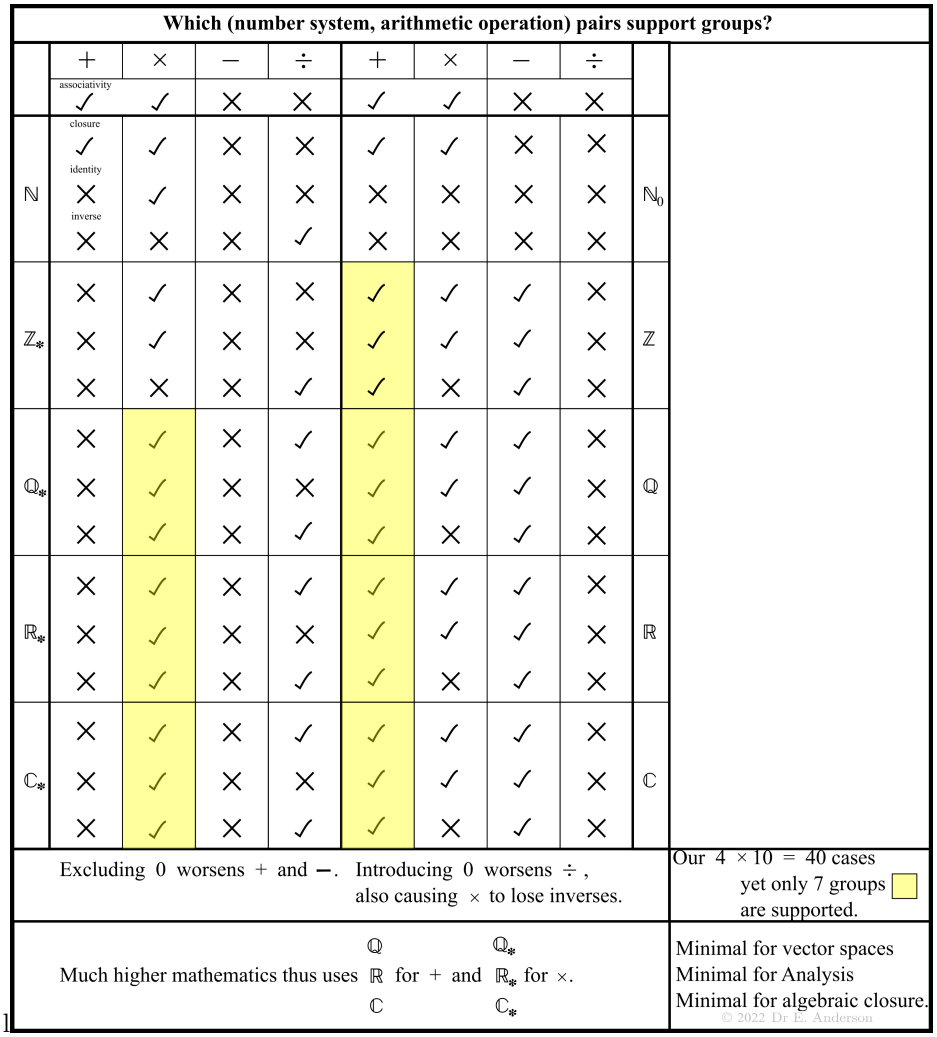


Figure 19.1:

19.2 Supporting Lemmas

Lemma 1 Groups identity is unique.

Proof For suppose not. Then there is an $e' \neq e$ obeying

$$e' g = g = g e \ \forall \ g \in G \ . \tag{19.3}$$

Then setting $g = e'$ in (19.1),

$$e e' = e \ . \tag{19.4}$$

But also by (19.3),

$$e e' = e' \ . \tag{19.5}$$

So equating (19.4) and (19.5),

$$e = e e' = e' \ .$$

Contradiction! □

Lemma 2 Each element's inverse in a group is unique.

Proof Suppose not, so that we can take $\left(g^{-1}\right)' \neq g^{-1}$ obeying

$$\left(g^{-1}\right)' g \;=\; e \;=\; \left(g^{-1}\right)' g \,. \tag{19.6}$$

Then

$$\left(g^{-1}\right)' \;=\; \left(g^{-1}\right)' e \;=\; \left(g^{-1}\right)' \left(g\, g^{-1}\right) \;=\; \left(g^{-1}\right)' g\,) \, g^{-1} \;=\; e\, g^{-1} \;=\; g^{-1} \,.$$

This is firstly by the identity property, secondly by (19.6), thirdly by associativity, fourthly by (19.2), and fifthly by the identity property again. Contradiction! □

Corollary 1 i)

$$(\, g_1\, g_2\,)^{-1} \;=\; g_2^{-1}\, g_1^{-1} \,.$$

ii)

$$\left(\prod_{i\,=\,1}^{n} g_i \right)^{-1} \;=\; \prod_{i\,=\,n}^{1} {g_i}^{-1} \,.$$

Exercise 2^{-} Prove Corollary 1.

19.3 Subgroups

Remark 1 The notion of subgroup straightforwardly uplifts to the abstract context, as follows.

Definition 2 Let $\langle\, G \,,\ * \,\rangle$ be a group and $H \;\subseteq\; G$. Then we say that $\langle\, H \,,\ * \,\rangle$ is an *(abstract) subgroup* if the following hold.

0) Its operation $*$ is indeed the same as G's (i.e. an *inherited operation*).

i)

$$\forall\, h_1,\, h_2 \;\in\; H \,,\ \ h_1\, h_2 \;\in\; H \ \ (\textit{closure}) \ \ .$$

ii) $e \;\in\; H$: the *identity is included in the subgroup.*

iii) $\forall\ h \;\in\; H \,,\quad \exists\ h^{-1} \;\in\; H$: *inverses are included in the subgroup.*

Exercise 3 Establish that the following criterion suffices to diagnose subgroups (it is more efficient than the above). $K \;\subseteq\; G$ is a subgroup if

a) $H \;\neq\; \emptyset$.

b)

$$\forall\, h_1,\, h_2 \;\in\; H \,,\ \ h_1\, {h_2}^{-1} \;\in\; H \,.$$

19.4 Further simple examples of groups

Worked Example 1 Show that $\langle\, C_n,\ \ +_n \ \rangle$ is a group. Where $C_n \;=\; \{\, 1,\, 2,\, \ldots,\, n \,\}$ and $+_n$ is addition (mod n).

Answer 1 Closure: $\forall\, a,\, b \;\in\; C_n,\, a \;+\; b \;=\; c \,(\text{mod}\ n) \;\in\; C_n$ too.

Associativity:

$$(\, a \;+\; b\,) \;+\; c \;=\; a \;+\; b \;+\; c \;=\; a \;+\; (\, b \;+\; c\,)$$

for standard addition. Taking residues (mod n) does not affect this.

Identity: $\exists\, 0 \;\in\; C_n$ such that

$$a \;+\; {}_n 0 \;=\; a \;=\; 0 \;+_n\; a \,.$$

Inverse: For each $a \in C_n$, $\exists\, n - a$ also $\in C_n$ such that

$$(n - a) +_n a = n = 0 = n = a +_n (n - a) .$$

Worked Example 2 Let $\langle G_1 , *_1 \rangle$ and $\langle G_2 , *_2 \rangle$ be groups. Show that $\langle G_1 \times G_2 , * \rangle$, for $*$ defined by

$$(g_1, g_1') * (g_2, g_2') = (g_1 *_1 g_1', g_2 *_2 g_2') , \tag{19.7}$$

is itself a group: the *direct product group* [109] of G_1 and G_2.

Answer 2 Closure: for $i = 1, 2$, $g_i, g_i' \in G_i$, as these are groups, $g_i *_i g_i' \in G_i$. Thus (19.7) $\in G_1 \times G_2$.

Associativity: For also $g_i'' \in G_i$, since these are groups,

$$g_i (g_i' g_i'') = (g_i g_i') g_i'' .$$

Thus

$$(g_1, g_2) * ((g_1', g_2') * (g_1'', g_2'')) = (g_1, g_2) * (g_1' *_1 g_1'', g_2' *_1 g_2'') =$$

$$(g_1 *_1 (g_1' *_1 g_1'') , g_2 *_2 (g_2' *_2 g_2'')) = ((g_1 *_1 g_1') *_1 g_1'', (g_1 *_2 g_2') *_2 g_2'') =$$

$$(g_1 *_1 g_1', g_2 *_2 g_2') * (g_1'', g_2'') = ((g_1, g_2) * (g_1', g_2')) * (g_1'', g_2'') .$$

Identity: As the G_i are groups, $\exists\ e_i$ such that $\forall\ g_i \in G_i, e_i g_i = g_i = g_i e_i$. Thus

$$(e_1, e_2) * (, g_1, g_2) = (e_1 *_1 g_1, e_2 *_2 g_2) = (g_1, g_2) =$$

$$(g_1 *_1 e_1, g_2 *_2 e_2) = (g_1, g_2) * (e_1, e_2) .$$

So (e_1, e_2) serves as identity for $G_1 \times G_2$.

Inverse: As the G_i are groups, for each $g_i \in G_i\ \exists\ {g_i}^{-1} \in G_i$ such that

$${g_i}^{-1} g_i = e_i = g_i {g_i}^{-1} .$$

Thus

$$\left({g_1}^{-1}, {g_2}^{-1}\right) * (g_1, g_2) = \left({g_1}^{-1} *_1 g_1, {g_2}^{-1} *_2 g_2\right) = (e_1, e_2) =$$

$$\left(g_1 *_1 {g_1}^{-1}, g_2 *_2 {g_2}^{-1}\right) = (g_1, g_2) * \left({g_1}^{-1}, {g_2}^{-1}\right) .$$

So $\left(g_1^{-1}, g_2^{-1}\right)$ serves as inverse $(g_1, g_2)^{-1}$ for $G_1 \times G_2$.

Lemma 3 The direct product group formed from a finite number of commutative groups is itself commutative.

Proof

$$(g_1, g_2) * (g_1', g_2') = (g_1 g_1', g_2 g_2') = (g_1' g_1, g_2' g_2) = (g_1', g_2') * (g_1, g_2) .$$

This is by the definition of product group's operation in the first and last steps, and by G_1 and G_2 being commutative in the middle step. A trivial induction then suffices for the finite product case. □

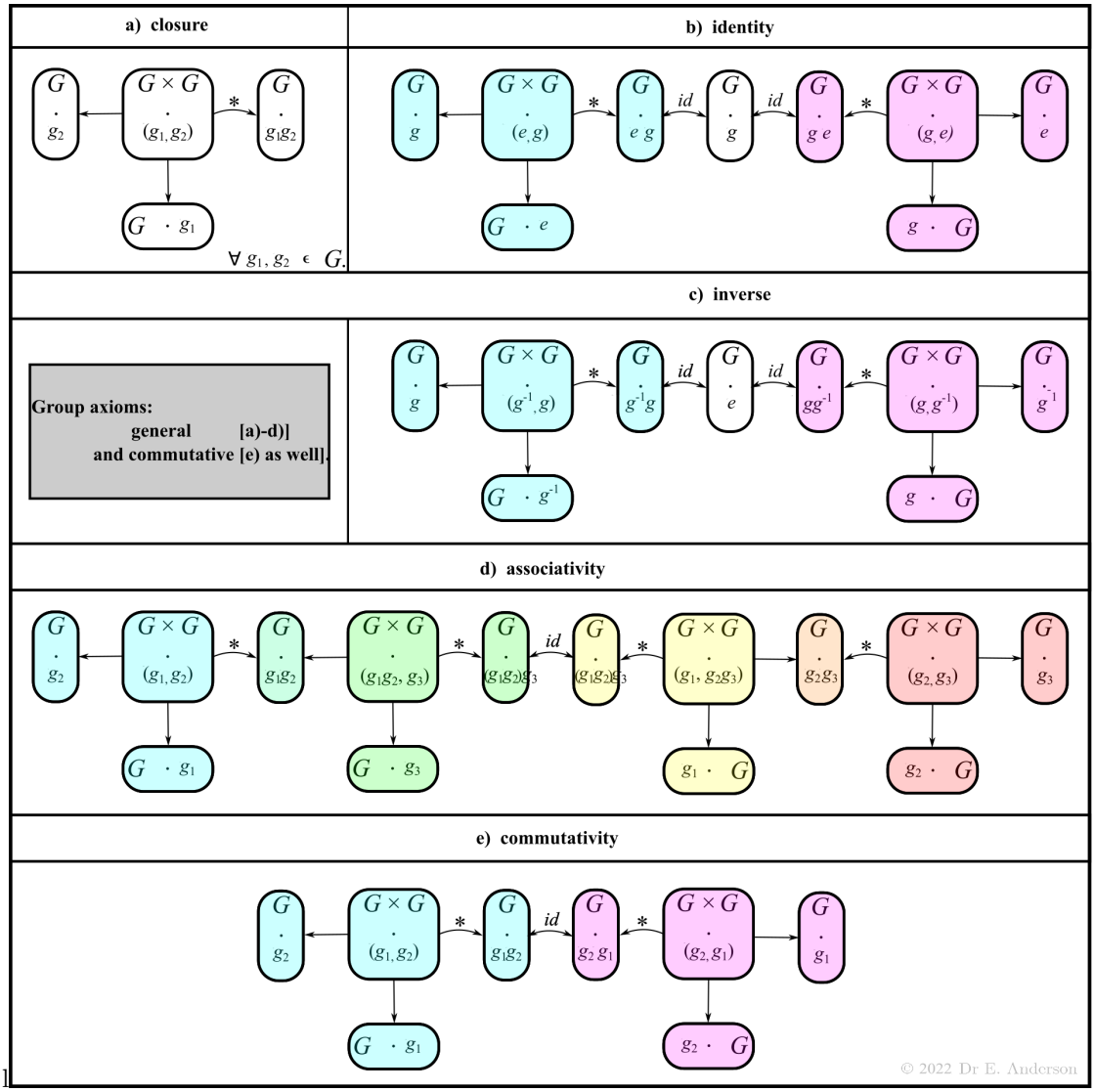


Figure 19.2:

19.5 Diagrammatic presentation of the group axioms

This is supplied in Fig 19.2.

Remark 1 b), c) and e) involve two copies of the product map combination of a); we colour one copy pink and the other blue. In b) the blue copy is the *left identity* while the pink copy is the *right identity*. In c) the blue copy is the *left inverse* while the pink copy is the *right inverse*. In d) the blue copy is the left-ordered product whereas the pink copy is the right-ordered product. d) involves four copies, with the blue copy now compounding into the green copy while the red copy compounds into the yellow copy. I use red for this, not pink, as it is a distinct product from the right version of the blue copy.

Remark 2 Some more general algebraic structures distinctly realize left and right identities, or inverses. And some realize just one of left or right (if any).

Remark 3 Also contrast bridging via an extra G (white) for b) and c) with doing so by direct

adjunction for d) and e). Finally the compounding in d) proceeds via deleting a copy of G, which we present neutrally coloured in blue + green = aquamarine and red + yellow = orange. [Extra G's are picked out by remaining uncoloured.]

Remark 4 The subgroup axioms are depicted in Fig 19.3.

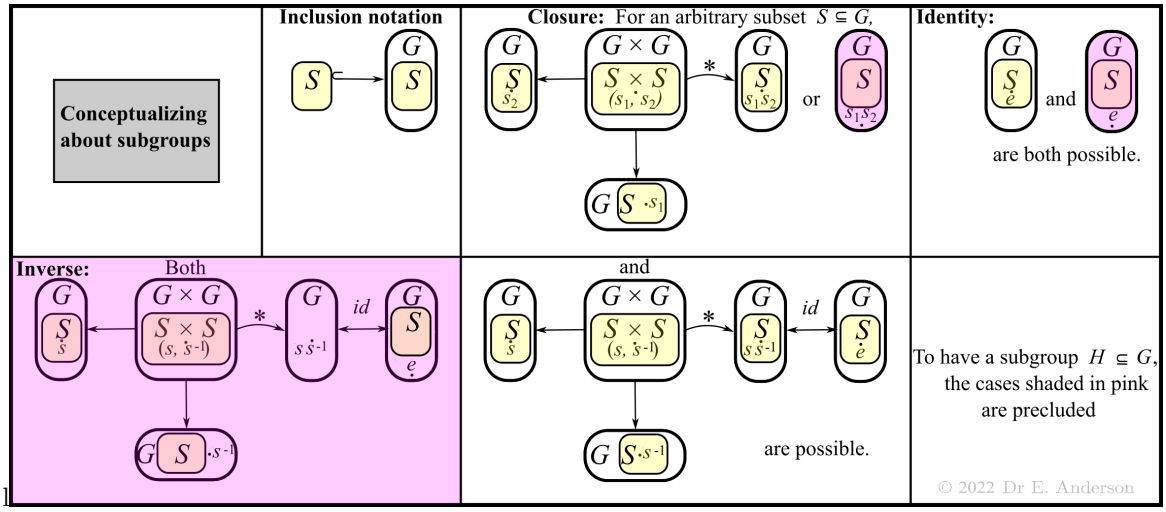


Figure 19.3:

Remark 5 The product group's composition diagram is in Fig 19.6. In realizing the group axioms, this block composes in the same way as Fig a)'s copies of many colours do.

19.6 Lie groups

These are groups that are also differentiable manifolds. Treating Lie groups in general is a more advanced subject which we shall not attempt in the current Volume.

Some examples of Lie groups are however very well-known and are often mentioned as examples of groups in elementary texts. These include the group of rotations in 2-d, $Rot(2)$, whose names as a Lie group include $SO(2)$ and $U(1)$. And the group of rotations in 3-d, $Rot(3)$, whose Lie group names include $SO(3)$ and $SU(2)$. $(S)U$ here stands for '(special) unitary' and SO for 'special orthogonal'.

Extending to include reflections, we arrive at the *orthogonal groups* $O(2)$ and $O(3)$. Orthogonal and unitary groups' basic Algebraic and geometric properties are further studied in Books 1 [441] and 2 [443]; see also [114, 314]. Special orthogonal and special unitary groups form two of the only three series of 'simple Lie groups'. The third consists of the symplectic groups, whose most commonly-encountered basic application is to the study of phase space in Dynamics. Though mathematicians additionally consider the orthogonal series to be split into distinct series for the cases with odd and even numbers of generators.

Including translations as well, one arrives at the *Euclidean group* of isometries of flat space. Including dilations as well gives the *similarity group* of flat space. These two groups are further mentioned in Sec 21.4.

19.7 Group homomorphisms and isomorphisms

For abstract groups, different circumstances can give rise to the same group including in initially distinct-looking forms.

Definition 1 Let $\langle G_i\ ,\ *_i \rangle$ be groups, for $i\ =\ 1,2$. We say that $\phi\ :\ G_1\ \longrightarrow\ G_2$ is a *(group) isomorphism* if the following both hold.

i) It is a bijection.

ii) It furthermore preserves the group structure, by (Fig 19.4.a)

$$\phi(\,g_1\ *_1\ g_1'\,)\ =\ \phi(\,g_1\,)\ *_2 \phi(\,g_1'\,)\ \ \forall\ \ g_1, g_1'\ \in\ G_1\ . \tag{19.8}$$

We then say that G_1 and G_2 are *isomorphic*, denoting this by

$$G_1\ \cong\ G_2\ .$$

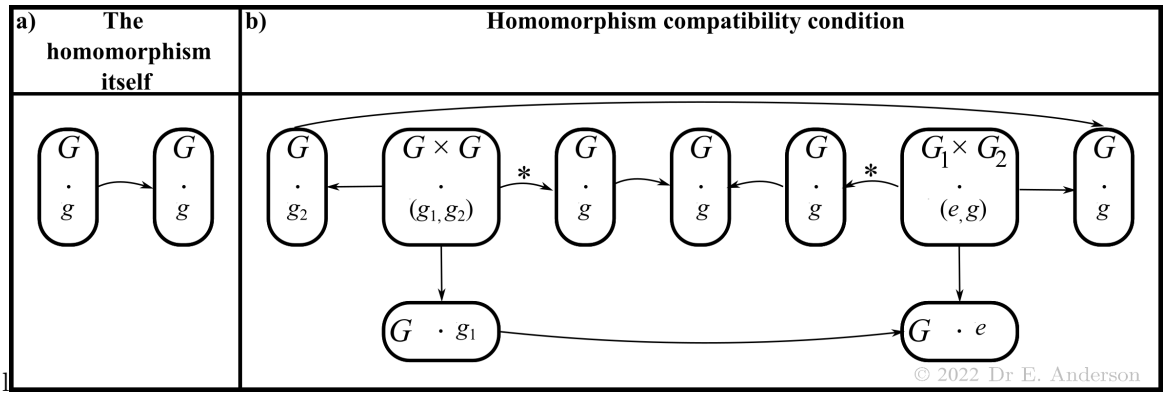


Figure 19.4:

Definition 2 (19.8) is itself termed the *homomorphism property*. If ii) holds without i) necessarily holding as well, then ϕ is said to be a homomorphism.

Remark 1 Together, i) and ii) constitute preserving both the underlying set and the group structure, which for a group are everything. So this is another example of equipped-set isomorphism (c.f. Chapter10). It remains to check that the homomorphism property fixes the identity and inverses of the target group G_2.

Lemma 4 i)

$$\phi(e_1)\ =\ e_2\ .$$

ii)

$$\phi(g)^{-1}\ =\ g^{-1}\ .$$

Proof i)

$$\phi(\,e_1\,)\ *_2\ \phi(\,e_1\,)\ =\ \phi(\,e_1\ *_1\ e_1\,)\ =\ \phi(\,e_1\,)\ .$$

This is by the homomorphism property and then the identity property in G_1. So

$$e_2\ =\ \phi(\,e_1\,)^{-1}\ *_2\ \phi(\,e_1\,)\ =\ \phi(\,e_1\,)^{-1}\ *_2\ (\,\phi(\,e_1\,)\ *_2\ \phi(\,e_1\,)\,)$$

$$= \left(\phi(e_1)^{-1} *_2 \phi(e_1) \right) *_2 \phi(e_1) = e_2 *_2 \phi(e_1) .$$

Thus i) holds.

ii) See Fig 19.5 This is by the homomorphism property as the outermost pair, next the inverse

Interaction between inverses and homomorphisms

$$\phi(g_1) *_1 \phi(g_1^{-1}) = \phi(g_1 *_1 g_1^{-1}) = \phi(e_1) = \phi(g_1^{-1} *_1 g_1) = \phi(g_1^{-1}) *_1 \phi(g_1^{-1})$$
$$\| \qquad\qquad\qquad\

and

$$SO(3) \;\cong\; SU(2)\;,$$

are two simple examples of this. These are established in e.g. Books 1 [441] and 3 [443] respectively.



Exercise 4^{-} Show that $\cong$ is an equivalence relation.

Exercise 5^{-} Show that

$$(\,G_1 \;\times\; G_2\,) \;\times\; G_3 \;\cong\; G_1 \;\times\; (\,G_2 \;\times\; G_3\,)\;,$$

by which direct product of groups is associative.

Worked Example 3 Construct the blob diagram for a product group's homomorphism property.

Answer 1 See Fig 19.6.

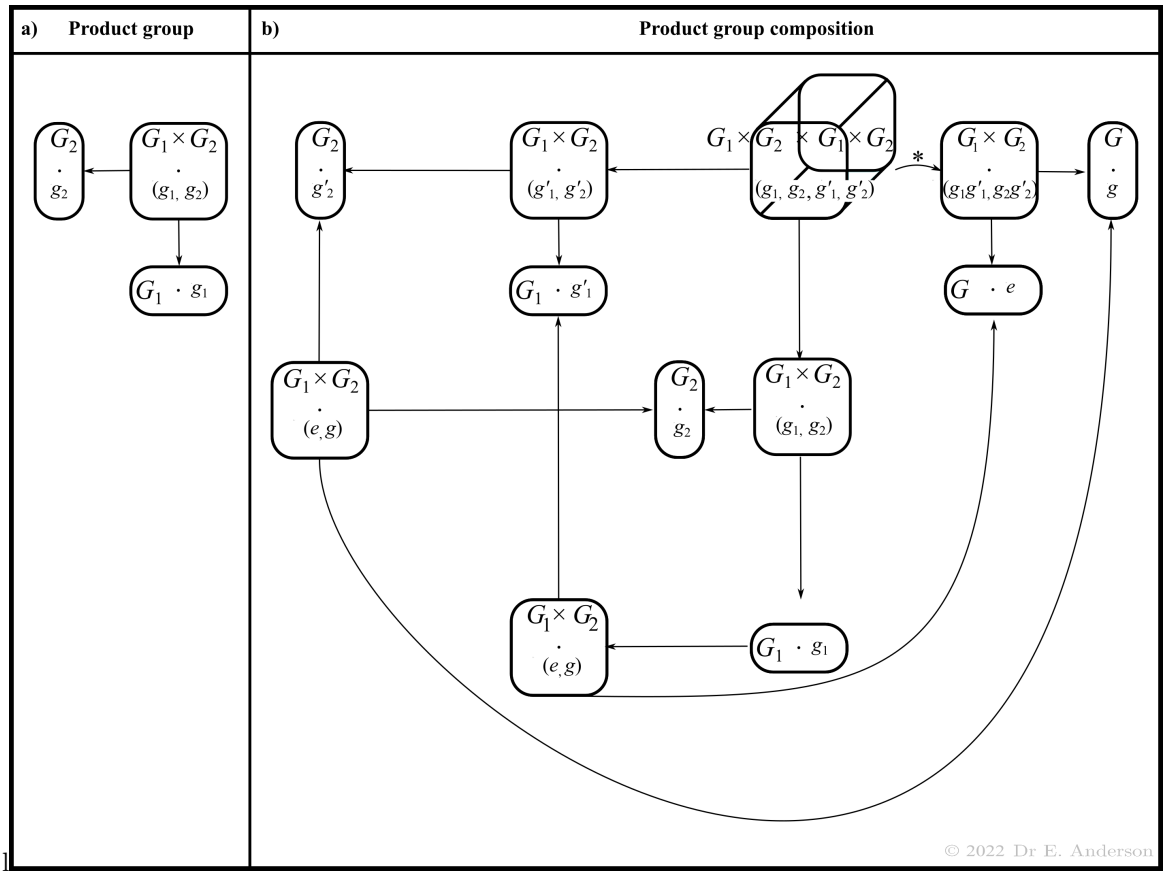


Figure 19.6:

Worked Example 4 Show that the power set $\mathfrak{P}(N)$ for an N-element set, as equipped with the symmetric difference operation, is a group. An is furthermore isomorphic as a group to the commutative product group ${C_2}^N$.

Answer 4 Closure: The elements of the power set of a given set are subsets of that set. Given any two of these, their symmetric difference is also a subset of the given set and thus also in the power set.

Associativity: we already demonstrated this in Part I.

Identity: For each $\mathfrak{A} \in \mathfrak{P}(\mathfrak{X})$,

$$\{\emptyset\} -_{\mathbf{S}} \mathfrak{A} = \mathfrak{A} = \mathfrak{A} -_{\mathbf{S}} \{\emptyset\} .$$

Inverse: For each $\mathfrak{A} \in \mathfrak{P}(\mathfrak{X})$,

$$\mathfrak{A} -_{\mathbf{S}} \mathfrak{A} = \{\text{elements in } \mathfrak{A} \text{ and in } \mathfrak{A} \text{ but not in both}\}\{\emptyset\} .$$

Abelianness: For each $\mathfrak{A}, \mathfrak{B} \in \mathfrak{P}(\mathfrak{X})$,

$$\mathfrak{A} -_{\mathbf{S}} \mathfrak{B} = \{\text{elements in } \mathfrak{A} \text{ and in } \mathfrak{B} \text{ but not in both}\}$$

$$= \{\text{elements in } \mathfrak{B} \text{ and in } \mathfrak{A} \text{ but not n both}\} = \mathfrak{B} -_{\mathbf{S}} \mathfrak{A} .$$

Let ${C_2}^N$ be generated by c_1 to c_n. Also label our finite set's elements x_1 to x_n.

Then

$$\phi : \prod_{i=1}^{n} c_i^{\alpha_i} \leftrightarrow \langle \cup_{i=1}^{n} \alpha_i x_i, i = 1 to n \rangle$$

for $\alpha_i = 0$ or 1 is a bijection.

Finally,

$$\phi\left(\prod_{i=1}^{n} c_i^{\alpha_i} \prod_{j=1}^{n} c_j^{\alpha'_j}\right) = \phi\left(\prod_{k=1}^{n} c_k^{\alpha_k + \alpha'_k}\right) = \{\cup_{k=1}^{n} (\alpha_k + \alpha'_k) x_k\} =$$

$$\{\cup_{i=1}^{n} \alpha_i x_i\} -_{\mathbf{S}} \{\cup_{j=1}^{n} \alpha'_j x_j\} = \phi\left(\prod_{i=1}^{n} c_i^{\alpha_i}\right) \phi\left(\prod_{j=1}^{n} c_j^{\alpha'_j}\right) .$$

Thus the homomorphism property is obeyed. We leave justifying passage from step 3 to step 4 here to the Reader (**Exercise 6**).



19.9 Abstract and concrete groups are equivalent

Theorem 1 [Cayley] Every abstract group is isomorphic to some concrete group.

Proof Consider the map

$$\sigma : G \longrightarrow G$$
$$\sigma_g(x) \mapsto g x .$$

We firstly need to show that this is an isomorphism.

Surjectivity:

$$\sigma_g\left(g^{-1} x\right) = g\left(g^{-1} x\right) = \left(g g^{-1}\right) x = e x = x .$$

Here we have used the definition of our map, then associativity, the inverse property and finally the identity property.

Injectivity: Suppose that

$$\sigma_g(x) \;=\; \sigma_g(y)\;.$$

Then by the definition of our map,

$$g\,x \;=\; g\,y\;.$$

So multiplying each side by g^{-1},

$$g^{-1}\,(\,g\,x\,) \;=\; g^{-1}\,(\,g\,y\,)\;.$$

Thus by associativity,

$$\left(g^{-1}\,g\right)\,x \;=\; \left(g^{-1}\,g\right)\,y\;.$$

So by the inverse property,

$$e\,x \;=\; e\,y\;.$$

Thus finally by the identity property,

$$x \;=\; y\;.$$

Homomorphism:

$$\sigma_g\,\sigma_{g'}(x) \;=\; \sigma_g\,(\,\sigma_{g'}(x)\,) \;=\; g\,(\,g'\,x\,) \;=\; (\,g\,g'\,)\,x \;=\; \sigma_{g\,g'}(x) \;\;\forall\;\; g, g' \;\in\; G \;\text{ and }\; x \;\in\; G\;.$$

We here use the definition of our map in the second step, associativity in the third, and then the definition of our map in reverse in the fourth step.

Secondly, we also need to check that our candidate map is indeed between the right two spaces. Meaning that, in particular, the σ_g are to form a subgroup of a permutation group (by the definition of concrete group).

Identity:

$$\sigma_e\,\sigma_g(x) \;=\; e\,(\,g\,x\,) \;=\; (\,e\,g\,)\,x \;=\; g\,x \;=\; (\,g\,e\,)\,x \;=\; g\,(\,e\,x\,) \;=\; \sigma_g\,\sigma_e(x)\;.$$

This is by the definition of our map in step 1, associativity in step 2, the identity property in step 3, and then these same three moves in reverse order.

Inverse: We claim that

$$\sigma_g{}^{-1} \;=\; \sigma_{g^{-1}}$$

serves as a such.

We continue our proof in Fig 19.7. This is by the definition of our map in step 1, associativity in step 2, the inverse property in step 3, the identity property in the downwards step and then the first three moves in reverse order.

Part of the working in proving Cayley's Theorem

$$\sigma_g{}^{-1}\,\sigma_g\,x \;=\; g^{-1}\,(\,g\,x\,) \;=\; \left(g^{-1}\,g\right)\,x \;=\; \begin{array}{c} e\,x \\ \| \\ x \end{array} \;=\; \left(g\,g^{-1}\right)\,x \;=\; g\,\left(g^{-1}\,x\right) \;=\; \sigma_g\,\sigma_g{}^{-1}\,x$$

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Figure 19.7:

Finally, closure already follows from the above homomorphism property. □

Naming Remark 1 A truer name for Cayley's Theorem is the *Abstract* $\Rightarrow$ *Concrete Theorem for Groups*.

Theorem 2 This can furthermore be built up to *the Abstract* $\Leftrightarrow$ *Concrete Theorem for Groups*.

Remark 1 This is by Exercise 6.7 showing that concrete groups obey abstract groups' axioms. Concrete groups thus coincide with abstract groups. Cayley's is moreover the harder of the two legs in this larger Theorem's proof.

19.10 Cayley's Theorem as a Representation Theorem

Remark 1 This is our Book Series' first example of a Representation Theorem. I.e. a Theorem by which all the objects of one kind can be viewed as cases of objects of an interestingly a priori different kind.

Remark 2 The Reader should take care not to confuse this with *Representation Theory*, in which more specifically every element of a given group is to be expressed as a square matrix of the same size. Group element composition can then simply be read off by use of matrix multiplication.

19.11 Orders of groups and of elements

Definition 1 The *order* of a group G is the number of elements its set contains. Standard notation for this is $|\,G\,|$.

Remark 1 Most of the groups in this Volume are finite.

Example 1

$$|\,C_n\,| \;=\; n\;.$$

Example 2

$$|\,S_n\,| \;=\; n!\;.$$

Example 3

$$|\,G_1 \;\times\; G_2\,| \;=\; |\,G_1\,|\,|\,G_2\,|\;.$$

This extends to

$$\left|\,\times_{\mathrm{i}\;=\;1}^{n}\,G_i\,\right| \;=\; \prod_{i\;=\;1}^{n}\,|\,G_i\,|$$

by a trivial induction.

♠ ◇ ♣

Definition 2 The *order* $|g|$ of an element $g \;\in\; G$ is the smallest number $n \;\in\; \mathbb{N}$ ($>\; 0$ in particular) such that

$$g^n \;=\; e\;.$$

If this supports no solutions for $n \;\neq\; 0$, g is said to have *infinite order*.

Example A

$$C_2 \;=\; \{\,e, a\,\} \;\text{ with }\; a^2 \;=\; e\;,$$

so the order of $a \;\neq\; e$ is 2.

Example B For p prime,

$$C_p \;=\; \left\{ e,\, a,\, a^2,\, \ldots,\, a^{p\,-\,1} \right\}$$

has all elements bar e of order p.

Exercise 7 What are the orders of the elements in C_n if n is not prime? Begin by considering C_4 and C_6.

♠ ◇ ♣

Lemma 5 i) If

$$|\,G_1\,| \;\neq\; |\,G_2\,| \;,$$

then $G_1 \cong G_2$.

ii) If $g \in G_1$ has order n and no $g' \in G_2$ does, then $G_1 \cong G_2$.

iii) If the list of orders of elements of G_1 (with multiplicities) does not coincide with that of G_2, then $G_1 \cong G_2$.

Remark 2 These provide two quick and one more detailed diagnostic for $\cong$ between groups. i) is 'global', whereas ii-iii) are 'local'.

Proof i) Bijections require equal cardinality of source and target spaces.

ii) Suppose that G_1 has an element g of order n but G_2 does not.

$$g^n \;=\; e \;\;\Rightarrow\;\; \phi\,(g^n) \;=\; \phi(e) \;=\; e\;. \tag{19.9}$$

Thus by a basic induction on the homomorphism property,

$$(\,\phi(g)\,)^n \;=\; \phi\,(\,g^n\,)\;. \tag{19.10}$$

Then equating (19.9) and (19.10),

$$(\phi(g))^n \;=\; e\;.$$

I.e. there is some element $g' \;=\; \phi(g)$ of order n in G_2. Contradiction!

iii) Use ii)'s argument on every element. If the two lists of element orders are not identical, a contradiction continues to be supported. □

Worked Example 1 Show that

$$\langle\, \mathbb{R},\, + \rangle \cong \langle\, \mathbb{R}_*,\, \cdot\, \rangle\;.$$

Answer 1 Firstly, observe that $-1 \in \langle\, \mathbb{R}_*,\, \cdot\, \rangle$ is of order 2: $-1 \neq 1$ in $\mathbb{R}$ and $(-1)^2 \;=\; 1$.

Secondly, suppose that $\langle\, \mathbb{R},\, +\, \rangle$ contains an element n that is of order 2. Then

$$n \;+\; n \;=\; 0 \;\Rightarrow\; 2\,n \;=\; 0\;.$$

So $n \;=\; 0$. But 0 is this group's identity and thus is of order 1. Thus this group has no element of order 2.

So by Lemma 5.ii), our two groups cannot possibly be isomorphic.

Exercise 8$^+$ Show that the *Klein 4-group*

$$V_4 \;:=\; C_2 \;\times\; C_2 \cong C_4$$

and yet

$$C_2 \;\times\; C_3 \;\cong\; C_6\;.$$

What about $C_3 \;\times\; C_3$ and C_9? $C_5 \;\times\; C_2$ and C_{10}? How far can you generalize?

19.12 Group times tables

Remark 1 Groups are characterized up to isomorphism by their times tables. This gives another way of establishing which groups are isomorphic (though this is only reasonably efficient for fairly small groups). The following Lemma is useful here.

Lemma 6 Given any 2 group elements, there is a unique element taking the one to the other by right-multiplication, and vice versa. I.e. for any

$$g, g' \in G , \quad \exists\ !\, j, k \in G$$

such that the following hold.

$$g\, j = g' , \quad g' k = g . \tag{19.11}$$

Proof View the first equation in 19.11 as an equation to be solved for j. To make j the subject, left-multiply by g^{-1}:

$$g^{-1} g' = g^{-1} (g\, j) = (g^{-1} g) j = e\, j = j .$$

The second step here uses associativity, the third the inverse property and the fourth the identity property. For the second equation, just switch around the primed and unprimed meaningless labels, while using k in place of j. □

Worked Example 2 Show that all groups of order ≤ 4 must be commutative.

Answer 2 For $|G| = 1$, $g\, h = h\, g$ is forced by g, h both having to be e. By which,

$$g\, h = e\, e = e = e\, e = h\, g .$$

For $|G| = 2$, the only product of unequals is

$$g\, e = g = e\, g$$

by the identity property.

For $|G| = 3$, let the distinct elements be e, g, h. The above argument deals with all cases bar the products of g and h. If $g\, h = g$, then Lemma 6 would give that $h = e$, so we must discard this possibility. If $g\, h = h$, then $g = e$ would ensue. This and switching around the meaningless g and h labels thus forces that

$$g\, h = e = h\, g .$$

For $|G| = 4$, let the distinct elements be e, g, h, j. $j = g\, h$ is now allowed. Lemma 6 precludes $h\, g$ from being h or g. However the group axioms permit both $h\, g = g\, h$ and $h\, g = e$. In this way, we establish that two non-isomorphic order-4 groups are possible.

Yet the first is manifestly commutative. While Lemma 6 forces the second case's g and h to be an inverse pair. So

$$h\, g = g^{-1} g = e = g\, g^{-1} = g, h ,$$

by which the second case is also Abelian.

Worked Example 3 What are the possible group times tables, and thus the possible groups, of order 1 to 4?

Answer 3 Let us use the notation 0 to $|G|$ for the elements of the group, with 0 denoting the identity. For $|G| = 1$, the identity property (or closure) forces the trivial (1×1) times table in Fig 19.8.a). For $|G| = 2$, the identity property fixes the first row and column of Fig 19.8.b). While Lemma 6 forces the last entry.

This observation about the first row and column applies to all groups' times tables. For $|G| = 3$, this leaves us with 4 elements to fix. Lemma 6 implies that each row and each column must be a permutation of $(1, ... |G|)$ This means that two of our remaining entries must be 0, and with these neither sharing a column nor a row. If these were in the leading diagonal, then the (1, 2) entry could not be 1 or 2, prohibiting this possibility. So the 0's are in the subleading diagonal. Lemma 6 then forces the leading diagonal's entries to be 2 and then 1 (Fig 19.8.c).

An alternative argument uses Exercise 7.14.b)'s symmetry, which can only be realized for a 2×2 block if the 0's are on the subleading diagonal. For $|G| = 4$, we are left with a 3×3 symmetric block to determine. This is large enough to permit the 0's to be on the lead diagonal. With Lemma 6 then forcing Fig 19.8.d). But it is also possible for two 0's to be off this diagonal. If they are on the subleading diagonal, Lemma 6 forces Fig 19.8.e) If placed elsewhere, contradictions like those in the $|G| = 3$ case ensue.

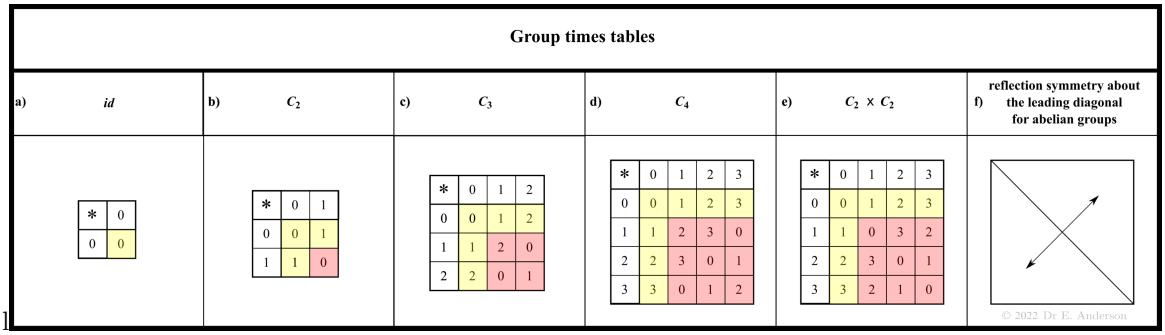


Figure 19.8:

♠ ◇ ♣

Exercise 9$^-$ Show that left works just as well as right as regards having its own version of Lemma 6.

Exercise 10$^-$ Show that for Abelian groups, a)

$$(\text{left } j) = (\text{right } j)$$

and similarly for k.

b) That the group times table must be reflection-symmetric about its leading diagonal. [Or, equivalently, that the corresponding matrix is symmetric.]

Exercise 11 Where in this Volume have you seen all but one of the Figure's tables before? Also find all the possible group times tables for groups of order 5, 6 and 7.

Chapter 20

Pillar 3: Symmetry groups

20.1 Introduction

Suppose that we are given an object obj or a space $\langle \mathfrak{X}, \boldsymbol{\sigma} \rangle$ (in each case in general carrying levels of mathematical structure σ). Then a matter of considerable practical interest is how to determine the corresponding symmetry group $Sym(\mathfrak{X}, \boldsymbol{\sigma})$. The current chapter considers this for a wide range of examples. It also establishes that this is equivalent to the concrete groups perspective (and thus also to the abstract groups perspective). In fact, while presented third in the current Volume, the symmetry group perspective was historically the first to be considered for groups. Indeed, Readers studying Physical or Chemical Sciences still often encounter this perspective first.

Definition 1 An object obj or space $\langle \mathfrak{X}, \boldsymbol{\sigma} \rangle$ is *asymmetric* if its symmetry group is *id*.

Naming Remark 1 Pure Mathematicians use rather the nomenclature *automorphism groups* for what more Applied texts call *symmetry groups*, with corresponding notation $Aut(\mathfrak{X}, \boldsymbol{\sigma})$

20.2 Group actions

♠ ◇ ♣

One subtlety to consider is that how a group G acts on an object obj or space $\mathfrak{X}$ is in general a source of ambiguity.

Definition 1 Let $\langle G, \ \square \ \rangle$ be a group. A *group action* of G on a space $\langle \mathfrak{X}, \sigma \rangle$ is a map

$$\alpha \ : \ G \times \mathfrak{X} \longrightarrow \mathfrak{X}$$

$$(g, x) \mapsto g \cdot x$$

with the following properties.

A.1)

$$e \cdot x \ = \ x \ (\ \text{ identity action } \ ) \ .$$

A.2)

$$g \cdot (g' \cdot x) \ = \ (g \ \square \ g') \cdot x \ (\ \text{ action–group composition compatibility } \ ) \ .$$

One can similarly define this upon an object obj rather than a space $\mathfrak{X}$.

Example 1 The *symmetric action*

$$\alpha_{\mathrm{S}} \ : \ S_n \times \mathfrak{X} \longrightarrow \mathfrak{X}$$

$$(\sigma, x) \mapsto \sigma(x) .$$

S_n is here the permutation group on n entities, here indeed acting by permuting elements $x \in \mathfrak{X}$.

Example 0 A simple subcase of the symmetric action is the *trivial action*

$$triv = \alpha_0 : S_n \times \mathfrak{X} \longrightarrow \mathfrak{X}$$

$$(\sigma, x) \mapsto x .$$

Here, every element $\sigma \in S_n$ just induces the identity permutation ι on $\mathfrak{X}$. The notion of trivial action furthermore straightforwardly extends to any abstract group, as follows.

$$\alpha_0 : G \times \mathfrak{X} \longrightarrow \mathfrak{X}$$

$$(g, x) \mapsto x .$$

Example 2 Groups can be thought of as acting on themselves according to the *multiplicative action*

$$\alpha_\times = \alpha_\square : G \times G \longrightarrow G$$

$$(g, g') \mapsto g \square g' .$$

Example 3 The group of symmetries of an object or space can act on particular parts thereof. For instance, in Sec 20.3 we shall see group actions on polygons that act on the vertices and/or edges by permuting them. In Sec 21.3, we shall extend this to group actions on polyhaedra, that act on whichever of the vertices, edges or faces by permuting them. Acting on the various notions of diagonal that these geometrical figures support is also entertained.

Remark 1 Let us now return to Example 2 and identify it as a group's multiplicative action on its own elements. One can then envisage such a multiplicative action on its own subgroups instead. Further chapters 24 and 26.5 will respectively extend this repertoire to on its own cosets and on its own quotients.

Example 4 A further section (24.2) will consider groups acting on themselves in a different way: by the *conjugacy action*.

♠ ◇ ♣

See Fig 20.1 for visualizations of group action and its two axioms.

♠ ◇ ♣

Remark 2 Example 2 illustrates to the maximal extent that group actions are capable of losing information. This is quite generally true (if not maximally realized by) Examples 1 and 3. This gives one reason to consider not $(G, \mathfrak{X})$ pairs but a fortiori $(G, \mathfrak{X}, \alpha)$ triples.

Remark 3 Which group action is involved in studying a given object or space is indeed almost always ambiguous even once given the group in question. For there is a minimal ambiguity between the *natural action*

$$\alpha_{\text{natural}} : G \times \mathfrak{X} \longrightarrow \mathfrak{X}$$

$$(g, x) \longrightarrow g(x) .$$

and the trivial action α_0. With only three very small exceptions, for which these two notions coincide. Namely $G = id$, $\mathfrak{X} = \text{pt}$ and $\mathfrak{X} = \emptyset$.

Exercise 1 Actions such that

$$g \square x = x \ \forall \ x \in \mathfrak{X} \ \Rightarrow \ g =$$

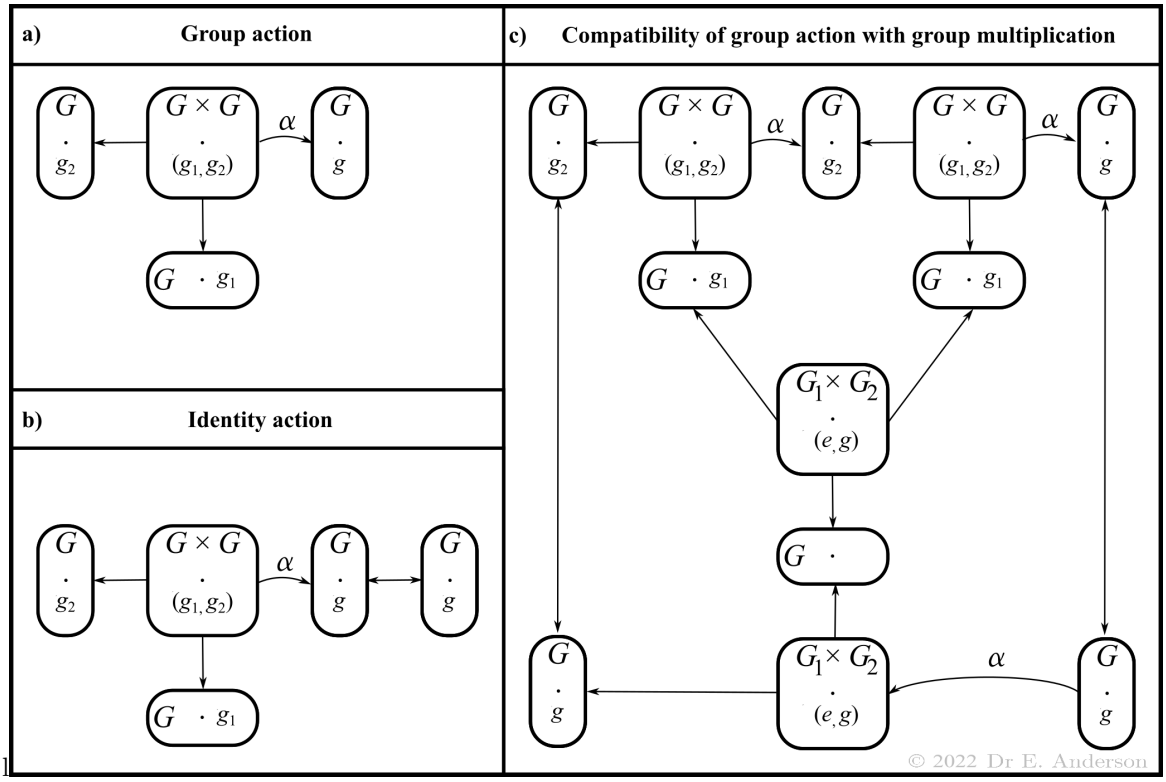


Figure 20.1:

are called *faithful* or *effective*.

a) Show that faithful actions lose no information at all.

An action is *free* if

$$(\text{ for } g, g' \in G\,, \quad \exists\ x \text{ such that } g \cdot x = g' \cdot x) \ \Rightarrow \ g = g'\,.$$

b) Show that free actions have at least one fixed point.

Remark 4 An action is *transitive* iff

$$\forall\ x, x'\ \exists\ g \text{ with } g \cdot x = x'\,.$$

Such an action takes any element $x \in \mathfrak{X}$ to any other x'.

Exercise 2 Draw blob diagrams for the faithful, free and transitive action conditions (including Exercise 1's version of the second).

Remark 5 We will have quite a lot more to say about faithful, free and transitive actions in Volume 5 [444].

20.3 Symmetries of regular polygons

In the current Volume, we assume that these reside in the Euclidean plane $\mathbb{E}^2$, as finite subregions bounded by a finite number of line segments.

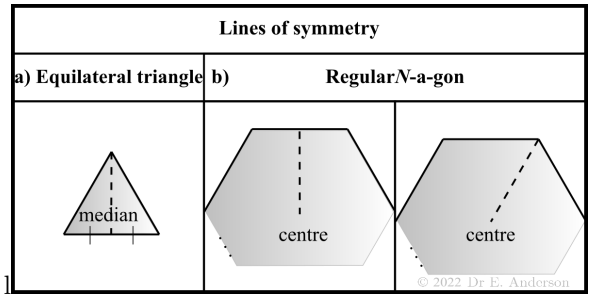


Figure 20.2:

Worked Example 1 What elements does the symmetry group of the equilateral triangle have?

Answer 1 i) The identity transformation ι.

ii) σ, corresponding to a reflection about an axis of symmetry (here realized by a median: Fig 20.2.a).

iii) ρ, corresponding to a 3-fold rotation – by $2\,\pi/3$ – about the centre of the triangle.

iv) ρ^2: a rotation by $4\,\pi/3$.

v) The composition $\rho\,\sigma$ yields a further distinct transformation.

vi) So does $\rho^2\,\sigma$.

Fiducially labelling the vertices A, B, C, we can count out that these labels can be placed in 6 different positions on our equilateral triangle. Thus the above six distinct transformations cover all possibilities (Fig 20.3.a). The group obtained in this way, when viewed as an abstract group, is denoted by D_3. This is furthermore isomorphic to S_3 since the above 6 different positions can be viewed as free permutations of the vertices. Thus we write

$$Sym(\mathrm{E}quilateral) \;=\; D_3 \;\cong\; S_3 \;.$$

Exercise 3 What are the 8 possible transformations of the square? Also provide their group times table. Give a geometrical argument why this group cannot be a permutation group. Also give a simple counting argument for this.

Worked Example 2 What elements does the symmetry group of the regular N-a-gon possess?

Answer 2 Fiducially label one vertex with the letter A. We can rotate this to any other vertex in our regular N-a-gon. Fiducially label the vertices adjacent to A by B and N. B has two possible placings: (anti)clockwise. This exhausts all remaining freedom in our N-a-gon. Two underlying operations are involved, as follows.

i) ρ: an N-fold-rotation – i.e. through an angle of $2\,\pi/N$ – about the regular N-a-gon's centre.

ii) σ: a reflection about an axis of symmetry. This is either an edge to centre line or a side bisector: Fig 20.2.b).

These obey

$$\sigma^2 \;=\; \iota \tag{20.1}$$

and

$$\rho^N \;=\; \iota \;. \tag{20.2}$$

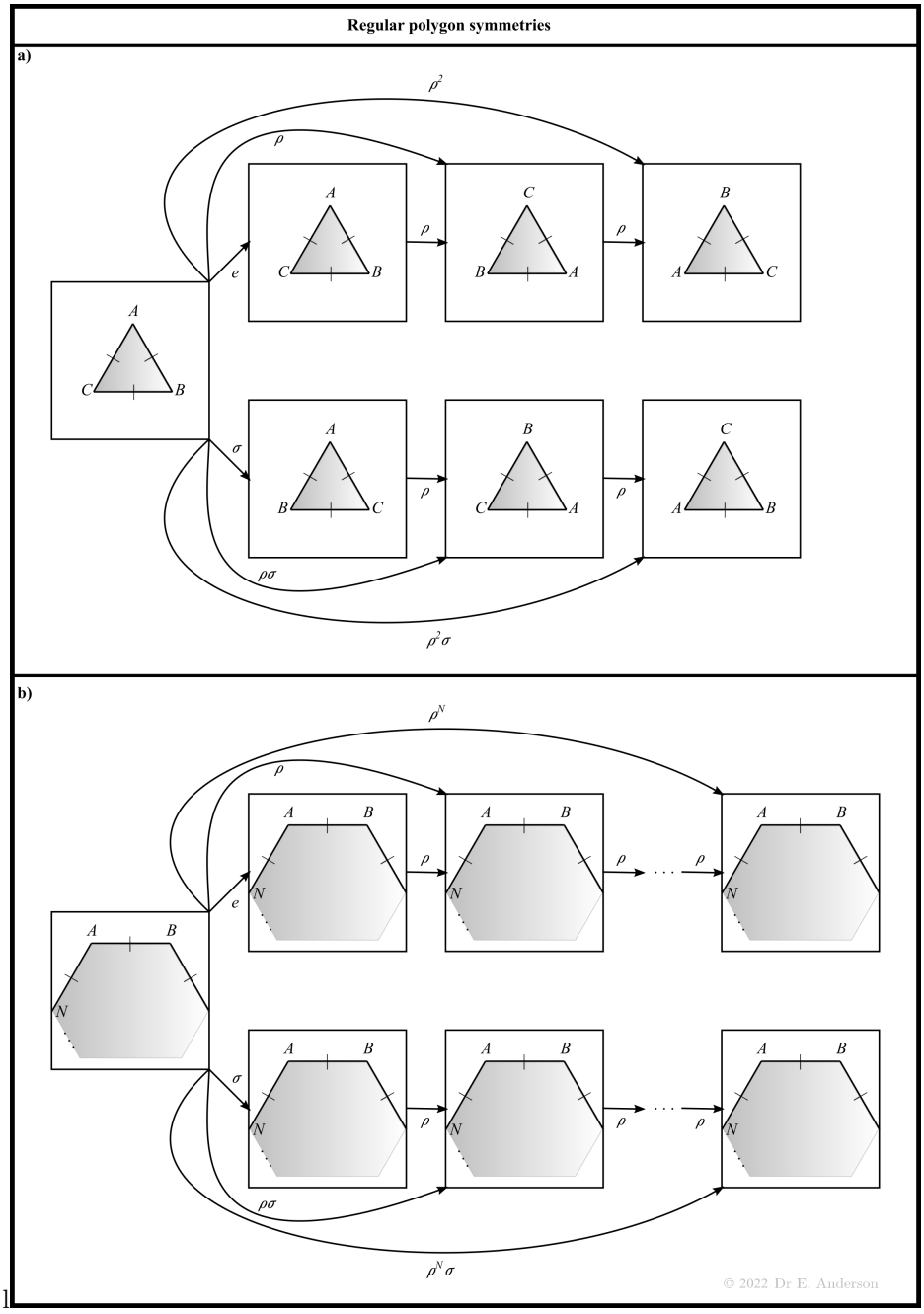


Figure 20.3:

ι, ρ^i, σ, $\sigma\,\rho^i$ for $i \;=\; 1$ to $N \;-\; 1$ cover all possible placings of our fiducial labels. By drawing sequences of five labelled regular N-a-gons, the Reader can indeed check that

$$\sigma\,\rho \;=\; \rho^{-1}\,\sigma\;. \tag{20.3}$$

It suffices to follow where A and N map to in each of the RHS and LHS combinations of transformations (Fig 20.3.b).

20.4 Dihedral group interpretation

Definition 1 The groups that we obtained above, which we now summarize as

$$D_n \;:=\; \left\langle\, \sigma, \rho \,\middle|\, \sigma^2 \;=\; 1\;,\;\; \rho^N \;=\; 1\;,\;\; \sigma\,\rho \;=\; \rho^{-1}\,\sigma \,\right\rangle\;,$$

are known as the *dihedral groups*.

Naming Remark 2 This can be taken to refer to its double-covering ('di') the single face ('haedron') of the regular polygon that it naturally acts upon.

Exercise 4 Show that the above group action of D_3 on the equilateral triangle is faithful, transitive but not free.

Remark 1 Since $D_3 \;\cong\; S_3$ – a permutation group – it is D_4 which is the smallest distinct dihedral group.

Exercise 5 Find a 2 to 1 homomorphism from D_n to C_n. Is it unique? Give a natural way in which the dihedral group double-covers the regular polygon it acts upon.

20.5 Generators, relators and presentations

Remark 1 We eventually term underlying operations' group elements (mentioned in Worked Example 2 above) *group generators*.

Remark 2 Also, we eventually term such relations between generators as (20.1, 20.2, 20.3) *relators*).

Remark 3 Definition 1's summary takes the form

$$G \;=\; \langle\, \{\,\text{generators}\,\} \,|\, \{\,\text{relators}\,\} \,\rangle\;,$$

which is termed a *presentation* of the group G.

Exercise 6$^-$ Write down presentations for C_n and $C_2 \;\times\; C_2$.

Remark 4 Many Lie groups have an uncountable infinity of elements, and yet possess a finite number of generators (such Lie groups are called *finitely-generated*). We alluded to this in Sec 17.4 as an example of how one aspect of a mathematical structure may be infinite while another aspect remains finite. For instance, $SO(2)$ has an uncountable infinity of elements corresponding to the real interval $[\,0,\, 2\,\pi\,)$ of values that can be taken by the angle of rotation. Yet $SO(2)$ is generated by a single nontrivial infinitesimal rotation about the origin. For rotations though all other angles can be built up by repeated application of this.

20.6 Symmetries of less regular polygons

Worked Example 3 (Triangles) What are the symmetry groups of isosceles and scalene triangles?

Isosceles triangles (Fig 20.4.b) have just the reflection symmetry σ left, so

$$Sym(\text{I}sosceles) \;=\; C_2 \;=\; C_{2\,\text{ref}}\;.$$

The ref subscript here denotes reflective action.

Scalene triangles (Fig 20.4.c) are asymmetric:

$$Sym(\mathrm{S}calene) \;=\; id \;.$$

Worked Example 4 (Quadrilaterals) What symmetry groups can non-square quadrilaterals exhibit?

Answer 4 *Oblongs* (Fig 20.4.e) – rectangles explicitly omitting the square – retain just the reflection symmetry σ and the order-2 rotation ρ^2. The Reader can check that these form the Klein 4-group:

$$Sym(\mathrm{O}blong) \;=\; V_4 \;=\; C_2 \;\times\; C_2 \;.$$

The same applies to *diamonds* (Fig 20.4.f) – rhombi explicitly omitting the square – so

$$Sym(\mathrm{D}iamond) \;=\; V_4 \;=\; C_2 \;\times\; C_2 \;.$$

Parallelograms (Fig 20.4.g) retain just the order-2 rotation, so

$$Sym(\mathrm{P}arallelogram) \;=\; C_2 \;=\; C_{2\mathrm{inv}} \;.$$

This inv subscript refers to the 2-fold rotation being equivalent to inversion about the centre of rotation, as the Reader can readily check.

Kites (Fig 20.4.h) including the arrowheads for our purposes) retain just the reflection, so

$$Sym(\mathrm{K}ite) \;=\; C_2 \;=\; C_{2\,\mathrm{ref}} \;.$$

Isosceles trapeziums (Fig 20.4.i) retain just the reflection, so

$$Sym(\mathrm{I}sosceles\text{-}\mathrm{T}rapezium) \;=\; C_2 \;=\; C_{2\,\mathrm{ref}} \;.$$

No other quadrilaterals have any symmetries. (For each of the above cases, we consider the corresponding crossed version to be included in each). That some of these asymmetric quadrilaterals have names – the remaining trapeziums, the (non-rectangular) cyclic quadrilaterals... – indicates that symmetry group is not everything in studying objects. That oblongs and diamonds share symmetry group also serves to show that symmetry groups do not necessarily suffice to discern between qualitatively distinct objects. The parallelograms, kites and isosceles trapeziums also share symmetry group, this now being C_2. In this case, the further ref and inv group action label serves to partly disambiguate.

20.7 Symmetry breaking

Example 1 Isosceles triangles can be viewed as arising by somewhat squeezing equilateral triangles while maintaining their axis of reflection.

This is an example of breaking a symmetry – the order-3 rotation – by carrying out a deformation of an object.

If such a deformation is performed without respecting the axis of reflection, a scalene triangle results. This corresponds to the most general deformation if the straightness of the triangle's edges is to persevere.

We thus arrive at a 3-chain of increasingly less symmetric triangles (Fig 20.5.a).

Example 2 For the quadrilateral, it is clear that symmetry can be broken in more than one way (Fig 20.5.b).

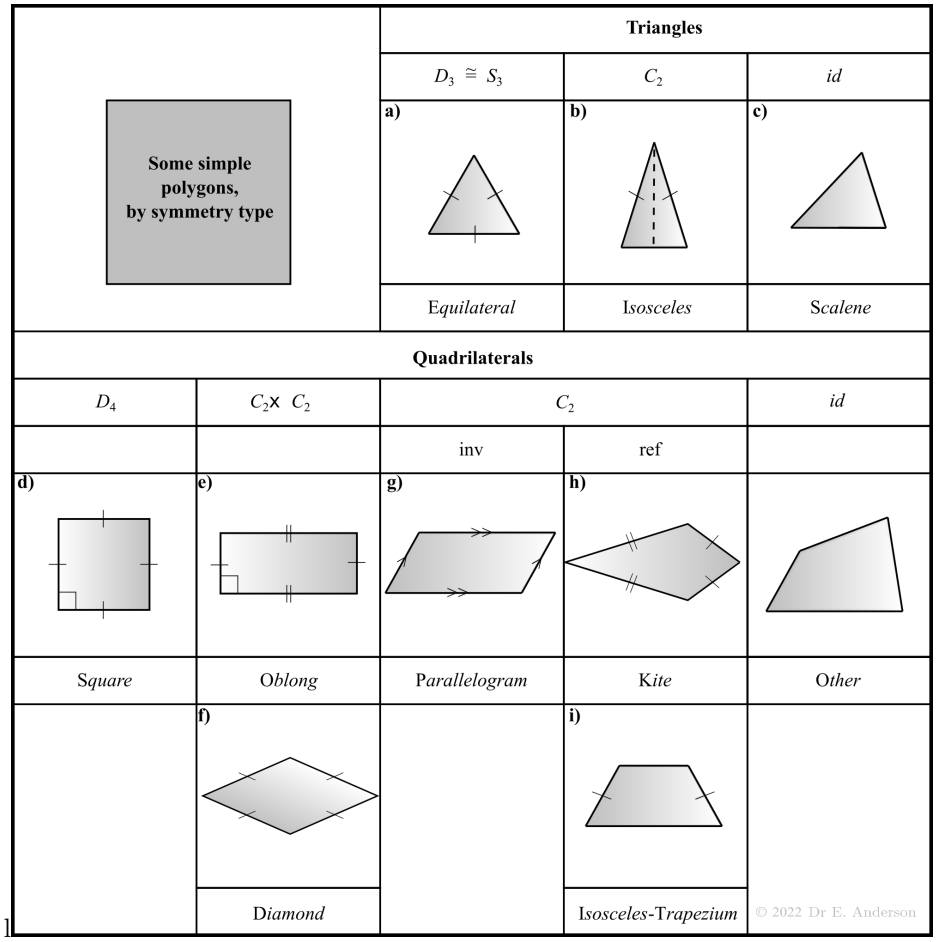


Figure 20.4:

Remark 1 Not all subgroups are realized in either example; C_3 is not in Example 1 and C_4 is not in Example 2.

Example 3 A distinct way of breaking symmetry is to add labels. For instance, the equilateral triangle supports three cases of this, as follows (Fig 20.6).

i) The indistinguishable vertex labelling (Fig 20.6.a), which preserves the D_3 symmetry.

ii) The totally-distinguishable vertex labelling (Fig 20.6.c) totally breaks the D_3 symmetry: returning *id*. The partly-distinguishable vertex labelling (Fig 20.6.b) breaks the rotational symmetry but preserves the reflective symmetry, yielding $C_{2\,\mathrm{ref}}$. These form the same chain of 3 subgroups as in Example 1 (Fig 20.6.d).

Example 4 However, for the square, the possibilities are as in Fig 20.6.e). These realize the same subgroups as in Example 2, yet not with the same multiplicities or orderings (Fig 20.6.f).

Remark 2 We can use more structured labels, like e.g. in Fig 20.6.g).

Remark 3 Symmetry can instead be broken by adding levels of mathematical structure. For

$$Aut(\mathfrak{X}, \sigma, \tau) \leq Aut(\mathfrak{X}, \sigma)$$

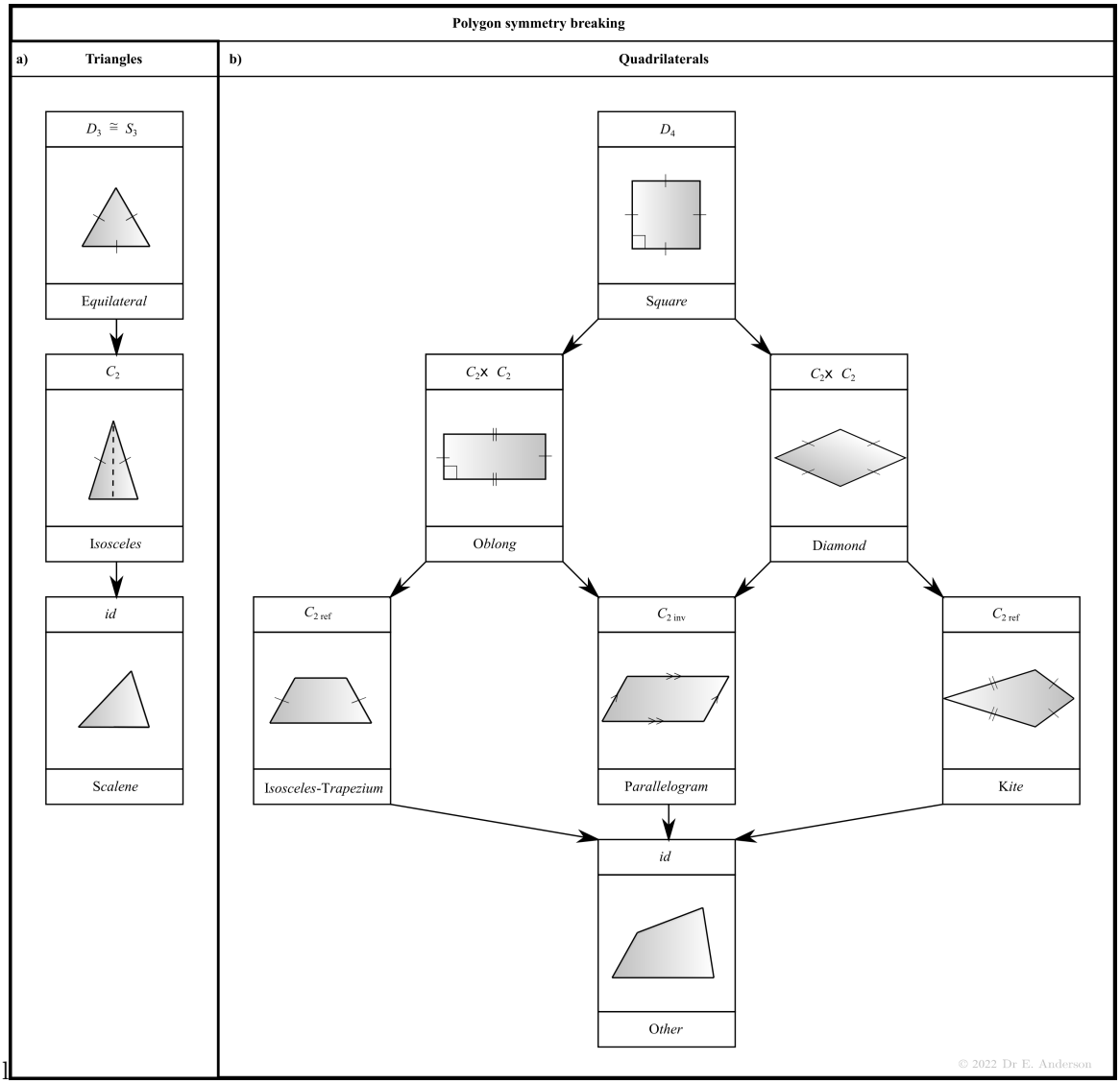


Figure 20.5:

since not all of the latter need to additionally respect the τ structure.

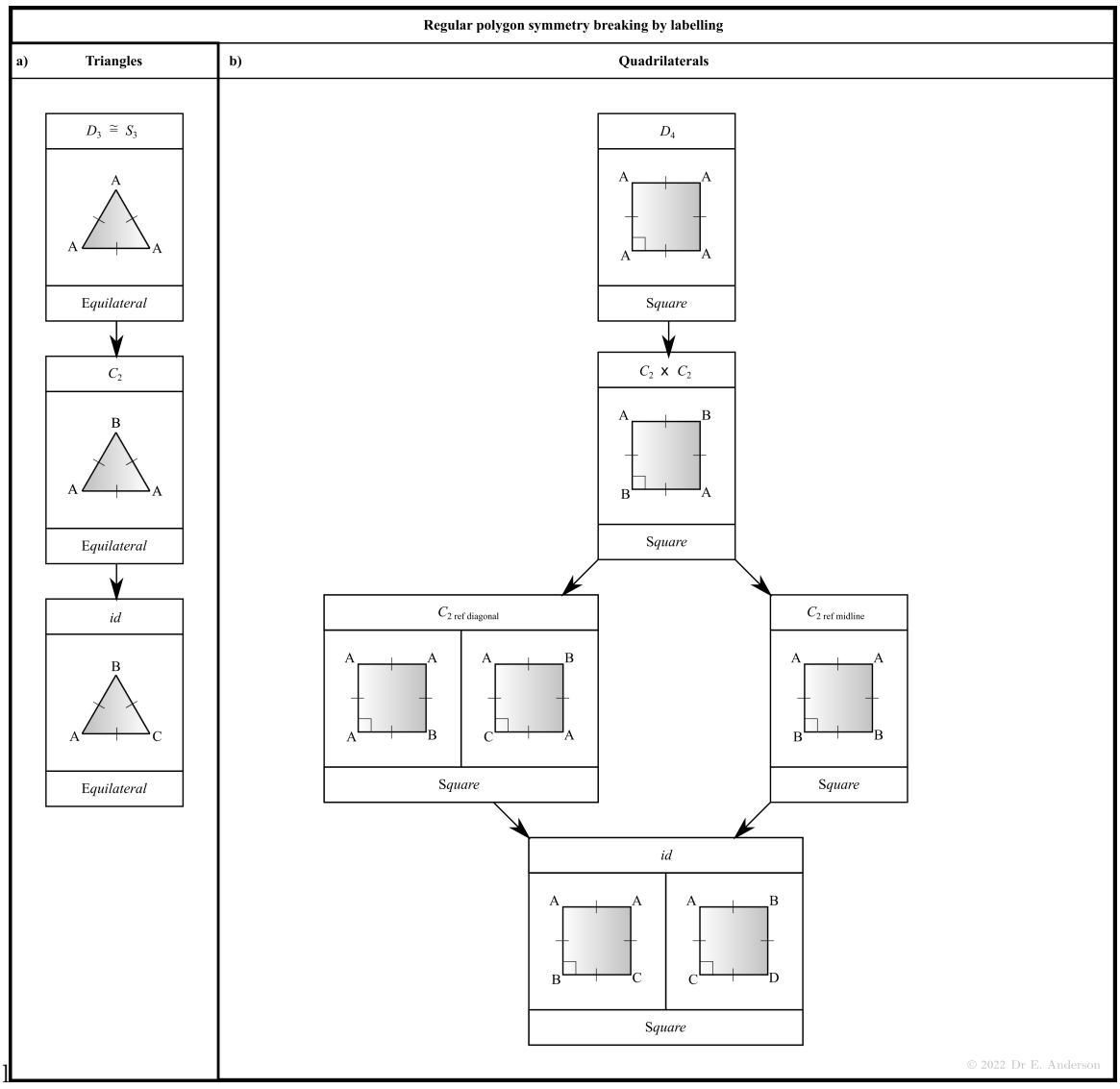

Regular polygon symmetry breaking by labelling
a) Triangles
b) Quadrilaterals
$D_3 \cong S_3$
Equilateral
C_2
Equilateral
id
Equilateral
D_4
Square
$C_2 \times C_2$
Square
$C_{2\text{ ref diagonal}}$
Square
$C_{2\text{ ref midline}}$
Square
id
Square
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Figure 20.6:

Chapter 21

Geometrical and Chemical examples of symmetry groups

21.1 Circular, cylindrical and spherical symmetry

Example 1 *Circular symmetry* is invariance under 2-d rotational symmetry (Fig 21.1.a). This corresponds to the group

$$Circ \;=\; Rot(2) \;=\; SO(2) \;=\; U(1) \;=\; C_\infty \;.$$

Using for instance the $U(1)$ version's natural complex-polar representation

$$R(\phi) \;=\; e^{i\,\phi}$$

for these rotations, commutativity immediately follows:

$$R\,(\phi_1)\;R\,(\phi_2) \;=\; e^{i\,\phi_1}\,e^{i\,\phi_2} \;=\; e^{i\,(\,\phi_1\,+\,\phi_2\,)} \;=\; e^{i\,(\,\phi_2\,+\,\phi_1\,)} \;=\; e^{i\,\phi_2}\,e^{i\,\phi_1} \;=\; R\,(\phi_2)\;R\,(\phi_1) \;.$$

Example 2 1-d *translation symmetry* (Fig 21.1.b) corresponds to the group

$$Tr(1) \;=\; \langle\,\mathbb{R},\,+\,\rangle\;;$$

let us use $\mathbb{R}$ as a shorthand for this below. Using $T(a)$ to denote the translation by amount a, we have

$$T(a)\,T(b) \;=\; T(\,a\;+\;b\,)\;.$$

$Tr(1)$ is also commutative, either from the precedding expression or from our previous context-free study of $\langle\,\mathbb{R},\,+\,\rangle$,

Example 3 *Cylindrical symmetry* in 3-d (Fig 21.1.c) corresponds to invariance under rotational symmetry in a plane and under translations along the corresponding axis. These being independent motions, the cylindrical symmetry group is

$$Cyl \;=\; Rot(2) \;\times\; Tr(1) \;=\; SO(2) \;\times\; \mathbb{R}\;.$$

Exercise 1$^-$ a) Prove that $Rot(2)$ is commutative using instead the natural 2×2 rotation matrix representation.

b) Prove that Cyl is commutative.

Example 4 *Spherical symmetry* (Fig 21.1.d). is invariance under the 3-d rotations. This corresponding to the group

$$Sphe \;=\; Rot(3) \;=\; SO(3)\;.$$

Exercise 2^{-} Set up 3-d rotation matrices about the x, y and z axes. Use select such to minimally demonstrate that $SO(3)$ is non-commutative.

♠ ◇ ♣

Exercise 3 For some purposes, we can include reflections in each of the above.

a) To

$$C_\infty \;=\; \langle\, \mathbb{Z}, + \,\rangle \;, \quad \text{shorthand} \;\; \mathbb{Z} \;,$$

being the *infinite cyclic group* – adding one conceptualization and name to Sec 19.6's list – the D_∞ obtained by including a reflection is the *infinite dihaedral group*. Show that this is non-commutative, and establish what relations characterize it. [Another name for this group is $O(2)$: the 2-d orthogonal group.]

b) Including a reflection, $SO(3)$ becomes $O(3)$. What relations involving this reflection does this group possess? [E.g. set up a matrix for the reflection with the xy plane as a mirror plane.]

c) Show that the infinite cylinder supports two independent reflections. This gives the *infinite dicyclic group* Dic_∞ (many applications use the more specialized point-group action notation D_h or D_d, rather). Having seen this, why can spherically-symmetric problems involve at most one independent rotation?

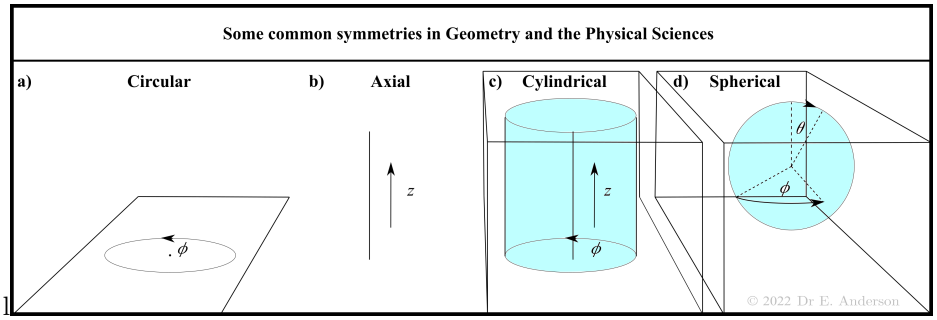


Figure 21.1:

Pointer 1 Circular, cylindrical and spherical symmetry are ubiquitously used in Physics. This is due to their considerably enhancing calculability, as Volume 1 lays out and exploits with further underlying reasons explained in Books 4 and 6.

Remark 1 These symmetries are also needed for the next section's account of molecular symmetries.

Remark 2 These symmetries are a subset of those which Euclidean and Similarity Geometry uses, as per Sec 21.4.

21.2 Molecular examples

Using for now point-particle models with line segment bonds, we give the following examples.

Example 1 Helium, present in Nature as individual atoms He, exhibits $O(3)$ symmetry (Fig 21.2.a).

Example 2 Carbon monoxide CO has symmetry group $O(2) \;=\; D_\infty$ (Fig 21.2.b).

Example 3 Carbon dioxide CO_2 has symmetry group Dic_∞ (Fig 21.2.c). Observe that while this symmetry group arose the previous section for an infinite cylinder, it has now also been realized by a finite object.

Example 4 Sulphur trioxide SO_3 has symmetry group D_3 (Fig 21.2.d).

Example 5 Methane CH_4 has symmetry group $T = S_4$ (Fig 21.2.e).

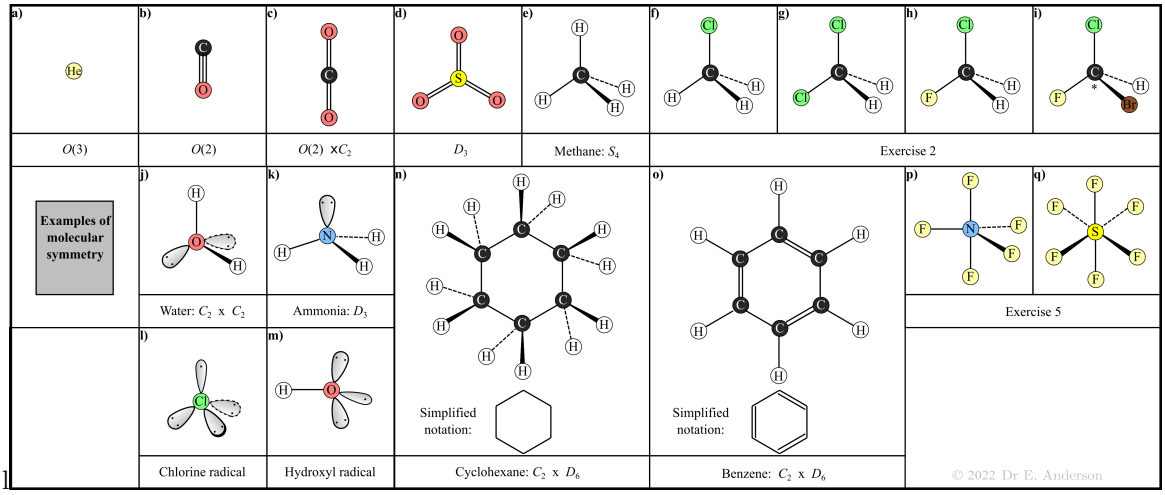


Figure 21.2:

♠ ◇ ♣

Exercise 4$^-$ (Symmetry breaking) a) What are the symmetry groups of $C H_3 Cl$, $C H_2 Cl_2$ $C H_2 F Cl$, and $C H F Cl Br$ (Fig 21.2.f-i)?

b) Order these up into symmetry breakings of methane.

[Fluorine F, chlorine Cl and bromine Br are the first three *halogen* atomic species.] This renders them suitable in this role as substitutes for hydrogen in symmetry-breaking methane.]

Exercise 5 For the above repertoire of molecules,

a) what happens to the symmetry groups if we model using small balls rather than point particles? (Small relative to the bond lengths.)

b) And if we use spherically-symmetrical quantum probability distributions in place of point-particles?

c) What if we allow the bonds to vibrate?

Remark 1 In determining molecular shapes and thus symmetries, one approximate guiding principle is as follows. To arrange the atoms bonded to a given atom so as to be maximally apart from each other.

Remark 2 This principle by itself fails in the case of water (H_2O). This is since oxygen here additionally has two *lone pairs* of electrons – outer-shell electrons that are not however participating in the bonding (Fig 21.2.j). Water molecules thus just have $C_2 \times C_2$ symmetry group, rather than

the straight-line D_∞. At some level of modelling, ammonia (NH_3) takes a pyramidal rather than planar-equilateral form (Fig 21.2.k). This leaves it with a D_3 symmetry.

Remark 3 Aside from lone pairs, some chemical and metastable species have a lone (unpaired) electron in their outer shells. Such species are known as *free radicals*; Fig 21.2.l-m) exhibits two such: the chlorine and hydroxyl radicals. Having such a lone electron tends to render free radicals highly reactive, in particular *catalytic*. This means that radicals cause other compounds to react without themselves getting used up in the process).

Remark 4 In the following discussion, let us use X, Y to denote any of a lone pair, a loose electron or a bonded atom. When the X around a given atom A are heterogeneous, placing these maximally apart is only approximately correct. This is because different atomic species attract a shared bond's electrons to differing extents ('electronegativity'). One does well to look up what the bond angles are, though for calculating symmetry groups, this is not necessary. Just use that angles $\angle XAY$ will not be exactly the same for each chemically-distinct XY pair around A.

♠ ◇ ♣

Exercise 6 [Symmetry breaking] a) Show that cyclohexane (Fig 21.2.o) and benzene (Fig 21.2.p) share the symmetry group D_6.

b) Attach atoms or sidechains to one or the other of these so as to realize the symmetry groups D_3, C_2, id, C_6 and C_3. Can any other symmetry groups be realized thus?

[Note here that the alternating double and single bonds here are no different from each other. Also, if you cannot prove the full scope of symmetry groups here, return to it after studying Chapter 24.3.]

c) Arrange the resulting molecules into a symmetry-breaking ordering for cyclohexane and a separate such for benzene.

Exercise 7 a) a) What are the symmetry groups of NF_5 and SF_6 (Figs 21.2 .p-q)?

b) Find examples of molecules exhibiting D_4 Dic_4, D_5, Dic_5, D_8 and Dic_8 symmetries.

Pointer 1 See e.g. [116] for further chemical examples.

Exercise 8 What are the symmetry groups of *naphthalene* (Fig 21.3.a) and of the single-layer sheet of *graphite* (Fig 21.3.b).

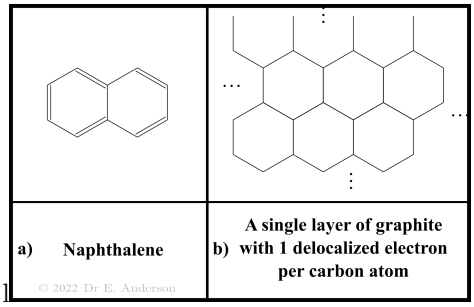


Figure 21.3:

21.3 The platonic solids

Worked Example 1 Show that $Sym(\mathrm{T}etrahaedron) \cong S_4$.

Answer 1 Fiducially label the vertices A, B, C and D. We can map A to any vertex, fiducially labelled W in the image. W has an axis of symmetry emanating from it, going through the centre S of the equilateral triangle formed by the other vertices (Fig 21.4.a). Rotating thereabout, B can be mapped to any of the remaining vertices, fiducially labelled X. Finally, reflecting in the WSX plane if needs be, we can send C to either of the remaining vertices. We have thus established the stated isometry.

Worked Example 2 Show that $Sym(\mathrm{C}ube) \cong S_4 \times C_2 \cong Sym(\mathrm{O}ctahaedron)$.

Answer 2 Fiducially label the cube's vertices A, B, C, D, E, F, G and H. A can be mapped to any other vertex, fiducially labelled W in the image. On this occasion, B – taken to be adjacent to A – can only be mapped to one of the 3 vertices adjacent to W, call it X in the image copy. Then D – also taken to be adjacent to A – can only be mapped to one of the remaining 2 vertices adjacent to W, call it Y. By this stage, where C and E to G map to has been fixed. This serves to establish that

$$|\, Sym(\mathrm{C}ube)\,| \;=\; 8 \times 3 \times 2 \;=\; 48\ .$$

To deduce further structure for this group, we need to consider its action on some further aspect – set of geometrical features – of the cube. It turns out that the four diagonals through the centre of the cube suffice to fully characterize $Sym(cube)$. All permutations of these can be realized by rotations. So an S_4 is included in $Sym(\mathrm{C}ube)$. Keeping track of their ends' labels, a single reflection about some plane through the centre is additionally necessary. Reflection is an involution, so this operation by itself forms a $C_2(\ =\ C_{2\,\mathrm{ref}})$ subgroup.

By following where the vertices go, the Reader can furthermore check that the above S_4 and C_2 simply combine in the manner of a direct product. The first claimed isometry is thus arrived at.

Observe that the centres of the faces of the octahaedron form a cube. the octahaedron. By the object acted upon being a cube, we have already established what its transformation group is. Thus the second claimed isometry has been established. This paragraph's strategy is moreover a *repeatedly useful* trick: a type of *duality argument*.

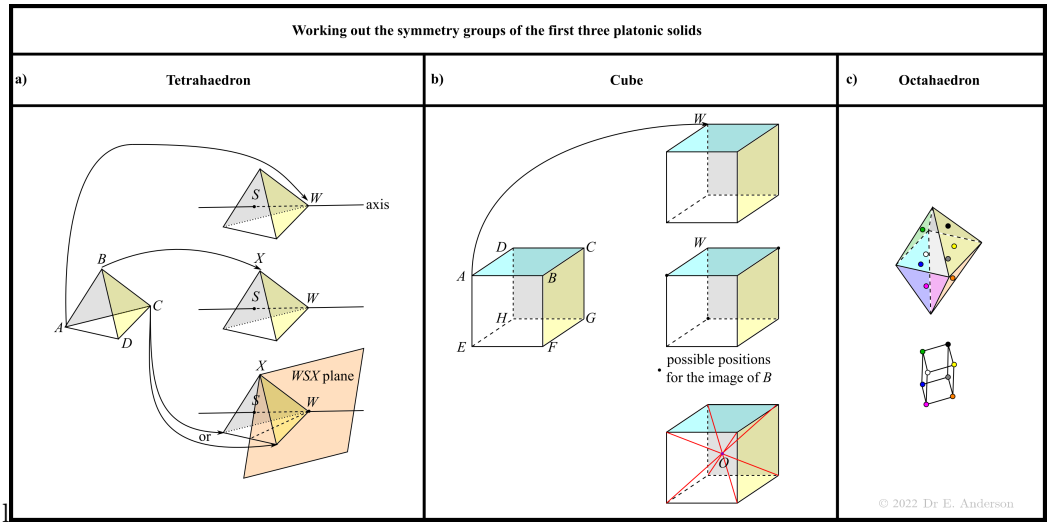


Figure 21.4:

Exercise 9 a) Show that the dodecahaedron and the icosahaedron form another dual pair, by which they also share symmetry group.

b) What is the dual of the tetrahaedron?

Pointer 2 See e.g. [161, 443] for what symmetry group it is that the dodecahaedron and the icosahaedron share.

Pointer 3 See also e.g. [172, 443] as regards proving that there are no further platonic solids in $\mathbb{E}^3$.

Exercise 10 In which ways can the symmetries of the tetrahaedron be broken? How about for the cube and the octahaedron?

21.4 Kleinian approach to Geometry, and to Mathematics more generally

Pointer 4 Symmetry groups give furthermore an approach to Geometry: the *Kleinian* alias *Erlangen* [83] approach (referring to a German city). Let us for now consider cases in flat space $Reals^d$.

Example 1 While we have already mentioned that the Euclidean transformations are a group $Eucl(d)$ consisting of (compositions of) translations, rotations and reflections, a *principle* picking these out is as follows. These are the transformations preserving $\mathbb{R}^n$'s usual notion of distance (see Appendix B). The Kleinian approach to Euclidean Geometry is to view it as the study of properties invariant under $Eucl(d)$.

Example 2 The Kleinian approach to Similarity Geometry is to view it as the study of properties invariant under the similiarity group $Sim(d)$. These are the transformations preserving angles and ratios of distances, which extend $Eucl(d)$ by additionally including dilations.

Remark 1 See Volumes 3 and 7 for further details of how these and various further types of Flat Geometry follow from considering further groups.

Remark 2 Circular symmetry can be viewed as a symmetry breaking of $Eucl(2)$ by modelling a privileged central point, which serves as a unique centre of rotation. This breaks translation-invariance while retaining rotations and reflections. Spherical symmetry is the 3-d counterpart of the above. Cylindrical symmetry corresponds to a distinct breaking in which a privileged axis is modelled. Now on the one hand, only the 2-d rotations in the plane perpendicular to this axis survive out of the more general 3-d rotations. But on the other hand, the translations parallel to our axis survive as well.

Exercise 11 Neglecting reflections, which other symmetry breakings are possible for $Eucl(2)$, $Eucl(3)$, $Sim(2)$ and $Sim(3)$?

[You may wish to check the coorrectness and completeness of your answer to this once you know how the translations, rotations and dilations 'fit together' to form $Sim(d)$.]

Pointer 5 In fact, the Kleinian approach extends to give a definition for Differential Geometry (invariance under diffeomorphisms: differentiable bijections with differentiable inverses). And for Topology (invariance under homeomorphisms: continuous bijections with continuous inverses). It even gives an approach to Mathematics, usefully covering most branches of Mathematics (at least classical branches within the Equipped Sets Foundational System). This paragraph's settings are various vast generalizations of the preceding paragraph's $\mathbb{R}^n$ example.

Chapter 22

Combinatorial applications and discussion

22.1 Partial labelling and selection

Remark 1 Let us first return to another source of variety that expand on the oversimplified dichotomy of labelled objects (denoted with an L) versus unlabelled objects (denoted with a blank). Namely, the extent to which objects' labels are distinguished We already introduced partial distinguishability. What we are now to ascertain is its connection to subgroups H of the permutation group $S_n \times S_n$ acting on all n labels. The first factor here being the distinguishability group to the second being the meaningfulness group. The above dichotomy corresponds respectively to the trivial subgroup and the whole-group subgroup of the S_n of distinguishabilities, in cases in which meaningfulness is trivial. So that the other possibilities are encompassed by the *proper* subgroups of $S_n \times S_n$ that are realizable by the labellings of our object.

Exercise 1 What are the labelling symmetry groups for

a) 4 objects carrying 2 pairs of distinguishable and yet meaningless labels?

b) All possible cases of partial distinguishability-and-meaningfulness for a set of 5 labelled objects? For a set of 6?

Remark 2 Labelling is even more generally parametrized by (σ, H), where σ' is the list of levels of structure of the labels. For instance, the labels could all be copies of some group K, in which case H is the distinguishability S_n itself. They could however be several distinct groups, so that H is a proper subgroup of the distinguishability S_n In the first case, if the underlying object has symmetry group G, the labelled version has symmetry group $G \times K \times S_n$.

Remark 3 To pass to counting arguments, we postpone until Sec 22.2, out of needing the intervening Chapters' theorization to do so.

♠ ◇ ♣

Structure 1 Modelling indistinguishability in selections. This I-label is really subgroup-valued. I is S_p-valued, whereas $\overline{I}$ is id-valued. But cases with partially-distinguishable labels involve symmetry groups on the selected set: a subgroup G of S_p: the symmetries of the selected set. N.B. that this now varies between possible p-object selections.

Notational Remark 1 We thus redeclare

$$\mathrm{Sel}\left(q \curvearrowright p;\, O,\, R,\, G^{\mathrm{ball}}_{\mathrm{dist}},\, G^{\mathrm{box}}_{\mathrm{dist}}\right) .$$

notation. This is for q choose p in an ordered manner O with the 3 states O, P or $_$ (denoting a blank entry). With replacement label taking the extended set of values $_$, $<$, $\leq$, $=$, $\geq$, $>$ and $\neq$. With $G_{\text{dist}}^{\text{ball}}$ the distinguishability group of the 'balls' and $G_{\text{dist}}^{\text{box}}$ that of the 'boxes'.

Remark 4 There is probably less applicational value to partial replacements, or to only ordering part of one's selection. Or to using a more general notion of ordering for all of, or part of, one's selection. So we halt our conceptual panorama at this culmination point. Partial distinguishability moreover plays quite a large role in Physics, Chemistry, and Biology. Multiple species and other multiplicities of kind give some indication of this, from atomic to molecular to cellular to genotypes to phenotypes to wholes species of organism... Immunology's distinguished subpopulations (infected, infectious, survivor, vaccinated...) constitute another example.

Remark 5 Barring one's O amounts to dividing by the order of S_p and multiplying by that of G. In the case of total distinguishability, $G = id$. So we just divide by the order of S_p, i.e. $p!$. Then

$$\bar{O}[R]\bar{I}(q,\, p) \;=\; \frac{O[R]\bar{I}(q,\, p)}{p!} \;. \tag{22.1}$$

In our general case,

$$\bar{O}[R](q,\, p,\, G) \;=\; \frac{O[R]\bar{I}(q,\, p) \;\times\; |G|}{p!} \;. \tag{22.2}$$

22.2 Combining partitioning with object structure

♠ ◇ ♣

Structure 2 In considering partitions, suppose that some of our compartments are distinguishable from each other but others are not. With two indistinguishable compartments, 2 1 can be realized in just one way. But if the compartments are labelled distinguishably, there are two ways of realizing this. Such examples involve interplay between partition states and symmetry structure. The symmetry group realized by each configuration (or shared by a family of configurations) provides a varying multiplicity of ways in which each partition can be realized

Examples 1-4 Our starlike, quadruped, monkey and scorpion tree graph examples in Secs 36.1 and 38.6 give Graph Theory examples of this phenomenon. Alongside what symmetry group and associated type of partial distinguishability is realized in each case. Adding metric information to these graphs then gives many further examples of symmetry breaking (Chapter 38).

Example 5 The opening sentence in the graduate-and-research level masterpiece [43] is "*Many of the objects we shall consider in this book will turn out to be parametrized by partitions*".

22.3 Pictorial automorphisms may not be all

Remark 1 Our polygon examples can be taken reside in geometrical $\mathbb{E}^2$. Our polyhaedron and molecular examples can be taken to reside in geometrical and physical $\mathbb{E}^3$ respectively.

Remark 2 Our partition and selection examples need not however be in the context of any such residency. For there is not usually any reality to how the 'bags' or 'boxes' in these examples' conceptualizations are arranged in relation to each other in physical or geometrical space. The partitions' Young diagram representations are specifically 2-d depictions, however, within some *abstract* $\mathbb{R}^2$. Small partitions furthermore do not have a particularly complicated theory of symmetry. The selection example's symmetry properties moreover reduce to (a labelled version of) the partition example's. It can thus be treated in terms of (labelled) Young diagrams as well.

Remark 3 We shall however see in Part V that graph automorphisms are a more considerable matter. For graphs are usually interpreted as having 'deformable' topological edges. They are thereby free from all of the following. A background space, fixed edge lengths, fixed edge-length ratios, fixed relative angles between edges, or even any guarantee of edge rectilinearity.

Some modelling situations do bring back in some of this structure. This returns us to such as the polygon, polyhaedron or molecular modelling regimes. But when not enough of such 'Euclidean', 'length' or 'similarity' structure is present, a gap of the following kind opens up.

$$Aut(\,\mathrm{Pict(G)}\,) \;<\; Aut(\,\mathrm{G}\,) \tag{22.3}$$

for Pict a depiction and G a graph. On $\leq$ 3 vertices, it turns out that we can always maximize pictorial symmetry in 2-*d* so as to coincide with the graph's own symmetry. On $\leq$ 4 vertices, the same applies in 3-*d*. For $\geq$ 5 vertices, however, (22.3) is in some cases forced.

Remark 4 While there can be a certain amount of 'visual intuition' in comparing objects as regards which 'looks more symmetric', the above example illustrates that the following kind of subtlety places limitations on this.

$$Aut(\,\mathrm{Pict}(\,\mathfrak{X},\,\sigma\,)\;<\;Aut(\,\mathfrak{X},\,\sigma\,)\;. \tag{22.4}$$

Pointer 1 How to effectively depict graphs is a subject in its own right [194]. Symmetry-maximizing depictions are one part of this subject; see Chapter 45 for another.

22.4 Equivalence between symmetry groups and abstract groups

Theorem 1 Symmetry groups are equivalent to abstract groups.

Proof ($\Rightarrow$) The Reader can straightforwardly check that symmetry transformations obey the abstract group axioms (**Exercise 2**).

($\Leftarrow$) It turns out that even some types of graph suffice to realize all possible symmetry groups. For instance, Frucht's First Theorem [137, 193] is to this effect for finite groups. We postpone proof of this to Volume 5, however. □

Remark 1 The second part of the above is the harder part to establish. It can be rephrased as 'automorphism groups represent abstract groups'.

Remark 2 The above is this Volume's second mention of a Representation Theorem.

22.5 More general discussion of naming

Naming Remark 1 This equivalence established, this is an opportune place to comment that the name 'group' was coined by Galois [80] (in French: 'groupe'). His conception of group was a concrete one. Now separated out from its polynomial-equation-solving application. This being the context in which previous work subsequently envisaged to be about groups, such as Lagrange's [79], was done. As well as being an application that Galois revolutionized the study of: Galois Theory [346, 288]. The general concept of concrete group was developed by Cauchy [81].

Naming Remark 2 The abstract conception of groups is due to Cayley [82] and Dyck [85]; see in particular [98] for the history of Group Theory up to the late 19th Century. The name 'abstract group' did not however appear until the early 20th Century. For 'abstract group' is but a subcase of 'Abstract Algebra', a truer name for the (then) 'Modern Algebra' developed by Artin, Hilbert and Noether. So these names in fact arose contemporaneously with 'abstract art'. More significantly,

the names 'abstract group' and 'concrete group' as a pair can be viewed as a subcase of the abstract versus concrete conceptualization of Philosophy by Quine [410]. Unpaired, abstraction was a philosophical theme developed in particular by Locke [338].

Remark 1 One should however bear in mind that a truer name for 'concrete group' is the rather more specific *subgroup of a permutation group*. While 'abstract group' is specifically a subcase of Abstract Algebra, in the sense of the first organized textbook on this [90] (for all that this was still called 'Modern Algebra'). Which heralded the end of conceiving of Algebra in terms of algebraic equations, to be replaced by conceiving in terms of *algebraic structures*.

Naming Remark 3 The current major application of groups to modelling symmetry is currently considered a third pillar of Group Theory. This foundation is however somewhat less deep than the philosophical dichotomy between concrete and abstract. It is also useful to recall that the original application of groups was to solving polynomial equations in radicals. So it might come to pass that the main application of Group Theory might change again, or become a completed rather than still actively-researched application.

♠ ◇ ♣

Pointer 2 Other basic structures in Abstract Algebra are [288] *fields* and *vector spaces*, and their respective generalizations to *rings* and *modules*. To groups being 1-operation algebraic structures, fields and rings) are 2-operation ones. Simple examples of fields are $\mathbb{R}$ and $\mathbb{C}$, each as equipped with $+$ and $\times$ Vector spaces are then linear spaces over fields. Such as $\mathbb{R}^n$ with multiplication of its vectors by scalars in the corresponding field. Modules are similar, but now over rings. By observing that commutative groups are already a type of module, we do not leave modules entirely mysterious to the Reader!

Pointer 3 Following on from Abstract Algebra, *Categories* [227] are, on the one hand, a subcase. Namely, a 2-axiom weakening of groups that keeps the identity and associativity properties that are innate to compositions of maps. On the other hand, Categories are a universally-applicable fusion of Abstract Algebra and the Erlangen program. I.e. that everything else is to be studied at the level of

$$(\text{ structure }, \quad \text{structure-preserving morphism })$$

pairs.

Pointer 4 *Arenization* can then be viewed as a universal second-order version of Abstract Algebra. I.e. that everything is to be studied at the level of spaces of spaces, or structures of structures. Vector spaces are moreover already arenas: of vector objects. And so the seed of arenization is already planted in these basic structures of Abstract Algebra (and their generalization to modules).

22.6 Restricted versus full Group Theories

Remark 1 $\mathbb{E}^2$ and $\mathbb{E}^3$ permit realization of but a limited repertoire of finite symmetry groups as subgroups. These do cover a number of very useful examples.

Pointer 5 The *molecular symmetry groups* A subset of the groups supported within Euclidean space are realized as the symmetry groups of at least one observed reasonably stable molecule. These are

0) *id* for asymmetric molecules.

1) Three infinite families:

C_n for $n \geq 2$, including ∞.

D_n for $n \geq 3$, including ∞.

Dic_n – the dicyclic groups – for $n \geq 2$, including ∞.

What cutoffs for large n bound what is known to be realized is an issue, though the infinite cases are each realized as well by very simple molecules (Sec 21.2).

3) Some loose extra cases: $S_4 = $ T*etrahaedron*, C*ube* $=$ O*ctahaedron*, I*cosahedron* $=$ D*odecahaedron*, and $O(3) = Sym(\mathbb{S}^2)$.

Pointer 6 *Line groups* are the groups manifested by repeating patterns in $\mathbb{E}$. These are just id and C_2.

Pointer 7 *Wallpaper groups* [103, 104] are the groups manifested by repeating patterns in $\mathbb{E}^2$. These turn out to be just the line groups alongside C_3, C_4, $C_2 \times C_2$, C_6, D_3, D_4, and D_6.

Pointer 8 *Crystallographic groups* [100, 115] are the groups manifested by repeating patterns in $\mathbb{E}^3$. These turn out to be just the wallpaper groups alongside $C_2 \times C_4$, ${C_2}^3$, $C_2 \times C_6$ $C_2 \times D^4$ $C_2 \times D_6$, A_4 $C_2 \times A_4$ S_4 and $C_2 \times S_4$. A_4 here denotes the alternating group, which we study in Sec 27.5.

Pointer 9 Intrinsic treatment of polygons and polyhaedra will feature in Books 3, 5 and 11.

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Pointer 10 There are further ambiguities concerning some groups being able to act in more than one way on their corresponding $\mathbb{E}^n$ space. And as regards *point groups* – fixing a point – versus larger unrestricted sets of groups. And also as regards subcases exhibiting a lower-dimensional regularity within a higher-d space, such as in the following question.

Question 0 *Friezes* are types of wallpaper pattern that only repeat in one of the two perpendicular directions supported by the plane. How many wallpaper groups are furthermore frieze groups?

Question 1 a) How many translationally-invariant tessellation groups (tilings) can you find for the plane?

b) What kinds of tessellation groups can you find for the sphere (using a single type of tile)? [Volume 3 [443] will permit *systematic* approaches to such questions.]

♠ ◇ ♣

Remark 2 The current Volume thus contains enough Group Theory to cover all of these applications with the sole exception of the study of the icosahaedron group, which completion is in Volume 3 [443].

Big Pointer 11 The above is a 'Small Group Theory'. Full Group Theory starts with Volume 5 [444], rather. This involves

$$(\text{abstract Group Theory}) = (\text{the automorphism groups of } \textit{any} \text{ mathematical stucture}).$$

Some areas needing this include the theory of continuous symmetries [114, 301, 94, 314, 102] , hyperbolic plane analogues [423] of what we mention above for Euclidean space and spheres, and Algebraic Topology [250, 260].

Questions 2-7$^+$ A separate matter is proving that there are no more groups in each of Pointer 2 to 7's situations. Make a start on this to whatever extent possible with what you know so far (and return to this after you have completed the next Part!)

Part IV

A first installment of Group Theory, with Combinatorial Applications

Chapter 23

Orbits and stabilizers

23.1 Introduction

Let G be a group acting on a set $\mathfrak{X}$.

Definition 1 The *orbit* [110, 113, 109] of $x \in \mathfrak{X}$ under G

$$\mathfrak{o}\mathrm{rb}(x) = \{ g\,x \,|\, g \in G \} .$$

I.e. the set of images in $\mathfrak{X}$ that a given point x is carried to under the totality of group elements $g \in G$ (see Fig 23.1.a).

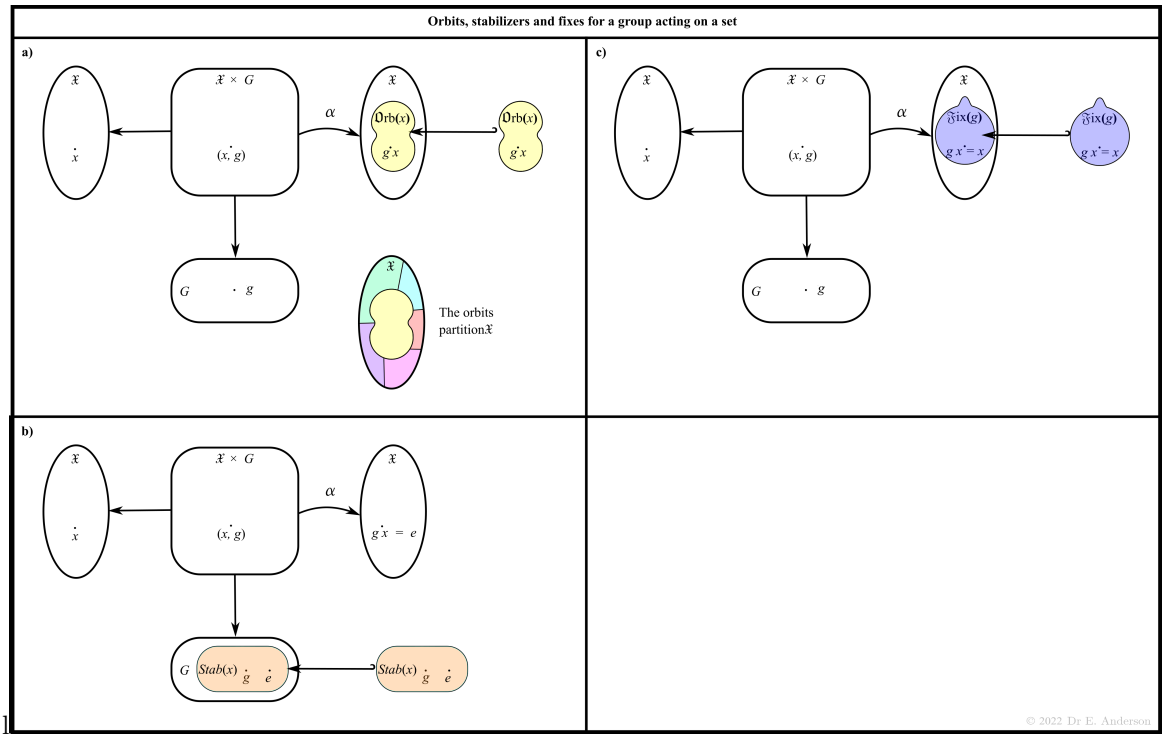


Figure 23.1:

Definition 2 The *stabilizer* [110, 113, 109] of $x \in \mathfrak{X}$ under G is the set

$$S\mathrm{tab}(x) = \{ g \in G \,|\, g\,x = x \} . \tag{23.1}$$

I.e. the set of group elements $g \in G$ that fix the element x of the set $\mathfrak{X}$ acted upon (see Fig 23.1.b).

Naming Remark 1 'Stable' means unchanging, with reference to the fixed elements.

Naming Remark 2 In Geometry, these are alias *isotropy groups* and in Theoretical Physics, *little groups*. This 'group' and 'little' terminology is justified as follows, while comparative discussion of notions of isotropy is postponed until Books 1, 3 and 4).

Lemma 1 The $S\mathrm{tab}(x)$ are indeed groups, more specifically subgroups

$$S\mathrm{tab}(x) \leq G \, .$$

[These are thus quite generally littler than G, and, in any case, are never larger than G.]

Naming Remark 3 Hence 'little' is meant *relative to* G. Though on occasion $S\mathrm{tab}(x)$ can be the whole of G itself (see Worked Example 2 below). So 'little' more accurately means $\leq$ in the subgroup sense.

Proof Closure: for $g_1, g_2 \in S\mathrm{tab}(x)$,

$$(g_1 g_2)(x) = g_1(g_2(x)) = g_1(x) = x \, .$$

This is by rebracketing, then $g_2 \in S\mathrm{tab}(x)$ and finally $g_1 \in S\mathrm{tab}(x)$. Thereby $g_1 g_2$ itself $\in S\mathrm{tab}(x)$.

Identity:

$$e\,x = x \Rightarrow e \in S\mathrm{tab}(x) \, .$$

Inverse: for $g \in S\mathrm{tab}(x)$,

$$x = e\,x = (g^{-1} g)\,x = g^{-1}(g(x)) = g^{-1}(x) \, .$$

This is by the inverse property, followed by rebracketing, and then $g \in S\mathrm{tab}(x)$. Thereby, g^{-1} itself $\in S\mathrm{tab}(x)$. □

♠ ◇ ♣

Lemma 2 i)

$$\bigcup_{x \in \mathfrak{X}} \mathfrak{o}\mathrm{rb}(x) = \mathfrak{X} \ (\textit{cover}) \, . \tag{23.2}$$

For $x_1, x_2 \in \mathfrak{X}$, either

$$\mathfrak{o}\mathrm{rb}(x_1) \cap \mathfrak{o}\mathrm{rb}(x_2) = \emptyset \ (\textit{disjointness}) \, . \tag{23.3}$$

Or

$$\mathfrak{o}\mathrm{rb}(x_1) = \mathfrak{o}\mathrm{rb}(x_2) \, . \tag{23.4}$$

The first two equations here can be summarized by

$$\bigcup_{x \in \mathfrak{X}} \mathfrak{o}\mathrm{rb}(x) = \mathfrak{X} \ (\textit{disjoint union}) \, . \tag{23.5}$$

Thereby, orbits *partition* the set $\mathfrak{X}$ acted upon.

Proof i) $e \in G$ and $x \in e\,x$. So $x \in \mathfrak{o}\mathrm{rb}(x)$. Thus

$$\bigcup_{x \in \mathfrak{X}} \mathfrak{o}\mathrm{rb}(x) \supseteq \bigcup_{x \in \mathfrak{X}} x = \mathfrak{X} \, . \tag{23.6}$$

But also $G \ : \ \mathfrak{X} \ \longrightarrow \ \mathfrak{X}$, so

$$\mathfrak{orb}(x) \ \subseteq \ \mathfrak{X} \ . \tag{23.7}$$

Thus taking unions of both sides and using that the union of multiple copies of the same thing is just that same thing,

$$\bigcup_{x \, \in \, \mathfrak{X}} \mathfrak{orb}(x) \ \subseteq \ \mathfrak{X} \ . \tag{23.8}$$

Together, (23.6) and (23.8) imply the equality i).

ii) Suppose that

$$\mathfrak{orb}(x) \ \cap \ \mathfrak{orb}(y) \ \neq \ \emptyset \ .$$

Then we can pick some $x_0 \ \in \ \mathfrak{orb}\,(x_1) \ \cap \ \mathfrak{orb}\,(x_2)$. Thus both of the following hold.

$$x_0 \ = \ g_1 \, x_1 \ \text{ for some } \ g_1 \ \in \ G \ . \tag{23.9}$$

$$x_0 \ = \ g_2 \, x_2 \ \text{ for some } \ g_2 \ \in \ G \ . \tag{23.10}$$

Equating (23.9) and (23.10),

$$g_1 \, x_1 \ = \ g_2 \, x_2 \ .$$

So

$$g_2{}^{-1} \, g_1 \, x_1 \ = \ g_2{}^{-1} \, g_2 \, x_2 \ = \ e \, x_2 \ = \ x_2 \ .$$

Thereby, $x_2 \ \in \ \mathfrak{orb}\,(x_1)$. So for some $g \ \in \ G$,

$$\bar{x} \ = \ g \, x_2 \ = \ g \, g_2{}^{-1} \, g_1 \, x_1 \ \in \ \mathfrak{orb}\,(x_1) \ .$$

Thus

$$\mathfrak{orb}\,(x_2) \ \subseteq \ \mathfrak{orb}\,(x_1) \ . \tag{23.11}$$

But this argument is invariant under reversal of 1 and 2 labels, so also

$$\mathfrak{orb}\,(x_1) \ \subseteq \ \mathfrak{orb}\,(x_2) \ . \tag{23.12}$$

Together, (23.11) and (23.11) imply (23.4). □

We then have the following preliminary Combinatorial result.

Corollary 1 For finite $\mathfrak{X}$ and $\mathfrak{orb}\,(x_k)$ with $k \ = \ 1$ to n distinct,

$$|\, \mathfrak{X} \,| \ = \ \sum_{i \, = \, 1}^{n} |\, \mathfrak{orb}(x) \,| \ . \tag{23.13}$$

Proof Let

$$\mathfrak{a} \ = \ \bigcup_{k \, = \, 1}^{n} \mathfrak{a}_k$$

for $\mathfrak{a}_k$ mutually disjoint. Then

$$|\, \mathfrak{a} \,| \ = \ \sum_{k \, = \, 1}^{n} |\, \mathfrak{a}_k \,| \ .$$

Finally applying this result to (23.2) yields (23.13) □.

Lemma 3 $x \ \tilde{} \ y$ iff $x \ = \ g \, y$ for some $g \ \in \ G$ is an equivalence relation, embodied by 'lying on the same orbit as each other'.

Proof Reflexivity: $x \ \tilde{} \ x$ by $x \ = \ e \, x$.

Symmetry: $x \tilde{} y$ means that $x = g\,y$. Then

$$g^{-1}\,x \;=\; g^{-1}\,(\,g\,x\,) \;=\; (\,g^{-1}\,g\,)\,x \;=\; e\,x \;=\; x\;.$$

Thus $y \tilde{} x$. Transitivity. If $x \tilde{} y$ and $y \tilde{} z$, $x = g\,y$ and $y = g'z$. Thus

$$x \;=\; g\,(\,g'\,z\,) \;=\; (\,g\,g'\,)z\;,$$

so by group closure of G, $x\tilde{z}$. □

Remark 1 $\mathfrak{o}\mathrm{rb}(x)$ is a subset of $\mathfrak{X}$, not of G. It thus stands no chance of being a subgroup of G.

Definition 3 The *fix* $\mathfrak{F}\mathrm{ix}(g)$ is the subset of $\mathfrak{X}$ that is fixed by the group element g (Fig 23.1.c).

23.2 Examples

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Worked Example 1 Let V_4 act on $\{\,1,\,2,\,3,\,4\,\}$ as follows.

$$a \;:\; \{\,1,\,2,\,3,\,4\,\} \;\longrightarrow\; \{\,2,\,1,\,3,\,4\,\}\;,$$

$$b \;:\; \{\,1,\,2,\,3,\,4\,\} \;\longrightarrow\; \{\,1,\,2,\,4,\,3\,\}\;,$$

Where does the last non-identity element map to? How may orbits are there? What are the corresponding stabilizers?

Answer 1

$$a\,b \;:\; \{\,1,\,2,\,3,\,4\,\} \;\longrightarrow\; \{\,1,\,2,\,4,\,3\,\} \;\longrightarrow\; \{\,2,\,1,\,4,\,3\,\}\;.$$

Two orbits are formed:

$$\mathfrak{o}\mathrm{rb}(1) \;=\; \{\,1,\,2\,\} \;=\; \mathfrak{o}\mathrm{rb}(2) \;\text{ and }\; \mathfrak{o}\mathrm{rb}(3) \;=\; \{\,3,\,4\,\} \;=\; \mathfrak{o}\mathrm{rb}(4)\;.$$

The corresponding stabilizers are

$$C_2 \;=\; \{\,e,\,b\,\}$$

and

$$C_2 \;=\; \{\,e,\,a\,\}$$

respectively (Fig 23.2.a).

Worked Example 2 Let C_3 act on $\mathbb{R}^2$ in the natural manner of rotations about the origin. What are the orbits and stabilizers?

Answer 2 $S\mathrm{tab}(0)$ is the whole of C_3, as all rotations in 2-d preserve the origin. For $x \neq 0$, $S\mathrm{tab}(x)$ is just e, however, as no nontrivial rotation preserves any other point. $\mathfrak{o}\mathrm{rb}(0)$ is just the origin itself. For $x \neq 0$, $\mathfrak{o}\mathrm{rb}(x)$ is a triple of open rays forming 120-degree angles about the origin (Fig 23.2.b).

Exercise 1 What are the orbits and stabilizers for D_3 acting on $\mathbb{R}^2$ in the manner of rotations and reflections?

Worked Example 3 Suppose that $\langle\,\mathbb{Z}\;,\;\;+\,\rangle$ acts on $\mathbb{R}$ by the *integral shift map* $x \;\mapsto\; x \,+\, n$. What are the orbits and stabilizers?

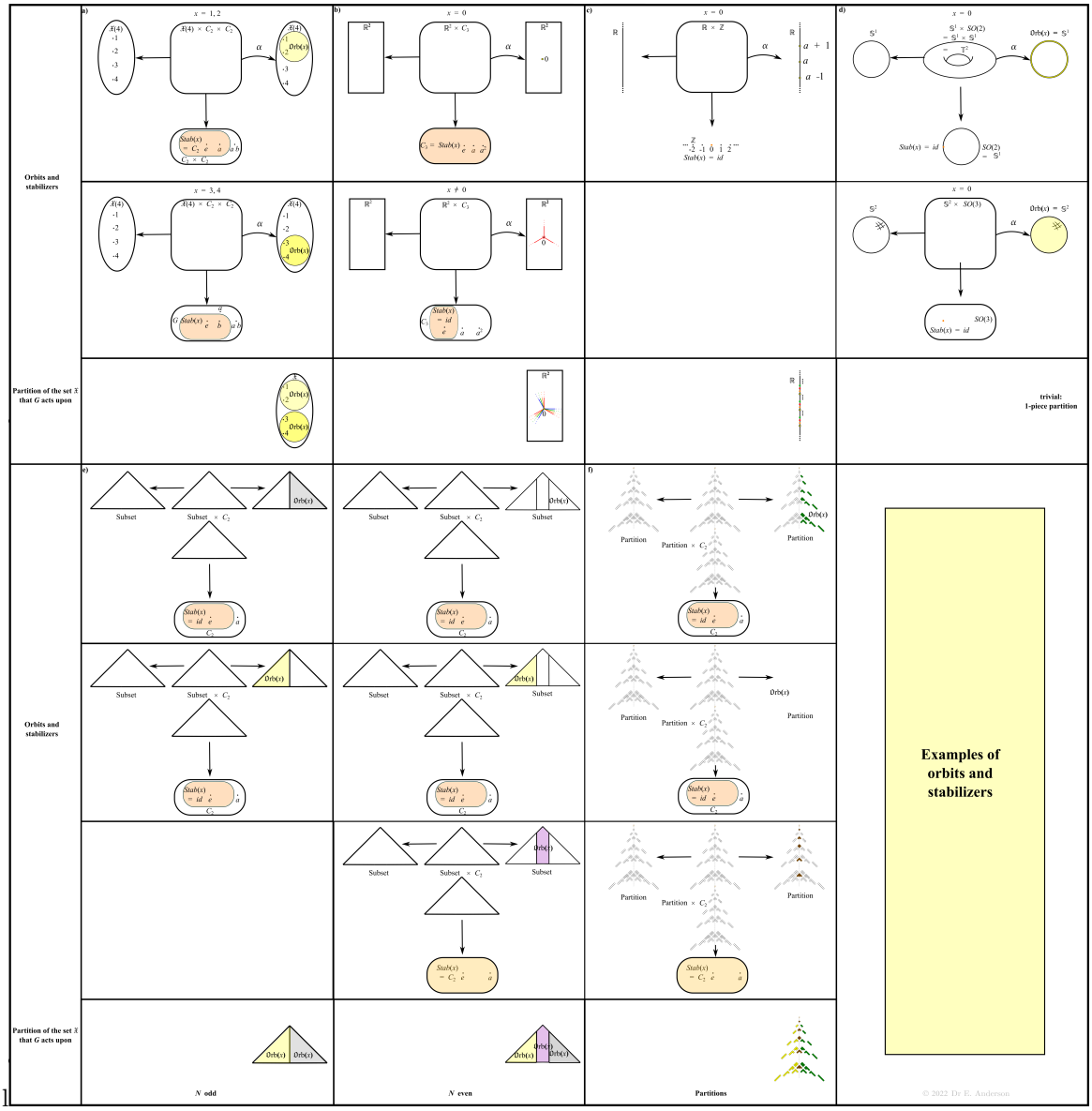


Figure 23.2:

Answer 3 Since no $x \in \mathbb{R}$ is preserved by non-identity shifts, the stabilizers are all id. Our group action sends x to $x + n$. And then to $x + mn$ recursively for $m \in \mathbb{N}_0$ Considering the inverse element as well, x is also sent to $x - n$ and to $x - mn$. Jointly summarizing, the orbits are $x + \mathbb{Z}$, each of which $\cong \mathbb{Z} \cong \mathbb{N}$. We have a distinct such for each $x \in [0, n)$ (say: Fig (Fig 23.2.c).

Worked Example 4 What are the orbits and stabilizers for $SO(2)$'s standard rotational action on $\mathbb{S}^1$? And for $SO(3)$'s on $\mathbb{S}^2$

Answer 4 Nontrivial rotations around a circle preserve no point thereon, while the identity rotation preserves all points. Thus $Stab(x) = id$ for every $x \in \mathbb{S}^1$. Correspondingly, all x lie on a single orbit: the whole circle (Fig 23.2.d). This analysis extends to rotations around a sphere, the single orbit now being the whole sphere (also in Fig 23.2.d).

Remark 1 More generally, in all cases in which a single orbit is realized, that orbit must be the

entirety of $\mathfrak{X}$ itself. This serves as a simple example of *transitive* group action.

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Worked Example 5 What are the orbits and stabilizers for subsets under complementation?

Subset complementation is a C_2 action. For N odd, every subset is heterocomplementary. So $Stab(\mathfrak{Y}) = id$ for all $\mathfrak{Y} \in \mathfrak{X}(N)$. Each subset is mapped to its complement, and acting again sends this back to the original; i.e. it is an involution. So all the orbits are of size 2, of the form $\{\mathfrak{Y}, \mathfrak{Y}\mathfrak{C}\}$.

For each N even, there is also one homocomplementary subset $\mathfrak{Z}$. These being invariant under our action, $Stab(\mathfrak{Z}) = C_2$. The corresponding orbits are clearly singletons $\{\mathfrak{Z}\}$. All the other subsets behave as in the odd-N case. We summarize in Fig 23.2.e).

Worked Example 6 What are the orbits and stabilizers for partitions under conjugation?

Answer 6 Homoconjugation and heteroconjugation are the only two possibilities. The first has $Stab(\mathrm{p}) = C_2$ and singleton orbits $\{\mathrm{p}\}$. The second has $Stab(\mathrm{p}) = id$ and doublet orbits $\{\mathrm{p}, \overline{\mathrm{p}}\}$. These two possibilities are distributed somewhat more irregularly over N than in Worked Example 5. We summarize the corresponding orbits for $N \leq 6$ in Fig 23.2.f).

23.3 Exercises

Exercise 2 Show that $SO(2)$'s standard rotational action on $\mathbb{R}^2$ gives concentric circles as orbits, alongside realizing the centre of rotation as an extra point that comprises a further orbit. Show that this working generalizes to $SO(3)$'s standard rotational action on $\mathbb{R}^3$ having an 'onion' of concentric spheres as orbits, with the point at the origin as a further orbit.

Exercise 3 a) Show that on a chess board of any size, the corner squares constitute an orbit.

b) How many other orbits are there on a 4×4 chess board?

c) And on a full-sized 8×8 chess board?

d) For each of this Exercise's orbits, what are the corresponding stabilizers?

Exercise 4 a) What are the orbits and stabilizers for a Rubik's cube if it is just modelled as immobile and uncoloured?

b) If mobile and coloured in the standard ways, how many configurations can the cube be in?

[Comparing the order of the corresponding group with that of those in the current Volume gives some indication of why it would be premature to consider orbits and stabilizers for this in the current Volume. Though there are additionally some structural subtleties with the Rubik's cube group as well.]

Chapter 24

Cosets, conjugation and Lagrange's Theorem

24.1 Cosets

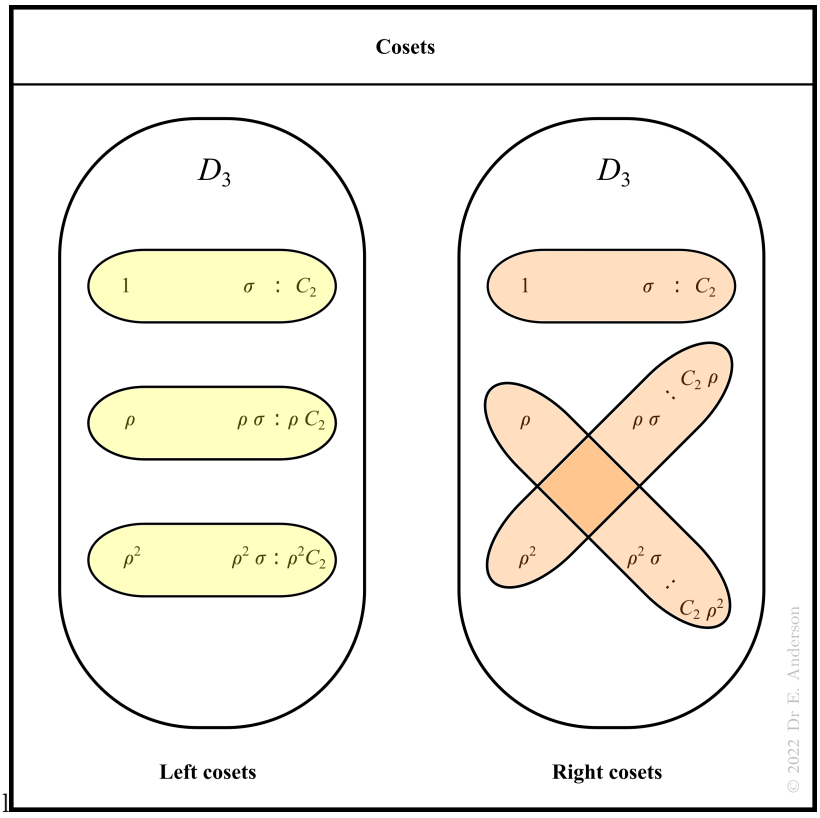


Figure 24.1:

Definition 1 Let $H \leq G$ and $g \in G$. Then

$$gH := \{ gh \,|\, h \in H \} \tag{24.1}$$

is a *left coset* [87, 110, 109] of H.

$$Hg := \{ hg \,|\, h \in H \} \tag{24.2}$$

is a *right coset* of H.

Remark 1 In some contexts, right and left cosets coincide, so we can there say just *cosets*. The most obvious examples of this coincidence are for commutative groups. For here

$$(G \text{ commutative }) \;\Rightarrow\; (H \text{ commutative }) \,,$$

and then

$$g\,H \;=\; \{\, g\,h \,|\, h \;\in\; H \,\} \;=\; \{\, h\,g \,|\, h \;\in\; H \,\} \;=\; H\,g \,.$$

Worked Example 1 To establish that this does not always occur (so that right and left cosets retain meritorious distinction as concepts), the minimal non-commutative group $D_3 \;=\; S_3$ turns out to suffice.

Answer 1 Claim: for

$$H \;=\; C_2 \;=\; \{\, e, \sigma \,\} \;<\; D_3 \,, \;\; C_2\,\sigma \;\neq\; \sigma\,C_2 \,.$$

This can be seen from

$$\rho\,C_2 \;=\; \{\, \rho, \rho\,\sigma \,\} \;\neq\; \{\, \rho, \rho^2\sigma \,\} \;=\; \{\, \rho, \sigma\,\rho \,\} \;=\; C_2\,\rho \,.$$

The middle equality here holds by the dihedral relation.

Remark 2 Sec 26.4 concerns the most general case beyond commutative groups, for which all right and left cosets coincide: normal subgroups.

Remark 3 For a fixed H and G, the following hold.

i) All resulting cosets are of the same size.

ii) All non-equal cosets are disjoint from each other.

iii) Cosets thus partition groups into equal-sized pieces.

We establish this in Sec 24.12 while proving Lagrange's Theorem.

Remark 4 This gives a straightforward pictorial presentation of cosets, as exemplified in Fig 24.1.

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Worked Example 2 What are the cosets a) for $C_2 \;<\; C_4$ and b) Worked Example 1?

a) is Abelian, so there is only one case to consider: Fig 24.1.a).

b) This exhibits left-right distinction, as per Fig 24.1.b) versus Fig 24.1.c).

Exercise 1 What are the cosets of $C_3 \;\in\; T_4$ and of $D_3 \;\in\; T_4$ for T_4 the regular tetrahaedron's symmetry group?

24.2 Conjugation

Definition 1 Let G be a group and $g_1,\, g_2 \;\in\; G$ Then g_1 and g_2 are *conjugate* if $\exists\, a \;\in\; G$ such that

$$g_1 \;=\; a\,g_2\,a^{-1} \,. \tag{24.3}$$

Remark 1 Compare

$$P\,A\,P^{-1} \tag{24.4}$$

in Matrix Algebra and Linear Algebra [441]; this is usually referred to as *similarity* in these subjects.

Remark 2 The notion of conjugation uplifts to subgroups, as follows.

Definition 2 Let G be a group and $H, K \leq G$. Then H and K are *conjugate* if $\exists\, a \in G$ such that

$$K = a\,H\,a^{-1}\ . \tag{24.5}$$

I.e.

$$K = \{\, a\,h\,a^{-1} \,|\, h \in H \,\}\ . \tag{24.6}$$

When thinking of conjugation as a binary relation, let us denote it by $\stackrel{c}{=}$.

Lemma 1 For H a subgroup,

$$K = a\,H\,a^{-1}\ . \tag{24.7}$$

is also a subgroup.

Proof Closure: Let $k_i \in K$ for $i = 1, 2$. So $k_i = a\,k_i\,a^{-1}$.

$$\Rightarrow\ \ k_1\,k_2 = (\,a\,h_1\,a^{-1}\,)\,(\,a\,h_2\,a^{-1}\,) = a\,h_1\,(\,a^{-1}\,a\,)\,h_2\,a^{-1} = a\,h_1\,e\,h_2\,a^{-1} = a\,h_1\,h_2\,a^{-1}$$

$$\Rightarrow\ \ k_1 k_2 \in K\ .$$

Identity:

$$a\,e\,a^{-1} = a\,a^{-1} = e\ .$$

Inverse:

$$\text{for}\ \ k \in K\ ,\ \ k = a\,h\,a^{-1}\ \ \text{for some}\ \ K\ .$$

Then

$$k^{-1} = (\,a\,h\,a^{-1}\,)^{-1} = (\,a^{-1}\,)^{-1}\,h^{-1}\,a^{-1} = a\,h^{-1}\,a^{-1}\ .$$

The second step here uses the inductive version of Sec 19.2's Corollary. □

Worked Example 1 Show that $\stackrel{c}{=}$ is an equivalence relation.

Answer 1 Reflexivity:

$$e\,g\,e^{-1} = e\,g\,e = g\ ,$$

so

$$g \stackrel{c}{=} g\ .$$

Symmetry. Suppose that

$$g_1 \stackrel{c}{=} g_2\ .$$

Then

$$g_1 = a\,g_2\,a^{-1}\ \ \text{for some}\ \ a \in G\ .$$

Thus

$$a^{-1}\,g_1\,a = a^{-1}\left(a\,g_2\,a^{-1}\right)a = \left(a^{-1}\,a\right)g_2\left(a^{-1}\,a\right) = e\,g_2\,e = g_2\ .$$

So $g_2 \stackrel{c}{=} g_1$ as well.

Transitivity. Now suppose also that

$$g_2 \stackrel{c}{=} g_3\ \ \Rightarrow\ \ g_2 = b\,g_3\,b^{-1}\ \ \text{for some}\ \ b \in G\ .$$

Then

$$g_1 \;=\; a\,g_2\,a^{-1} \;=\; a\,(\,b\,g_3\,b^{-1}\,)\,a^{-1} \;=\; (\,a\,b\,)\,g_3\,(\,b^{-1}\,a^{-1}\,) \;=\; (\,a\,b\,)\,g_3\,(a\,b\,)^{-1}\;.$$

The last step is by the pairwise version of Sec 19.2's Corollary.

The subgroup version of this result holds similarly. □

Naming Remark 1 'Coset' – Miller's term [87] for a notion first used by Galois [80] – can be a confusing word, by seemingly making reference to sets while actually being a Group-Theoretic notion. Burnside [86] called these 'conjugate groups'. But, aside from H itself, the other cosets are not themselves subgroups since they do not contain the identity. And yet *conjugate to a subgroup* is a truer name, by reference to how cosets are formulated, as well as to the right branch of Mathematics, without however implying that they are groups themselves.

Naming Remark 2 By itself, the word 'conjugation' does not distinguish between related pairs and more general related subsets. The first applies to conjugate partitions, which we made clear by talking about *conjugate pairs* in that setting. But in the current Group Theory setting, more general related subsets is meant.

Exercise 2 Prove that permutations have the same cycle type iff they are conjugate to each other.

Theorem 1 Points on the same orbit have conjugate stabilizers.

I.e. for x_i, x_j in the common orbit

$$\mathfrak{orb}\,(x_1) \;=\; \mathfrak{orb}\,(x_2)\;,$$

$$S\mathrm{tab}(x_i) \;=\; g\,S\mathrm{tab}(x_i)\,g^{-1}\;,$$

so

$$S\mathrm{tab}(x_i) \;=\; S\mathrm{tab}(x_j)\;. \tag{24.8}$$

Proof Claim: for $g_1 \in S\mathrm{tab}\,(x_1)$, $x_1 \in \mathfrak{orb}\,(x_2)$ and $g_2 \in S\mathrm{tab}\,(x_2)$,

$$g \;\stackrel{c}{=}\; g_2\;. \tag{24.9}$$

For $x_1 \in \mathfrak{orb}\,(x_2)$ means that

$$x_1 \;=\; g\,x_2 \;\text{ for some }\; g \in G\;.$$

But also $g_2 \in S\mathrm{tab}x_1$ iff

$$g_2\,x_1 \;=\; x_1 \;\text{ iff }\; g_2\,g\,x_2 \;=\; g\,x_2 \;\text{ iff }\; g^{-1}\,g_2\,g\,x_2 \;=\; g^{-1}\,g\,x \;=\; e\,x \;=\; x \;\text{ iff }\; g^{-1}\,g_2\,g \in S\mathrm{tab}\,(x_2)\;.$$

Relabelling, also $\bar{g} \in S\mathrm{tab}\,(x_1)$ obeys

$$\bar{g} \;\stackrel{c}{=}\; g_2\;. \tag{24.10}$$

Finally

$$g \;\stackrel{c}{=}\; \bar{g} \tag{24.11}$$

follows from (24.9) and (24.10) by transitivity. □

Naming Remark 3 and Pointer 1 This can be interpreted as a toy *Rigidity Theorem*. Such consist of a surprising structural limitation manifested as a conjugation. Volume 5 [444] contains a second toy such toy: *Sylow's Third Theorem* for finite groups. Rigidity Theorems remain mostly a matter of research in Geometry, however. Some famous ones include *Bieberbach's Rigidity Theorem* [88, 115] for crystallographic groups, *Mostow's Rigidity Theorem* [97, 101] for groups acting on hyperbolic space, and *Weil's local Rigidity Theorem* [93] for the lattice of subgroups of a Lie group.

24.3 Lagrange's Theorem

Theorem 2 (Lagrange) Let $H \leq G$ for G finite. Then

$$|H| \,\Big|\, |G| \,. \tag{24.12}$$

In words, *a subgroup's order divides that of the group.*

Proof Consider the w.l.o.g. left cosets gH of H.

Claim 1.

$$G = \bigcup_{g \in G} gH \,.$$

[We have already established the counterpart of this for orbits.]

$$g = ge \in gH \,.$$

So

$$G = \bigcup_{g \in G} g \subseteq \bigcup_{g \in G} gH \subseteq G \,,$$

establishing Claim 1.

Claim 2. Cosets either coincide or are disjoint. [Again, we are aided by having already established the counterpart of this for orbits.]

For take

$$g_1 H \cap g_2 H \neq \emptyset \,.$$

Then we can pick a

$$g_3 \in g_1 H \cap g_2 H \,.$$

So

$$g_1 h_1 = g_3 = g_2 h_2 \text{ for some } h_1, h_2 \in H \,.$$

Thus

$$g_1 = g_2 h_2 h_1^{-1} \,.$$

So for some $g \in G$,

$$c \in gH \Rightarrow c = g_1 h \text{ for some } h \in H \,.$$

Thus

$$c = g_2 h_2 h_1^{-1} h \,.$$

So

$$c \in g_2 H \,.$$

Thus

$$g_1 H \subseteq g_2 H \,. \tag{24.13}$$

But the roles of g_1 and g_2 can be exchanged in this proof. So also

$$g_2 H \subseteq g_1 H \,. \tag{24.14}$$

Together, (24.13) and (24.14) imply that

$$g_1 H = g_2 H \,.$$

The previous two claims combine to give

$$G = \coprod_{g_k \in G} g_k H \,.$$

k here is to include disjoint cosets covering G. Thus paralleling Sec 23's Corollary 1 about orbits,

$$|\,G\,| \;=\; \sum_{g_k \,\in\, G} |\,g_k\,H\,| \;. \tag{24.15}$$

Claim 3.

$$|\,g_i\,H\,| \;=\; |\,H\,| \;. \tag{24.16}$$

I.e.

$$\#(\,\text{elements in a coset}\,) \;=\; \#(\,\text{elements in the underlying subgroup}\,) \;.$$

For

$$g\,h \;=\; g\,h' \;\;\Rightarrow\;\; h \;=\; h' \;.$$

Thus the map

$$\begin{array}{ccc} H & \longrightarrow & g\,H \\ h & \mapsto & g\,h \end{array}$$

is a bijection. It thus preserves the size of finite sets, establishing Claim 3.

Finally substitute Claim 3 in (24.15) to obtain the following.

$$|\,G\,| \;=\; \sum_{g_k \,\in\, G} |\,g_k\,H\,| \;=\; \#(\,\text{disjoint cosets}\,)\,|\,H\,| \;.$$

Thereby,

$$|\,H\,| \;\Big|\; |\,G\,| \;,$$

so Lagrange's Theorem has been established. □

Remark 1 Claim 3 is *not* in general true for orbits, which can differ in size from element to element given a fixed G and $\mathfrak{X}$ equal So the simplicity of counting the sizes of cosets, and the consequent Combinatorially-useful presence of Lagrange's Theorem has no orbit counterpart. In this way, counting out the sizes of orbits is in general a harder endeavour.

Theorem 2′ (Lagrange) For any element $g \;\in\; G$ finite,

$$|g| \;\Big|\; |\,G\,| \;. \tag{24.17}$$

Proof For each $g \;\in\; G$,

$$H_g \;:=\; \{\,g^a \,|\, a \;\in\; \mathbb{Z}\,\}$$

is a commutative subgroup of G. Indeed, $H \;\cong\; C_n$ for the smallest $n \;\in\; \mathbb{N}$ such that $g^n \;=\; 1$, conforming to the definition of order of g, $|g|$ Which must exist since G is finite, so there is no room for H to be infinite. (24.17 then follows from Theorem 2. □

Chapter 25

Further combinatorial applications

25.1 Simple applications of Lagrange's Theorem

Exercise 1^{-} Classify all groups of prime order.

Exercise 2 Classify all the groups up to order 6. What orders of subgroup do each of these support? And with which multiplicities (i.e. how many copies are there of each)?

Remark 1 We have thereby exhaustively covered up to order 7.

Exercise 3 Keen people who have not done this before are welcome to furthermore try orders 8 and 9; which distinctive group do you discover in the process? Continue to keep track of the orders and multiplicities of subgroups supported.

Exercise 4 a) Classify the groups of order 10.

b)$^{+}$ Where have you already seen the pattern exhibited by the classification of groups of order 9 and 10? Hypothesize and prove a generalization of this observation. Consequently list which groups of order < 100 require further classificatory work.

Pointer 1 It is advised that you wait until you have studied Sylow's Theorems [92, 96] – structurally enlightening – before you continue much further with such exhaustions.

25.2 The Orbit–Stabilizer Theorem

Let G be a group acting on a set $\mathfrak{X}$, and consider any element $x \in \mathfrak{X}$.

Exercise 5^{-} Show that

$$|\,S\mathrm{tab}(x)\,| \;\Big|\; |\,G\,| \;. \tag{25.1}$$

I.e. that the order of each stabilizer divides the order of the group.

Remark 1 More interestingly, the following two statements turn out to hold for G finite.

Orbit–Stabilizer Theorem i)

$$|\,\mathfrak{o}\mathrm{rb}(x)\,| \;\Big|\; |\,G\,| \;. \tag{25.2}$$

I.e. the order of each stabilizer divides the order of the group.

ii) Indeed,

$$|\,G\,| \;=\; |\,\mathfrak{o}\mathrm{rb}(x)\,|\,|\,S\mathrm{tab}(x)\,| \;. \tag{25.3}$$

Lemma 1 $\exists$ a bijection

$$\mathfrak{o}\mathrm{rb}(x) \;\leftrightarrow\; g\,S\mathrm{tab}(x) \text{ [or just as well to } S\mathrm{tab}(x)g \text{ when distinct]} \;. \tag{25.4}$$

Proof i) By Lemma 1 of Sec 23,

$$S\mathrm{tab}(x) \;\leq\; G \;.$$

Thus the notion of $S\mathrm{tab}(x)$ having cosets is well-defined.

Let us first look at the map in the more straightforward backward direction. Suppose that

$$g_1\,S\mathrm{tab}(x) \;=\; g_2\,S\mathrm{tab}(x) \tag{25.5}$$

holds. Then

$$g_2{}^{-1}\,g_1\,S\mathrm{tab}(x) \;=\; g_2{}^{-1}\,g_2\,S\mathrm{tab}(x) \;=\; e\,S\mathrm{tab}(x) \;=\; S\mathrm{tab}(x) \;.$$

Thus

$$g_2{}^{-1}\,g_1 \;\in\; S\mathrm{tab}(x) \;.$$

So, by the definition of stabilizer,

$$g_2{}^{-1}\,g_1\,x \;=\; x \;.$$

This enables the last step of the following.

$$g_1\,x \;=\; e\,g_1\,x \;=\; \left(g_2\,g_2^{-1}\right)\,g_1\,x \;=\; g_2\,(\,g_2^{-1}\,g_1\,x\,) \;=\; g_2\,x \;.$$

Thus

$$g_1\,x \;=\; g_2\,x \;.$$

We next consider the forward direction.

$$g_1\,x \;=\; g_2\,x \;\Rightarrow\; g_2{}^{-1}\,g_1\,x \;=\; g_2{}^{-1}\,g_2\,x \;=\; e\,x \;=\; x \;.$$

Thus

$$g_2{}^{-1}\,g_1 \;\in\; S\mathrm{tab}(x) \;.$$

And so

$$g_2{}^{-1}\,g_1\,S\mathrm{tab}(x) \;=\; S\mathrm{tab}(x) \;.$$

This permits the following rearrangement.

$$g_1\,S\mathrm{tab}(x) \;=\; e\,g_1\,S\mathrm{tab}(x) \;=\; \left(g_2\,g_2{}^{-1}\right)\,g_1\,S\mathrm{tab}(x) \;=\; g_2\,\left(g_2{}^{-1}\,g_1\right)\,S\mathrm{tab}(x) \;=\; g_2 S\mathrm{tab}(x) \;.$$

We thus arrive at (25.5). □

Proof of the Orbit–Stabilizer Theorem itself. Set $H \;=\; S\mathrm{tab}(x)$ in the Coset-Counting Lemma to obtain the following.

$$|\,G\,| \;=\; \sum_{k\,=\,1}^{n} |\,g_k S\mathrm{tab}(x)\,| \;=\; k\,|\,S\mathrm{tab}(x)\,| \;. \tag{25.6}$$

k here runs over disjoint cosets covering G.

By Lemma 1,

$$|\,\mathfrak{o}\mathrm{rb}(x)\,| \;=\; \#(\text{ disjoint cosets }) \;=\; k \;. \tag{25.7}$$

Substituting (25.7) in (25.6) we arrive at i), and ii) immediately follows. □.

Exercise 6 Check that the Orbit–Stabilizer Theorem holds for each Example and Exercise in Chapter 23 [This also roughly halves the amount of working needed there.]

25.3 The orbit-counting formula

Remark 1 This section is our third installment of Group Theory applied to Combinatorics.

Proposition 1 (orbit-counting formula)[1] Let G be a finite group acting on a finite set $\mathfrak{X}$. Then

$$\#(\,\text{orbits}\,) \;=\; \frac{1}{|\,G\,|}\sum_{g\,\in\,G}\#(\,\mathfrak{X}\text{-elements fixed by }\;g\,) \;=\; \langle\,\#(\,\mathfrak{X}\text{-elements fixed by }\;g\;)\rangle_G\;, \quad (25.8)$$

where $\langle\;\;\rangle_G$ denotes *group average*.

Remark 2 Group averaging is a more widely useful construct and trick; see Chapter 28 and Books 3 and 5 for further applications.

Proof Firstly consider a Combinatorial double-count for

$$|\,\{\,(g,\,x)\;=\;G\;\times\;\mathfrak{X}\,|\,g\,x\;=\;x\,\}\,|\;.$$

This is both the quantity featuring in the Proposition,

$$\sum_{g\,\in\,G}\#(\,\mathfrak{X}\text{-elements fixed by }\;g\;) \;=\; \sum_{g\,\in\,G}|\,\mathfrak{Fix}(g)\,|\;.$$

And

$$\sum_{x\,\in\,G\mathfrak{X}}\#(\,G\text{-elements for which }x\text{ is fixed }\;) \;=\; \sum_{x\,\in\,\mathfrak{X}},|\,S\mathrm{tab}(x)\,|\;.$$

We can thus apply the Orbit–Stabilizer Theorem to obtain that

$$\sum_{g\,\in\,G}\#(\,\mathfrak{X}\text{-elements fixed by }\;g\;) \;=\; \sum_{x\,\in\,\mathfrak{X}}\frac{|\,G\,|}{|\,\mathfrak{orb}(x)\,|} \;=\; |\,G\,|\sum_{x\,\in\,\mathfrak{X}}\frac{1}{|\,\mathfrak{orb}(x)\,|}\;, \quad (25.9)$$

since we can pull $|\,G\,|$ outside of the sum over $\mathfrak{X}$.

Secondly,

$$\mathfrak{X} \;=\; \amalg_{k\,=\,1}^{n}\,\mathfrak{orb}_k(x) \quad (25.10)$$

for k indexing disjoint orbits covering $\mathfrak{X}$. So we can dissect

$$\sum_{x\,\in\,\mathfrak{X}}\;\text{into}\;\sum_{k\,=\,1}^{n}\;\sum_{x\,\in\,\mathfrak{orb}_k(x)}\;: \quad (25.11)$$

the 'orbit-by-orbit sum'. Thus

$$\sum_{x\,\in\,\mathfrak{X}}\frac{1}{|\,\mathfrak{orb}(x)\,|} \;=\; \sum_{k\,=\,1}^{n}\;\sum_{x\,\in\,\mathfrak{orb}_k(x)}\frac{1}{|\,\mathfrak{orb}(x)\,|} \;=\; \sum_{k\,=\,1}^{n}\frac{1}{|\,\mathfrak{orb}(x)\,|}\;\sum_{x\,\in\,\mathfrak{orb}_k(x)}1\;=$$

$$\sum_{k\,=\,1}^{n}\frac{|\,\mathfrak{orb}(x)\,|}{|\,\mathfrak{orb}(x)\,|} \;=\; \sum_{k\,=\,1}^{n}1 \;=\; k \;=\; \#(\,\text{orbits}\,)\;. \quad (25.12)$$

[Here the second step amounts to pulling a constant outside of a sum.]

[1] This result is often known as either 'Burnside's Lemma' or 'the Lemma that is not Burnside's'. As a word of explanation, it was already known to Cauchy (1845) though it only became widely known to more modern audiences though Burnside's 1897 book.

Substituting 25.12 in 25.9, we arrive at

$$\sum_{g \in G} \#(\,\mathfrak{X}\text{-elements fixed by } \ g\ ) \ = \ |G|\,\#(\,\text{orbits}\,)\ .$$

Finally divide by $|G|$ to obtain our desired formula. □

Applications This is a *counting principle*, of 'enumerative Combinatorics' use; see Chapter 28 for applications of this to basic Geometry and Chemistry. Graph Theory, Applied Geometry and Algebraic Topology make some further use of similar enumerations.

Chapter 26

Groups from a relational point of view

A. Preamble

26.1 Relative quantities for groups

Definition 1 The *relative order* of element $g_1 \in G$ relative to element $g_2 \in G$ is

$$|\, g_1 \,| - |\, g_2 \,| \ .$$

Definition 2 The *unsigned relative order* of elements $g_1, g_2 \in G$ is

$$|\, |\, g_1 \,| - |\, g_2 \,|\, | \ .$$

The outer straight lines here are mod signs, while the inner ones denote element order.

♠ ◇ ♣

Structure 1 The *maximal order* that a group element can have is

$$\max_{g \in G} (\, |\, g \,|\,) \ .$$

This can be infinite. It is not necessarily the group's maximal cycle length. For some less regular product of elements might take longer to return to the identity.

Remark 1 The most immediately opposite bound is

$$\min_{g \in G} |\, g \,| = |\, e \,| = 1 \ .$$

This is far less interesting, since it is shared by all groups and thus cannot distinguish between them.

Structure 2 One could however ask instead about the nontrivial such bound, meaning to the exclusion of the identity:

$$\min_{e \neq g \in G} |\, g \,| \ .$$

The following further relative quantities can now be defined.

Definition 3 The *upper-shifted order* of a group element is

$$|\, g \,|_{\mathrm{u}} := \max_{g \in G} (\, |\, g \,|\,) - |\, g \,| \ .$$

Definition 4 The *nontrivial lower-shifted order* of a group element is

$$|\, g \,|_{\mathrm{l}} \; := \; |\, g \,| \; - \; \min_{e \neq g \in G} (\, g \,) \; .$$

Structure 3 A further single characteristic number for each group is given by

$$\Delta \, |\, g \,| \; = \; |\, g \,|_{\mathrm{u}} \; - \; |\, g \,|_{\mathrm{l}} \; .$$

This is not necessarily finite nor necessarily well-defined.

♠ ◇ ♣

Structure 4 The *group-averaged element order*

$$\langle |\, g \,| \rangle_G \; = \; \frac{1}{|\, G \,|} \sum_{g \, \in \, G} |\, g \,| \; .$$

Remark 2 As already mentioned in the previous Chapter, group averages already have a long history as useful constructs

Remark 3 Some further relative quantities are as follows.

Definition 5 The *group element order fluctuation* is

$$g \; - \; \langle |\, g \,| \rangle_G \; .$$

Definition 6 This has unsigned counterpart

$$|\, g \; - \; \langle |\, g \,| \rangle_G \,| \; .$$

26.2 Ratio quantities for groups

Definition 1 The *element-to-group order ratio* is

$$\frac{\max_{g \, \in \, G} (\, |\, g \,| \,)}{|\, G \,|} \; .$$

Remark 1 This is well-defined for finite groups. It is bounded above by 1 since the order of the group limits the order of each of its elements.

Remark 2 The reciprocal of this is $\mathbb{N}$-valued by Lagrange's Theorem.

Remark 3 We can similarly normalize maximal, minimal and average element order by group order. Normalizing element, minimal and average orders by maximal element order may on some occasions also be of use.

Remark 4 One way of forming relational quantifiers of element order is to normalize relative quantifiers by $\Delta |\, g \,|$.

26.3 Subgroup versions of the preceding

Exercise 1 Under the substitution of (general) subgroup for (general) element, all of the above have counterparts. Spell this out. Also specialize the above to obtain relational quantities for stabilizers. With orbits being subsets rather than subgroups, obtain relational quantities for orbits by specializing Chapter 1's structures instead.

Structure 1 Among these, the ratio of subgroup order to group order is widely used in the literature, under the name of *index* of H in G. This is denoted

$$[\, G \,:\, H \,] \;,$$

in clear allusion to its ratio nature.

Remark 1 'Little group' in Sec 23.1 also carries relational connotations: a subgroup is often littler than (and never larger than) the whole group.

Remark 2 The quotient groups and quotients by groups developed below can also be viewed as relational structures.

B. Deeper structure

26.4 Normal subgroups

Structure 1 Let G be a group and $H \leq G$. Let us next contemplate whether a group structure can be placed on the family of (say left) cosets $g\,H$.

$$g_1\,H \cdot g_2\,H \;=\; (\, g_1\,g_2\,)\,H$$

offers a natural possibility for coset multiplication. This turns out however not to be possible for all G and H, as follows.

Proposition 1 The following are equivalent. i)

$$\left.\begin{array}{l} g_1\,H \;=\; g_1'\,H \\ g_1\,H \;=\; g_1'\,H \end{array}\right\} \;\Rightarrow\; (\, g_1\,g_2 \,)\,H \;=\; (\, g_1'\,g_2' \,)\,H \;:$$

well-definedness of natural coset multiplication, in the sense of representative-independence.

ii)

$$g\,h\,g^{-1} \;\in\; H \;\text{ whenever }\; g \;\in\; G,\, h \;\in\; H \;:$$

homoconjugatedness of H in G.

Proof i) $\Rightarrow$ ii) Let $h \in H$;

$$h\,H \;=\; H \;=\; e\,H$$

$\Rightarrow$ for $g \in G$,

$$h\,g\,H \;=\; g\,H \;=\; e\,g\,H \;\Rightarrow\; g^{-1}\,h\,g\,H \;=\; H \;=\; e\,H$$

$$\Rightarrow\; g^{-1}\,h\,g \;\in\; H \;\;\forall\;\; h \;\in\; H\;,\;\; g \;\in\; G\;.$$

[Setting

$$\gamma \;:=\; g^{-1}\;,$$

this is equivalent to

$$g\,h\,g^{-1} \;\in\; H\;.]$$

ii) $\Rightarrow$ i) Suppose that

$$g_i\,H \;=\; g_i'\,H\;,\;\; i \;=\; 1,\,2\;. \tag{26.1}$$

Then

$$g_i' \;=\; g_i'\,e \;\in\; g_i'\,H \;=\; g_i\,H\;,$$

using (26.1) in the last step. Furthermore,

$$g_1' \, g_2' \;=\; g_1 \, h_1 \, g_2 \, h_2 \;=\; g_1 \, \left(g_2 \, g_2^{-1} \right) \, h_1 \, g_2 \, h_2 \;=\; g_1 \, g_2 \, \left(g_2^{-1} \, h_1 \, g_2 \right) \, h_2 \; .$$

The last bracket is some element of H, so the last four terms in the last formula belong to H.

Thus

$$g_1' \, g_2' \;\in\; g_1 \, g_2 \, H \; .$$

So

$$g_1' \, g_2' \, H \;=\; g_1 \, g_2 \, H \; ,$$

as required. □

Remark 1 We eventually depict coset multiplication in Fig 26.3.

♠ ◇ ♣

Definition 1 For G a group and $H \;\leq\; G$, H is said to be *normal in* G [109, 106] (more occasionally alias *invariant*) if the following holds.

$$g \, h \, g^{-1} \;\in\; H \;\; \text{whenever} \;\; g \;\in\; G \;\; \text{and} \;\; h \;\in\; H \; . \tag{26.2}$$

Notation 1 We then write $H \;\lhd\; G$.

Naming Remark 1 *homoconjugate* is itself a reasonable explanatory truer name for normal in the present context, and *heteroconjugate* for non-normal subgroups.

Remark 2 A common repackaging of this definition involves replacing (26.2) with the equivalent

$$g \, h \, g^{-1} \;\subseteq\; H \;\; \forall \;\; g \;\in\; G \; . \tag{26.3}$$

Remark 3 There is some scope for 'normal in' to be a nontrivial specification when one is considering multiple subgroups at once. This is since some may or may not be normal in each other; we leave details of this to Volume 5 [444].

♠ ◇ ♣

Lemma 1

$$H \;\lhd\; G \;\; \text{iff} \;\; g \, H \;=\; H \, g \; \forall \; g \;\in\; G \; . \tag{26.4}$$

I.e. normal subgroups also characterize precisely when left and right cosets coincide.

Proof Suppose that (26.3) holds. Then

$$g^{-1} \, (\, g \, H \, g^{-1} \,) \, g \;\subseteq\; g^{-1} \, H \, g \;\; \forall \;\; g \;\in\; G$$

$$\Rightarrow \;\; (\, g^{-1} \, g \,) \, H \, (\, g^{-1} \, g \,) \;=\; e \, H \, e \;=\; H \;\subseteq\; g^{-1} \, H \, g \tag{26.5}$$

$$\Rightarrow \;\; g \, H \, g^{-1} \;\subseteq\; H \; . \tag{26.6}$$

Together, (26.5) and (26.6) $\Rightarrow$

$$g \, H \, g^{-1} \;=\; H \;\; \forall \;\; g \;\in\; G \; .$$

$\Rightarrow$ (26.4).

Finally, this argument can be taken in reverse. □

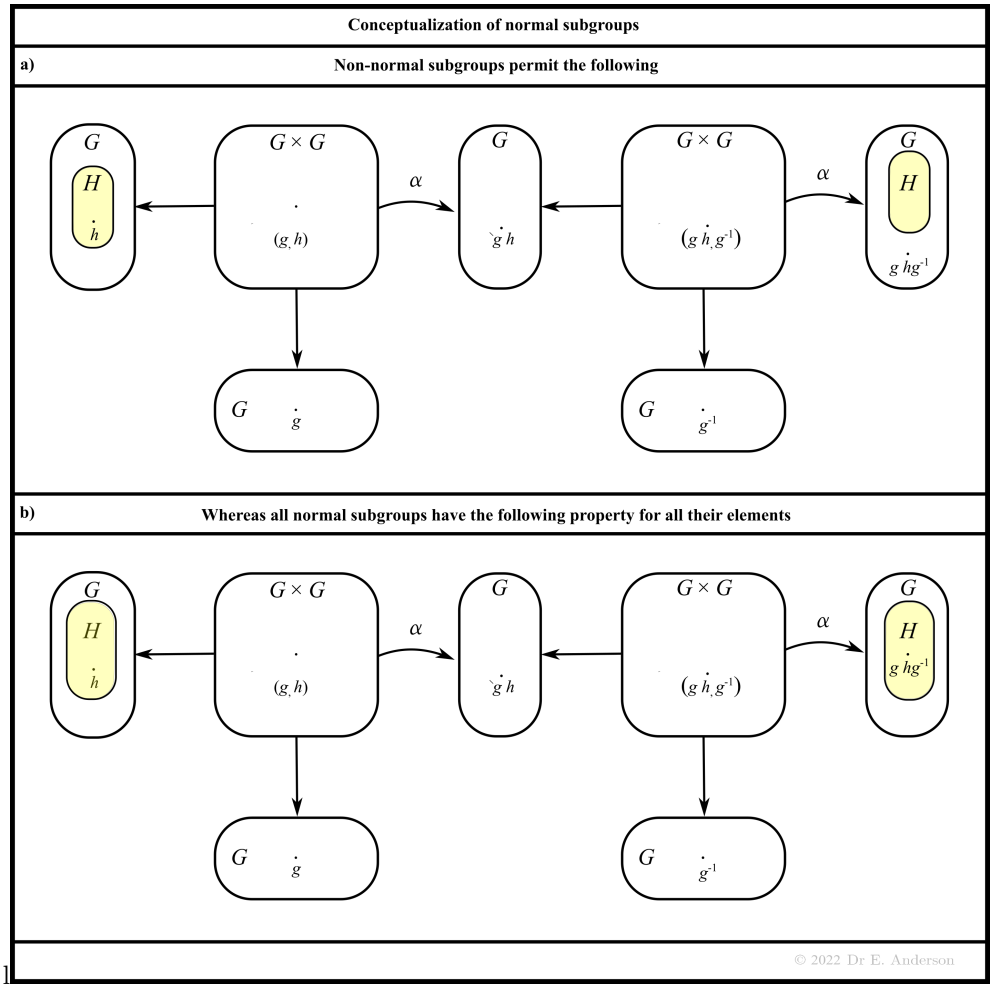


Figure 26.1:

26.5 Quotient groups

Remark 1 Normal subgroups are what we need in order to have well-defined natural coset multiplication, by Proposition 1. In this case, (the set of cosets $g\,H$ as equipped with coset multiplication constitutes a group: the *quotient group* [109], alias *factor group*, of G by H. This is denoted by G/H or $\frac{G}{H}$.

Proposition 2 Quotient groups are groups.

Proof We have a candidate composition law which closes. It thus remains to check the following.

Identity:

$$g\,H \circ e\,H \;=\; (g\,e)\,H \;=\; g\,H \;=\; (e\,g)\,H \;=\; e\,H \circ g\,H\;.$$

This is by coset multiplication, then the right-identity property, followed by these same moves in reverse now making use of the left-identiy instead.

Inverse:

$$g\,H \circ g^{-1}\,H \;=\; (g\,g^{-1})\,H \;=\; e\,H \;=\; H \;=\; e\,H \;=\; (g^{-1}\,g)\,H \;=\; g^{-1}\,H \circ g\,H\;.$$

This is by coset multiplication, then the right-inverse property, and then the identity property, followed by these same moves in reverse now making use of the left-inverse instead.

Associativity:

$$g_1 H \circ (g_2 H \circ g_3 H) \ = \ g_1 H \circ (g_2 \, g_3) \ H \ = \ (g_1 \ (g_2 \, g_3)) \ H \ = \ g_1 \, g_2 \, g_3 \, H$$

$$= \ ((g_1 \, g_2) \ g_3) \ H \ = \ (g_1 \, g_2) \ H \circ g_3 \, H \ = \ (g_1 \, H \circ g_2 \, H) \circ g_3 \, H \ .$$

This is by coset multiplication twice, followed by group associativity, and then these moves again in reverse order, now using instead the other possible bracketing. □

Remark 2 This is rather more straightforward than finding the context in which G/H is well-defined.

Remark 3 While id and $G \ \triangleleft \ G$,

$$\frac{G}{1} \ = \ G \ \text{ and } \frac{G}{G} \ = \ id \ .$$

So we need nontrivial normal subgroups

$$1 \ \not\triangleleft H \ \not\triangleleft G$$

for quotient groups to be interesting.

26.6 Basic Examples

Example 1

$$H \ \triangleleft \ A$$

whenever A is commutative. I.e. all subgroups of any commutative group are normal:

$$a \, h \, a^{-1} \ = \ a \, a^{-1} \, h \ = \ e \, h \ = \ h \ \in \ H \ .$$

Thus any

$$\frac{A}{H}$$

is a quotient group, and for H nontrivial, these are nontrivial quotient groups.
As a first subcase,

$$C_2 \ \triangleleft \ C_4 \ \text{ and } \ \frac{C_4}{C_2} \ = \ C_2 \ .$$

As a second subcase,

$$C_2 \ \triangleleft \ C_2 \ \times \ C_2 \ \text{ and } \ \frac{C_2 \ \times \ C_2}{C_2} \ = \ C_2 \ .$$

Example 2 What about non-commutative groups? For $S_3 \ = \ D_3$, no order-2 subgroups are normal. This follows e.g. from their right and left cosets not coinciding.

Exercise 2$^-$ Show that the right and left cosets of C_3 in D_3 do however coincide, so that $C_3 \ \triangleleft \ D_3$. What is the corresponding quotient group?

Exercise 3$^-$ Show that $G_1, G_2 \ \triangleleft \ G_1 \ \times \ G_2$. What are the corresponding quotient groups?

Exercise 4 Show that $H \ \triangleleft \ G$ whenever

$$| \, H \, | \ = \ \frac{| \, G, |}{2} \ .$$

I.e. that

$$[\, G \ : \ H \,] \ = \ 2 \ .$$

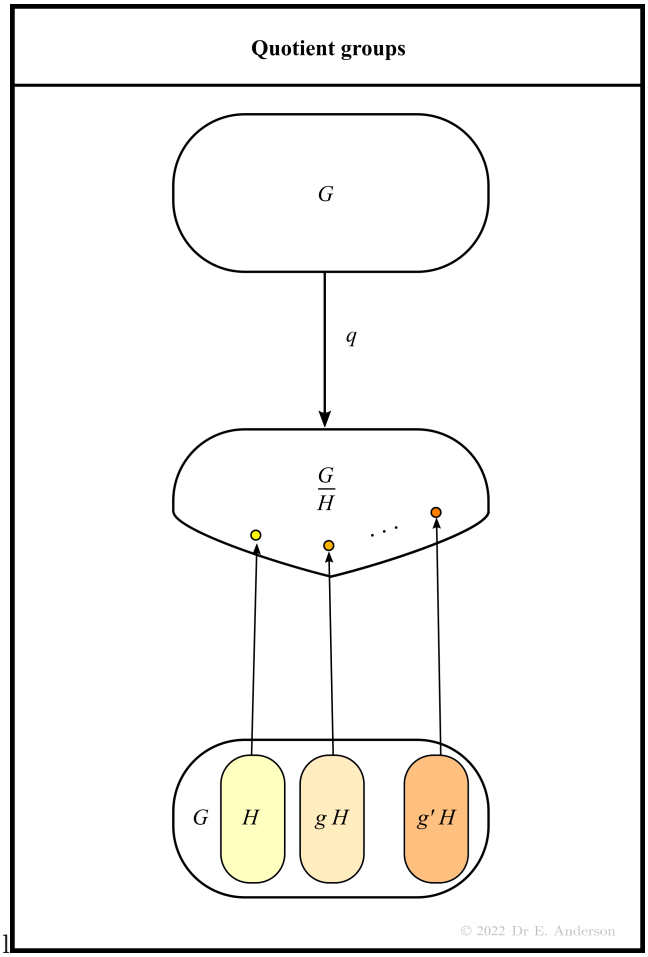


Figure 26.2:

26.7 (Co-)homology groups as further examples of quotient groups

Pointer 1 An eventually significant set of examples is that Algebraic Topology's homology and co-homology groups arise as quotient groups. For instance, homology groups $H_n(\Sigma)$ take the form of quotients of groups of cycles $Z_n(\Sigma)$ in a given space Σ by the group of boundaries $B_n(\Sigma)$ in that space. I.e.

$$H_n(\Sigma) \ = \ \frac{Z_n(\Sigma)}{B_n(\Sigma)} \ .$$

This can furthermore be viewed as in terms of the image and kernel of two maps, as follows.

$$H_n(\Sigma) \ = \ \frac{\ker(\partial_{n\,+\,1})(\Sigma)}{\text{image}(\partial_n)(\Sigma)} \ . \tag{26.7}$$

See Fig 26.4.

Example 1 of Sec 8.4 then specializes to give the corresponding count. We will however postpone explaining what the ∂-maps are until Volume 1 [441].

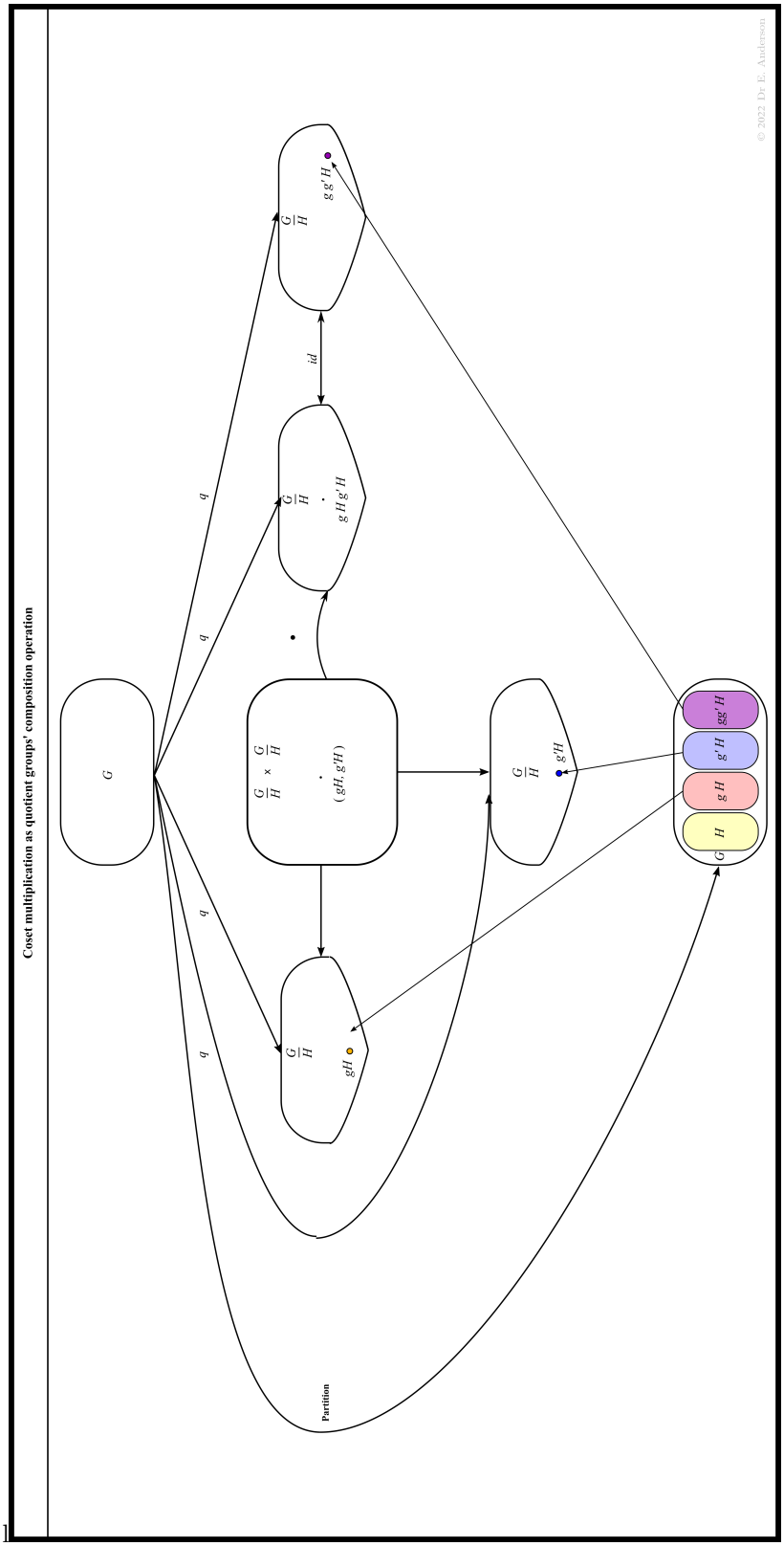

Coset multiplication as quotient groups' composition operation
G
q
q
q
q
G/H × G/H
(gH, g'H)
G/H
gH
G/H
g g' H
id
G/H
g H g' H
G/H
g'H
gg' H
g' H
g H
H
G
Partition
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Figure 26.3:

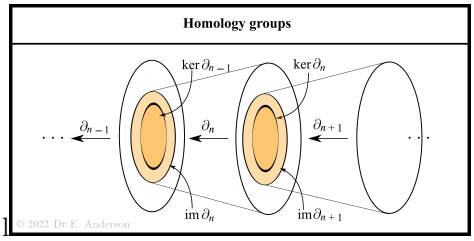


Figure 26.4:

C. Quotients by groups of spaces other than groups

26.8 Example 1) Meaningless labels

Let us now illustrate this notion (introduced in Sec 10.5) with some Combinatorics and arenas examples.

Structure 1 Meaningless labels. Our second example in Sec 8.7 has 2 labels, up and down, and is dealt with by quotienting out a C_2 group. The third has 3 – red, blue and green – and is dealt with by quotienting out a S_3 group. To be clear, up and down labels are individually meaningless, but *the distinction between* them is real. This affects a given configuration's symmetry group, thus requiring care with one's counting procedure.

This is significant for instance because Fundamental Physics does not come with the means of meaningfully labelling particles. Ideas like 'the blue point', or 'the blue planet' or 'the bigger planet' cease to make sense when applied to fundamental particles of a given species, say electrons. Electrons all have the same mass, charge, magnitude of spin *and no other intrinsic properties* that could serve as distinguishing labels.

Remark 1 The Introduction's view of Pascal's triangle as the arena of combinations gives one example of quotienting by $\mathfrak{C}_2$: reflection about the middle value of the number of selected objects. Chapter 8's consideration of each of sets of subsets, and families give two examples of quotients by $\mathbb{Z}_2$, both involving the removal of C_2 acting by a complementation. We commented in Sec 10.8 that family complementation does not transcend to any of covers, partitions or topologies. This is a further example of symmetry breaking of a C_2 (sub)group. Chapter 15 documented however a different C_2 quotient supported by partitions: conjugation. See Chapters 29, 33, 39, 49 and 64 for yet further quotients by discrete groups, still most usually C_2.

Philosophical Aside [Leibniz's Identity of Indiscernibles] This is that 'objects that are indiscernible from each other are to be treated as identical'.

Observe the idea that two objects can be physically-equivalent (even philosophically equivalent) and yet be mathematically distinct. In such cases, a deeper modelling involves quotients, with the actual physical states being mathematically realized in 1:1 correspondence by the quotient's equivalence classes.

Remark 1 Indistiguishability, and label-meaninglessness are two types of notion of indiscerniblity.

26.9 Example 2) Quarters and eighths of families

Remark 1 For $N \geq 3$ families, a singleton can reside inside or outside a doublet. The families with one singleton less are thus not in 1:1 correspondence with those with one singleton more. The only

subsets that are exempt from such distinctions between families not containing them and families containing them are $\mathfrak{X}$ and $\emptyset$. For all other subsets have to reside in $\mathfrak{X}$ while no other subset can reside in $\emptyset$.

Structure 1 Furthermore, it turns out to be useful to observe that for $N \geq 1$, each family comes in 4 copies. Namely the fullest one, containing both $\mathfrak{X}$ and $\emptyset$, the one containing $\mathfrak{X}$ but not $\emptyset$, the one containing $\emptyset$ but not $\mathfrak{X}$ and the simplest one, containing neither $\emptyset$ nor $\mathfrak{X}$. Figs 26.5 and 26.6 picks these out within Chapter 8 's presentation. And then Figs 26.7 and 26.8 teases apart the four quarters. This being much more lucid, all further developments are depicted in this way.

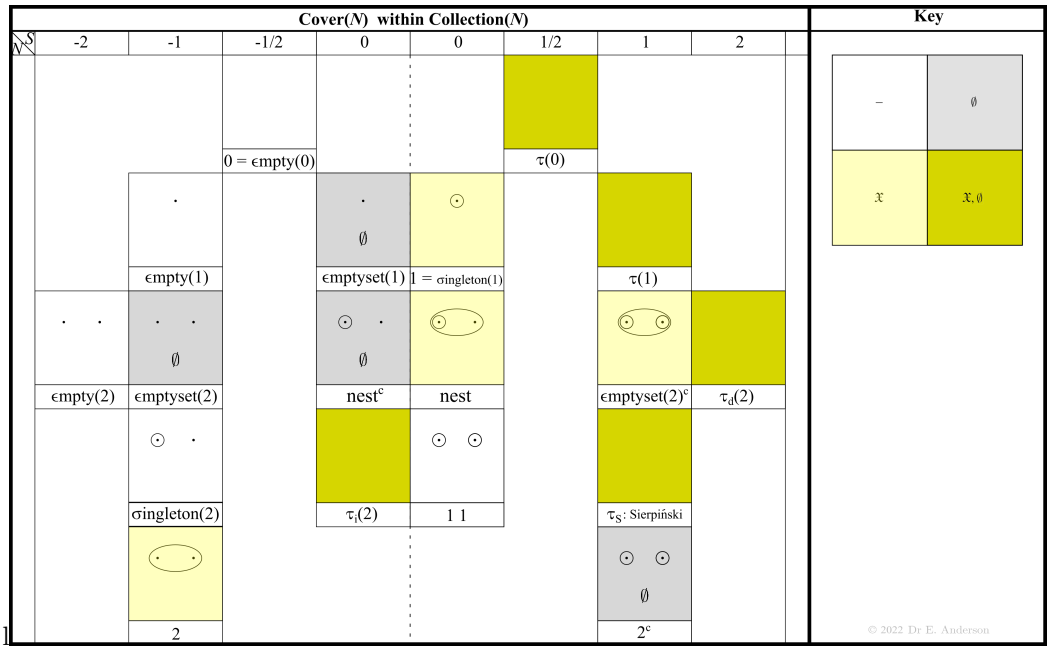


Figure 26.5:

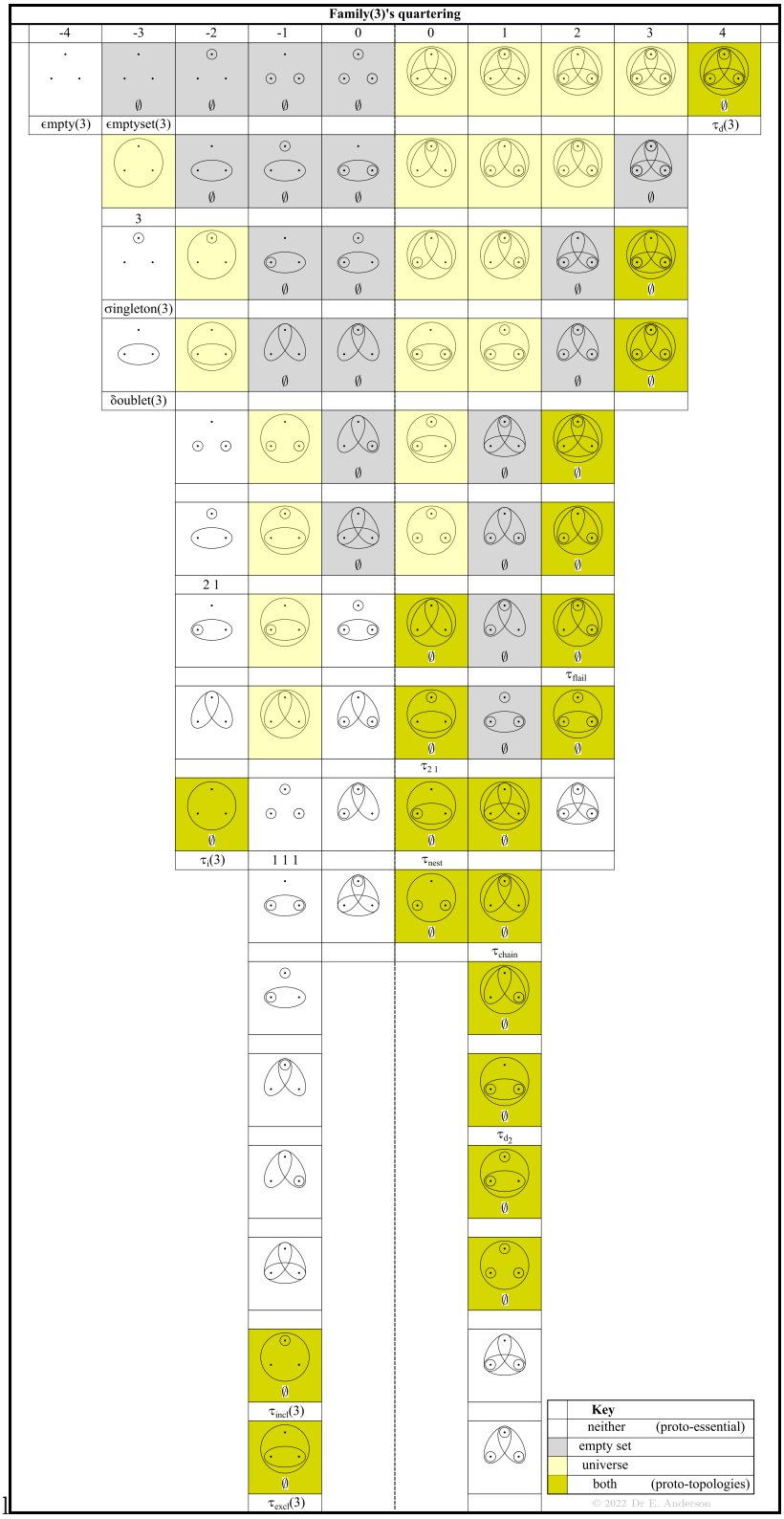

Family(3)'s quartering
-4
-3
-2
-1
0
0
1
2
3
4
ϵmpty(3)
ϵmptyset(3)
τd(3)
3
σingleton(3)
δoublet(3)
2 1
τflail
τ2 1
τl(3)
1 1 1
τnest
τchain
τd2
τincl(3)
τexcl(3)
Key
neither (proto-essential)
empty set
universe
both (proto-topologies)
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Figure 26.6:

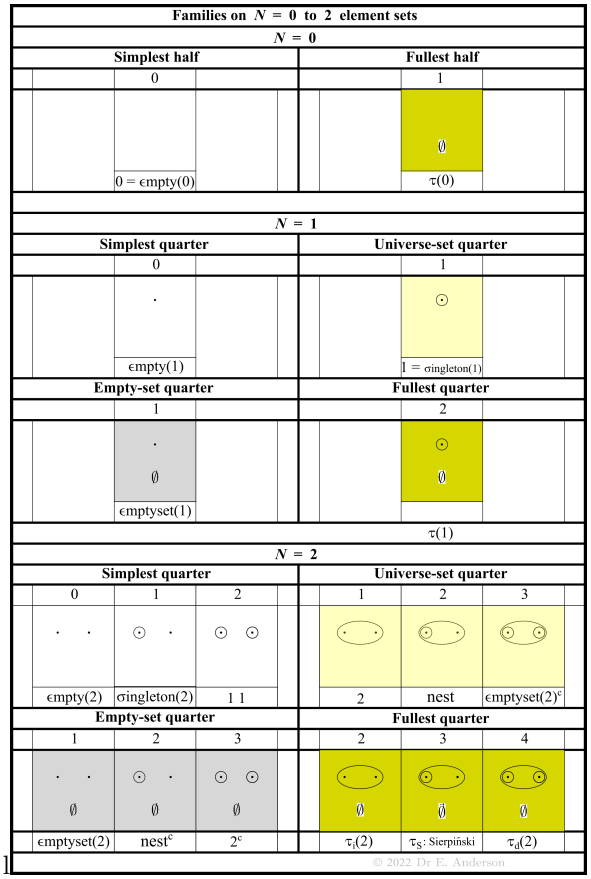


Figure 26.7:

Remark 2 $N = 0$ is not included in this quartering description, since here the empty set *is* the universe set, so there is a collapse down to two copies.

Application 1 This quartering is helpful in our study of $\mathfrak{F}$amily(3). A priori, $|\,\mathfrak{F}\text{amily}(3)\,| = 80$: well outside the scope of this Volume's order and graph analysis. But its order-20 quotient is manageable.

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Remark 2 The family complementation quotient is already included in this quartering quotient, as underpinned with V_4's 3 C_2 subgroups.

Structure 2 Within each of these 4 copies – and thus more locally – there is however a further notion of complementation. Namely, that involving listing all the subsets not in the family bar evoking $\mathfrak{X}$ or $\emptyset$.

Corresponding relative centering variables include

$$R_{\pm} := n - 2^{N-1} \pm 1 \,.$$

R_+ is for the copy with neither $\mathfrak{X}$ nor $\emptyset$. R_- is for the copy with both of these present. The same R as before will do in the two other copies. With the caveat that all three variables' ranges are now $1 - 2^{N-1}$ to $2^{N-1} - 1$. These variables come into play in Figs 26.9 and 26.10.

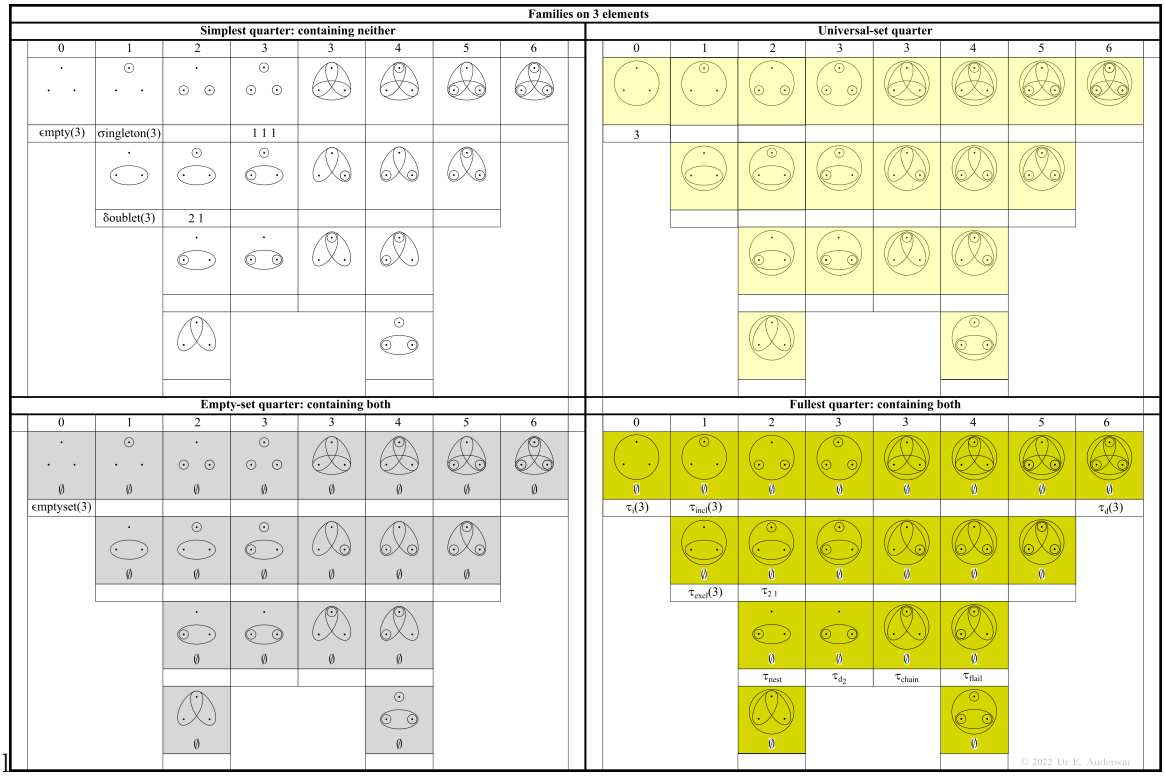


Figure 26.8:

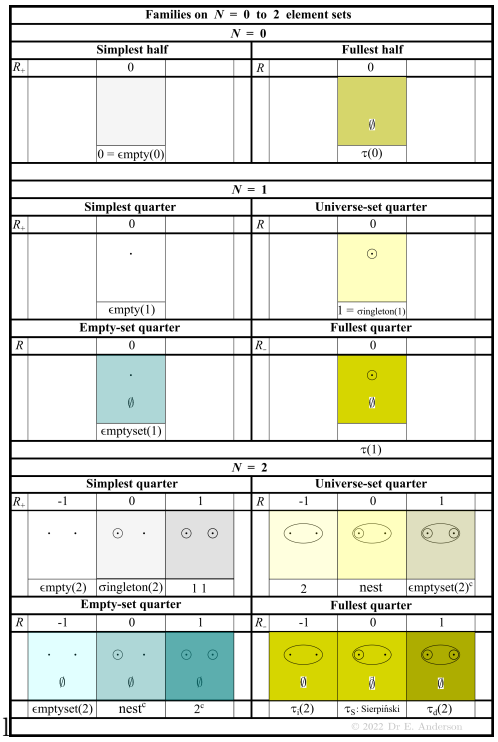


Figure 26.9:

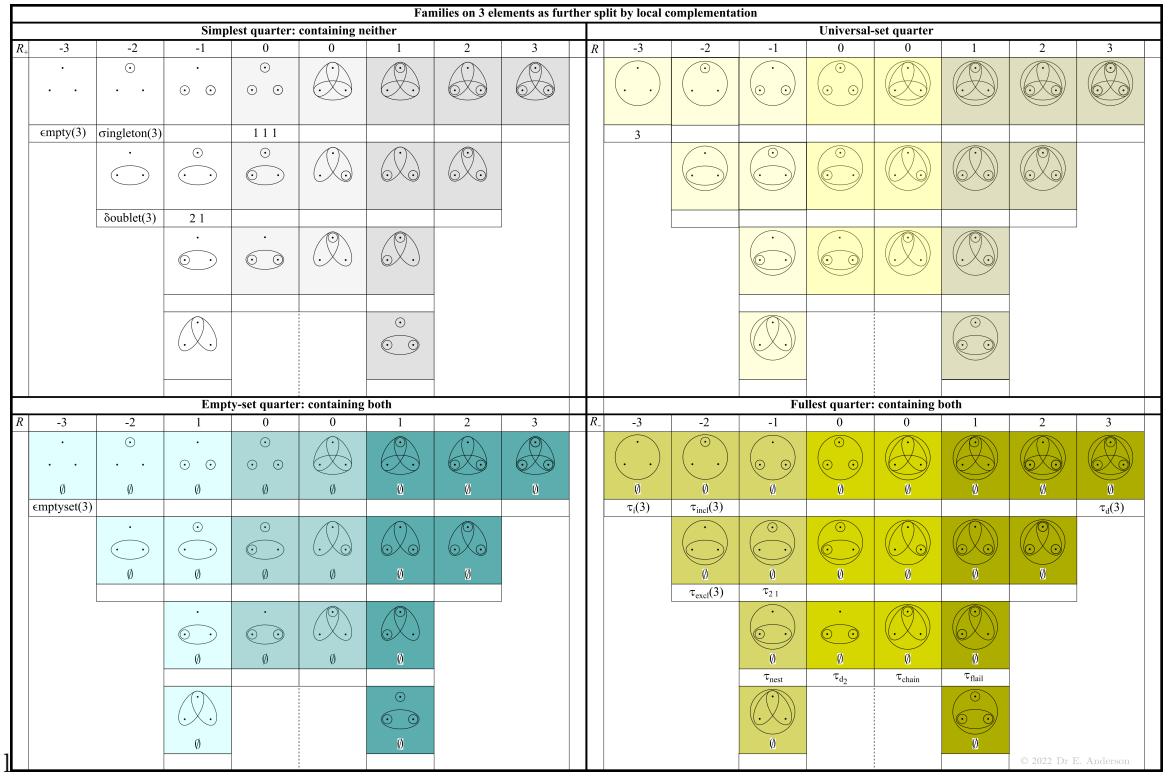


Figure 26.10:

Remark 3 The 8-fold quotient is provided in Fig 26.11. For some purposes, this quotient can furthermore be combined with the local family complementation quotient, thus producing an eightfold quotient.

Application 2 This eightfold quotient is realized purely heterocomplemantarily for each $N \geq 3$, establishing that

$$|\,\mathfrak{F}\text{amily}(N)\,| \;=\; 0 \;(\text{mod } 8)$$

for these N.

Remark 4 For the two 8-quotiented 3-families for which there was no name before, I introduce σnake and ρattlesnake. Fig 26.11.b) draws the sparse presentation, while rattlesnake refers to the dense complement as exhibited in Subfigure a).

$N = 0$ and 1 must be excluded since they are the sole exceptions to heterocomplementarity. Thereby, the complementation quotient has no effect for these N.

Exercise 5 a) Give a symmetry-breaking diagram for each $\mathfrak{F}\text{amily}(N)$ for each $N = 1$ to 3.

b) How do each of these spaces split into orbits of $C_2 \times C_2 \times C_2$?

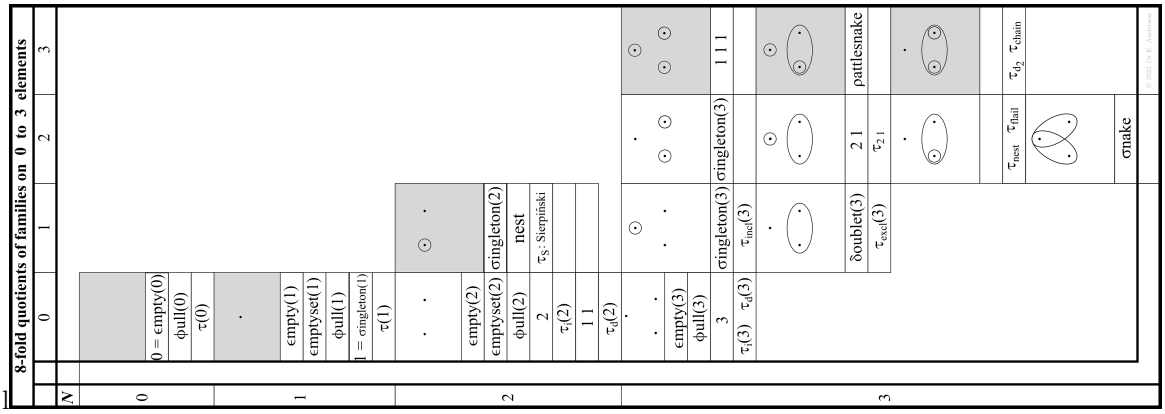


Figure 26.11:

26.10 Example 3) Essential covers

Definition 1 An *essential cover* is one that contains neither the universe set $\mathfrak{X}$ nor the empty set $\emptyset$.

Remark 1 Thus these all live in one of the 4 constituent copies of $\mathfrak{F}$amily (Fig 26.13.a). For $N \geq 2$, not all of this copy consists of covers, however. The same applies to the $\emptyset$ copy, whose heat map of covers (Fig 26.13.b) is identical to that of the essential covers In contrast, the entirety of the $\mathfrak{X}$ copy and $\mathfrak{X}, \emptyset$ copy are covers [Fig 26.13.c)-d)]. For any family containing $\mathfrak{X}$ automatically meets the cover condition. By forming a proper subspace of the 4-fold quotient arena of families, essential covers are somewhat more interesting than these other covers.

Remark 2 It is the essential covers which are picked out in red in our Figures, to the other covers' orange. Observe that $N = 2$ is minimal for there to be any essential covers. There is precisely one here: the pair of singletons. Which is also the discrete cover on two elements, and the smallest partition with more than one part: $1 \,|\, 1$.

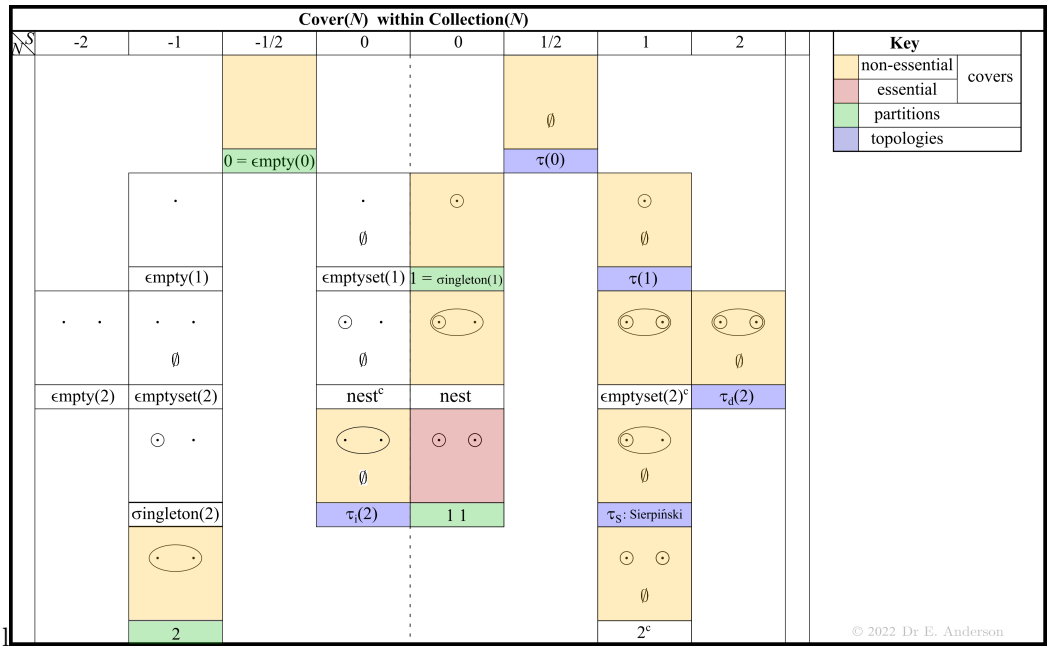


Figure 26.12:

Exercise 6 How many essential covers are supported on the 3-element set?

[The answer to this is marked likewise in red overleaf, in Fig 26.13's heat map.]



Figure 26.13:

26.11 Incompatible quotients

Pointer 2 It is possible for some groups to admit two further sets of generators either of which consistently extend the group and yet which are not mutually consistent. Quotienting by such then leads to the phenomenon of *incompatible quotients*. Let us postpone giving examples of this to Sec 62.2 and subsequent Volumes listed there.

Chapter 27

The Group Homomorphism Theorem

27.1 Preliminaries

Remark 1 The group-specific versions of some of the definitions phrased acategorically in Chapter 6 are as follows.

Definition 1 Let $\eta \ : \ G \longrightarrow G'$ be a homomorphism.

Then its *image* is

$$\mathrm{Im}_\eta(G) \ := \ \{ \, \eta(g) \, | \, g \ \in \ G \, \} \ .$$

Also its *kernel* is

$$\mathrm{Ker} \, \eta \ := \ \eta^{-1}(e) \ = \ \{ \, g \ \in \ G \, | \, \eta(g) \ = \ e \, \} \ .$$

Lemma 1 i)

$$\mathrm{Im}_\eta \ \leq \ G' \ .$$

ii)

$$\mathrm{Ker} \, \eta \ \leq \ G \ .$$

iii) A fortiori

$$\mathrm{Ker} \, \eta \ \lhd \ \leq \ G \ .$$

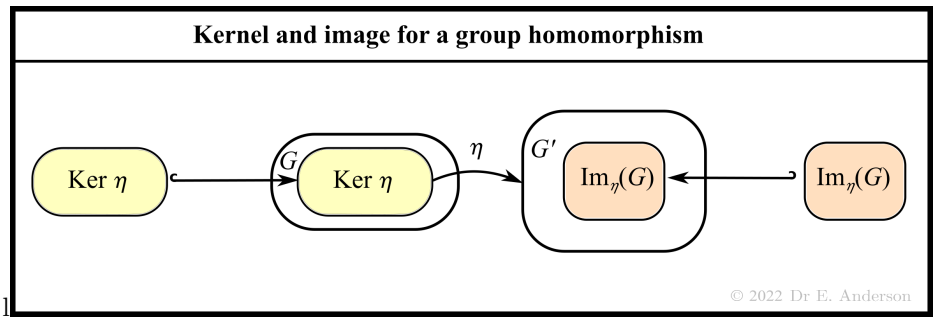


Figure 27.1:

Proof i) By the group homomorphism property,

$$\eta\,(g_1)\;\eta\,(g_2)\;\;=\;\;\eta\,(g_1\,g_2)\;\;\in\;\;\eta(G)\;.$$

So $\eta(G)$ is closed.

Identity:

$$e\;\;=\;\;\eta(e)\;,$$

which is $\in\;\eta(G)$.

Inverse:

$$\eta(g)\,\eta(g)^{-1}\;\;=\;\;e\;\;=\;\;\eta(e)\;\;=\;\;\eta\left(\,g\,g^{-1}\,\right)\;\;=\;\;\eta(g)\,\eta\left(g^{-1}\right)\;.$$

This is by the inverse property in G', then (27.1), next the inverse property in G and finally the homomorphism property. Consequently, by the uniqueness of inverses for groups,

$$\eta(g)^{-1}\;\;=\;\;\eta\left(\,g^{-1}\,\right)$$

and this $\in\;\eta(G)$.

ii) Closure: Suppose that $g_i\;\in\;\mathrm{Ker}\,\eta$ for $i\;=\;1$ and 2. Then $\eta(g_i)\;=\;e$. Using this after the homomorphism property,

$$\eta\,(\,g_1\,g_2\,)\;\;=\;\;\eta\,(g_1)\;\eta\,(g_2)\;\;=\;\;e\,e\;\;=\;\;e\;.$$

Thus $g_1\,g_2\;=\;\ker\eta$ too.

Identity: $\eta(e)\;=\;e$, so $e\;\in\;\ker\eta$.

Inverse: $g\;\in\;\ker\eta$, so $\eta(g)\;=\;e$. Thus

$$\eta\left(g^{-1}\right)\;\;=\;\;(\,\eta(g)\,)^{-1}\;\;=\;\;e\;. \tag{27.1}$$

So $g^{-1}\;\in\;\ker\eta$ too.

iii) Let $g\;\in\;G$ and $k\;\in\;\ker_\eta$. Then

$$\eta\left(\,g\,k\,g^{-1}\,\right)\;\;=\;\;\eta(g)\,\eta(k)\,\eta\left(\,g^{-1}\,\right)\;\;=\;\;\eta(g)\,e\,\eta\left(\,g^{-1}\,\right)\;\;=\;\;\eta(g)\,e\,\eta\left(\,g^{-1}\,\right)\;\;=\;\;\eta\left(\,g\,g^{-1}\,\right)\;\;=\;\;\eta(e)\;\;=\;\;e\;.$$

Here, the first expression's bracket is what to probe with in investigating normality. The first step uses the group homomorphism property, the second the definition of kernel, and the third the identity property in G'. The fourth step uses the group homomorphism property again, the fifth the inverse property in G and the sixth (27.1). Thus

$$g\,k\,g^{-1}\;\in\;\mathrm{Ker}\,\eta$$

And so the definition of normal is met. □

Remark 2 The following interrelates quotient groups and homomorphisms.

27.2 The Theorem

Group Homomorphism Theorem Let $\eta\;:\;\;G\;\longrightarrow\;G'$ be a group homomorphism. Then

$$\frac{G}{\ker\eta}\;\cong\;\mathrm{Im}_\eta\;. \tag{27.2}$$

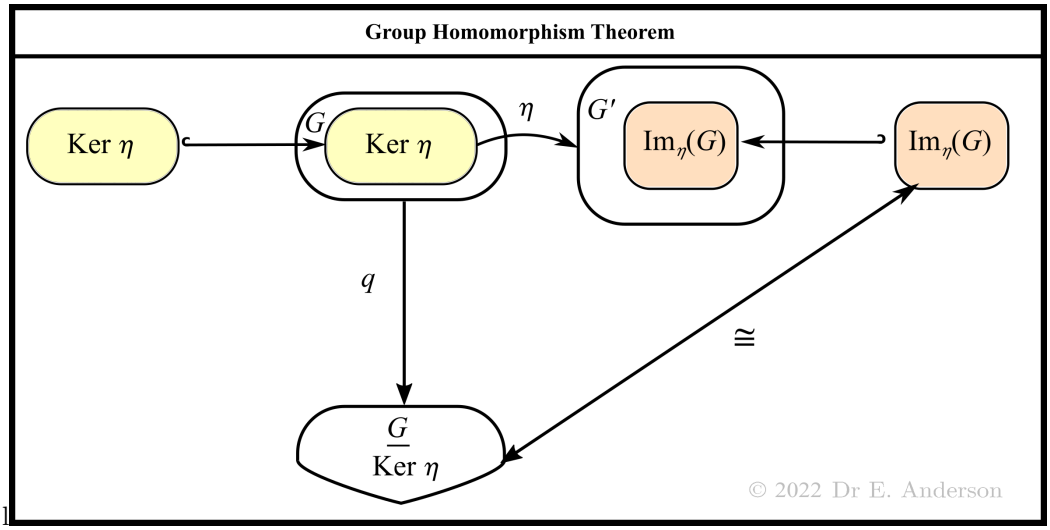


Figure 27.2:

Naming Remark 1 This is often alias ' Group Isomorphism Theorem', or 'First Group Isomorphism Theorem' in texts advanced enough to contain further such. 'Group Homomorphism Theorem' is however a truer name for this particular Group Isomorphism Theorem.

Proof Let φ be a map

$$\frac{G}{\ker \eta} \longrightarrow \mathrm{Im}_\eta \ ,$$

$$g \ker \eta \mapsto \mathrm{Im}_\eta(g) \ .$$

Well-definedness. $\ker \eta \lhd G$ by Lemma 1.iii).

Surjectivity: Let $i \in \mathrm{Im}\, \eta$. Then $i = \eta(g)$ for some g. I.e. $\varphi(\, g \ker \eta\,)$ realizes our i. All of Im_η is thus realized.

Injectivity: Suppose that

$$\varphi(\, g \ker \eta\,) = \varphi\,(\, g_2 \ker \eta\,) \ .$$

So

$$\eta\,(g_1) = \eta\,(g_2) \ .$$

Inserting this into the identity property for G', we obtain the first two steps of the following.

$$e' = \eta\,(g_2)\ \eta\,(g_2)^{-1} = \eta\,(g_1)\ \eta\,(g_2)^{-1} = \eta\,(g_1)\ \eta\left(g_2^{-1}\right) = \eta\left(g_1\, {g_2}^{-1}\right) \ .$$

The third step uses (27.1) while the fourth step uses the homomorphism property. Consequently,

$$g_1\, g_2^{-1} \in \ker \eta \ .$$

Also

$$g_1 = g_1\, {g_2}^{-1}\, g_2 \in \ker \eta\, g_2 = g_2 \ker \eta \ .$$

Thus

$$g_1 \ker \eta = g_2 \ker \eta$$

by distinctness of disjoint cosets.

Homomorphism property:

$$\varphi\,(\,g_1 \ker\eta \cdot g_2 \ker\eta\,) \;=\; \varphi\,(\,g_1\,g_2 \ker\eta\,) \;=\; \eta\,(\,g_1\,g_2\,) \;=\; \eta\,(g_1)\;\eta\,(g_2) \;=\; \varphi\,(\,g_1 \ker\eta\,)\;\varphi\,(\,g_2 \ker\eta\,)\;.$$

This is by coset multiplication, then the definition of φ, next the group homomorphism property and finally twin uses of the definition of φ again. □

27.3 Examples

Worked Example 1 Show that

$$\eta \;:\; \langle\,\mathbb{Z}\;,\;\;+\,\rangle \;\longrightarrow\; C_n$$

$$m \;\mapsto\; a^m$$

is a homomorphism. Show furthermore that

$$\ker\eta \;=\; n\,\mathbb{Z}\;,$$

$$\mathrm{Im}_\eta \;=\; C_n$$

and the quotient is C_n.

Answer 1 For the homomorphism property:

$$\eta(\,l \;+\; m\,) \;=\; a^{\,l\,+\,m} \;=\; a^l\,a^m \;=\; \eta(l)\,\eta(m)\;.$$

The kernel is

$$\ker\eta \;=\; \{\,m \;\in\; \mathbb{Z}\,|\,a^m \;=\; e\,\} \;=\; \{\,m\,|\,m \;\equiv\; 0 \pmod n\,\} \;=:\; n\,\mathbb{Z}\;.$$

The image is

$$\mathrm{Im}\,\eta \;=\; \eta(\mathbb{Z}) \;=\; \left\{\,a^m\,\middle|\,m \;\in\; \mathbb{Z}\,\right\} \;=\; \left\{\,a^m\,\middle|\,m \;\in\; [\,0,\,1,\,\ldots,\,n \;-\; 1\,)\,\right\} \;=\; C_n\;.$$

Finally, the quotient is

$$\frac{\mathbb{Z}}{n\,\mathbb{Z}} \;=\; \mathrm{Im}_\eta \;=\; C_n\;.$$

Worked Example 2 Show that

$$\eta \;:\; D_3 \;\longrightarrow\; C_2$$

$$\rho^m\,\sigma^n \;\mapsto\; a^m$$

is a homomorphism: the *orientation homomorphism*. Show furthermore that

$$\ker\eta \;=\; C_3\;,$$

$$\mathrm{Im}_\eta \;=\; C_2$$

and the quotient is C_2.

Answer 2 Observe that

$$\rho^m\,\sigma\,\rho^n\,\sigma \;=\; \rho^m\,\rho^{-n}\,\sigma^2 \;=\; \rho^{\,m\,-\,n}\,e \;=\; \rho^{\,m\,-\,n}\;.$$

Thus for o, p separately taking values of 0 or 1,

$$\rho^m \sigma^o \rho^n \sigma^p \;=\; \eta\left(\rho^{m\,-\,n}\right) \;.$$

The homomorphism property then ensues by applying η to both sides:

$$\eta\left(\rho^m\sigma^o\rho^n\sigma^p\right) \;=\; \eta\left(\rho^{m\,-\,n}\right) \;=\; a^{m\,+\,n} \;=\; a^m\,a^n \;=\; \eta\left(\rho^m\,\sigma^o\right)\,\eta\left(\rho^n\,\sigma^p\right) \;.$$

The second step here uses the definition of η, the third a basic property of exponents, while the fourth is a twin use of the definition of η again.

The kernel is

$$\ker\eta \;=\; \{\,\rho^m\,\sigma^o \,|\, \eta(\,\rho^m\,\sigma^o\,) \;=\; e\,\} \;=\; \{\,\rho^m\,\sigma^o \,|\, a^m \;=\; e\,\} \;=\; \{\,e\;,\;\;\rho\;,\;\;\rho^2\,\} \;\cong\; C_3 \;.$$

The image is

$$\mathrm{Im}_\eta \;=\; \{\,1\;,\;\;a\,\} \;\cong\; C_2 \;.$$

Finally, the quotient is

$$\frac{D_3}{C_3} \;\cong\; \mathrm{Im}_\eta \;\cong\; C_2 \;.$$

Exercise 1$^-$ Generalize Worked Example 2 from D_3 to D_n.

Exercise 2 Let p_1 and p_2 be distinct primes.

a) Show that the only homomorphism $\eta \;:\; C_{p_1} \;\longrightarrow\; C_{p_2}$ is the trivial one.

b) Show however that nontrivial homomorphisms $C_{p_1\,p_2} \;\longrightarrow\; C_{p_1}$ are possible. What are the image, kernel and quotient here?

27.4 More on Permutation Groups

Lemma 1 Permutations consisting of single nontrivial disjoint cycles commute with each other. Thereby, 'the order in which one carries out' disjoint cycles does not matter.

Proof Suppose that 2 cycles σ_{l} and σ_{g} are disjoint. Then the elements cycled by each of these cycles are disjoint subsets, say $\{\,a, \ldots z\,\}$ and $\{\,\alpha, \ldots \omega\,\}$ respectively. It then makes no difference to the description of the group's action whether we first list the action on the Latin elements or that on the Greek ones. □

Definition 1 The *signature* of a permutation is a homomorphism

$$\eta \;:\; \; S_n \to C_2 \;\; \text{viewed as} \;\; \langle\,\{-1,\,1\},\;\;\times\;\,\rangle \;.$$

Remark 1 This is a nontrivial proper homomorphism for $n \;\geq\; 3$.

Example 1 The signature of a transposition is -1.

Example 2 More generally, the signature of a length-m cycle is $(-1)^{m\,-\,1}$.

Remark 2 We will prove that this is the form that signature takes by showing that there is a unique such proper nontrivial homomorphism, in Volume 1.

Exercise 3$^-$ Given that signature is an invariant, prove that the product of an even number of transpositions cannot equal the product of odd number of transpositions.

27.5 Exercises introducing the alternating groups

Exercise 4 The *alternating group* A_n is the $ker\,\eta$ corresponding to the signature homomorphism.

a) Locate this within S_n by showing that the even-signature permutations themselves form a group: the *alternating group* A_n. But that the odd-signature ones do not. What structure do these constitute, rather?

Exercise 5 Show that a) $n = 4$ is minimal for A_n not to be isomorphic to some group that we have seen before. In the process, identify what each of A_1, A_2 and A_3 are isomorphic to.

b) $A_n \triangleleft S_n$.

Exercise 6 a) Show that $V_4 \triangleleft A_4$ Describe the corresponding homomorphism with image and quotient C_3

b$^+$) Show that A_5 is however *simple*, meaning that it is bereft of nontrivial proper normal subgroups. Which deep mathematical features does this observation lead to?



Chapter 28

Pólya's Enumeration Theorems

We now give our third, largest and final installment of Group Theory applied to Combinatorics.

28.1 Cycle-index and Pólya's First Enumeration Theorem

Let $\sigma \in S(\mathfrak{X}_n)$ be of cycle type $[a_1, a_2, \ldots, a_l]$: the longhand notation introduced in Sec 6.10). We take these permutations to be acting naturally on $\mathfrak{X}_n$.

Definition 1 The *cycle-index* [49, 58, 108] of σ,

$$Z(\sigma; \underline{\boldsymbol{x}}) := \prod_{i=1}^{n} {x_i}^{a_i} . \tag{28.1}$$

$\underline{\boldsymbol{x}}$ is here a length-n vector of dummy variables.

Notational Remark 1 Z stands for 'Zyklenzeiger' – cycle-index in German – as used by Pólya, who introduced this notation.

Remark 1 This is a *multi-monomial*, meaning a single additive power term in multiple variables. It is also *monic*, meaning that its (lead and here sole) coefficient is 1. This corresponds to considering precisely one element from our permutation group.

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Structure 1 For the whole permutation group $S(\mathfrak{X}_n)$, the *cycle-index* is

$$Z(S(\mathfrak{X}_n); \underline{\boldsymbol{x}}) := \langle Z(\sigma; \underline{\boldsymbol{x}}) \rangle_{S(\mathfrak{X}_n)} := \frac{1}{n!} \sum_{\sigma \in S(\mathfrak{X}_n)} Z(\sigma, \underline{\boldsymbol{x}}) := \frac{1}{n!} \sum_{\sigma \in S(\mathfrak{X}_n)} \prod_{i=1}^{n} {x_{i_\sigma}}^{a_i(\sigma))} .$$

So this is set up to be the *group-average* of the individual elements' cycle-indexes. The second formula makes this explicit, while the third concatenates with (28.1).

Remark 2 This is a multi-polynomial, i.e. multiple additive power terms in multiple variables. Its coefficients count types of element. It is again monic, since the lead coefficient now just counts out that each permutation group $S(\mathfrak{X}_n)$ has just one identity element.

Remark 3 We can extend this to general group actions as follows. Let $\alpha : G \times \mathfrak{X} \longrightarrow \mathfrak{X}$ be a group action encoded by $g \mapsto \sigma(g)$. Now $\mathfrak{X}$ can be some $\mathfrak{X}_m$ for $m \neq n$; $\mathfrak{X}_m$ may furthermore admit more than one $S(\mathfrak{X}_n)$-action. Recalling from Sec 20.2 that information can in general be lost by group actions, the following notation is opportune.

Definition 2 The *cycle-index* [49, 58, 108] of $g \in G$ with action α on a given finite set $\mathfrak{X}_m$ is

$$Z(g, \mathfrak{X}_m, \alpha; \underline{\boldsymbol{x}}) := \prod_{i=1}^{n} x_i^{a_i(\sigma(g))} . \tag{28.2}$$

Structure 2 For the whole group G, the *cycle-index* is

$$Z(G, \mathfrak{X}_m, \alpha; \underline{\boldsymbol{x}}) := \langle Z(g, \mathfrak{X}_m, \alpha; \underline{\boldsymbol{x}}) \rangle_G := \frac{1}{|G|} \sum_{g \in G} Z(\sigma, \underline{\boldsymbol{x}}) := \frac{1}{|G|} \sum_{g \in G} \prod_{i=1}^{n} {x_{i_g}}^{a_{i_g}(\sigma(g))} . \tag{28.3}$$

♠ ◇ ♣

Theorem 1 (Pólya's Enumeration Theorem) Let

$$c : \mathfrak{X} \longrightarrow \mathfrak{p}$$

be colouring functions for a set $\mathfrak{X}$'s elements. Using a palette $\mathfrak{p}$ that is nontrivial (i.e. of $P \geq 2$ colours). Furthermore, let $\mathfrak{C}$olourFn be the arena of all such colouring functions (Fig 28.1.a). Also let G be a permutation group acting upon $\mathfrak{X}$. Then

$$\#(\text{colour patterns admitted by } \mathfrak{X}) , \quad \#C = Z(G, \mathfrak{X}\alpha; P, P, \ldots, P) . \tag{28.4}$$

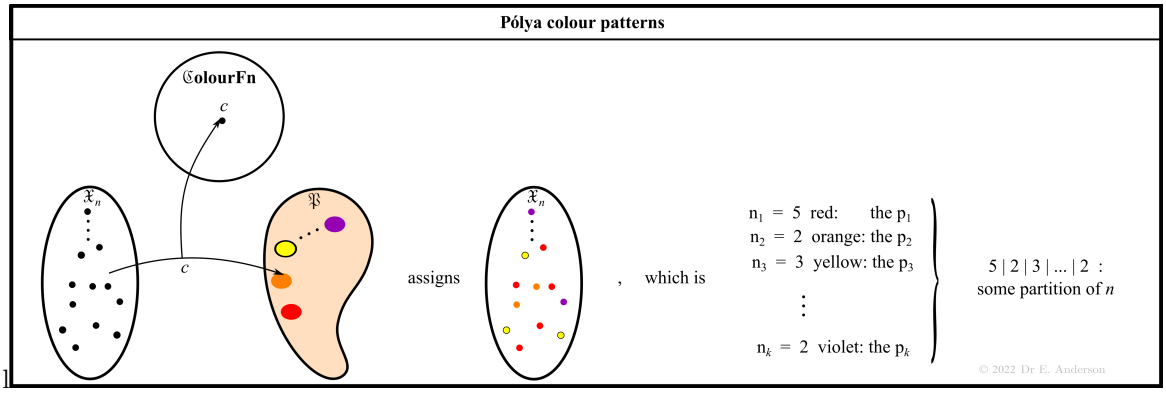


Figure 28.1:

Exercise 1$^+$ Prove this along the following lines.

a) Find a suitable isomorphism between G and the group G_c of permutations that commute with the colouring function.

b) Use this to convert the orbit-counting formula for G_c to a statement about G.

c) Show that c is constant for each cycle of each permutation, and establish a suitable converse statement to this.

d) Plug the count corresponding to a given permutation cycle type into your answer to b).

28.2 Examples of cycle-index

Example 1 For the trivial group id with the trivial action α_0 as S_1 on the 1-element set $\mathfrak{X}_1$,

$$Z(\,id;\,x\,) \;=\; Z(\,id,\,\mathfrak{X}_1,\alpha_0;\,x\,) \;=\; Z(\,id,\,\{\mathrm{pt}\},\alpha_0;\,x\,) \;=\; x\;.$$

But when acting naturally on the n-element set, rather, we have

$$Z(\,id,\,\mathfrak{X}_n,\alpha_0;\,\underline{\boldsymbol{x}}\,) \;=\; {x_1}^n\;.$$

This is our first example indicating that cycle index depends on the acted-upon set as well as on the group. One here often writes out the component variables explicitly: $(\,x_1,\,...,\,x_n\,)$ in place of specifying $\mathfrak{X}_n$ to indicate that $\underline{x}$ is a vector of length n. There is still however room for confusion when the set acted upon has further meaning, e.g. by being a set of vertices, edges or faces, in the geometrical examples below.

Worked Example 1 a) When acting naturally as S_2, show that

$$Z(\,C_2,\,\mathfrak{X}_2,\,\alpha_{\mathrm{natural}};\,\underline{\boldsymbol{x}}\,) \;=\; \frac{1}{2}\,\left(\,{x_1}^2 \;+\; x_2\,\right)\;.$$

b) Generalize this to

$$Z(\,C_p,\,\mathfrak{X}_p,\,\alpha_{\mathrm{natural}};\,\underline{\boldsymbol{x}}\,) \;=\; \frac{1}{p}\,(\,{x_1}^p \;+\; (\,p\;-\;1\,)\,x_p\,)$$

when acting on a p-element set, whenever p is prime.

Answer 1 a) C_2 has a trivial-cycle element, with 2 elements 1-cycling to themselves and thus a ${x_1}^2$ contribution. It also has a 2-cycle element, contributing x_2. We divide by 2, since this is the order of the group.

b) C_p has a trivial-cycle element, with p elements 1-cycling to themselves and thus a ${x_1}^p$ contribution. Each order-p element is a p-cycle, contributing x_p. There are $p\;-\;1$ of these, so their total contribution is $(\,p\;-\;1\,)\,x^p$. We finally divide by p, since this is the order of the group.

Worked Example 2 Explain why, however,

$$Z(\,C_4,\,\mathfrak{X}_4,\,\alpha_{\mathrm{natural}};\,\underline{\boldsymbol{x}}\,) \;=\; \frac{1}{4}\,\left(\,{x_1}^4 \;+\; {x_2}^2 \;+\; 2\,x_4\,\right)\;.$$

Answer 2 This reflects that this cyclic group now has one element of order 2 as well as two elements of order 4. It is a minimal counterexample to the above simple formula for prime orders extending to non-prime orders.

The identity and the two elements of order 4 contribute in the previous pattern. The remaining element of order 2 involves 2 2-cycles. And thus contributes ${x_2}^2$ rather than a third x_4.

Exercise 2^+ Prove that

$$Z(\,C_n,\,\mathfrak{X}_n,\,\alpha_{\mathrm{natural}};\,\underline{\boldsymbol{x}}\,) \;=\; \frac{1}{n}\sum_{d\,|\,n}\varphi(d)\,{x_d}^{\frac{n}{d}}\;.$$

Here, the d are the divisors of n, while Euler's totient function $\varphi(d)$ appears in the role of counting the individual types of contribution. [Proceed via further concrete examples if you first need to 'gain a feel for' how this works out.]

Worked Example 3 Show that cycle index is compatible with direct product of groups, according to the following.

$$Z(G_1 \times G_2; x_1, \ldots, x_n) = Z(G_1; x_1, \ldots, x_{m_1})\, Z(G_2; x_1, \ldots, x_{m_2}) .$$

Answer 3 Let $g_i \in G_i$ for $i = 1$ to 2. Suppose that the cycle type of g_i is

$$[a_{i\,j_1} \ldots a_{i\,j_{m_i}}] .$$

P4 With say g_2 involving cycles at least as long as those of g_1. Then the cycles of $G_1 \times G_2$ can be represented as the list of cycles in G_1 followed by that of cycles in G_2.
Thus its cycle type is

$$[C_1 \ldots C_n] = [a_{1\,j_1} + a_{2\,k_1}, \ \ldots\ a_{1\,j_l} + a_{2\,k_l},\ a_{2\,j_{l+1}},\ \ldots\ a_{2\,j_m}] . \tag{28.5}$$

So

$$Z(G_1 \times G_2) = \frac{1}{|G_1 \times G_2|} \sum_{k \in G_1 \times G_2} \prod_{I=1}^{n} x_I^{C_I}$$

$$= \frac{1}{|G_1 \times G_2|} \sum_{g_1 \in G_1} \sum_{g_2 \in G_1} \prod_{p=1}^{m_1} x_p^{a_{1\,j_p} + a_{2\,j_p}} \prod_{r=m_1+1}^{m_2} x_r^{a_{2\,l_r}}$$

$$= \frac{1}{|G_1|\,|G_2|} \sum_{g_1 \in G_1} \sum_{g_2 \in G_1} \prod_{p=1}^{m_1} x_p^{a_{1\,j_p}} \prod_{q=1}^{m_1} x_q^{a_{2\,k_q}} \prod_{r=m_1+1}^{m_2} x_r^{a_{2\,l_r}}$$

$$= \left(\frac{1}{|G_1|} \sum_{g_1 \in G_1} \prod_{p=1}^{m_1} x_p^{a_{1\,j_p}} \right) \left(\frac{1}{|G_2|} \sum_{g_1 \in G_2} \prod_{s=1}^{m_2} x_s^{a_{2\,k_s}} \right)$$

$$= Z(G_1; x_1, \ldots x_{m_1})\ Z(G_2; x_1, \ldots x_{m_2}) .$$

The first step here applies the definition of cycle index to the product group. The second step substitutes (28.5) in. While also using that the order of the product group is the product of the orders of the input groups. And that summing over the product-group elements is the same as double-summing over the input groups. The third step separates this into G_1 and G_2 factors. The fourth step concatenates the last two products into one. The last step then makes one use per input group of the definition of cycle index in reverse.

Exercise 3 Evaluate $Z(C_2 \times C_2, \mathfrak{X}_4, \alpha_{\text{natural}}; \underline{\boldsymbol{x}})$, $Z(\prod_{i=1}^{n} G_i; \underline{\boldsymbol{x}})$ and $Z({C_2}^n, \mathfrak{X}_{2^n}, \alpha_{\text{natural}}; \underline{\boldsymbol{x}})$.

♠ ◇ ♣

Worked Example 4 Show that

$$Z(D_4, \mathfrak{X}_4, \alpha_{\text{natural}}; \underline{\boldsymbol{x}}) = \frac{1}{8} \left({x_1}^4 + 2\,{x_1}^2 x_2 + 3\,{x_2}^2 + 2\,x_4 \right) .$$

Answer 4 This contains a C_4 subgroup, which contributes as above. It also contains 2 further elements of order 2, σ and $\rho^2\,\sigma$. Whose only nontrivial cycle is a 2-cycle, thus contributing $x_2\,{x_1}^2$ each. And 2 further elements of order 4. These each exhibit 2 2-cycles. Thus boosting ${x_2}^2$'s coefficient from C_4's 1 to D_4' 3.

Exercise 4 Show that i)

$$Z(D_3; \text{equilateral}; \alpha_{\text{natural}}; \underline{\boldsymbol{x}}) = Z(S_3; \mathfrak{X}_3; \alpha_{\text{natural}}; \underline{\boldsymbol{x}}) = \frac{1}{6} \left({x_1}^3 + 3\,x_2\,x_1 + 2\,x_3 \right) .$$

ii)

$$Z\left(D_6; \text{regular hexagon, } \alpha_{\text{natural}}; \underline{\boldsymbol{x}}\right) = \frac{1}{12}\left({x_1}^6 + 3{x_1}^2{x_2}^2 + 4x_2^3 + 2x_3^2 + 2x_6\right) .$$

Exercise 5$^+$ i) Show that

$$Z\left(D_n, \text{ regular } n\text{-a-gon, } \alpha_{\text{natural}}; \underline{\boldsymbol{x}}\right)$$

picks up a contribution of

$$\frac{1}{2}Z(C_n, \text{ regular } n\text{-a-gon, } \alpha_{\text{natural}}; \underline{\boldsymbol{x}})$$

from its rotational elements.
ii) Show furthermore that the elements involving reflections contribute

$$\frac{1}{2}x_1\,{x_2}^{\frac{n-1}{2}} \text{ for } n \text{ odd} .$$

And

$$\frac{1}{4}\left({x_1}^2{x_2}^{\frac{n-2}{2}} + {x_2}^{\frac{n}{2}}\right) \text{ for } n \text{ even} .$$

♠ ◇ ♣

Worked Example 5 Show that

$$Z(\text{rotations of tetrahaedron vertices}) = \frac{1}{12}\left({x_1}^4 + 8x_1x_3 + 3{x_2}^2\right) .$$

But that

$$Z(\text{rotations of tetrahaedron edges}) = \frac{1}{12}\left({x_1}^6 + 3{x_1}^2{x_2}^2 + 8{x_3}^2\right) .$$

What is

$$Z(\text{rotations of tetrahaedron faces})?$$

Answer 5 The first of these encapsulates having 1 identity, 8 $2\pi/3$ rotations preserving 1 vertex (Chapter 20) and 3 π rotations exchanging vertices in pairs.

$$Z(\text{rotations of tetrahaedron faces})$$

manifests vertices $\rightarrow$ faces uuality of this and so shares cycle-index with the vertex case.
However, since we are now acting on 6 objects, the identity must now contribute ${x_1}^6$. Also now the 8 $2\pi/3$ rotations execute two disjoint 3-cycles of the edges, while the 3 π rotations transpose two edge pairs while preserving the remaining edge pair. So the coefficients 1, 3 and 8 remain the same, but the cycle content, and thus multi-mononomial terms, differ as indicated.

Remark 1 This is our second, more substantial, example of cycle index depending on group action as well as on the underlying group itself.

Worked Example 6 Show that

$$Z(\text{rotations of cube vertices}) = \frac{1}{24}\left({x_1}^8 + 9{x_2}^4 + 8{x_1}^2{x_3}^2 + 6{x_4}^2\right) .$$

Answer 6 The identity fixing all 8 vertices gives the first term. The last term follows from the $\pm\pi/2$-rotation about the line through the midpoints on opposite faces. Which act as 2 4-cycles. The second term is the superposition of two contributions. For the 3 rotations through π about lines

joining midpoints of opposite faces transpose 4 pairs of vertices. As do the 6 rotations through π about lines joining midpoints of opposite sides.

The third term, corresponds to the 8 rotations though $\pm 2\,\pi/3$ about each of the longest diagonals. Which act by two 3-cycles while leaving the other 2 vertices fixed.

Exercise 6 a) What are the edge and face cycle indices for the cube? And the vertex, edge and face cycle indices for the octahaedron?

b) Why is there no vertex-edge distinction in evaluating cycle indexes for regular n-a-gons?

28.3 Weighted orbit-counting formula

Exercise 7 a) Show that

$$\sum_{x \,\in\, \mathfrak{Fix}(g)} 1 \;=\; \sum_{x \,\in\, \mathfrak{X}} \chi_{Stab(x)} \;\times\; 1 \;,$$

where χ denotes characteristic function. And that equality continues to hold here if the 1's are replaced by a weighting function $w(x)$ that is constant on each orbit.

b) Deduce that

$$\sum_{g \,\in\, G} \sum_{x \,\in\, \mathfrak{Fix}(g)} w(x) \;=\; \sum_{x \,\in\, \mathfrak{X}} w(x)\,|\,Stab(x)\,|\;.$$

c) Finally generalize the proof of the orbit-counting Lemma to include weighting of this kind.

28.4 Pólya's Pattern-Inventory Enumeration Theorem

Definition 1 A *Pólya pattern inventory* encodes the possible patterns via a generating function that is a multi-variable polynomial in the colouring functions.

$$\mathfrak{I}\text{nventory}\,(\,G;\,c\,(P_1)\,,\,\ldots\,,c\,(P_k)\,)\;\;=\;\;\sum_{\sum_{i\,=\,1}^{k}\,n_i\,=\,n} \#c\,(\,n_1,\,\ldots\,,\,n_k\,)\,\prod_{i\,=\,1}^{k}\,c(P_i)^{n_i}\;.$$

The n_i here are non-negative; the sum that they form indicates the corresponding partition of $\mathfrak{X}$'s elements into colour classes. The $\#c$ are counts of colourings that have n_i elements of $\mathfrak{X}$ in the colour $c(P_i)$ (Fig 28.1). These colourings are the usual interpretation given to the patterns in question.

Theorem 2 (Pólya's Pattern-Inventory Enumeration Theorem)

$$\mathfrak{I}\text{nventory}\,(\,G,\,c\,(P_i)\,,\,\ldots\,,\,c\,(P_k)\,)\;\;=\;\;Z\,(\,G,\,x_1,\,\ldots\,,\,x_n\,)\big|_{x_j\,=\,\sum_{i\,=\,1}^{k}\,c(P_i)^j}\;\;,\;\;\;j\;=\;1\;\;\text{to}\;\;n\;.$$

Here, $|_z$ denotes 'evaulated at z', where we fix x_j for every j from 0 to n.

Exercise 8$^+$ Upgrade your proof of Pólya's Pattern Enumeration Theorem to run with the Weighted Orbit-Counting Lemma, so as to prove Pólya's Pattern-Inventory Enumeration Theorem.

28.5 Further Exercises

Exercise 9 The Jewelitanians wish to issue commemorative 4-segment necklaces with each segment bearing a ruby, a sapphire, an emerald or a diamond. What is the the largest number of distinct patterns that they could have in their inventory?

Exercise 10 a) Classical Gifts have a make-your-own bead bracelet activity in which one can pick a combined total of five red, blue or green beads to string together. How many different types of bead bracelet can be made there?

b) Quantum Gifts run a similar activity on a much smaller level, with the further distinctive feature that their notions of 'red', 'blue' and 'green' are mutually-distinguishable and yet meaningless labels. How many different types of bead bracelet can be made there?

Exercise 11 a) How many types of chlorofluorobenzene molecule are possible? (Allow for each of benzene's six hydrogen atoms to be substituted by chlorine or fluorine atoms, in any proportion).

b)$^+$ With reference to Fig 21.3, how many types of chlorofluoronaphthalene are possible?

Exercise 12 $^+$ The local tabletop gaming club have some of the usual 4-sided, 6-sided, 8-sided and 10-sided role-playing dice. They wish to decorate them using three types of paint that they have in surplus: troll-grey, orc-green and skeleton-white. They wish for each to have at least one face of each colour. How many rotationally different ways could they paint each of them?

28.6 Naming and Pointers

Naming Remark 1 A truer name for Pólya's Enumeration Theorems (and similar) is Polynomial Enumeration Theorems. For the first of these (in fact previously presented by Redfield [89]), one does well to evoke the cycle-index content of the polynomial: *Cycle-Index Enumeration Theorem.* For the second of these, the conceptually-distinctive feature is that it is evaluating inventory patterns, by which a more specific name is *Inventory-Pattern Enumeration Theorem.*

Pointer 1 See [95, 36, 107] for further variants of Polynomial Enumeration Theorem

Pointer 2 [36] also briefly discusses algorithmically implementing Pólya enumeration.

Part V

Introduction to Modelling with Graphs

Chapter 29

Simple graphs

29.1 Point clouds

Definition 1 A *point cloud*

$$\mathrm{Pt}_n$$

is a set of points bereft of inter-relating structure.

Remark 1 The points in a point cloud can furthermore be labelled. This allows for fully and partly distinguishable versions.

29.2 Point cloud morphisms

Definition 1 *Point-cloud-morphisms* are maps between point clouds,

$$\phi \ : \ \mathrm{Pt}_n \ \longrightarrow \ \mathrm{Pt}_m \ .$$

Definition 2 *Point-cloud-isomorphisms* are point-cloud-preserving maps.

Remark 1 Labelled point cloud (iso)morphisms can also be defined, by restricting to those which respect the labelling.

Remark 2 For finite point clouds, isomorphisms only make sense for $n \ = \ m$ and must be some permutations of the points. If the points have no labels, *id* is the sole morphism. In the case of no or fully-indistinguishable labelling, this is the permutation group on n elements, S_n. In the fully distinguishable case, it is just *id*. For cases in between, it is some subgroup of S_n.

29.3 Examples of point clouds

Example 0 In this context, the empty set $\emptyset$ can be thought of as consisting of the *unpoint*, $\mathfrak{U}$.

Example 1 The single point pt, alias the *unit point*.

Example 2 n disjoint points, pt^n for whichever $n \ \in \ \mathbb{N}$.

Remark 1 Indeed, counting the points suffices to distinguish between unlabelled point clouds, by which pt^n is also a suitable notation for Pt_n.

Remark 2 If labelling is present, counting the points and their types of label suffices. We denote this by $\mathrm{Pt}_n(L)$, where L is in general a multi-index of labels and labelling multiplicities. Matters

become more interesting – structured – if one has instead a graph: the main topic of the current Chapter.

29.4 Motivation for point clouds

Some applications of point clouds are as follows.

Motivation 1 Putting names in a hat, each on a separate piece of paper, is a memorable prototype. There could however be further information on each point: putting applicants' CVs into a folder in no particular order. Data Science makes use of point clouds to model *unstructured data*. In particular, this is unstructured in the sense that no inter-relations between the points – individual chunks of data – enter the modelling. The points could however be internally labelled – the names – or even multi-labelled: the contents of each CV.

Motivation 2 Relations between chunks of data can subsequently be built into our modelling. For instance, in the form of graphs below. Which provide a more general model of data structures, with point clouds now being a trivial included subcase.



29.5 Graphs

Definition 1 Let

$$\mathrm{G} \;=\; (\,\mathfrak{V}, \mathfrak{e}\,)$$

be a *graph* [190, 183, 178, 193]: a family of vertices v, w, ... that form the *vertex set* $\mathfrak{V}$ some of which are joined by edges $e \;=\; v\text{-}w$ that form the *edge set* $\mathfrak{e}$. It is of *order*

$$V \;:=\; |\,\mathfrak{V}(\mathrm{G})\,| \;=\; |\,\mathrm{G}\,| \;=\; N$$

and *size*

$$E \;:=\; |\,\mathfrak{e}(\mathrm{G})\,| \;.$$

Remark 1 In the current Chapter, restrict ourselves to one edge per distinct-vertex pair.[1]

Naming Remark 0 'Graph' comes from the Greek $\gamma\rho\alpha\phi\omega$, one of whose meanings is a scratching. Which is a way of making a figure. Compare 'lithography': making markings on stones, and 'geography': drawing maps as figures to represent parts of the Earth.

Structure 1 Suppose that vertices u, v are joined by an edge, $u\,v$. Then we also say that u and v are *adjacent*, which as a relation on vertices we denote by

$$u \;\mathcal{A}\; v \;.$$

Structure 2 Suppose that $v\,w$ is another edge. By sharing vertex v, these two edges are also said to be *adjacent*. Let us denote this relation on edges by

$$u\,v \;\;\mathcal{A}_1\;\; v\,w \;.$$

Structure 3 If the edge e has an end at vertex v, then we say that e is *incident on* v.

Exercise 1 Show that $\mathcal{A}$ and $\mathcal{A}_1$ are both symmetric, but that only the second is reflexive for simple graphs. Is either transitive?

[1] I.e. the *simple graphs*, as opposed to multi-graphs and/or loop-graphs and/or arrowed graphs. See Sec 29.14, Chapters 33 and 39 and [193] for more about these.

Remark 2 Finally observe that $\mathcal{I}$ is a *heterogeneous* binary relation

$$\mathcal{I} \ : \ \mathfrak{e} \times \mathfrak{V} \longrightarrow \{ 0, 1 \} ,$$

where the outputs can be interpreted as 'no' and 'yes'. We shall be encountering further notions of incidence in Volume 3 on Flat Geometry [443].

29.6 Graph morphisms

Definition 1 A *graph morphism* is a map between graphs,

$$\mu \ : \ \mathrm{G} \longrightarrow \mathrm{G}' .$$

Definition 2 Graphs G and G′ are *isomorphic* if there is a bijection

$$\phi \ : \ \mathfrak{V}(\mathrm{G}) \longrightarrow \mathfrak{V}(\mathrm{G}')$$

that additionally preserves adjacency relations.

Remark 1 I.e.

$$v \ \mathcal{A} \ w \quad \Leftrightarrow \quad \phi(v) \ \mathcal{A} \ \phi(w) \ \forall \ v, w \in \mathfrak{V}$$

and

$$e \ \mathcal{A}_1 \ f \quad \Leftrightarrow \quad \phi(e) \ \mathcal{A}_1 \ \phi(f) \ \forall \ e, f \in \mathfrak{e} .$$

Remark 2 We need graph isomorphisms so as to be able to distinguish different graphs from different presentations of the same graph.

Proposition 1 Isomorphic graphs share order and size.

Proof Preserving vertex set $\mathfrak{V}$ includes preserving its order V.

Preserving adjacency relations includes preserving the edge set's size E. □

Remark 3 To establish two given presentations of graphs to be isomorphic, exhibit the bijection. Less formally, we can deform some depictions of graphs into others. Such operations preserve $\mathfrak{V}$ and $\mathfrak{e}$ while letting edges slide over each other (but not be torn); see Exercises 31.9-14 and Chapters 45, 46 and 48 for examples.

Remark 4 In contrast, to establish that two given presentations of graphs are non-isomorphic, one is to show that they differ in some invariant property. Values of V and E are but a small start on such properties; most Chapters in this Part add more.



29.7 The smallest graphs

Example 1 = Naming Remark 1 For $N = 1$, there is just pt, now alias the discrete graph on 1 vertex, D_1

Example 2 = Naming Remark 2 For $N = 2$, there is $\mathrm{pt}^2 = \mathrm{D}_2$ but also the first nontrivial graph, K_2 alias P_2. The notation K_2 refers to this being the *complete graph* on 2 vertices, a complete graph being one in which every vertex is adjacent to every other. The notation P_2 refers to this here also coinciding with the *path graph* on 2 vertices, a path graph being one in which the vertices are ordered into a line.

Remark 1 See rows 1 and 2 of Fig 29.2 for depictions of the graphs on 1 and 2 vertices respectively. With larger N in mind, we adopt the strategy of ordering our graphs by increasing edge number from left to right.

Worked Example 1 Find all the graphs on 3 vertices.

Answer 1 = Naming Remark 3 First observe that any N admits a D_N and a K_N graph. So far these have been the only possibilities. These are now realized by graphs with 0 and 3 edges respectively. This points to there being 2 further possibilities: the graphs with 1 and 2 edges. This is the smallest case for which P_3 and K_3 are distinctly realized.

With 1 edge, we have the $K_2 = P_2$ graph alongside a loose point. With 2 edges, we have the 3-path graph, P_3. This notation extends to *path graphs* P_N for any N. We can now also re-envisage K_2 as P_2, and indeed $D_1 = K_1$ as P_1.

We also observe K_3 to have a new feature: it contains – and indeed is – a cycle. Let us celebrate by also using the notation C_3 for this, which extends to *cycle graphs* C_N for any $N \geq 3$. When neutral as to complete or cycle connotations, we denote this graph by Triangle.

See row 3 of Fig 29.2 for depictions of all these graphs. In this figure, we order our graphs lexicographically: firstly by order and then by size.

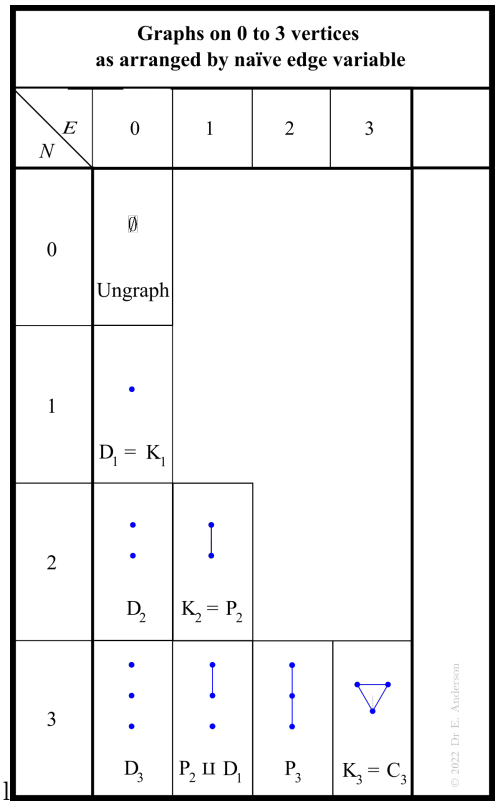


Figure 29.1:

29.8 Edge count for simple graphs

Remark 1 A simple graph of order N has maximum size

$$E_{\max} \;=\; \left(\begin{array}{c} N \\ 2 \end{array} \right) \;=\; \frac{N\,(\,N\;-\;1\,)}{2}\;.$$

Thus

$$0 \;\leq\; E \;\leq\; E_{\max}\;. \tag{29.1}$$

Counting graphs is hard [78].

Exercise 2 Counting the fully distinguishably vertex-labelled graphs should however now be straightforward for the Reader.

Definition 1 For $N \;\geq\; 2$, the *density* of a graph G is

$$\rho(\mathrm{G}) \;:=\; \frac{E}{E_{\max}} \;\overset{\text{simple graphs}}{=}\; \frac{2\,E}{N\,(\,N\;-\;1\,)}\;. \tag{29.2}$$

Remark 1 (29.1) can now be rewritten as

$$0 \;\leq\; \rho \;\leq\; 1\;.$$

This is saturated below by D_N alone and above by K_N alone.

29.9 Graph complements

Structure 1 The *complement* of a graph is the graph obtained by switching edges and non-edges between each vertex pair. Most graphs occur in complementary pairs, though a few are homocomplementary.

Lemma 1 a)

$$E \;=\; E_{\mathrm{crit}} \;=\; \frac{E_{\max}}{2} \;\overset{\text{simple graphs}}{=}\; \frac{N\,(\,N\;-\;1\,)}{4}\;.$$

is a necessary but not sufficient condition for graph homocomplementarity.

$$N \;=\; 0 \;\text{ or }\; 1 \;\pmod 4$$

is necessary so as to have homocomplementarity.

Proof The switched and unswitched numbers of edges have to coincide, which is only possible if both are equal. The sum of these two edge counts moreover includes all the edges. Thus each of these two equal portions of edges must be of size $E_{\max}/2 \;=\; E_{\mathrm{crit}}$ as claimed.

It is not sufficient by e.g. the S_3 and $\mathrm{K}_3 \;\amalg\; \mathrm{D}_1$ heterocomplementary pair.

$$N\,(\,N\;-\;1\,) \;=\; 4\,k \;\;\Rightarrow\;\; N \,|\, 4 \;\text{ or }\; (\,N\;-\;1\,) \,|\, 4 \;\;\Rightarrow\;\; N \;=\; 4\,p \;\text{ or }\; 4\,p\;+\;1\;.\;\square$$

Remark 1 Suppose that

$$E(\mathrm{G}) \;<\; E_{\mathrm{crit}} \;:=\; \frac{E_{\max}}{2} \;\overset{\text{simple graphs}}{=}\; \frac{N\,(\,N\;-\;1\,)}{4}\;.$$

Then the complement has fewer edges, by which it is more structurally sparse and presentationally convenient.

Remark 2 A somewhat finer analysis of regimes supported is as follows.

The *dense regime* is characterized by

$$E \simeq E_{\max} \overset{\text{simple graphs}}{=} \frac{N(N-1)}{2} \simeq \frac{N^2}{2} .$$

The *sparse regime* by

$$E \ll E_{\max} \overset{\text{simple graphs}}{=} \frac{N(N-1)}{2} \simeq \frac{N^2}{2} .$$

The *critically dense regime* by

$$E \simeq E_{\text{crit}} \overset{\text{simple graphs}}{=} \frac{N(N-1)}{4} \simeq \frac{N^2}{4} .$$

♠ ◇ ♣

Definition 1 A [*graph*] [232] is a graph modulo complementation.

Example 1 Each D_N and K_N are a complementary pair. Except for $N = 1$, for which $\mathrm{D}_1 = \mathrm{K}_1$ is homocomplementary.

Example 2 P_3 and P_2 alongside a point constitutes the remaining complementary pair of graphs on 3 vertices.

Exercise 3 Show that if $\mathrm{G}_1 \cong \mathrm{G}_2$, then $\bar{\mathrm{G}_1} \cong \bar{\mathrm{G}_2}$, so that isomorphisms between graphs are really isomorphisms between [graphs].

29.10 Further examples of small graphs

Worked Example 2 Find all the graphs on 4 vertices.

Answer 2 = Naming Remark 4 For $\mathrm{N} = 4$, there is a K_4–D_4 pair and a P_4. The Reader can quickly check that P_4 is in fact homocomplementary. There is now room for another connected graph with no cycles in it – a tree – that is not just a path. This is the *claw graph* Claw. We see that while there already were 0, 1 and 2-stars, these are just other names for

$$\mathrm{D}_1 = \mathrm{K}_1 = \mathrm{P}_1 ,$$

$$\mathrm{K}_2 = \mathrm{P}_2$$

and P_3.

Also now the cycle graph is no longer the same as the complete graph C_4. One can additionally have a cycle with a sidechain: the Paw graph alias 3-Pan. One can add a diagonal to the 4-cycle to form the Diamond alias 2-Fan graph, Fan_2. (1-Fan is just another name for

$$\mathrm{K}_3 = \text{Triangle} = \mathrm{C}_3).$$

Adding the other diagonal, one has exhausted all the simple graph possibiities for edges, thus arriving at K_4. See Fig 29.2 for K_4's 4-cycle manifesting presentation and uncrossed-edges presentation. (We will later justify calling these 'Hamiltonian' and 'planar' presentations respectively.)

$N = 4$ is minimal for order-and-size lexicographic order not to suffice to uniquely characterize graphs. This is what we subsequently mean by 'minimal graphs'. Some ways of further ranking graphs include by largest component, the most cycles, the largest cycles, or the highest degree content.

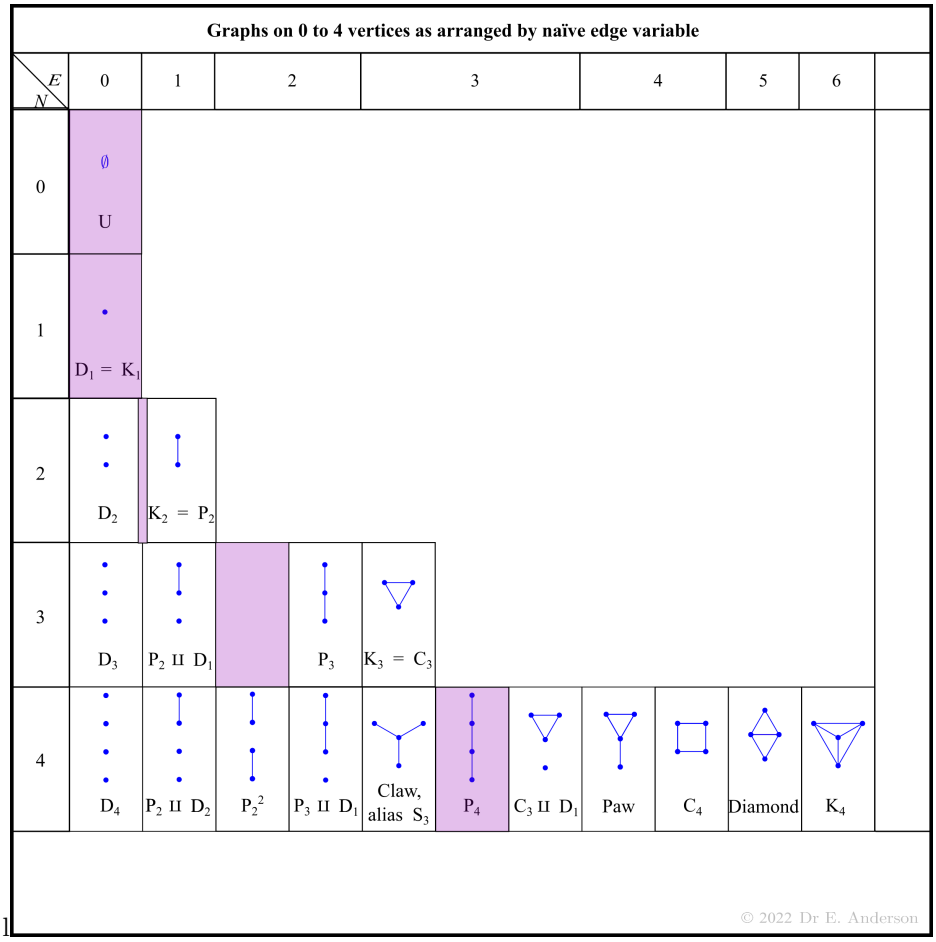


Figure 29.2:

Remark 1 The complements are as indicated in the Figure; homocomplementary graphs' boxes are picked out in yellow. N.B. that there are 2 heterocomplementary graphs of the same size as the homocomplementary graph.

Exercise 4 Prove the above Lemma.

Exercise 5 a) Show that $Aut(\mathrm{G}) = Aut(\bar{\mathrm{G}})$, so that 'graph automorphisms' are in fact a [graph] structure.

b) Find $Aut(\mathrm{G})$ for each G on 0 to 4 vertices.

Exercise 6 Find all the graphs on 5 vertices. Also show that none of these are asymmetric.

[Due to its subsequent use in this Volume, the answer is provided at the end of the next Chapter.]

29.11 Trees and forests

Definition 1 A graph that is not (path-)connected is said to have multiple *components*, each of which is maximally connected.

Definition 2 A *tree* is a connected graph that contains no cycles.

Naming Remark 5 The name 'tree' in the context of graphs was coined by Cayley in the 19th Century.

Example 1 Paths are trees. This includes $D_1 = K_1 = P_1$ and $K_2 = P_2$ but no further D_n or K_n.

Exercise 7 a) An (n-pointed) *star* graph S_n is an $(n+1)$-tree consisting of a distinguished point from which edges to all other points emanate. Under which other names have you so far encountered S_0, S_1, S_2 and S_3? What is the smallest tree that is neither a path nor a star?

b) Find all the trees for $N = 6$ and 7. How many are not stars?

Remark 1 The smallest tree which is neither a path nor a star is the chair graph, Chair (see Fig 29.3).

Exercise 8 Find all the trees on 6 and 7 vertices.

[Since we make subsequent use of these in this Volume, we provide the answer to Exercise 8 in Fig 29.4.]

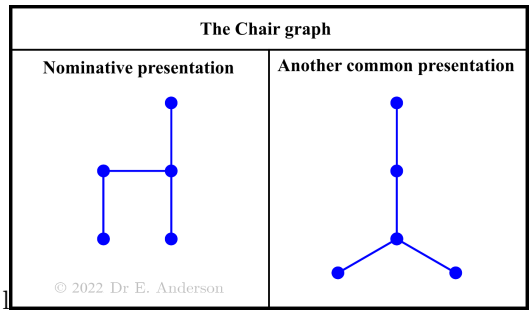


Figure 29.3:

♠ ◇ ♣

Definition 3 If a graph is acyclic but not necessarily connected, it is a *forest*.

Naming Remark 6 It is named thus since each of its components is a tree.

Exercise 9 How many forests are there on $N = 1$ to 6 vertices?

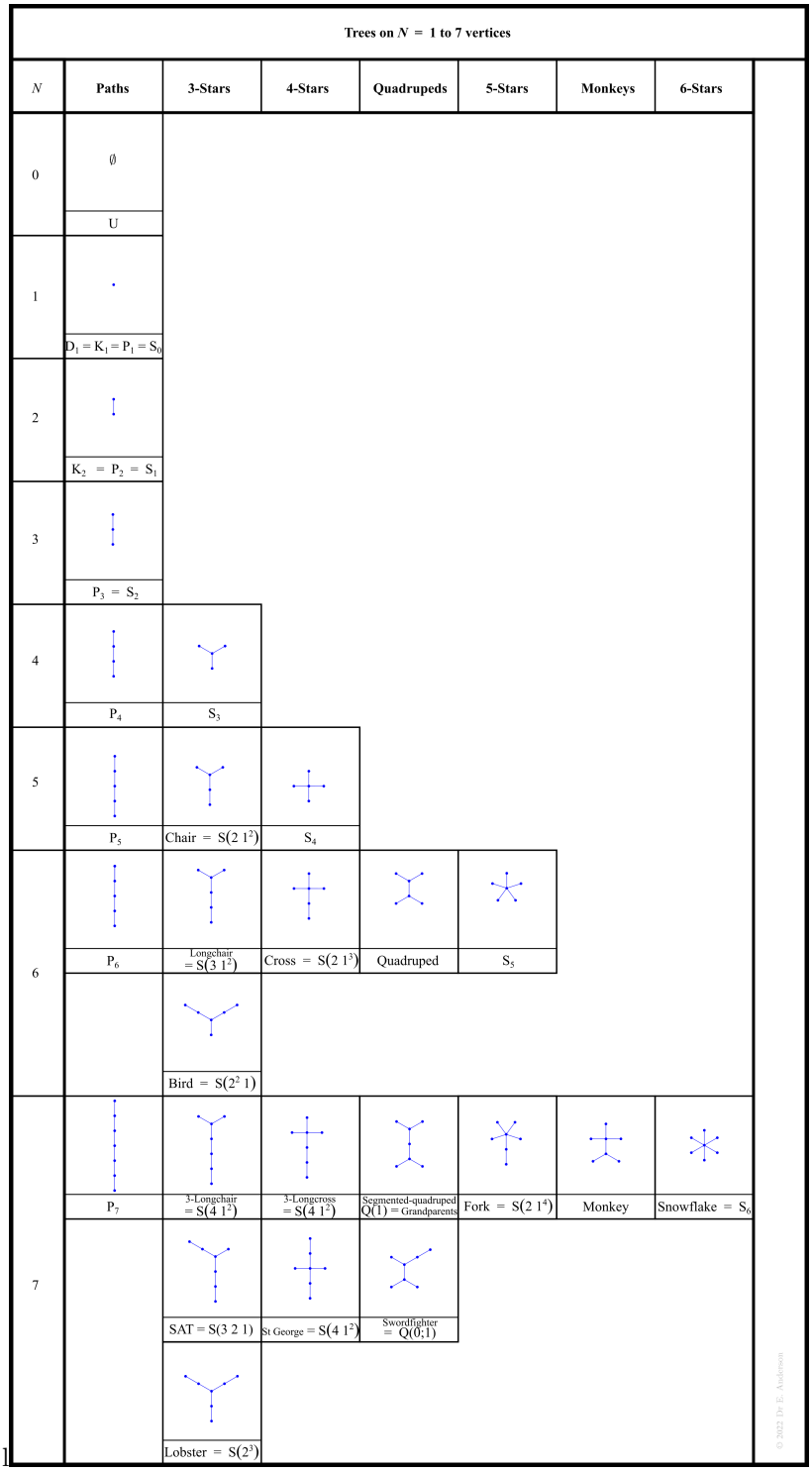


Figure 29.4:

29.12 Some further simple families of graphs

We have so far introduced D_n, K_n, P_n, S_n and C_n.

Examples 1 to 4 = Naming Remark 7 Worked Example 2 introduces you to the n-Pan family. And Exercise 6's answer to the wheel W_n fan F_n, and (mountain) range R_n and families. All of whose general members are depicted in Fig 29.5.a)-d). Let us furthermore adopt the convention that the first member of a family that is not conflatedly a member of a more basic family has the same name as its series. So for instance, the first nontrivial wheel is W_4, for which we write Wheel. And the first nontrivial range is R_3, for which we write Range.

Exercise 10$^-$ a) What are more basic names for W_3, R_2, R_1, F_2, and F_1? Are there such graphs as Wheel_2 and Wheel_1?

b) The reason that we do not call Gem Fan is because the current Volume also makes use of Strut (Fig 29.5.e). Check that Gem is Strut_3 as well. What are $\text{Fan}_2 = \text{Strut}_2$ and $\text{Fan}_1 = \text{Strut}_1$?

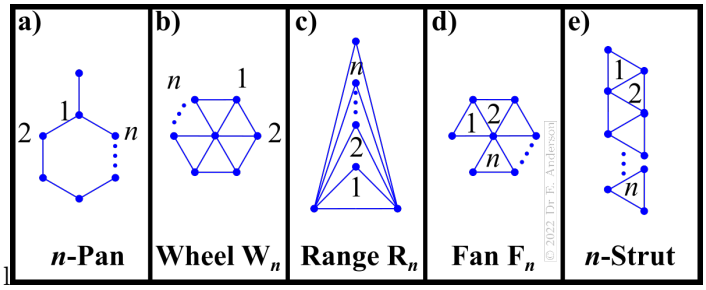


Figure 29.5:

♠ ◇ ♣

Exercise 11, including Naming Remark 8 Also observe within Exercise 6's answer the notation $K_{3,2}$: an example of a complete bi-partite graph. A graph is *bi-partite* if its vertex set can be partitioned into two parts, neither of which have any internal edges. So the only possible edges are mutual ones: from a vertex in one part to a vertex in the other. A bi-partite graph is *complete* if all such mutual edges are present.

i) Under what names are the complete bi-partite graphs $K_{1,1}$, $K_{2,1}$, $K_{3,1}$, $K_{4,1}$, and $K_{2,2}$ hiding in Figures 29.2 and 30.3?

ii) Three new houses need connecting to the water mains, power grid and internet (Fig 29.6). Can you find a way of doing this such that the edges (pipes, cables) do not cross over?

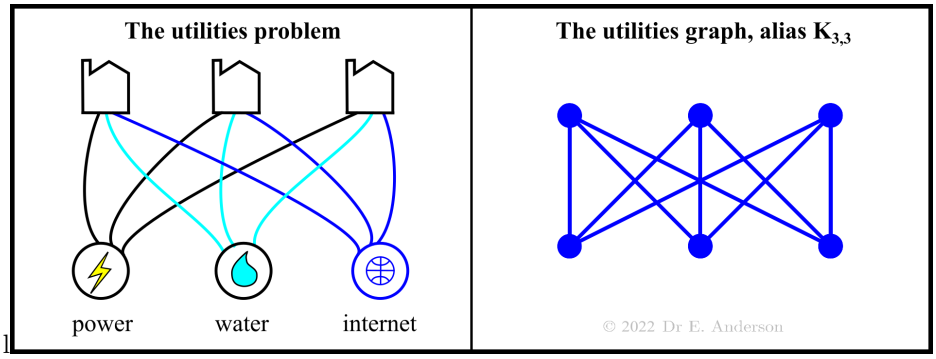


Figure 29.6:

[The graph modelling this situation is thereby known as the *utilities graph*, Utilities. It is the complete bi-partite graph $K_{3,3}$, where one of the triples of vertices models the houses and the other models the connection points to each of the utility grids.]

Exercise 12 A graph is *k-multi-partite* if its vertex set can be partitioned into k parts, none of which have any internal edges. So the only possible edges are mutual ones: now from a vertex in one part to a vertex in *some other* part. A multi-partite graph is *complete* if all such mutual edges are present.

i) What are the complete 3-, 4- and 5-partite graphs on up to 5 vertices?

ii) What is the complement of $K_{3,3}$? $K_{p,q}$? $K\left(\underline{p}\right)$ for $\underline{p}$ an n-vector?

Remark 1 The list of distinct simple families of graphs grows with N as further families get to exhibit their first distinguishing cases.



29.13 Motivation for graphs

Motivation 1 Chemical formulae contain graph information. Though not just: they also contain angular and length-ratio information out of being interpreted within ambient physical 3-space.

Motivation 2 Some of the more specialized tree graphs in Chapter 36 are useful for further modelling data structures in Computer Science [36, 442].

Motivation 3 Particle Physics processes are also usefully modelled by graphs: Feynman diagrams (Secs 40.6–40.8).

Motivation 4 So are binary relations (Sec 40.5).

Motivation 5 So are *polytopes* in Geometry [161, 170, 192, 184], in the sense of the dimensional sequence that begins as follows. Points in 0-d, line segments in 1-d, polygons in 2-d, polyhaedra in 3d,...

Motivation 6 So are 2-d Euclidean Geometry line-and-circle constructs [443] (for all that this has not hitherto been fully exploited).

Motivation 7 *Networks* [186, 450, 197] can usually be taken to refer to, or at least be modelled by, large graphs. These have many applications: electronics and computer circuitboards, the Internet, social media...

Motivation 8 Triangulations, complexes, unit cells, knots, and some proofs, in Topology [250, 449]. Series 1 shall cover many further uses of graphs in Topology

See also our Conclusion Chapter. Books 4, 5, 6, 9 and 11 shall cover yet further applications of graphs to Group Theory, Differential Geometry, Lie Theory, the Principles of Dynamics' configuration spaces, and Applied Geometry. The current Volume only considers small to middle-sized examples of graphs, concentrating on Science applications. 1) to 5) and 7) knot shadows are represented in the current Volume. Though far more material on 3), 4) and yet further applications of graphs are postponed to other books in the current series.

29.14 Modelling assumptions: simple graphs and alternatives

Remark 1 Until Chapter 33, we restrict ourselves to the following.

Definition 1 A graph is *simple* if the following three modelling conditions hold.

A) It has at most one edge between distinct vertices.

B) It has no *loops*: edges from a vertex to itself.

C) Its edges carry no arrows or other directional structure.

Remark 2 For contrast and use in subsequent work – many applications require some part of this – relaxing whichever of A)–C) gives the following.

Pointer 1 *Multi-graphs* are graphs which can contain more than one edge between one or more given pairs of distinct vertices.

Pointer 2 Graphs which can contain loops are termed *loop-graphs*.

Structure 1 Graphs in which both of these can occur are called *loop-multi-graphs*. See Chapter 39 for more.

Pointer 3 Graphs all of whose edges carry one-way arrows are called *oriented graphs* or *ori-graphs*. See Chapter 33 for these and further arrowed-edge-containing variants.

Chapter 30

From relational variables for graphs to a start on graph arenas

Remark 1 Our starting package for building relative, relational and dilatational quantities is thus $V = N$, E, and two particular fixed values thereof, $E_{\max}$ and E_{crit}. We deal with graphs first, eventually covering also [graphs], and then the spaces formed by each of these,

$$\mathfrak{G}\text{raph}$$

and

$$[\mathfrak{G}\text{raph}]$$

respectively (see Chapter 59 for more about these).

30.1 Ratio variables for graphs

Remark 1 Our starting package thus supports one ratio variable (see Appendix A.2) as follows.

Definition 1 The *size-to-order ratio*

$$\eta(\mathrm{G}) \ := \ \frac{E}{|\,\mathrm{G}\,|} \ = \ \frac{E}{N} \ .$$

Remark 2 The last form here is used whenever the graph being referred to is clear. In subsequently introducing objects, we skip making reference to the particular graph as a default.

Remark 3 An underlying point in introducing this is that in comparing graphs of different orders, some kind of normalization should be used. For now, we propose 'per unit order' as a such – edge density is size per unit order – though we introduce further normalizations.

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Remark 4 Our starting package also supports two particular normalizations, as follows.

Structure 1 *Edge density*

$$\rho \ := \ \frac{E}{E_{\max}} \ \overset{\text{simple graphs}}{=} \ \frac{2\,E}{N\,(\,N \ - \ 1\,)} \ .$$

Values of this run from 0 to 1: standardized-range.

Structure 2 The *Ramsey density* ratio variable

$$\sigma \ := \ \frac{E}{E_{\rm crit}} \ \stackrel{\text{simple graphs}}{=} \ \frac{4\,E}{N\,(\,N \ - \ 1\,)} \ = \ 2\,\rho \ .$$

Values of this thus run from 0 to 2.

Remark 5 The *dense regime* is then

$$\rho \ \lessapprox \ 1 \ .$$

Remark 6 The *sparse regime* is

$$\rho \ \ll \ 1 \ .$$

Remark 7 Finally, the *critical regime* is

$$\rho \ \simeq \ \frac{1}{2} \ .$$

Exercise 1 a) What are the counts for each relative size for the graphs on each of $N \ = \ 1$ to 5 vertices? Plot the realized value of densities for the distinct graphs for each of these N.

b) Show that for any tree T_N,

$$\rho(\mathrm{T}_N) \ = \ \frac{2}{N} \ .$$

And that, for any cycle C_N

$$\rho(\mathrm{C}_N) \ = \ \frac{2}{(\,N \ - \ 1\,)} \ .$$

What is $\rho(\mathrm{K}_{p,q})$?

30.2 Relative variables for graphs

Remark 1 One relative variable (see Appendix A.1) is also supported, as follows.

Definition 1 The 1-*d Euler characteristic* relative variable

$$\chi_1 \ := \ V \ - \ E \ = \ N \ - \ E \ . \tag{30.1}$$

Remark 2 Its name is explained in Sec 45.15. One non-relative total variable of note is also supported, as follows.

Definition 2 The *sum of qualitative cases* [433]

$$\Sigma \ := \ V \ + \ E \ . \tag{30.2}$$

Remark 3 Aside from Definition 1 being relationally significant, it is Topologically-significant, extending in particular to

$$\chi \ := \ V \ - \ E \ + \ F \tag{30.3}$$

in cases in which face count is meaningful.

Remark 4 For a simple graph of fixed N, χ runs from 0 to

$$\frac{N\,(\,N \ + \ 1\,)}{2} \ .$$

Σ runs from N to

$$\frac{N\,(\,N \ + \ 1\,)}{2} \ .$$

A limitation on this difference variable in the context of comparing between dense graphs is that V is extensive but E is quadratically super-extensive.

Exercise 2 What is the Euler characteristic for path graphs? Cycle graphs? Pans? Fans? Wheels?

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Remark 5 Graph edge counts support three shifted variables of note, as follows.

Definition 3 The *maximal edge difference relative variable*

$$MED \ := \ E_{\max} \ - \ E \ .$$

Definition 4 The *critical edge difference* alias *centring* alias (honorifically) *Ramsey relative variable*

$$CED \ := \ R \ := \ E \ - \ E_{\rm crit} \ .$$

Definition 5 The *unsigned centring* or *Ramsey relative variable* is

$$U \ := \ | \, E \ - \ E_{\rm crit} \, | \ .$$

30.3 Relational (Dilatational) variables for graphs

The following ratios of relative quantities: *dilatational quantities* – a type of relational quantity – (Appendix A.3) can then be formed.

Definition 1 The *Euler ratio* is the relational quantity

$$\mathcal{E} \ := \ \frac{\chi_1}{\Sigma} \ = \ \frac{V \ - \ E}{V \ + \ E} \ = \ \frac{1 \ - \ \eta}{1 \ + \ \eta} \ .$$

Remark 1 For simple graphs, this runs over the range

$$\left(\frac{1 \ - \ N}{1 \ + \ N} \ , \ \ 1 \right) \ .$$

To standardize range to $(\, 0, \, 1 \,)$, we can shift and rescale to

$$P_{\mathcal{E}} \ := \ (\, 1 \ + \ N^{-1} \,)(\, 1 \ + \ \eta^{-1} \,)^{-1} \ .$$

Definition 2 The *complementary edge density* relational quantity is

$$\mathcal{M} \ := \ \frac{MED}{E_{\max}} \ = \ \frac{E_{\max} \ - \ E}{E_{\max}} \ = \ 1 \ - \ \rho \ .$$

Remark 2 This runs from 0 to 1.

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Definition 3 The *relative* alias *Ramsey* ratio is

$$\mathcal{R} \ = \ \frac{R}{E_{\rm crit}} \ . \tag{30.4}$$

Remark 3

$$\mathcal{R} \ := \ \frac{E_{\rm crit} \ - \ E}{E_{\rm crit}} \ = \ 1 \ - \ \sigma \ = \ 1 \ - \ 2 \, \rho \ .$$

It runs from -1 to 1. Computationally for simple graphs,

$$\mathcal{R}(\gamma) \ = \ 1 \ - \ \frac{4 \, E(\gamma(N))}{N \, (\, N \ - \ 1 \,)} \ .$$

Remark 4 For [graphs], the key quantity is E_{crit}, supporting the ratio σ and the following.

Definition 4 The *unsigned Ramsey relational variable*[1]

$$\mathcal{U} \;=\; \left| \frac{E_{\text{crit}} \;-\; E}{E_{\text{crit}}} \right| \;=\; |\, 1 \;-\; \sigma \,| \;=\; |\, 1 \;-\; 2\,\rho \,| \;.$$

Remark 5 This is a standard-range function of note in the [graph] setting [232].

The surviving density regimes are sparse,

$$\mathcal{U} \;\ll\; 1 \;,$$

and critical,

$$\mathcal{U} \;\lessapprox\; 1 \;.$$

Exercise 3 What is the Ramsey unsigned relational variable averaged over the graphs on each of $N \;=\; 1$ to 4 vertices?

30.4 Point-cloud and graph arenas

Definition 1 Let us denote the *arena of n-point clouds* by

$$\mathfrak{Point}\mathfrak{Cloud}(n) \;=\; [\,1,\, \ldots,\, 1\,] \;.$$

And the *arena of distinctly-labelled n-point clouds* by

$$\mathfrak{LPoint}\mathfrak{Cloud}(n) \;=\; [\,1,\, \ldots,\, n\,] \;.$$

Definition 2 Let us denote the *arena of graphs of fixed order N* by

$$\mathfrak{Graph}(N) \;.$$

And the *arena of distinctly-labelled graphs on N* vertices by

$$\mathfrak{LGraph}(n) \;.$$

Graph arenas are much more complex than point-clouds arenas; see Part VIII for more about graph arenas.

30.5 Applying relational variables toward forming graph arenas

Remark 1 Our first relationally-motivated move toward forming the arena of graphs is to parametrize graphs in terms of the unsigned Ramsey variable, as per Fig 30.1. This acts to line up the homo-complementary graphs in the centre, which we shade in yellow.

Remark 2 Complementary pairs of graphs arrange mirror-symmetrically about this. By this, for some purposes we only need to deal with 'half' of the graphs, i.e. graphs up to complementation. Though which half we work with can vary (see Fig 30.2 for examples).

Remark 3 We have 2 choices of order of presentation: group the cycles (Fig 30.1.a) or group the connected graphs (Fig 30.1.b).

Remark 4 For N larger, however, it ceases to be the case that the complement of a connected graph is non-connected, or that cycles live solely on one side of the line of reflection. As we shall see

[1] As far as I am aware, all the quantities named after people in these past 3 sections are honorific apart from the Euler relative variable.

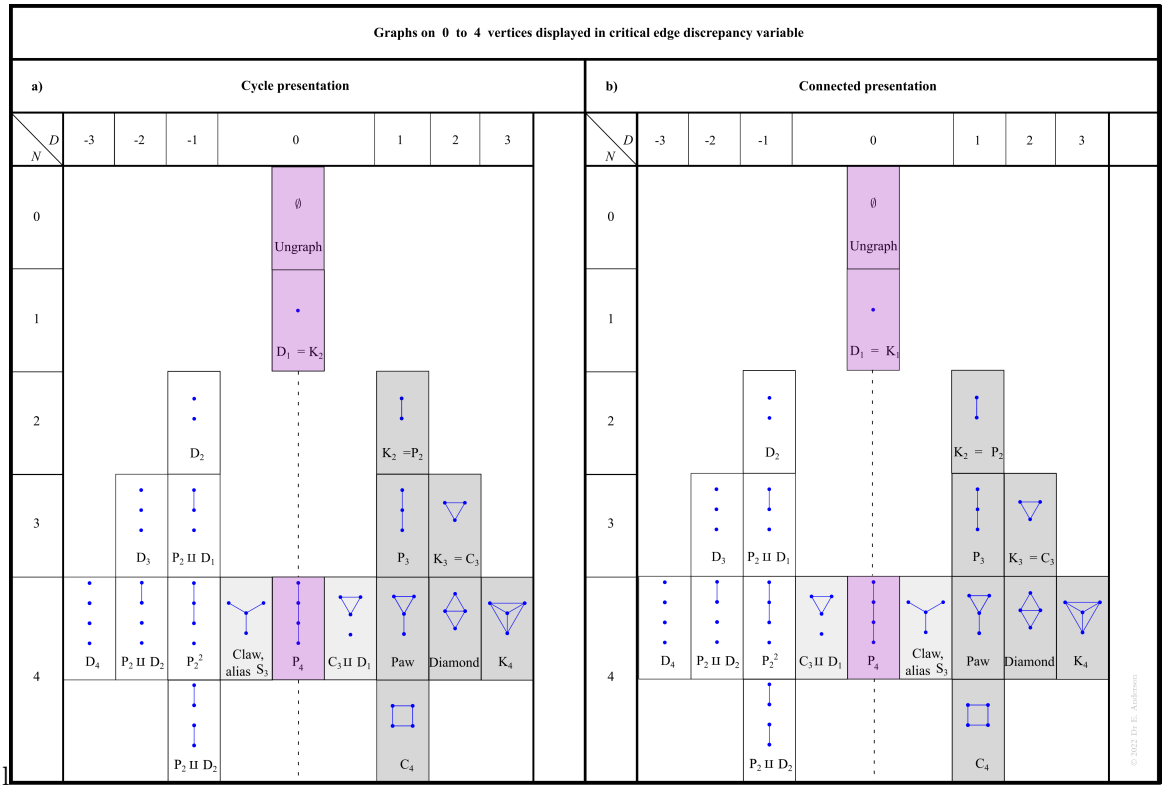


Figure 30.1:

below, already for $N = 5$ some complementary pairs have both graphs connected or contain cycles in their sparser graph.

Remark 5 The above 'halving' is modulo homocomplementary graphs that are taken to live on the reflection line itself.

$N = 1$'s D_1 is homocomplementary and thus lies on the line. So do P_4, C_5 and the Bull 5-graph.

Structure 1 See Fig 30.2 for the [G](N) with $N = 0$ to 4.

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Remark 6 Finishing off the above-mentioned centring additionally requires splitting out the homocomplementary graphs from the heterocomplementary pairs which also have E_{crit} edges. Such a splitting has been carried out in all three Figures presented so far. In the first two, it is non-unique as regards which member of each pair is depicted. In the third, it is non-unique as regards which representative is drawn on each side of the central yellow line.

Remark 7 The sparse presentation puts less cycles and then less branching points to the left.

30.6 [𝔊raph] on 0 to 4 vertices

Remark 1 The connected presentation is specific to ≤ 4 vertices: it does not carry over more generally.

Remark 2 Representatives of [G] can be made unambiguous is using the Ramsey presentation. I.e. depicting both the graph and its complement at once, using red and blue edges in some order. Red and blue being meaningless colour labels, there is no difference between red-dominated graphs and their blue-dominated colour reverse-colourings. While this is unamibiguous in terms of choice of representative, it is not necessarily presentationally unambiguous. In that K_N can in general be shaded partly red and partly blue to represent the same pair in visually different presentations.

Remark 3 As graphs get bigger, the sparse presentation is the one that remains visually clearest, by being the least cluttered with edges. While this will eventually rule out continuing to draw Ramsey presentations for large enough N, even in the more advanced Volume [452], we will never go above six vertices in depicting $\mathfrak{G}\mathrm{raph}(N)$ or $\mathfrak{G}\mathrm{raph}[N]$. Within this regime, the Ramsey presentation is practicable.

Remark 4 The symmetric versus planar Ramsey presentation ambiguity is just based on specifically K_4 having distinct manifestly-symmetric and planar depictions. For $N \leq 3$, the two coincide. And of course for $N \geq 5$, no planar presentation of K_N is possible. It is thus the symmetric Ramsey represesentation, based on bi-colouring the manifestly-symmetric presentation of K_n, which has the most future out of these variants of Ramsey presentation.

Remark 5 In the Ramsey presentation, we use not $[\mathrm{D}_N] = [\mathrm{K}_N]$ but M_N, standing for order-N *monochromatic* Ramsey-represented graph. With the exception of writing A for $[\mathrm{D}_1] = [\mathrm{K}_1] =$ pt, since this is the sole *achromatic* Ramsey-represented graph.

Remark 6 While it would be more consistent to use sparse presentation notation, at least for $N \leq 4$ it is usually more convenient to refer to non-sparse graph representatives, since these have more distinctive names.

30.7 Further Exercises

Exercise 4 Expanding on Exercise 29.6, organize the possible 5-graphs into a centred table.

Exercise 5 How many [graphs] are there on 5 vertices? Tabulate these using the unsigned U variable, in sparse presentation and Ramsey presentation. Show that a connected presentation is not possible now.

Exercise 6 What is the automorphism group of each graph on 5 vertices?

Remark 1 Due to further use in the current Volume, partial answers to the above Exercises are provided in the Figure-with-Namings overleaf.

Exercise 7 [Long] Show that for moderately large-order graphs, a size-to-order ratio η of around 1 corresponds to the linearly-sparse regime. Survey the distribution of η in the nonplanar graphs being produced in Part VIII's figures. These are the ones for which a presentation in the plane with no crossings cannot be found. Conclude that in studying Part VIII's graphs, we will need linearly-sparse regime techniques.

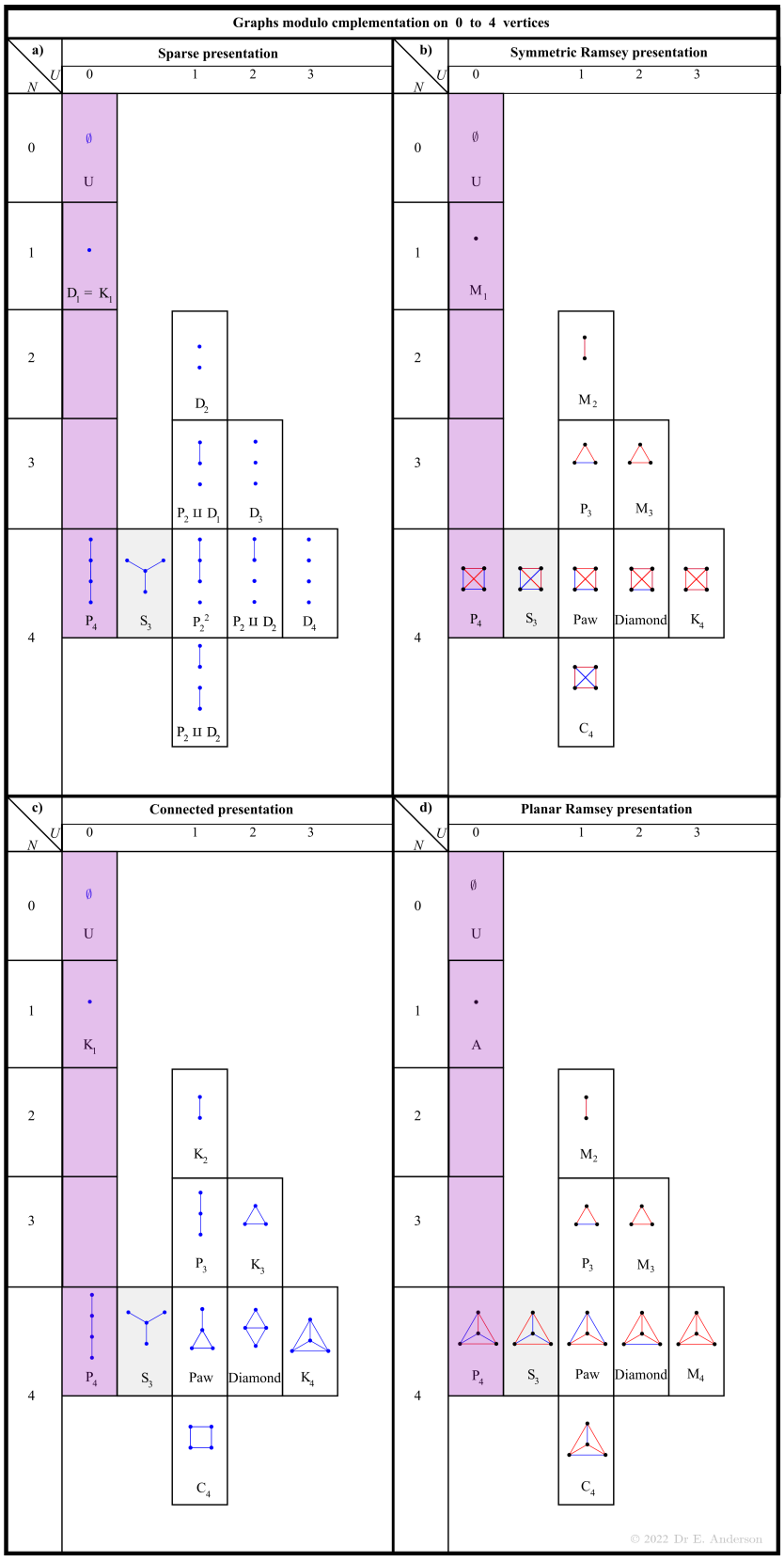

Graphs modulo cmplementation on 0 to 4 vertices
a) Sparse presentation
b) Symmetric Ramsey presentation
c) Connected presentation
d) Planar Ramsey presentation
N
U
0
1
2
3
4
∅
U
D1 = K1
D2
P2 ⊔ D1
D3
P4
S3
P2^2
P2 ⊔ D2
D4
C4
M1
M2
M3
P3
Paw
Diamond
K4
K1
K2
K3
A
M4
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Figure 30.2:

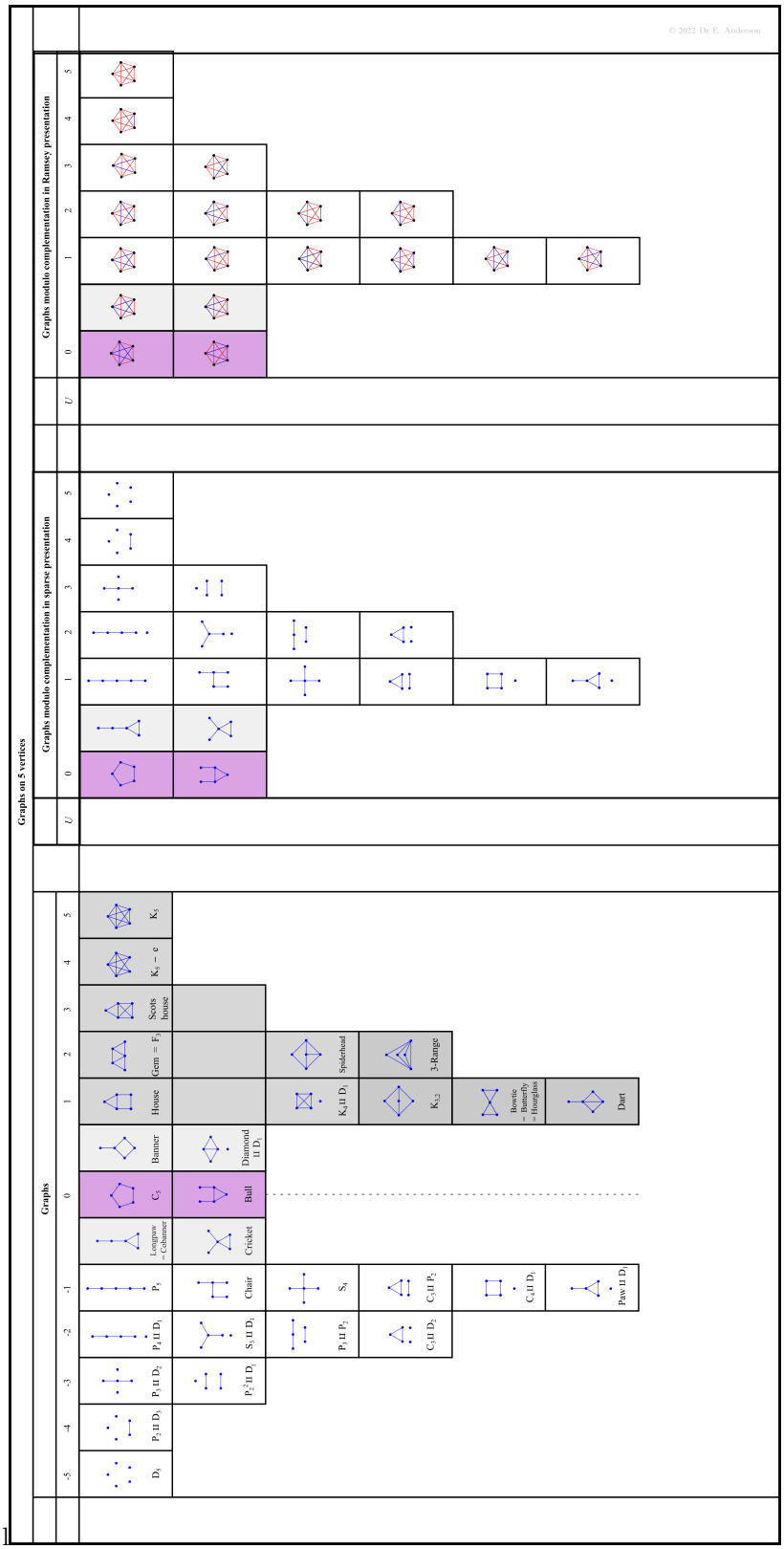

Graphs on 5 vertices
Graphs modulo complementation in sparse presentation
Graphs modulo complementation in Ramsey presentation
Graphs
U
© 2022 Dr E. Anderson
K5
K5 − e
Scots house
Gem = F3
House
Banner
C5
Lengyaw = Cobanner
P5
P4 ⊔ D1
P3 ⊔ D2
P2 ⊔ D3
D5
Diamond ⊔ D1
Bull
Cricket
Chair
S3 ⊔ D2
P2'⊔ D3
K4 ⊔ D1
Spidertoad
K3,2
3-Range
S4
P3 ⊔ P2
C3 ⊔ D2
Bowtie = Butterfly = Hourglass
C4 ⊔ D1
Dart
Paw ⊔ D1
C3 ⊔ P2

Figure 30.3:

Chapter 31

Degrees

31.1 Introduction

Definition 1 The *degree* $d(w)$ alias *valency* v_w of a vertex v is the number of edges adjacent to it.

Naming Remark 1 Valency is in particular the chemical name for degree.

Remark 1 For instance, hydrogen H has valency 1, and oxygen O almost always has valency 2. Nitrogen N and phosphorus P have valency 3 or 5. Carbon C – the defining ingredient for Organic Chemistry – has valency 4. The further types of atom that we make use of in the current Volume are sulphur S, completing the above set, which has valency 2 or 6, completing what is commonly encountered in Biochemistry. And the first three halogens, fluorine F, chlorine Cl, and bromine Br, which have valency 1.

Exercise 1$^-$ Check that these valencies hold in every compound in Fig 21.2.

Naming Remark 1 In Chemistry, the ending -mer is often short for 'isomer'. The Greek $\epsilon\nu\alpha\nu\tau\iota o\varsigma$ means 'opposite', so 'enantiomers' means (mirror-image) opposite isomers. $\sigma\tau\epsilon\rho\epsilon o\varsigma$ means 'solid'; taken in the sense of 'in space', 'stereoisomers' are isomers differing though spatial arrangement. 'Diasteromer' is a contraction of 'diastereoisomer', meaning 'isomer through spatial arrangement about two centres'. Though this word has extended meaning to two *or more* centres.

Remark 1 C HFCl Br has two enantiomers because the four atoms around the carbon 'pin each other in place', and such configurations do not coincide with their mirror images.

Exercise 2 a) What is the smallest hydrocarbon that comes in enantiomeric pairs?

b) Draw out a set of diastereomers which each have the minimal number of atoms present.

c) Likewise, but with three optically-active centres (in celebration of the above extended meaning).

Definition 2 A graph's *minimal degree* is

$$\delta \ := \ \min_{w \ \in \ \mathfrak{v}(\mathrm{G})} \ (\, d(w) \,) \ . \tag{31.1}$$

Its *maximal degree* is

$$\Delta \ := \ \max_{w \ \in \ \mathfrak{v}(\mathrm{G})} \ (\, d(w) \,) \ . \tag{31.2}$$

31.2 Regularity for graphs

Definition 1 A graph is *k-regular* if all its vertices have valency k.

Remark 1 Here $\delta = d(v) = \Delta$ for each vertex v.

Example 0 The 0-regular graphs are precisely the point clouds.

Example 1 The 1-regular graphs are precisely the disjoint unions of P_2's.

Example 2 The 2-regular graphs are precisely the disjoint unions of cycles (and the infinite path if infinite graphs are also under consideration).

Example 3 The first truly interesting examples of regular graphs can thus only be the *cubic* alias 3-*regular* graphs. The smallest such is K_4. This first opportunity for interesting regular graphs to occur is vindicated. For instance, some of the platonic solids can be represented by cubic graphs. More generally, convex polytope graphs – a rich subject – happen to be cubic. The problem of finding *cages* is about cubic graphs with a particular further property (Sec 38.8). Also e.g. the famous problem of finding snarks is about a more specialized class of cubic graphs (see Chapter 47.

Example 4 The following are all quartic = 4-regular. Pure carbon structures, the octahaedron (Fig 31.1.c), Feynman diagrams for internal processes of ϕ^4-Theory (Sec 40.6), and rubber band shadows (Sec 48.15).

Example 5 The icosahaedron graph (Fig 31.1.e) is 5-regular.

Exercise 3 Can you find a way of drawing each Platonic solid in the plane without crossed edges? Additionally using only straight lines?

[The answer to this is available overleaf in Fig 31.1.]

Exercise 4 a) What is the smallest k-regular graph for each k?

b) Find all the cubic graphs on each of 5 to 8 vertices.

c) Find all the quartic graphs on 5 and 6 vertices.

31.3 Degree vectors

Exercise 5$^-$ The *degree vector* of a graph is a list of its degrees, in descending order to put first the highest degree vertex information. This is

$$\text{a } |\,\mathrm{G}\,|\text{-vector, all of whose entries are integers} \quad \in [\,0, |\,G\,| - 1\,]\,. \tag{31.3}$$

Find the degree vectors of all graphs on 1 to 4 vertices, and of all connected graphs on 5 vertices.

Exercise 6 a) What is the shortest and numerically-smallest vector meeting description (31.3) that is not the degree vector of any graph?

b) What are the smallest graphs that fail to be distinguished by their degree vectors?

[Throughout the current Volume, 'smallest graph' is to be interpreted lexicographically, with order counting first and size counting second.]

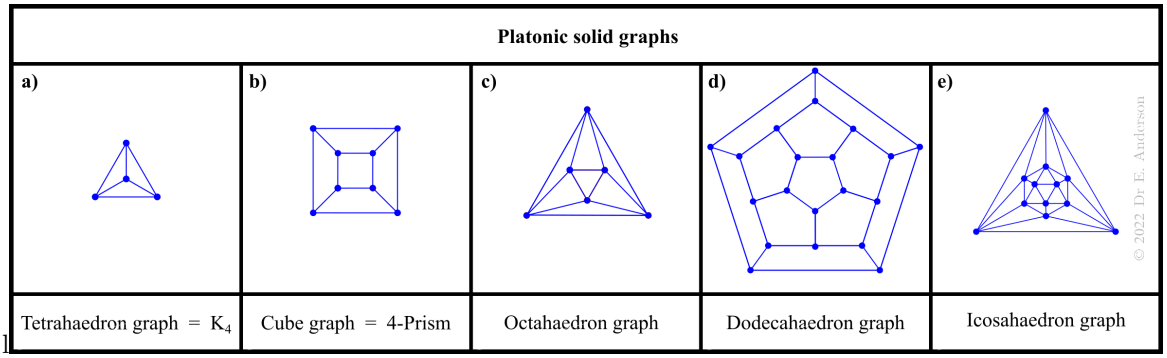


Figure 31.1:

31.4 Graph neighbourhoods

Degrees are a type of *local* information about graphs, while e.g. order and size are global information: about the graph as a whole. What does a graph look like 'up close'?

Definition 1 Given a vertex $v \in \mathrm{G}$, its *neighbourhood* consists of v itself alongside the vertices adjacent to a vertex $w \in \mathrm{G}$

$$\mathfrak{N}(w) := \{ w\, v \in \mathrm{G} \,|\, v \mathcal{A} w \} \tag{31.4}$$

of w.

Remark 1 Degree or valency can thus be rephrased as

$$d(w) = |\mathfrak{N}(w)| : \tag{31.5}$$

a notion of *local neighbourhood size*.

Exercise 7$^-$ Show that $v \in \mathfrak{N}(w)$ iff $w \in \mathfrak{N}(v)$.

Remark 2 δ and Δ can now also be interpreted as minimum and maximum neighbourhood size respectively.

Worked Example 1 [Visualizing graph neighbourhoods.] What are the neighbourhoods for the Fish graph in Fig Fish.a)?

Answer 1 Each neighbourhood is a star (marked in red in Figs 31.2.b-g). By including the other vertices, these are a 'closed' version of the more general notion of neighbourhood (c.f. [268]).

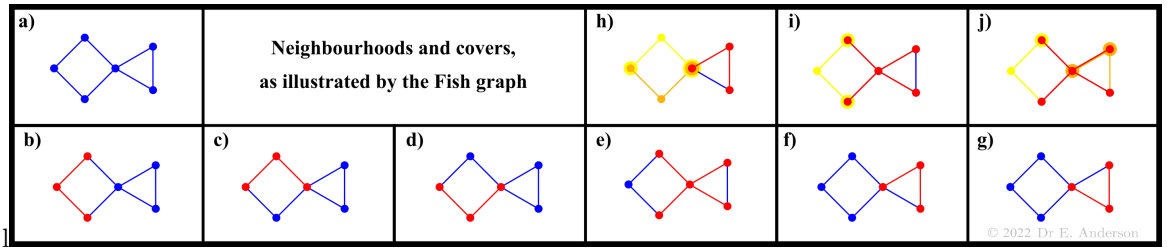


Figure 31.2:

Remark 3 Recollect from Chapter 9 that a *cover* is a set of subsets whose union is the whole underlying set. Fig 31.2.h-j) gives examples of covers. Observe that the one in Fig 31.2.i) consists of just two subsets. The Reader may wish to check that this is not unique, as well as giving an argument why there can be no cover of Fish by even fewer neighbourhoods. Fig 31.2.i)'s cover is thereby a nonunique minimal cover by neighbourhoods.

Exercise 8$^-$ What is the minimal cover by neighbourhoods of Bowtie? Banner? $K_{3,3}$? The cube graph in Fig 31.1.b)?

31.5 Exercises on graph isomorphisms

Remark 1 This is an opportune point at which to start practicing identifying isomorphic graphs among somewhat larger graphs. This is useful as regards recognizing (key sub)graphs in applications.

Exercise 9 Show that each of the graphs in Fig 31.3.a) is isomorphic to Utilities, and that each graph in Fig 31.3.b) is isomorphic to K_5.

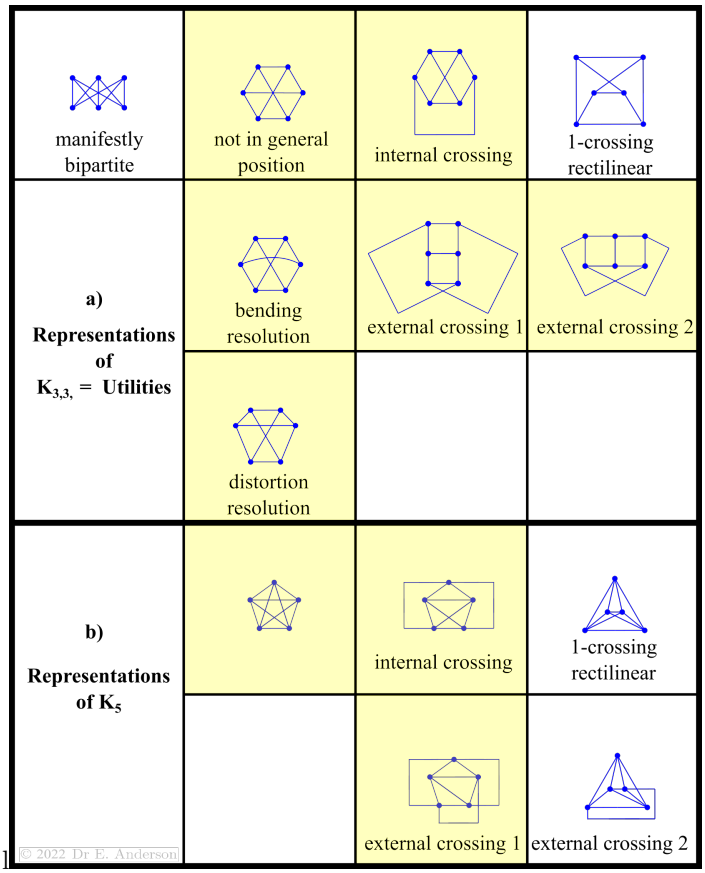


Figure 31.3:

Exercise 10 What name and presentation have we previously used for each graph in Fig 31.4?

Exercise 11 Which of the 6-graphs in Fig 31.5 are isomorphic? Which are isomorphic as [graphs]?

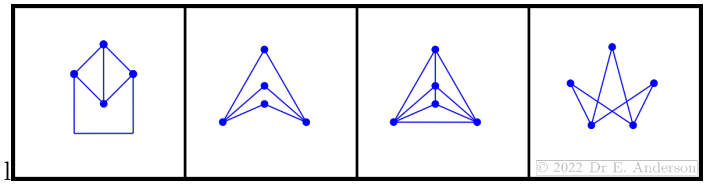


Figure 31.4:

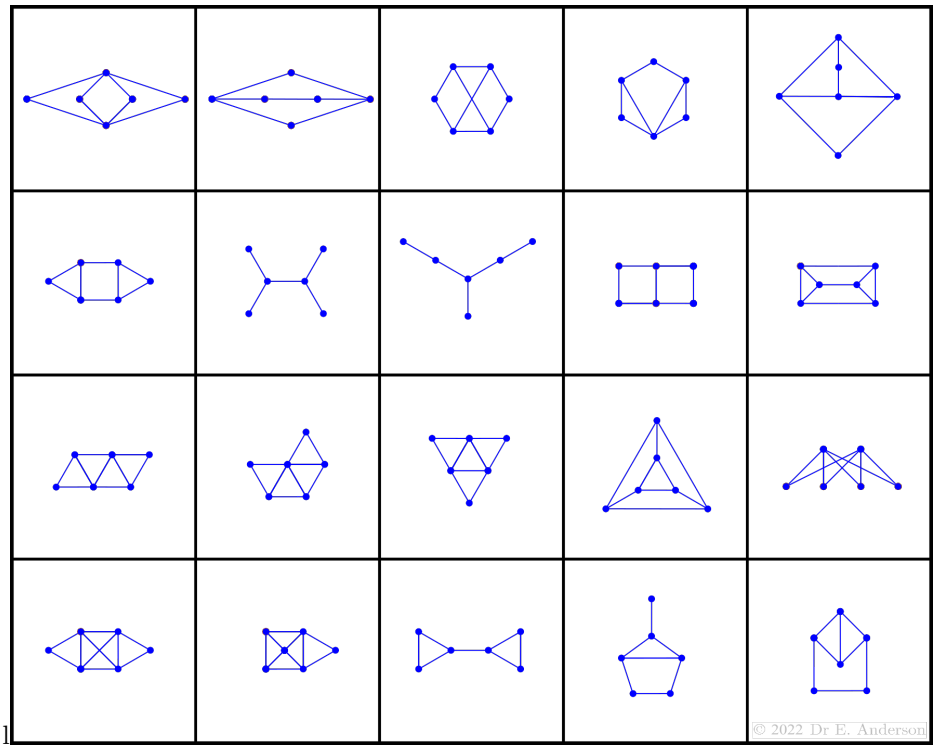


Figure 31.5:

Exercise 12 Show that last graphs in the last two columns of Fig 64.9 second graph are isomorphic to each other.

Exercise 13 Establish a separate graph isomorphism between the bottom two graphs in each of the following. Figs 60.12, 60.11 and 60.4.

Exercise 14^{+} [Long] Establish graph isomorphisms between each of the following pairs of presentations of graphs.

a) Fig 62.3.b)'s bottom two graphs, and again for Fig 62.3.c)'s bottom two graphs.
b) Fig 63.2.b)'s i) to iv).

c) Fig 64.2.c)'s bottom two graphs.

d) The bottom two graphs in Fig 64.6.c), and again for the bottom two in Fig 64.9.c).

e) The bottom two graphs in Fig 64.12.b)'s second column.

31.6 Local isomorphisms of graphs

A *local isomorphism* of graphs is a graph isomorphism that furthermore preserves each graph neighbourhood.

Example 1 P_3 and P_4 are related by a local isomorphism that splits P_3's interior vertex.

Example 2 There is no corresponding local isomorphism to go from Claw to Chair however, since Chair has a vertex of degree 2.

Exercise 15 Which 7-trees admit local isomorphisms to smaller trees?

31.7 Topologies on graphs

Structure 1 The discrete topology will not usually do, since it pairs vertices that do not have edges between them. It thus does not in general respect the adjacency structure of the graph. There are two exceptions: the complete graph does have edges between all vertices. And the discrete graph works as well, since it has *unedges* between all vertices, and unedges can for many purposes be envisaged as just edges of a different colour. So the discrete graphs, alias point clouds, can indeed be given the discrete topology.

Structure 0 The indiscrete topology can be given to any graph, though this is not structured enough to be of much interest.

Structure 2 Form a list of the graph's neighbourhoods. Then adjoin any unions of these, intersections of these, the whole graph and the empty graph to obtain a topology on the graph.

Worked Example 2 What is the topology thus induced on Claw?

Answer 2 The neighbourhood of each degree-1 vertex consists of it and the degree-3 vertex. The degree-3 vertex's neighbourhood is the whole graph. The degree-3 vertex by itself then arises as an intersection. Unions give all three 3-paths supported. Finally, the empty set must also be included (Fig 31.6).

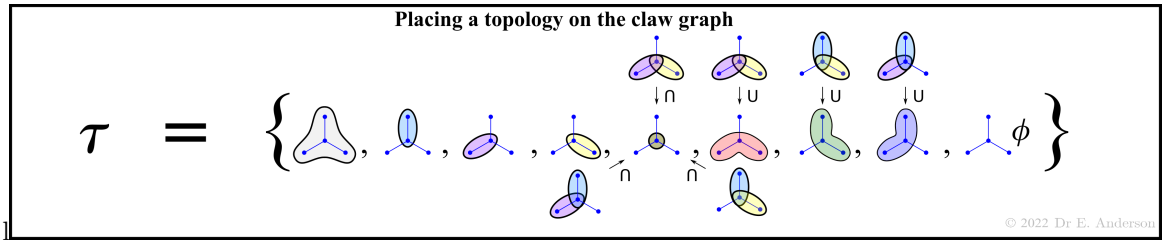


Figure 31.6:

Exercise 16 What is the topology thus induced on each of the Bird 6-tree and the Spiderhead 5-graph?

Structure 3 One could instead view each edge as a copy of (the image of) the real unit interval $[0, 1]$. For finite graphs, this is an example of *not* treating them in a finite way. Though as finite graphs still have a finite number of edges ($\leq E_{\max}$), finite and arbitrary unions remain equivalent.

31.8 Degree ratios

Let us next continue at the local level with our theme of ratios, relative quantities and relational variables.

Definition 1 The *fractional degree within a* graph is

$$f(v) \ = \ \frac{d(v)}{\Delta} \ .$$

Definition 2 The *realized degree ratio* of a given graph is

$$\mathcal{D} \ = \ \frac{\delta}{\Delta} \ .$$

Remark 1 This is 1 iff the graph is nontrivial and regular.

Definition 3 The *degree per unit size*

$$u \ := \ \frac{d(v)}{N} \ .$$

31.9 Relative degrees

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Definition 1 The *realized degree spread* of a fixed graph is

$$\vartriangle \ = \ \Delta \ - \ \delta \ .$$

Remark 1 Let us also introduce the following non-relative quantity.

Definition 2

$$\Sigma \ = \ \Delta \ + \ \delta \ .$$

Remark 2 This is quite often scaled by $\frac{1}{2}$ so as to pick out the centre of the spread.

♠ ◇ ♣

Remark 3 Two shifted degrees of note are as follows.

Definition 3 The *lower-shifted degree*

$$d_{\mathrm{l}}(v) \ := \ d(v) \ - \ \delta \ .$$

Definition 4 The *upper-shifted degree*

$$d_{\mathrm{u}}(v) \ := \ \Delta \ - \ d(v) \ .$$

Remark 4 There is of course one dependency between the above relative quantities,

$$d_{\mathrm{l}} \ + \ d_u(\Delta \ - \ d) \ + \ (d \ - \ \delta) \ = \ \Delta \ - \ d \ + \ d \ - \ \delta \ = \ \Delta \ - \ \delta \ = \ \vartriangle \ .$$

By this, we only make use of one of the shifted degrees.

Remark 5 A further non-relative quantity of note is as follows.

Definition 5 The *average degree*

$$\langle d \rangle \ := \ \frac{1}{|\,\mathrm{G}\,|} \sum_{v \, \in \, \mathrm{G}} d(v) \ = \ \frac{1}{N} \sum_{v \, \in \, \mathrm{G}} d(v) \ .$$

Definition 6 The '*unnormalized fluctuation*' is then

$$fluc(v) \ := \ d(v) \ - \ \langle d \rangle \ .$$

Remark 6 For graphs of a fixed regularity, however, degree is intensive so the degree per unit size variable makes little sense. Realized degree spread and degree fluctuation are also both zero in this regime.

31.10 Relational notions of degree

Definition 1 The '*average-normalized degree*'

$$\hat{d} \ = \ \frac{d}{\langle d \rangle} \ .$$

Definition 2 The *degree increment*

$$d_{\mathrm{I}} \ := \ \frac{\Delta}{\Delta} \ = \ \frac{(\Delta \ - \ \delta)}{\Delta} \ = \ 1 \ - \ \mathcal{D} \ .$$

Definition 3 The *half-normalized spread*

$$HNS \ := \ \frac{\Delta}{\Sigma} \ = \ \frac{\Delta \ - \ \delta}{\Delta \ + \ \delta} \ = \ \frac{1 \ - \ \mathcal{D}}{1 \ + \ \mathcal{D}} \ .$$

Definition 4 The *upper-shifted degree per maximal degree exhibited* is

$$\frac{\Delta \ - \ d(v)}{\Delta} \ = \ 1 \ - \ f(v) \ .$$

Definition 5 The *degree fluctuation per average degree*

$$f_{\mathrm{n}} \ := \ \frac{(\langle d \rangle \ - \ d)}{\langle d \rangle} \ = \ 1 \ - \ \hat{d} \ .$$

Remark 1 To run from 0 to 1 standard range, one would use instead the *shifted degree fluctuation*

$$\frac{\langle d \rangle \ - \ d \ - \ \delta}{\Delta} \ .$$

Remark 2 Our given notions of degree fluctuation provide unnormalized and normalized measures of inhomogeneity for the individual vertices of a graph. To quantify inhomogeneity as a property for the whole graph, however, we make use of the following.

♠ ◇ ♣

Definition 6 The *degree variance* of a graph is

$$\mathrm{Var}_d \ = \ \langle d^2 \rangle \ - \ \langle d \rangle^2 \ .$$

Remark 3 Perfect homogeneity in this sense is of course regularity. The question of maximal irregularity is however posed in the process of conceiving in terms of degree variance.

Exercise 17 a) Show that degree variance is a [graph] property.

b) Find the most irregular graphs on each of $N \ = \ 3, 4, 5$ vertices.

Exercise 18 Consider the following families of graphs. P_N, S_n, n-Pan, n-Fan, n-Range, n-Wheel, $\mathrm{K}_{p,q}$.

a) What are their average degrees,
their degree fluctuations per average degree, and their degree variances?

b) What happens to the variances of the first six families as $N \longrightarrow \infty$?

c) By a discussion in Appendix A, to compare irregularity between graphs with different N, *variance per unit order*,

$$\mathrm{var}_d(\mathrm{G}) := \frac{\mathrm{Var}_d(\mathrm{G})}{N^2}$$

is far more appropriate. What happens to this as $N \longrightarrow \infty$ for the first six families above?

d) Show that stars cease to dominate high irregularity for moderate N.

Exercise 19$^+$ a) Show that $0 \leq \mathrm{var}_d \leq$.

b) [Readers may benefit here from knowing a bit of Calculus] For the 2-parameter family $K_{p,q}$, show that the maximal irregularity var_d as $N \longrightarrow \infty$ is

$$b := \frac{3}{16} + \frac{1}{4\sqrt{2}} \simeq 0.364277 .$$

Thus $\mathrm{var}_d \ll 1$ gives a highly-homogeneous regime in the sense of low regularity. While $\mathrm{var}_d \lessapprox 1$ gives a highly-inhomogeneous regime, which continues to be realized at least to within a factor of 3 as $N \to \infty$. We also know that the maximal value for large N lies somewhere within $[\, b, 1\,)$.

Exercise 20 Define *size per unit order*

$$\eta(\mathrm{G}) := \frac{e(\mathrm{G})}{|\,\mathrm{G}\,|} .$$

Derive that

$$\eta = \frac{\langle\, \mathrm{d}\,\rangle_{\mathrm{G}}}{2} ,$$

$$\eta_{\max} = \frac{\Delta_{\max}}{2} ,$$

$$\eta_{\mathrm{crit}} = \frac{\Delta_{\mathrm{crit}}}{2} ,$$

and

$$\langle \delta \rangle_{\mathrm{G}} = \Delta_{\max} \rho .$$

31.11 Further shifted degree variables

Definition 1 A *branching point* for an arbitrary graph is a vertex of degree ≥ 3.

Structure 1

$$\text{(Branching point strength)} , \quad b := d - d_{\max}(\text{non- branching point}) = d - 2 .$$

Structure 2

$$\text{(Excess branching point strength)} , \quad e := d - d_{\min}(\text{branching point strength}) = d - 3 .$$

Remark 1 We make further use of this section's notions in Chapters 36, 38 and 44.

Chapter 32

Basic Combinatorics of graphs

32.1 Multi-regularity and sub-regularity

A graph is is $\underline{k}$-*multi-regular* if its vertices' degrees solely take the values of components of the 'multi-regularity vector' $\underline{k}$. It is k-*sub-regular* if all vertices' valencies $\leq k$.

Remark 1 All graphs are Δ-sub-regular; this is useful whenever our graph is finite. Multi-regularity can be particularly useful when the number of possible degrees involved is rather small.

32.2 Applications

Example 0 Some types of tree obey regularity conditions. For instance, paths are (1, 2)-regular and stars are (1, n)-regular. See Secs 36.1, 38.6, 36.8, 36.9 and 44.4 for further examples of trees with particular regularity conditions.

Exercise 1 [Chemistry] *Hydrocarbons* are chemical compounds consisting of hydrogen H and carbon C only. Let us for now consider simple graphs formed from these. These thus have to be (1 , 4)-regular *Alkanes* are those hydrocarbons that contain neither multiple bonds nor cycles.

a) Show that alkanes have molecular formulae of the form $C_k H_{2k+2}$. How many distinct alkanes are there for each of $k = 1$ to 7? Look up their chemical names. Via taking the corresponding carbon skeletons (omitting the hydrogen atoms in the case of hydrocarbons), which graphs we have encountered before are thus supplied with chemical aliases? What is the last 4-chain carbon skeleton alkane?

b) Show also that the single-cycle *cycloalkane* hydrocarbons have molecular formulae of the form $C_k H_{2k}$. What are the possible such on 3 to 7 vertices?

[Just work at the Graph-Theoretical level ignoring the possibility of some molecules being too strained to be (stably) realized. Unless you are doing Chemistry, or have done a degree in Chemistry, in which case you are also welcome to alter these counts accordingly to 'chemically allowed' counts.]

Example 1 [Physics: Quantum Field Theory] For reasons given in Sec 40.8 Feynman diagrams have valencies 3, 4 and possibly some terminal points (valency 1). Thus they are 4-sub-regular and, more precisely, (1, 3, 4)-regular.

Remark 1 (Multi-)regularity is thus a first characterizer of this Volume's 13 main examples of applications of graphs. See Chapters 39 and 44 for the other two main sources of diversity.

32.3 Euler's Degree-sum Theorem

Theorem 1.

$$2\,e \;=\; \sum_{w\,\in\,\mathrm{G}} d(w)\;. \tag{32.1}$$

For a k-sub-regular graph, this can furthermore be expressed in terms of degrees as follows.

$$2\,e \;=\; \sum_{i\,=\,1}^{k} i\,d(i)\;. \tag{32.2}$$

Here

$$d(i) \;:=\; \#(\text{ valency-}i\text{ vertices in }\mathfrak{V}(\mathrm{G}))\;.$$

Corollary 1 (Hand-shaking Lemma). In any party, the number of hands shaken must be even.

Corollary 2 i)

$$\delta(\mathrm{G}) \;\leq\; \left\lfloor \frac{2\,e(\mathrm{G})}{N} \right\rfloor\;. \tag{32.3}$$

ii)

$$\delta(\mathrm{G}) \;\geq\; \left\lceil \frac{2\,e(\mathrm{G})}{N} \right\rceil\;. \tag{32.4}$$

Corollary 3 The number of odd-degree vertices in a graph must be even. Proof

More generally, Euler's degree sum can be used to show that many hypothetical pairs of V, E do not permit regular graphs (or generalizations).

All of the above is set as **Exercise 2**; consider $N \;=\; 4,\,5,\,6$ K-regularities specifically.

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Exercise 3 a) Prove the Theorem Combinatorially.

[Hint: count out the degree sum vertex by vertex, but also find a way of obtaining this by counting up the number of edges.]

b) Deduce the above Corollaries.

Exercise 4 Show that regularity is a [graph] property.

Exercise 5 Which regularities can $N \;=\; 7, 8$ and 9 support? For each of these, provide an example or a nonexistence proof based on Euler's Degree Theorem for each.

Exercise 6^{-} Show that for $N \;>>\; k$, k-regular graphs are linearly-sparse. As are k-sub-regular graphs.

Exercise 7 How many entirely distinctly vertex-labelled simple graphs are there on N vertices?

Exercise 8 Find at least one homocomplementary graph on each of 8 and 9 vertices.

Chapter 33

Arrow-graphs: ori- and di- graphs

33.1 Introduction

Let us now return to Chapter 29's modelling considerations so as to set up further variants. There we introduced point clouds, and then passed to simple graphs by starting to allow edges between the points. We now permit our edges to carry one-way and two-way arrows, producing the following modelling variants. This installs modelling variant C) of Sec 29.14.

Definition 1 A *(purely) oriented graph* or *ori-graph* is a graph all of whose edges are *ordered* pairs, now termed *arcs*, the totality of which we denote by $\mathfrak{a}$.

Definition 2 The *reverse* of an arrow on an edge is the arrow pointing in the opposite direction along that edge.

Definition 3 If both arcs and their reverses are concurrently present on all edges, one has a *(pure) di-graph*.

Remark 1 Point clouds, simple graphs, ori-graphs and di-graphs are completed to an octet by the mixed-edge-type possibilities in Fig 33.1.

Remark 2 An arrow-graph of any of these kinds is *proper* if it nontrivially realizes at least one edge of each kind that it is named after. In studying arenas of mixed-edge-type graphs, however, we shall find it advantageous to include the trivial cases.

Definition 4 For use thoughout the current Series, let us introduce the term *arrow-graphs* to refer to the above octet, *plain graphs* for its arrowless case, and *nontrivial arrow-graphs* for its seven other cases.

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Remark 3 Forgetting the arrows, we return to an arrow-graph's underlying plain graph skeleton. For di-graphs, there is a choice of interpreting this as a 2-multi-graph (see Chapter 39). This choice entails remembering which edges carried two-way arrowheads.

Remark 4 Arrow-graphs can be envisaged as a subcase of labelling edges. This is moreover in excess of the usual unstructured colouring of edges, since directionality confers a further kind of structure.

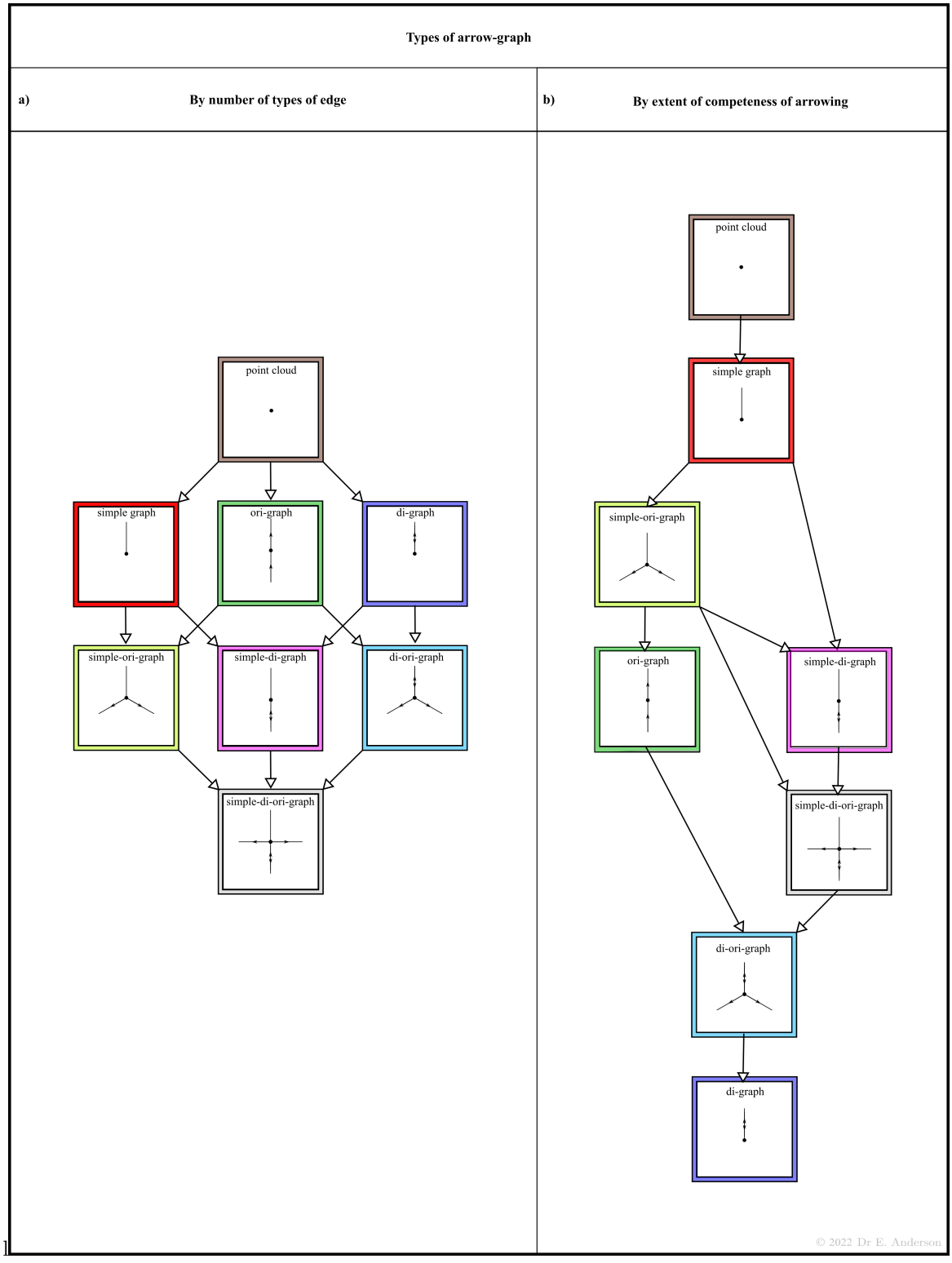


Figure 33.1:

Remark 5 One could furthermore label whichever of vertices and edges in other ways on top of deciding whether to arrow the edges. This pushes us up from an octet of possibilities to a $2^5 = 32$-

fold of modelling regimes (all the while disregarding the proper-improper distinction). Though as we know from Secs 8.7, 10.2, 13.9 and 22.1, such labels have their own sources of variety.

33.2 Motivation for arrow-graphs

Motivation 1 Flow charts, systems of reactions, Markov chains, decision processes, and classificatory keys can be viewed as directed graphs

Pointer 1 Many practical applications of trees, including in Computer Science, are directed or *rooted* [36]. Whenever a tree is rooted, it can be taken to have arcs flowing up from the root (if depicted at the bottom) or down from the root (if as more commonly depicted at the top).

Pointer 2 The possible outcomes in *tournaments* in which each of N contestants plays every other once can be modelled by considering all the ways in which the complete graph K_N can be oriented (Secs 33.11-33.13).

Pointer 3 Feynman diagrams for quantum particle processes are, more specifically, ori-graphs, with e.g. directed electron and positron edges and yet undirected photon edges.

Pointer 4 Posets can be viewed as a subcase of ori-graphs, by which almost everything in Part VII may be viewed as an ori-graph.

Modelling Limitation 1 Structural chemical formulae are however undirected. So are the most straightforward ways of associating graphs with Euclidean Geometry constructs and polytopes.

Pointer 5 While some models of knots use plain graphs, others use ori-graphs (see Sec 48.15).

33.3 Arrow-graph morphisms

Definition 1 An *arrow-graph morphism* is a map between arrow-graphs.

Definition 2 Arrow-graphs A and A' are *isomorphic* if there is a bijection

$$\phi \ : \ \mathfrak{V} \longrightarrow \mathfrak{V}'$$

that additionally preserves adjacencies to and from.

Definition 3 Arrow-graphs A and A' are *anti-isomorphic* if there is a bijection

$$\phi \ : \ \mathfrak{V} \longrightarrow \mathfrak{V}'$$

that additionally switches adjacencies to with adjacencies from.

I.e. an *arrow-reversing* morphism.

33.4 The smallest arrow-graphs

Example 1 There are only trivial ori-graphs on 0 and 1 vertices (rows 0 and 1 of Fig 33.2). [Nontrivial here requires that the ori-graph contains at least 1 edge.]

Example 2 There is only 1 nontrivial ori-graph on 2 vertices (row 2 of Fig 33.2).

Example 3 and Naming Remark 1 On 3 vertices, there are 6, as per row 3 of Fig 33.2.

Exercises 1-6 On 2 to 4 vertices, find all the connected nontrivial members of each of modelling classes ii) to vii).

Exercise 7 Find all the tree ori-graphs on 5 and 6 vertices.

33.5 Reversals

Definition 1 The *reversal* of an arrow-graph is the same underlying graph skeleton as in the original, but with all arcs on it reversed ('with arrows now pointing oppositely'). Let us use

$$A_1 \ \uparrow\longleftrightarrow\downarrow \ A_2$$

as a relation notation for this; as a subscript label, let us use 'rev'.

Definition 2 An arrow-graph is *homoreversing* if it is isomorphic to its own reversal. If not, it is one of a pair of *heteroreversing* arrow-graphs.

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Definition 3 An ⟨arrow-graph⟩ ⟨A⟩ is an arrow-graph A modulo arrow reversal. Everything in Fig 33.1 provides a more specialized version of this notion, which we denote similarly.

Notation Remark 1 Notice the arrowlike sides of this bracket, the more usual square brackets notation for quotients also already being taken up by graph complementation.

Remark 1 This gives a further example of C_2 quotient.

Remark 2 Due to its subsequent use, we now present our answer to Exercises 1 to 6 as figures organized according to behaviour under arrow-reversal. See here therefore for the smallest examples of connected homoreversing (yellow) and heteroreversing (grey and white pairs) arrow-graphs.

Exercise 8 a) Show that arrow-graph automorphisms are, more precisely, ⟨arrow-graph⟩ automorphisms.

b) Find the automorphism groups of each arrow-graph in these three Figures.

Remark 3

Di-graphs have the same diversity as graphs ,

differing from these just in having double-arrows on all edges.

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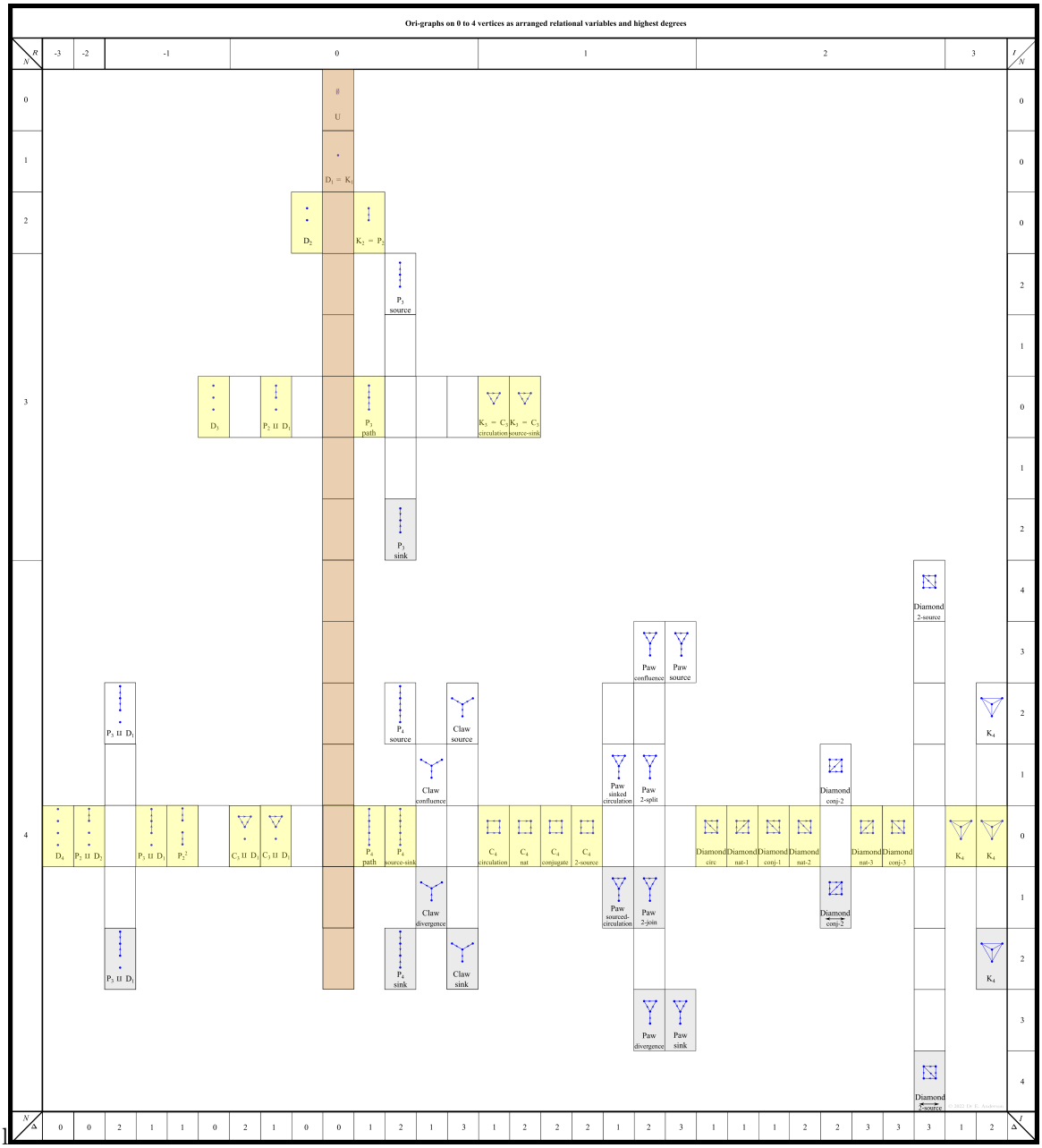


Figure 33.2:

33.6 Incidence, adjacency, and degrees for arrow-graphs

Definition 1 Suppose that an arc a's arrow points from vertex u to vertex v. Then we say that a is *incident from* u and *incident to* v. These give relations $\mathcal{I}^{\pm}$. Suppose that arc a is incident to some vertex v and arc b is incident from the same v. Then we say that b is adjacent from a and that a is adjacent to b. These give relations $\mathcal{A}^{\pm}$.

Definition 2 The *in-degree* $d_{-}(w)$ of a vertex w in an ori-graph is the number of arcs adjacent to w. The *out-degree* $d_{+}(w)$ of a vertex w in a ori-graph is the number of arcs adjacent from w.[1]

[1]Some authors use the names 'indegree' and 'outdegree', though as the diversity of modelling situations goes up,

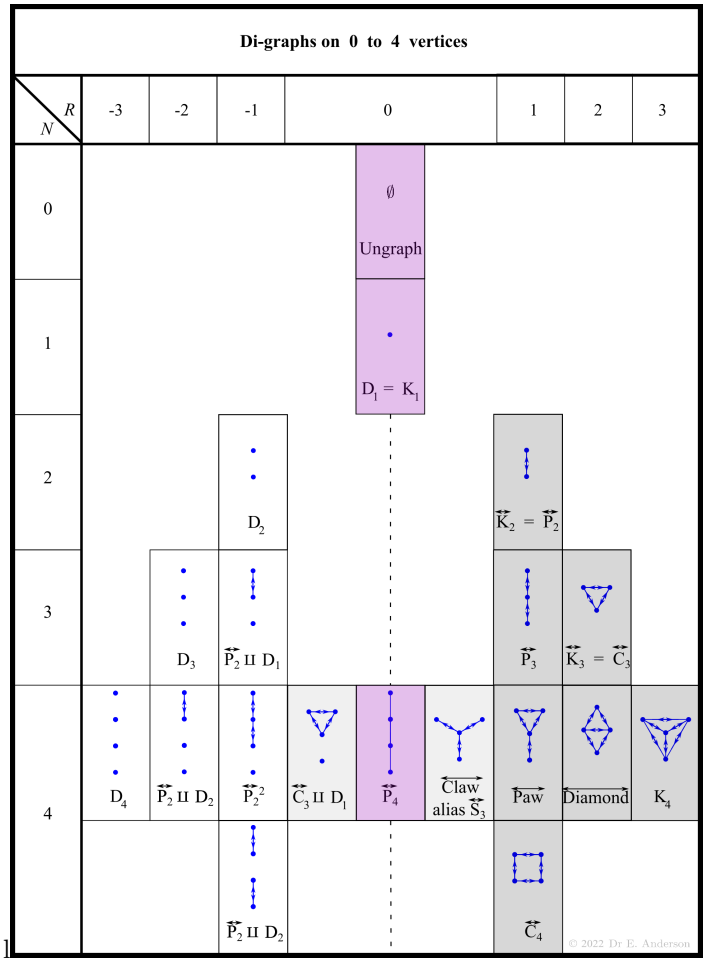


Figure 33.3:

Definition 3 An ori-graph's *minimal out-degree* and *in-degree* are

$$\delta_{\pm}(\mathrm{G}) \ := \ \min_{w \ \in \ \mathfrak{v}(\mathrm{G})} \ (\ d_{\pm}(w) \ ) \ .$$

Its *maximal out-degree* and *in-degree* are

$$\Delta_{\pm}(\mathrm{G}) \ := \ \max_{w \ \in \ \mathfrak{v}(\mathrm{G})} \ (\ d_{\pm}(w) \ ) \ .$$

Exercise 9 a) Find the shortest and numerically-smallest vector in positive integers that is not the in-degree vector of any arrow-graph.

b) What are the smallest graphs that fail to be distinguished by their in-degree and out-degree vectors?

Definition 4 The *in-neighbourhood* $\mathfrak{N}_{-}(v)$ of the vertex v in an ori-graph consists of the set of vertices adjacent to v alongside v itself. The *out-neighbourhood* $\mathfrak{N}_{+}(v)$ of the vertex v in an ori-graph consists of the set of vertices adjacent from v alongside v itself. When considered as punctured neighbourhoods – now excluding v itself – these partition the underlying graph skeleton's neighbourhood.

using (multi-)hyphenated prefixes just keeps on getting neater.

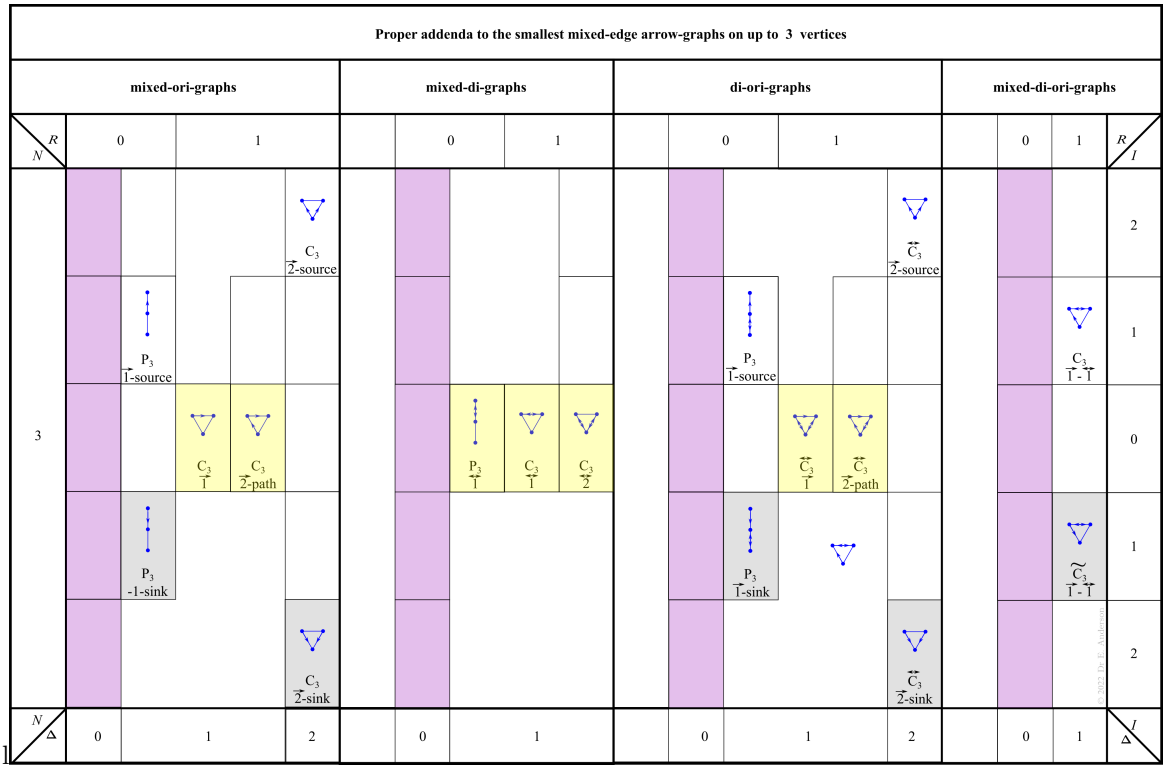


Figure 33.4:

I.e.

$$\mathfrak{N}_* \;=\; \mathfrak{N}_{*+} \;\amalg\; \mathfrak{N}_{*-}\;,$$

where the * here denotes puncturing (compare $\mathbb{C}_*$: the punctured complex plane).

Exercise 10 What is the topology induced by the out-neighbourhoods of the confluence divergence, source, and sink orientations of Claw?

♠ ◇ ♣

Structure 1 Further notions of degree supported by arrow-graphs are tabulated in Fig 33.5.

33.7 Maximal arc and edge counts for arrow-graphs

Remark 1 Maximal arc counts for ori-graphs coincide with maximal edge counts for simple graphs.

Remark 2 Maximal di-graph edge counts are double, since a single arc is allowed in each direction between two vertices. Thus here

$$E_{\max} \;=\; N\,(\,N\;-\;1\,)\;.$$

So also

$$E_{\mathrm{crit}} \;=\; \frac{E_{\max}}{2} \;=\; \frac{N\,(\,N\;-\;1\,)}{2}\;.$$

Unlike for simple graphs, these expressions always return integers, and so are always realized. Some consequent regimes are as follows.

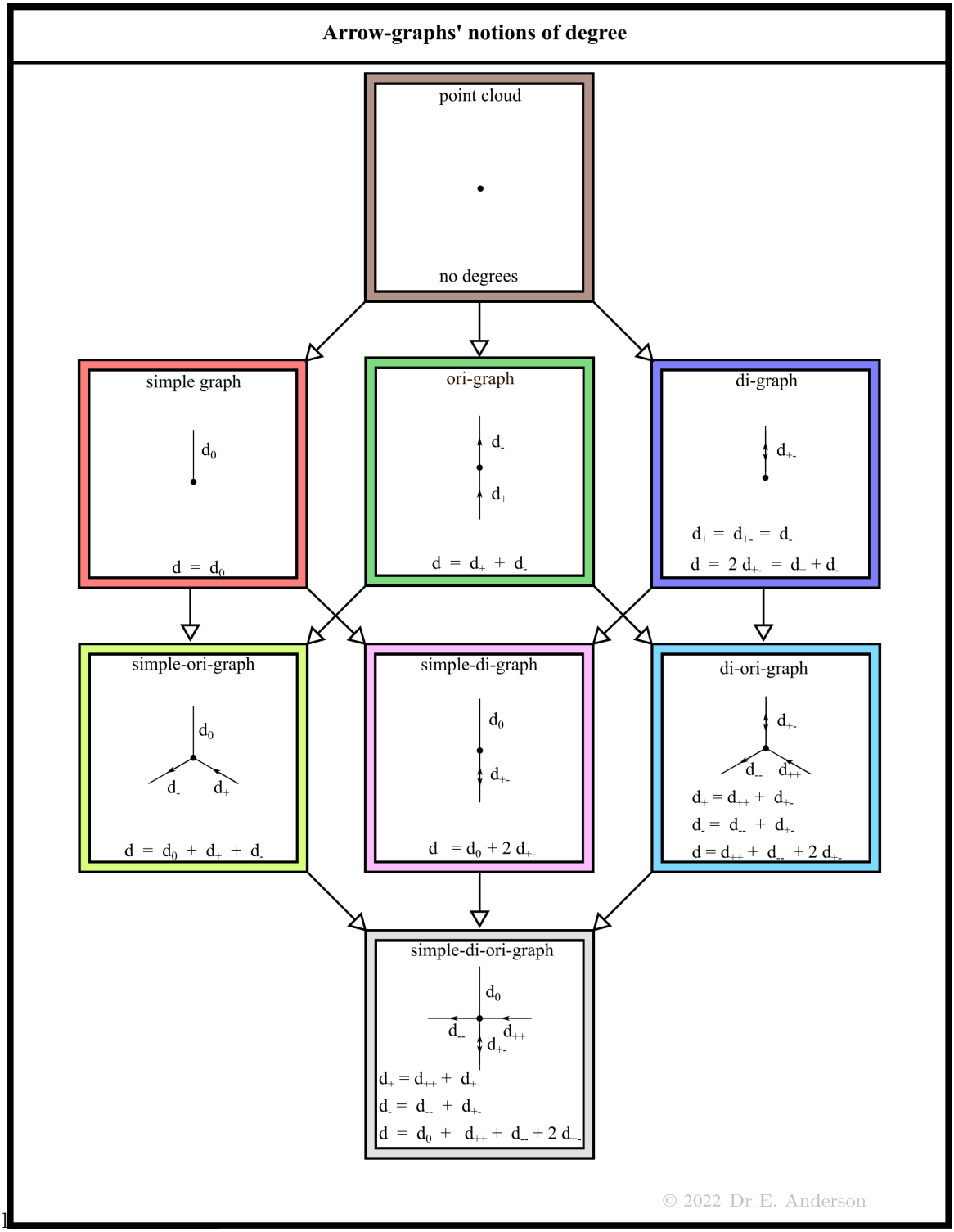


Figure 33.5:

1) The *dense regime*, for which

$$E \simeq E_{\max} \overset{\text{di-graphs}}{=} N(N-1) \simeq N^2 .$$

2) The sparse regime, for which

$$E \ll E_{\max} \overset{\text{di-graphs}}{=} N(N-1) \simeq N^2 .$$

3) The *critically-dense regime*, for which

$$E \simeq E_{\text{crit}} \overset{\text{di-graphs}}{=} \frac{N(N-1)}{2} \simeq \frac{N^2}{2} .$$

Exercise 11 a) What has replaced K_n as the maximally-edged graph for each of i) ori-graphs, ii) di-graphs, iii) simple-ori-graphs and iv) simple-di-graphs?

b) Find a homocomplementary graph among each of the i) to iv) on each of the following simple graph skeletons. I) P_4, II) K_4, III) C_5 and IV) Bull.

c) For each of the I.i) to IV.iv) hexadecuplet, can you find a homocomplementary graph that is also homoreversing?

33.8 Relational quantities for ori- and di-graphs

Aside from formulae where everything is now given a plus subscript, or everything is now given a minus subscript, for instance the following significant quantities ensue.

Structure 1 Unsigned Ramsey density now takes the form

$$U := |E - E_{\text{crit}}| \overset{\text{ori-graphs}}{=} \left| E - \frac{N(N-1)}{2} \right| .$$

Structure 2 The '*net flux index*' at a given vertex is

$$f := d_+ - d_- . \tag{33.1}$$

Structure 3 The *in-out ratio* is

$$\iota := \frac{d_+}{d_-} .$$

Structure 4 f pairs with d in a sum–difference way (c.f. the Euler ratio) to form the *normalized flux*

$$n := \frac{f}{d} = \frac{d_+ - d_-}{d_+ + d_-} = \frac{\iota - 1}{\iota + 1} . \tag{33.2}$$

Exercise 12 Formulate your own notions of relational variables for arrow-graphs more generally.

Of particular note,

$$(\text{arrowed–plain degree difference}) = d_+ + d_- - d_0$$

shall have counterparts with commentary in Volume 1 [441].

33.9 Basic Combinatorics of arrow-graphs

Theorem 1 (Euler's degree sum for arrow-graphs). We state this in the form of the modelling-case-wise equations in Fig 33.6.

Remark 1 It is not possible for in-degree sum to differ from out-degree sum: a matter of conservation.

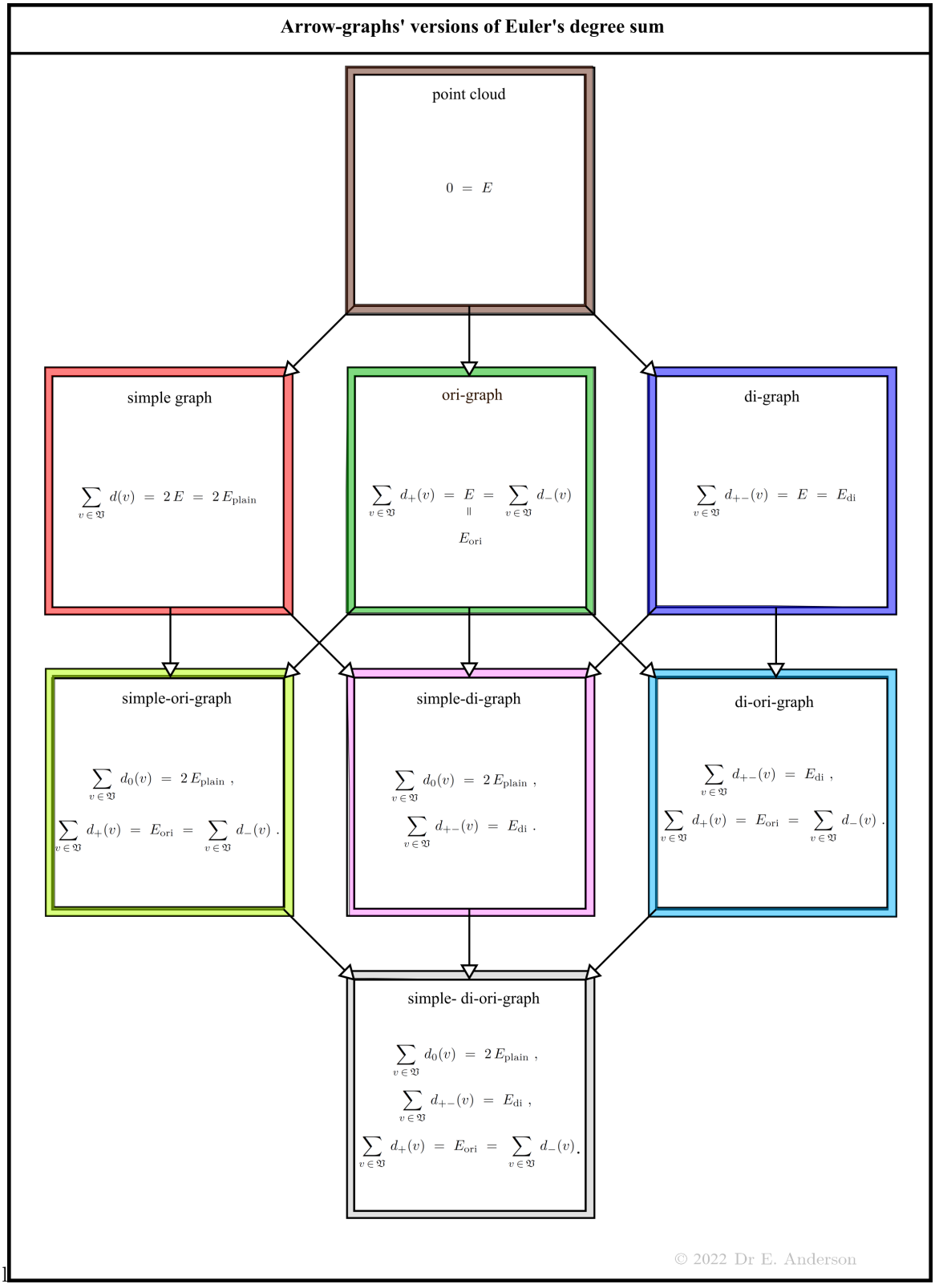

Arrow-graphs' versions of Euler's degree sum
point cloud
$0 = E$
simple graph
$\sum_{v \in \mathfrak{V}} d(v) = 2E = 2E_{\text{plain}}$
ori-graph
$\sum_{v \in \mathfrak{V}} d_+(v) = E = \sum_{v \in \mathfrak{V}} d_-(v)$
E_{ori}
di-graph
$\sum_{v \in \mathfrak{V}} d_{+-}(v) = E = E_{\text{di}}$
simple-ori-graph
$\sum_{v \in \mathfrak{V}} d_0(v) = 2E_{\text{plain}}$,
$\sum_{v \in \mathfrak{V}} d_+(v) = E_{\text{ori}} = \sum_{v \in \mathfrak{V}} d_-(v)$.
simple-di-graph
$\sum_{v \in \mathfrak{V}} d_0(v) = 2E_{\text{plain}}$,
$\sum_{v \in \mathfrak{V}} d_{+-}(v) = E_{\text{di}}$.
di-ori-graph
$\sum_{v \in \mathfrak{V}} d_{+-}(v) = E_{\text{di}}$,
$\sum_{v \in \mathfrak{V}} d_+(v) = E_{\text{ori}} = \sum_{v \in \mathfrak{V}} d_-(v)$.
simple- di-ori-graph
$\sum_{v \in \mathfrak{V}} d_0(v) = 2E_{\text{plain}}$,
$\sum_{v \in \mathfrak{V}} d_{+-}(v) = E_{\text{di}}$,
$\sum_{v \in \mathfrak{V}} d_+(v) = E_{\text{ori}} = \sum_{v \in \mathfrak{V}} d_-(v)$.
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Figure 33.6:

Exercise 13 a) Prove the remaining cases ii) to vii).

b) Which of these cases possess analogues of the Hand-shaking Lemma?

Exercise 14 a) Show that for each case in Fig 33.5 for which d_+ and d_- are defined,

$$\sum_{v \in \mathfrak{V}} d_+(v) \;=\; E_{\text{arrowed}} \;=\; \sum_{v \in \mathfrak{V}} d_-(v) \;.$$

[E_{arrowed} here being the number of arrowed edges of whichever kind.]

b) Show that for each case in Fig 33.6,

$$\sum_{v \in \mathfrak{V}} d(v) \;=\; 2\,E \;.$$

c) While a) and b) are useful checks-and-balances, why do they not however constitute further independent equations?

33.10 Notions of regularity for arrow-graphs

Definition 1 An ori-graph is *strongly k-regular* if all its vertices have in-degree and out-degree k. It is (-)-*k-regular* if all its vertices have the same in-degrees and (+)-*k-regular* if all its vertices have the same out-degees. These notions readily generalize to di-graphs and di-ori-graphs, by using the corresponding varieties of degree and their indices as prefixes of types of regularity. To include mixed-edge-type graphs, for strong k-regularity, we require all of in-degree, out-degree and plain-degree to be k throughout. An arrow-graph is (0)-*k-regular* alias *plain-k-regular* if all of its vertices have the same plain-degrees.

Remark 1 Each of the above notions of regularity has as obvious counterparts, notions of multi-regularity and of sub-regularity.

Exercise 15 Re-express each case of Euler's relation(s) in Fig 33.6 as an expansion in degrees (a particularly useful rearrangement whenever our arrow-graph is of low multi-regularity).

Exercise 16 i) Find an example of an ori-graph that is (+)-regular and yet not (-)-regular.

ii) Show that a graph is both (+)- and (-)-regular iff it is regular.

Remark 2 Notions like δ, Δ and average degree straightforwardly generalize to each type of degree.



33.11 Tournaments

Structure 1 *Tournaments* are ori-graphs obtained by assigning arrows to all edges of a complete graph K_n. Here every 'player' or 'team' contesting the tournament plays against every other precisely once. Our ori-graph model of this encodes wins by the directions in which the arrows point.

This model additionally assumes no draws: such are to be avoided by the likes of tie breaks, playoffs or penalty shootouts. These modelling assumptions correspond, more precisely, to a *round-robin tournament*.

Structure 2 Given a tournament, one can read off each participant's score, S. If wins contribute 1 point while losses contribute none, then the score is the out-degree of the team's vertex. Ordering these by non-decreasing size gives the tournament's ranking table.

33.12 Smallest examples of tournaments

Exercise 17 a) What are all the possible 2 to 5 team tournaments?

b) Which are homocomplementary?

c) Which are transitive?

[A tournament is *transitive* when A beats B and B beats C together imply that A beats C. These are acyclic orientations of complete graphs.]

33.13 Relational variables for tournaments

Structure 1 The *maximum score* of a team in the above tournament is

$$S_{\max} \;=\; N \;-\; 1 \;.$$

The *critical score* is

$$S_{\text{crit}} \;:=\; \frac{S_{\max}}{2} \;=\; \frac{N \;-\; 1}{2} \;.$$

The *total score* by all teams combined is

$$S_{\text{total}} \;=\; |\,\mathrm{K}_n\,| \;=\; E_{\max} \;=\; \frac{N\,(\,N \;-\; 1\,)}{2} \;.$$

The *average score* of a team is

$$\langle\, S \,\rangle \;:=\; \frac{S_{\text{total}}}{N} \;=\; \frac{N \;-\; 1}{2} \;=\; S_{\text{crit}} \;.$$

This provides a directly-meaningful interpretation for S_{crit}.

Structure 2 The *relative score* is

$$R \;=\; S \;-\; S_{\text{crit}} \;=\; S \;-\; \langle\, S \,\rangle \;,$$

thus being also the unnormalized score fluctuation. The *unsigned relative score* is

$$U \;=\; |\,S \;-\; S_{\text{crit}}\,| \;=\; |\,S \;-\; \langle\, S \,\rangle\,| \;.$$

Structure 3 The *proportion of the maximum score* is

$$s \;:=\; \frac{S}{S_{\max}} \;.$$

The *score fractions* – adding up to 1 – for the ith team are the alternative natural normalization

$$s_i \;:=\; \frac{s_i}{S_{\text{total}}} \;.$$

Structure 4 The *relational score* is

$$r \;:=\; \frac{S \;-\; S_{\text{crit}}}{S_{\text{crit}}} \;=\; \frac{S \;-\; \langle\, S \,\rangle}{\langle\, S \,\rangle} \;=\; 2\,s \;-\; 1 \;,$$

by the second form of which it is also the average-normalized fluctuation. This runs from -1 to 1.

The *unsigned relational score* is

$$u \;=\; \frac{|\,S \;-\; S_{\text{crit}}\,|}{S_{\text{crit}}} \;=\; \frac{|\,S \;-\; \langle\, S\,\rangle\,|}{\langle\, S\rangle} \;=\; |\,2\,s \;-\; 1\,|$$

This runs over the standard range.

♠ ◇ ♣

Remark 1 The above are variables for individual teams, and not for whole tournaments.

Structure 5 One way of qualifying whole tournaments is by the extent to which their scores are uniform. A relative quantifier for this is the extent to which the tournament scores depart from reflecting into themselves about the average score.

Structure 6 Another quantifier for whole tournaments is the variance of the scores. When this is 0, the tournament is perfectly uniform, corresponding to a 'perfectly level playing field' among the teams. So as to be able to compare uniformities between tournaments with different numbers of teams in each, we can use

$$var(s) \;:=\; \frac{Var(s)}{N}\;.$$

Remark 2 These and $s_{\max}$ turn out to suffice to parametrize all possible tournaments for up to 5 teams.

33.14 Vertical notation for tournaments

Structure 1 Here we only draw a subset of arrows; all arrows not drawn flow down the page. Some particular cycle-forming downward arrows are kept. By their sparseness, these depictions are rather more readable than depictions including all the arrows would be.[2]

Structure 2 Departure from reversibility, variance and the number of arrows in the vertical depiction serve to parametrize tournaments on 0 to 5 vertices. See Fig 33.7.a) for these tournaments and Fig 33.7.b) for the corresponding ⟨tournaments⟩.

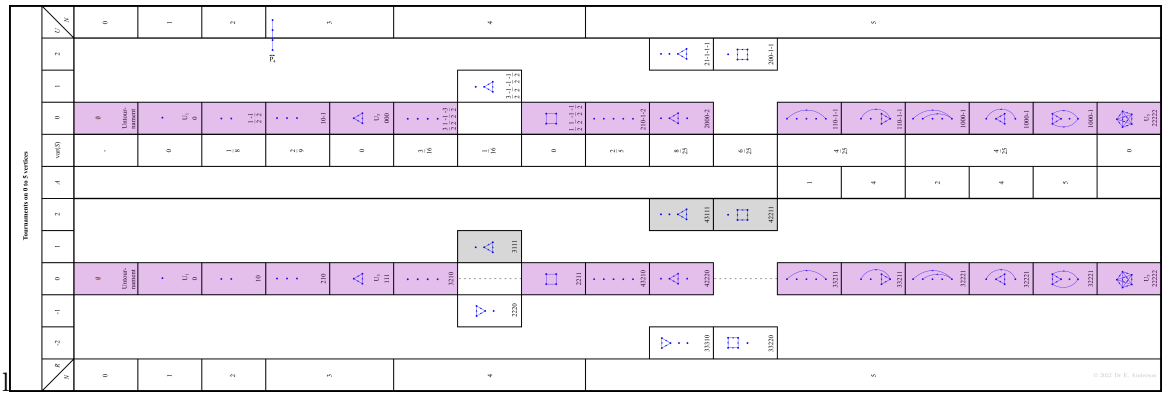


Figure 33.7:

[2] [182] credits M. Goldberg's 1966 Thesis when displaying this notation.

33.15 Exercises on tournaments

Exercise 18 How many labelled tournaments are there on each of 2 to 5 vertices?

Exercise 19 Show that the above three variables do not however suffice to parametrize the tournaments on $N = 6$ vertices.

Exercise 20 Extend this model of tournaments to allow for draws by use of simple as well as oriented edges. Now how many tournament outcomes are possible between five teams?

Exercise 21 Extend our tournament model instead to model each of the following.

a) 4 tennis players from different cities to play each other on both a home court and an away court.

b) 4 football teams who each have their own stadium to play each other under football league rules both at home and away.

[Tennis has tie breaks, while you can take football league rules to permit draws. In each case, count up all the possible tournament outcomes. For b), what are the possible final league points tables if one point is awarded for a draw and i) 2 for a win? ii) 3 for a win?]

Chapter 34

New graphs from old

34.1 Subgraphs

Definition 1 $G' = (\mathfrak{V}', \mathfrak{e}')$ is a *subgraph* of $G = (\mathfrak{V}, \mathfrak{e})$ if $\mathfrak{V}' \subseteq \mathfrak{V}$ and $\mathfrak{e}' \subseteq \mathfrak{e}$. We then write $G' \subseteq G$.

Definition 2 Suppose that a subgraph G' contains all edges of G that join vertices of $\mathfrak{V}'$. Then G' is called the subgraph *induced by* $\mathfrak{V}'$, $G[\mathfrak{V}']$.

Remark 1 The condition for a subgraph to be an induced subgraph is thus

$$G' = G[\mathfrak{V}'] \ .$$

♠ ◇ ♣

Example 1 The connected components of a graph are subgraphs thereof.

Example 2 Bi-partite graphs are built upon a choice of two-piece partition of the graph's vertices. These can be viewed as a D_p subgraph and a D_q subgraph, where $p + q = N$.

Example 3 Every graph contains tree subgraphs. And every connected graph G contains at least one *spanning tree*: with the same number of vertices as G, yet usually with fewer edges. See Sec 36.14 for more about spanning trees.

Exercise 1 a) What is the smallest graph admitting nonisomorphic spanning trees?

b) How many non-isomorphic spanning trees does the octahaedron graph of Fig 31.1.c) contain?

Exercise 2 Show that bi- and multi-partiteness are hereditary.

34.2 Paths, trails, walks, cycles, circuits, tours

Definition 1 A *path* is a graph P whose edges are of the form $w_0\, w_1$, $w_1\, w_2$, ... , $w_{n-1}\, w_n$, where the underlying vertex set is $\mathfrak{V} = \{\, w_i \,|\, i = 0 \text{ to } n \,\}$. $w_0\, w_1 \ldots w_n$ is a common shorthand. w_0 and w_n are the path's *end-vertices* alias *terminal vertices*.

Definition 2 A *trail* is a graph T which is the generalization of path in which some of the vertices can recur, while its edges however remain distinct.

Definition 3 A *walk* is the further generalization in which not all edges are necessarily distinct.

Remark 1 If edges recur, then so must their endpoint vertices. Thus one can not obtain a fourth notion by allowing for edges to recur while requiring that vertices do not.

♠ ◇ ♣

Definition 4 A *cycle* is a walk with 3 or more edges, $w_0 = w_n$, and w_i for $0 < i < n$ all mutually distinct and distinct from w_0.

Definition 5 A *circuit* is a closed trail, i.e. one with $w_0 = w_n$.

Definition 6 A *tour* is a closed walk.

Remark 2 All of this carries over to ori-graphs, provided that the arrows on the edges involved line up head to tail.

♠ ◇ ♣

Worked Example 1 What are the smallest nontrivial trail, walk, circuit and tour? What is the smallest nontrivial tour which includes a nontrivial cycle?

Answer 1 To be nontrivial, a trail should be neither a path nor a cycle. It is not a path by making two uses of at least one vertex. It cannot however just be a loop, since this would be a cycle. Nor a two-way arrow, since this would be both a cycle and a repeat use of an edge, which meets the definition of a walk, rather. It can contain a 3-cycle, though this cannot be its entirety so as to avoid being solely a cycle. So the smallest possibility is the directed Paw graph of Fig 34.1.a).

To be a nontrivial walk, it must not be a trail or a tour. This precludes the loop and the 2-way arrow. P_3 with an arrow leading to a 2-way arrow will however do (Fig 34.1.b).

To be a nontrivial circuit, it must not be a cycle or a tour. This produces multiple cycles. Were any of these 1- or 2-cycles, one would have a tour. 3-cycles are however unrestricted by our considerations. The minimal possibility is thus to have two of these sharing a single vertex. I.e. a directed graph on the Bowtie graph skeleton (Fig 34.1.c).

The above argument points to a single vertex carrying a double-arrowed loop as constituting the smallest nontrivial tour (Fig 34.1.d).

To be a nontrivial tour containing a nontrivial cycle, an oriented C_3 sub-arrow-graph needs to be present. As does a two-way arrow. Double-arrowing one of the cycle's edges is however in breach of starting and ending at the same vertex (Fig 34.1.e). Double-arrowing two of the cycle's edges still will not do (Fig 34.1.f). Double-arrowing all three would do (Fig 34.1.g), but there are two more minimal possibilities. Firstly, triple-arrowing one of the cycles (Fig 34.1.h). Secondly, within the confines of at most double-arrowing edges, we could add a double-arrowed pan-handle to our oriented cycle (Fig 34.1.i).

Naming Remark 1 Paths are linearly-ordered: Graph Theory's name for chains. *Trail-blazing* sets up subsequent points to be found, but not the trajectory that one is to move along on in finding them. Walks can 'use the same bridge' or involve going in and out of somewhere along the same piece of lane. Each of paths, trails and walks are generally from one place to another, rather than ending where they start.

Circuit training can visit a given piece of gym apparatus more than once, while walking around between such is unrestricted and thus generically not along identical edges. For this to be a graph

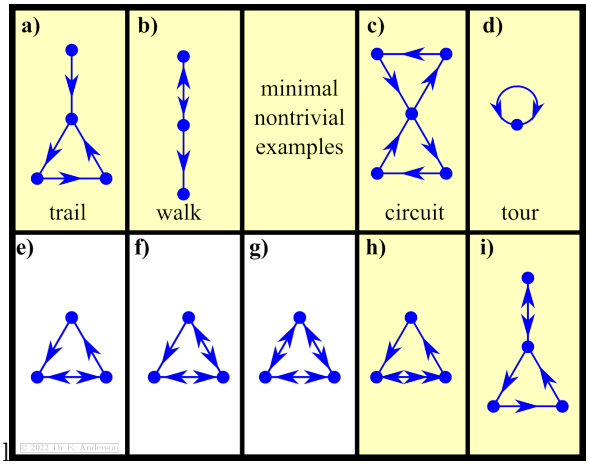


Figure 34.1:

circuit, it is to be in a gym with one door, which is then a (distinguished) common start-and-end point *Tourist tours* start and end where the bus is parked. They can involve using the same road to get to and from one of the more isolated locations, say a village church in what is elsewise a tour of a town. While these statements might help you remember which is which, everyday language is less precise. It is thus possible to come up with similar sayings which do not however match up with the precise Graph Theory terminology.

Exercise 3^{-} a) How many nontrivial path subgraphs are there in the graph of Fig 34.2.a)?

b) How many undirected circuit subgraphs are there in the graph of Fig 34.2.b) that make use of each edge at most once?

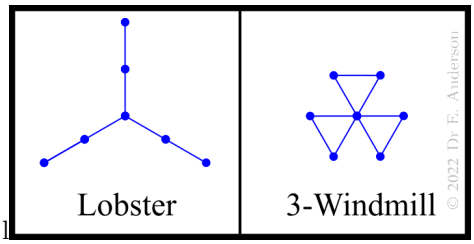


Figure 34.2:

Exercise 4 What is the smallest circuit that is not a cycle? What is the smallest trail that is neither a path nor a circuit?

34.3 Relative quantifiers for subgraphs

Definition 1 The *relative order*

$$\mathfrak{V}(\mathrm{G} - \mathrm{H}) = |\mathrm{G}| - |\mathrm{H}| .$$

Definition 2 The *relative size*

$$E(\mathrm{G} - \mathrm{H}) = E(\mathrm{G}) - E(\mathrm{H}) .$$

Remark 1 In each case add mod bars for the unsigned version, which is then symmetric between two not necessarily nested graphs.

34.4 Ratio quantities

Definition 1 The *order ratio*

$$\mathfrak{V}(\,\mathrm{H}\,|\,\mathrm{G}\,) \;=\; \frac{|\,\mathrm{H}\,|}{|\,\mathrm{G}\,|}\;.$$

Definition 2 The *relative size*

$$E(\,\mathrm{H}\,|\,\mathrm{G}\,) \;=\; \frac{E(\mathrm{H})}{E(\mathrm{G})}\;.$$

Remark 1 Here our own notation $(\,\mathrm{H}\,|\,\mathrm{G}\,)$ reads 'of H within G.

Definition 3 The *symmetric order ratio* of two graphs G, H is as follows.

$$\mathfrak{V}_{\mathrm{S}}(\,\mathrm{H}\,|\,\mathrm{G}\,) \;:=\; \frac{1}{2}\left(\frac{|\,\mathrm{H}\,|}{|\,\mathrm{G}\,|} \;+\; \frac{|\,\mathrm{H}\,|}{|\,\mathrm{G}\,|}\right)\;.$$

Remark 2 The factor of $1/2$ here ensures that this expression returns 1 when $\mathrm{G} \;=\; \mathrm{H}$.

Structure 1 The *symmetric order discrepancy* of two graphs G, H is as follows.

$$V_{\mathrm{D}}(\,\mathrm{H}\,|\,\mathrm{G}\,) \;:=\; V_{\mathrm{S}}(\,\mathrm{H}\,|\,\mathrm{G}\,) \;-\; 1 \;=\; \frac{1}{2}\left|\sqrt{\frac{|\,\mathrm{H}\,|}{|\,\mathrm{G}\,|}} \;-\; \sqrt{\frac{|\,\mathrm{H}\,|}{|\,\mathrm{G}\,|}}\right|^2 \;=\; \frac{1}{2}\left(\frac{(\,\sqrt{|\,\mathrm{H}\,|} \;-\; \sqrt{|\,\mathrm{G}\,|}\,)^2}{|\,\mathrm{H}\,|\,|\,\mathrm{G}\,|}\right)\;.$$

Remark 3 Symmetric size ratio and discrepancy are defined similarly.

34.5 Forced and forbidden subgraph pointers

Quite a large proportion of Graph Theory's significant results concern subgraphs. In particular, many results are about forbidden subgraphs that various modelling assumptions imply. We postpone giving examples of this until Chapter 43.



34.6 Union and Disjunction graphs

Remark 1 The union of 2 graphs consists of the union of their vertices and of their edges.

Structure 1 Let G_1 and G_2 be graphs. Then so is $\mathrm{G} \;=\; \mathrm{G}_1 \;\amalg\; \mathrm{G}_2$: the *disjoint* union of G_1 and G_2. G here realizes a partition into G_1 and G_2.

Exercise 5$^-$ What is the effect of disjoint unions on the number of components?

34.7 Cones and suspensions of graphs

Exercise 6 The *cone* C(G) *over a graph* G is a graph consisting of the following. G alongside one extra vertex o incident to all of G's vertices. Let us also call o the *cone point* of C(G). From the definition of cone, it follows that $d(o) \;=\; |\,\mathrm{G}\,|$.

a) What are the cones over the point clouds, the paths, the cycles, the stars, and the complete graphs?

d) Which graphs on 1 to 5 vertices can be viewed as cones over other graphs?

Exercise 7 The *suspension* S(G) *over a graph* G is a graph consisting of the following. G alongside two extra vertices s_i, $i = 1, 2$ incident to all of G's vertices but not to each other. We also call the s_i the *suspension points* of S(G). From the definition of suspension, it follows that $d(s_i) = |\,G\,|$.

a) What are the suspensions over the point clouds, the paths, the cycles, and the complete graphs?

d) Which graphs on 1 to 5 vertices can be viewed as suspensions over other graphs?

Pointer 1 Manifold and topological space counterparts of this notion of cone are quite prominent, as are suspensions on topological spaces [261].

34.8 Product graphs. I. Cartesian product □

Definition 1 Let G_1 and G_2 be graphs. The *Cartesian product* of G_1 and G_2 is a graph $G_1 \,\square\, G_2$ such that the following hold.

i)
$$\mathfrak{V}(G_1, G_2) = \mathfrak{V}(G_1) \times \mathfrak{V}(G_2) .$$

ii) The product vertices (v_1, v_2) and (w_1, w_2) are adjacent if either of the following hold.

a) $v_1 = w_1$ and $v_2 \mathrel{\mathcal{A}} w_2$ in G_2.

b) The $1 \leftrightarrow 2$ of this.

Exercise 8 Show that □ is both commutative and associative.

♠ ◇ ♣

Example 1
$$D_1 \,\square\, D_1 = D_1 .$$

Example 2
$$G \,\square\, D_1 = G ,$$

by which D_1 is this product's unit.

Example 3
$$P_2 \,\square\, P_2 = C_4 .$$

Example 4
$$P_2 \,\square\, P_n = L_n :$$

the *n-rung ladder graph* (Fig 34.3.d). $L_1 = P_2$ and $L_2 = C_4$; the smallest nontrivial ladder graph is $L_2 =$ Domino $=$ Theta.

Example 5
$$P_2 \,\square\, C_n = \text{Prism}_n :$$

the *n-prism graph* (Fig 34.6.a).

Example 6
$$P_n \,\square\, P_m = \text{Grid}(m, n) :$$

the *n by m grid graph* (Fig 34.3.f). $\text{Grid}(1, n) = P_n$ and $\text{Grid}(2, n) = \text{Ladder}_n$. The smallest grid that is not a ladder is
$$\text{Grid}(3, 3) = \text{Windowpane} .$$

Exercise 9 What are $P_3 \,\square\, C_3$, $P_m \,\square\, C_n$, $P_2 \,\square\, G$, $C_3 \,\square\, C_3$, $C_3 \,\square$ Paw and $K_4 \,\square\, K_4$?

Worked Example 2 a) Draw the *hypercube graphs*

$$\text{Hypercube}(n) \;=\; \square_{i\,=\,1}^{n} P_2$$

for $n = 1$ to 4. Under which names have you seen the first three of these before? [Also take Hypercube(0) to be just D_1: an interpretation of the *empty* $\square$ product.] Explore Hypercube(4) by giving multiple qualitatively-distinct presentations of it.

b) The *hypergrid graphs* are

$$\text{Hypergrid}(\underline{b}) \;=\; \square_{i\,=\,1}^{n} P_{b_i\,+\,1}\ .$$

Under which names have Hypergrid(m), Hypergrid(2, n), Hypergrid(m, n) and Hypergrid ($\underline{2}$) already featured in this Volume? Draw the three smallest-order Hypergrid ($\underline{b}$) that have not yet featured in this Volume.

Answer 2 See Fig 34.3.

Remark 1 Among these hypergrids that are not just grids, we name a further simple family Boxstack(n). For

$$\text{Boxstack}(2) \;:=\; \text{Hypergrid}(3,\,2,\,2) \;=\; P_3 \,\square\, C_4$$

will play a role in Part VIII. We also squash flat this stack of boxes to obtain a useful planar presentation.

Exercise 10 What are Aut(Boxstack(2)), Aut(Boxstack(3)), Aut(Hypergrid(3, 3, 2)) and Aut(Hypercube(4))?

34.9 II. $\times$ product

Remark 1 Graphs support a second natural and simple product, as follows.

Definition 2 The *direct product* of G_1 and G_2 is a graph $G_1 \times G_2$ such that i) and the following hold.

iii) $(v_1, v_2) \mathcal{A} (v_1', v_2')$ iff $v_1 \mathcal{A} v_1'$ in G_1 and $v_2 \mathcal{A} v_2'$ in G_2

Exercise 11 Show that this $\times$ operation is also commutative and associative.

♠ ◇ ♣

Example 0

$$D_1 \times D_1 \;=\; D_1\ .$$

Example 1

$$G \times D_1 \;=\; G\ ,$$

so D_1 is also this product's identity.

Example 2 As a first distinction between $\square$ and $\times$,

$$P_2 \times P_2 \;=\; {P_2}^2 \;\neq\; C_4 \;=\; P_2 \,\square\, P_2\ .$$

In fact, $P_n \times P_2 = {P_n}^2$, realized as a bi-partite double cover.

Example 3 $C_n \times P_2$ is similarly a 'dull double' ${C_n}^2$ for n even, while it is just $C_{2\,n}$ for n odd.

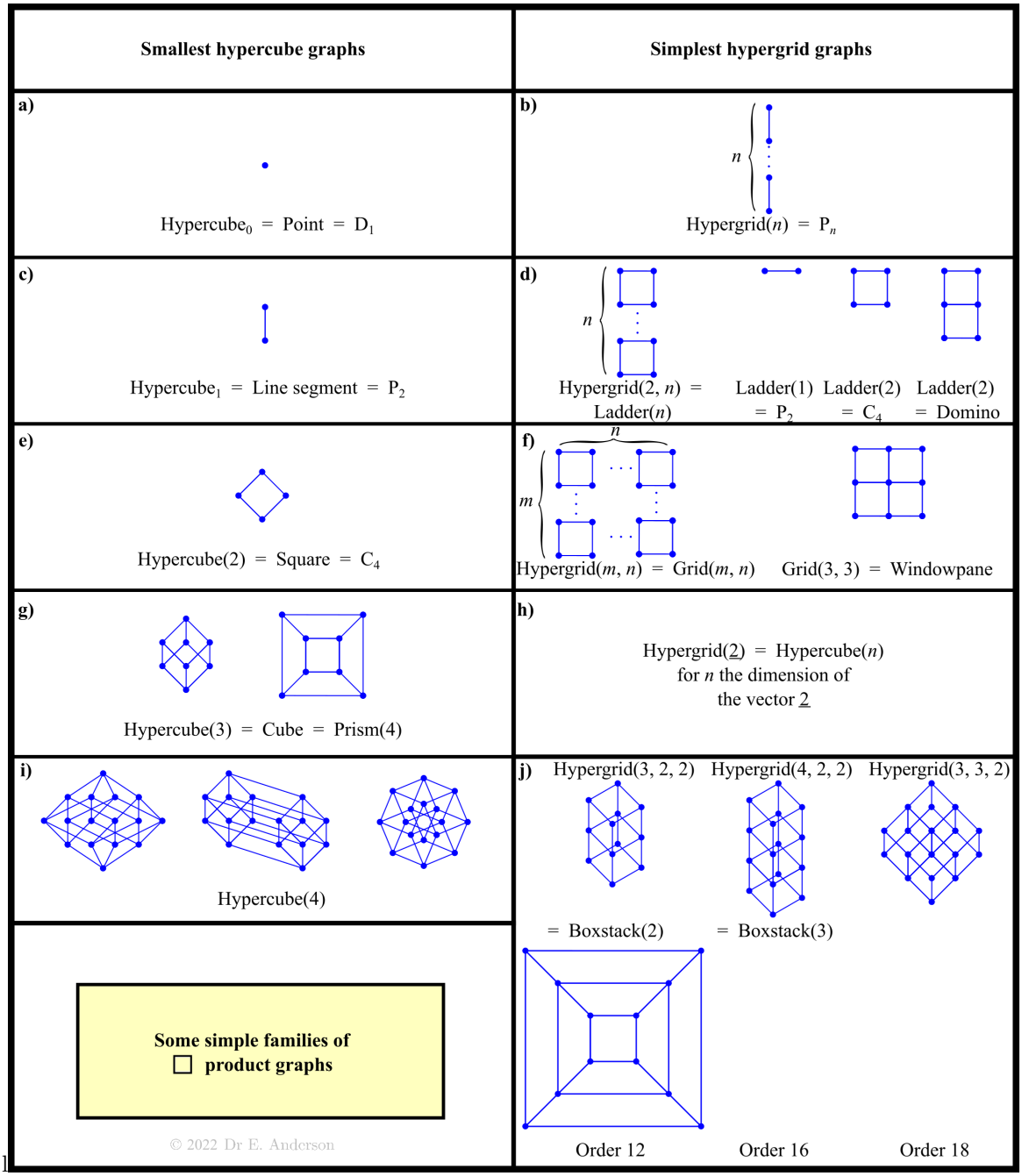


Figure 34.3:

Example 4 $K_4 \times P_2 = K_{4,4}$ minus a perfect matching's worth of edges, as per Fig 34.4.a). This gives the Crown(4) graph, a clearer presentation of which is given in Fig 34.4.b). $K_n \times P_2$ is one of the ways of defining the Crown(n) graph.

Exercise 12$^-$ Show that $P_2 \times G$ is a bi-partite double cover.

Exercise 13 a) Figure out what the 3-crown graph is also known as. Find b) $P_3 \times P_3$, c) $C_3 \times P_3$ and d) $C_3 \times C_3$.

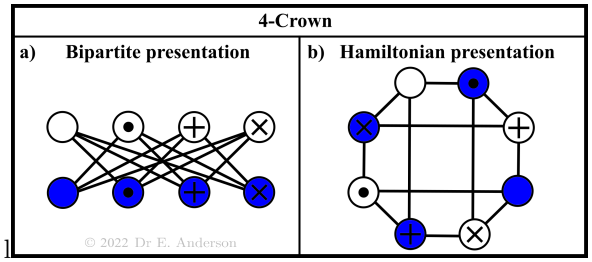


Figure 34.4:

34.10 Concatenations

Structure 1 The *concatenation* of two paths is another path, formed as follows. Add an edge (black in Figure 34.5.a.1) from the last vertex of the first path (blue) to the first vertex of the second path (red). Similar applies for trails in place of paths.

Structure 2 The *concatenation* of two circuits (blue and red in Fig 34.5.b.0) that pass through a common vertex v (purple) is a circuit consisting of the following. Sweep around the first circuit starting and ending at the common point. Follow this by sweeping around the second circuit starting and ending at the common point as well.

Suppose that one of our circuits forms more than one cycle about the common vertex (as the red one does in our Figure). Then sweep around these in the concatenated circuit in the same order as in the original circuit.

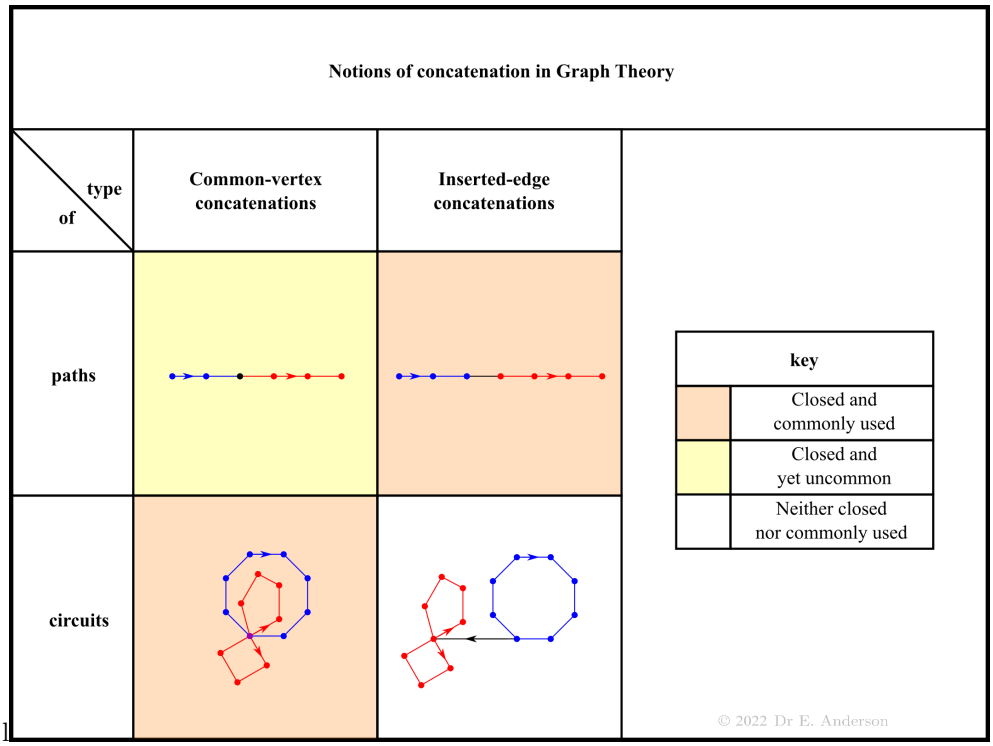


Figure 34.5:

Exercise 14$^-$ How many of Exercise 4's paths and circuits can be expressed as nontrivial concatenations of smaller paths and circuits respectively?

Naming Remark 2 Observe that Structure 2 is not a generalization of Structure 1. One could align the two definitions by having paths concatenate via sharing a common end (Fig 34.5.a.0). This is however far more rarely done in concatenating words or data strings as modelled by paths. Completing the conceptual square, concatenating circuits with an edge addition is providing a slightly more complicated network: Fig 34.5.b.1), which is also far more seldom done. It particular, this does not produce a circuit, and thus is not a closed binary operation on circuits. In contrast, both of our notions of concatenation of paths provide paths, and are thus closed binary operations on paths. It still however makes sense to call Structure 1 an *inserted-edge concatenation* and Structure 2 a *common-vertex concatenation*.

Exercise 15 Given two graphs G_1 and G_2, their *join* $G_1 \vee_G G_2$ is obtained as follows. Firstly form the disjoint union $G_1 \coprod G_2$. Secondly link G_1 to G_2 by forming all mutual edges.

a) Formulate Formulate coning and suspension operations as simple subcases of join.

b) Formulate any complete bi-partite graph as a suspension and as joins more generally.

c) Draw out $P_2 \vee_G P_2$, $P_3 \vee_G P_3$, $P_3 \vee_G C_3$, and $C_3 \vee_G C_3$. Identify these graphs in terms of previously introduced conceptual types of graph.

34.11 Quotient graphs and covering graphs

Structure 1 To define a notion of quotient for a graph G, firstly partition its vertices. Secondly, identify all of the vertices in each block of the partition. Two partition blocks are to be adjacent if any of the vertices of the one block are G-adjacent to any of the vertices of the other. The *quotient graph*'s vertexes are the partition blocks, with edges as determined by this adjacency relation.

Example 1 The quotient of Prism_4 (Fig 34.6.a) by identifying one of its nested squares is the 4-wheel, W (Fig 34.6.b).

Definition 1 A graph G_2 *covers* a graph G_1 if there is a surjection $\mathfrak{V}(G_2) \to \mathfrak{V}(G_1)$ that is also a local isomorphism.

Remark 1 If G_2 covers G_1, then G_2 can also be viewed as some quotient graph $G_1/\sim$.

Exercise 16$^-$ a) Describe the n-star S_n and Windmill$(n-1)$ as quotients of the *windmill graph* Windmill(n) of Fig 34.6.c).

b) Describe the (n, m)-*web graph* Web(n, m) of Fig 34.6.d) as a cover of the m-Wheel Wheel_m.



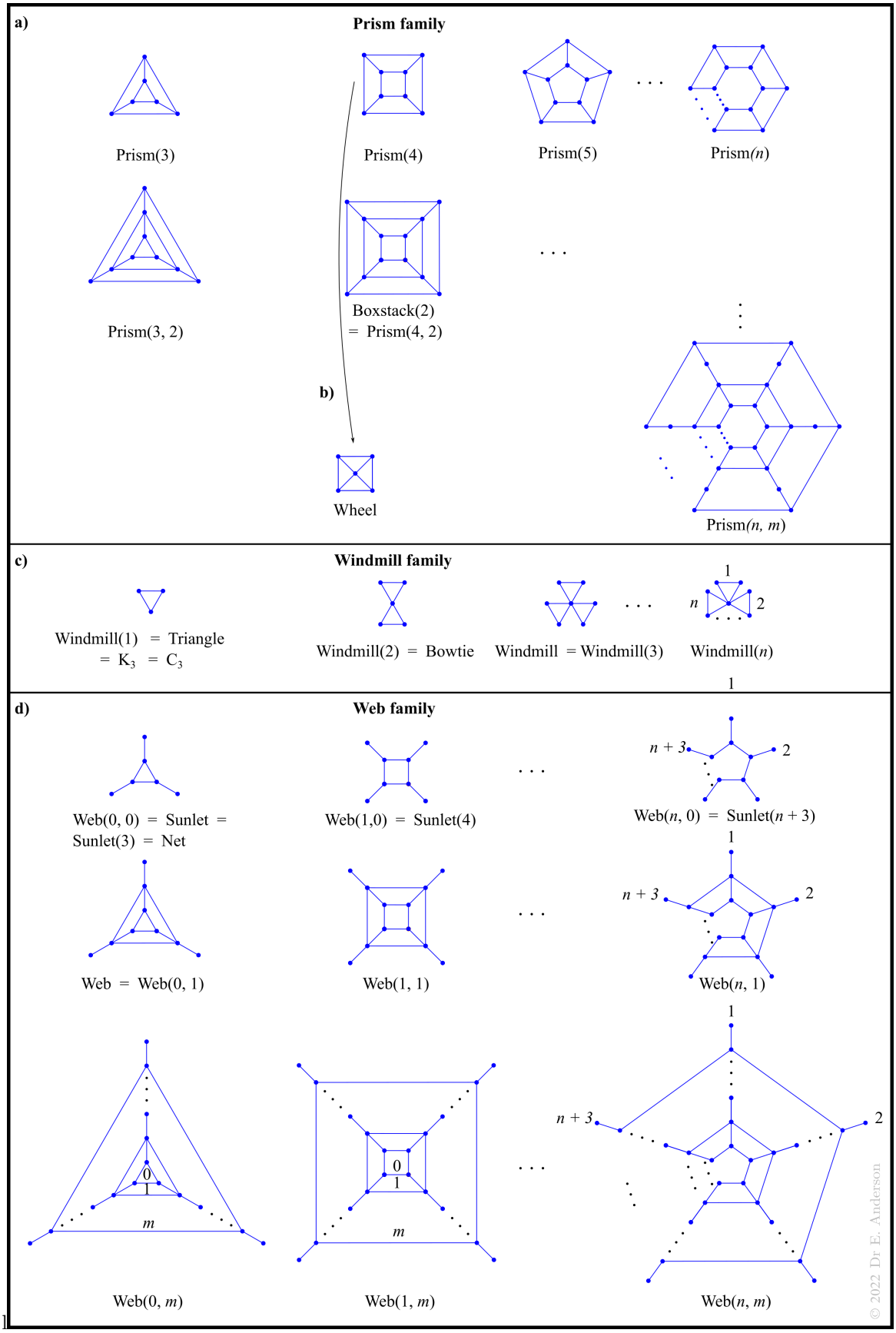

a)
Prism family
Prism(3)
Prism(4)
Prism(5)
Prism(n)
Prism(3, 2)
Boxstack(2)
= Prism(4, 2)
b)
Wheel
Prism(n, m)
c)
Windmill family
Windmill(1) = Triangle
= K3 = C3
Windmill(2) = Bowtie
Windmill = Windmill(3)
Windmill(n)
d)
Web family
Web(0, 0) = Sunlet =
Sunlet(3) = Net
Web(1,0) = Sunlet(4)
Web(n, 0) = Sunlet(n + 3)
Web = Web(0, 1)
Web(1, 1)
Web(n, 1)
Web(0, m)
Web(1, m)
Web(n, m)
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Figure 34.6:

Chapter 35

Graph connectivity

35.1 Introduction

Definition 1 A (simple) graph G is *connected* [183] if there is a path between any two of its vertices, v and w.

Definition 2 It is *disconnected* if it is not connected.

Remark 1 Definition 1 is a path-connectedness concept, while Definition 2 is a definition by exclusion.

Definition 3 It is *totally disconnected* if no pair of its vertices are connected.

Remark 2 So it has no edges, and can also be thought about in simpler terms as a point cloud.

$$D_n \ = \ {D_1}^n \ = \ \{\text{pt}\}^n \ .$$

Exercise 1$^-$ Draw paths between all pairs of vertices in Wheel Spiderhead (Fig 30.3).

35.2 Components of a graph

Definition 1 A *connected component* of a graph is a maximally-connected subgraph thereof.

Remark 1 If this returns the whole graph, then the graph is connected. I.e. nonempty connected graphs correspond to component number

$$c \ = \ 1 \ .$$

$$c \ > \ 1$$

corresponds to a nontrivial partition of our graph into multiple disjoint components.

Remark 2 Graphs are partitioned into connected components. Graphs can be characterized by number of connected components and sizes of each component; this is far from necessarily a unique characterization. This supersedes Chapter 29's account of graphs coming in single and multiple chunks.

35.3 Cut vertices and cut edges

Definition 1 A *cut vertex* alias *articulation vertex* is a vertex whose deletion would increase the number of components.

Definition 2 A *cut edge* alias *bridge* or *isthmus* is an edge whose deletion would increase the number of components.

Example 1 Any point in a tree is a cut vertex.

Example 2 Any edge in a tree is a cut bridge.

Exercise 2 How many graphs on 6 vertices contain a cut-vertex? A cut edge?

Example 3 The degree-4 point of Bowtie is a cutvertex (Fig 35.1.a).

Example 4 The middle edge in Bridge is indeed a bridge (Fig 35.1.b).

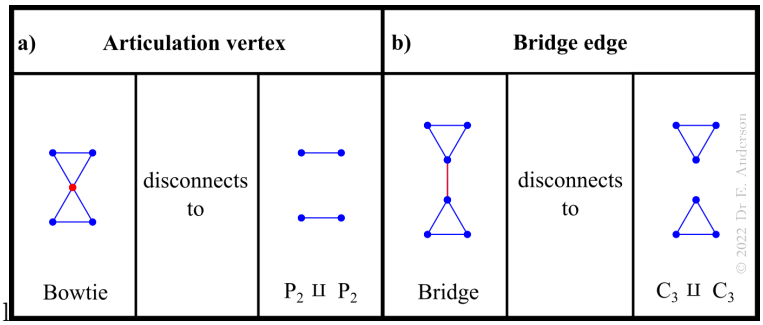


Figure 35.1:

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Definition 3 The *articulation number Art* of a graph is its number of articulation points.

Definition 4 The *bridge number Bri* of a graph is the number of bridges that it contains.

35.4 Separating sets

Definition 1 A set of vertices $\mathfrak{W} \in \mathfrak{V}(C)$ is a *(vertex-)separating set* if $C - \mathfrak{W}$ is disconnected.

Definition 2 A set of edges $\mathfrak{F} \in \mathfrak{E}(C)$ is an *(edge-)separating set* alias *disconnecting set* if $C - \mathfrak{F}$ is disconnected.

Definition 3 Let us denote the set of all vertex separating sets by

$$\mathfrak{Separ} .$$

Definition 4 Let us denote the set of all edge-separating sets $=$ disconnecting sets by

$$\mathfrak{Discon} .$$

Example 1 Cut vertices are each 1-point separating sets.

Exercise 3 What is the smallest connected graph with a 2-point separating set?

35.5 Connectivity numbers

Definition 1 The *(vertex-)connectivity number* of a graph G,

$$\kappa(\mathrm{G}) \;:=\; \min_{\mathfrak{W} \in \mathfrak{S}_{\mathrm{epar}(\mathrm{G})}} (\,|\,\mathfrak{W}\,|\,) \;. \tag{35.1}$$

Definition 2 The *edge-connectivity number* of a graph G,

$$\kappa_1(\mathrm{G}) \;:=\; \min_{\mathfrak{F} \in \mathfrak{D}_{\mathrm{iscon}(\mathrm{G})}} (\,|\,\mathfrak{F}\,|\,) \;. \tag{35.2}$$

Remark 3 Some texts denote this by λ. In our choice of notation, the 1 stands for the dimensionality of edges.

Worked Example 1 Show that $\kappa = 1$ for paths on ≥ 3 vertices. And that $\kappa = 2$ for paths on ≥ 3 vertices. What is κ for n-Pan? Deduce that κ is not hereditary, to the extent that $\kappa(\mathrm{H}) > k''(\mathrm{G})$ and $\kappa(\mathrm{H}) < k''(\mathrm{G})$ are possible.

Answer 1 Remove any non-terminal vertex to disconnect these paths. Remove any two distinct vertices to disconnect any cycle. Remove the handle attachment vertex on the pan graph's cycle to disconnect the remainder of this cycle from from the handle vertex. If κ were hereditary, then the cycle subgraph of the pan would have the same value of κ, but they are 2 and 1. And the n-path subgraph of the n-cycle would have the same value of κ, but they are 1 and 2.

Exercise 4 What is κ for each of n-Fan, n-Wheel, K_n and $\mathrm{K}_{p,q}$?

Exercise 5$^+$ A *cut-set* alias *bond* is a disconnecting set no subset of which is disconnecting. I.e. a minimally-sized disconnecting set.

a) What bonds are present in the following graphs? Prism(n), Web(n, m), Hypergrid(n, m) and Hypercube(4)

b) Show that an edge cut C in a graph G is a bond iff $\mathrm{G} - C$ has precisely two components.

Proposition 1 (Whitney's inequalities) For any graph G,

$$\kappa \;\leq\; \kappa_1 \;\leq\; \delta \;. \tag{35.3}$$

Proof 2) Observe that in any graph G

$$(\,\text{set of edges incident to a minimum-degree vertex}\,) \;=\; \{\, e \,|\, e \;\mathcal{I}\; (\,\text{any particular}\;\; v \;\;\text{with}\;\; d(v) \;=\; \delta(\mathrm{G})\,\}$$

constitutes a disconnecting set $\mathfrak{F}$. Thus the minimality in the definition of κ_1 guarantees our second inequality:

$$\kappa_1(\mathrm{G}) \;:=\; \min_{\mathfrak{F} \in \mathfrak{D}_{\mathrm{iscon}}(\mathrm{G})} (\,|\,\mathfrak{F}\,|\,) \;\leq\; \delta(\mathrm{G}) \;.$$

1) The following three cases clearly exhaust all possibilities.

A) Suppose that $\kappa_1(\mathrm{G}) = 0$. Then G is just some point cloud $\mathrm{pt}^n = {D_1}^n = D_n$. [This case readily includes realizing the ungraph as the unpoint if we wish to include this.] Thus $\kappa(\mathrm{G}) = 0$ too.

B) Suppose that $\kappa_1(\mathrm{G}) = 1$. Then G contains a bridge. Thus G possesses a cut vertex. So $\kappa(\mathrm{G}) = 1$ too.

Finally suppose that $\kappa_1(\mathrm{G}) > 1$. Then deleting $\kappa_1(\mathrm{G}) - 1$ edges from the graph is a well-defined and nontrivial operation. One way this can be performed 'concentrates firepower' so as to leave a

particular edge of the resulting G' with a bridge. This is clear from the definitions of κ_1 and of bridge.

A trick we can use here is as follows. We are free to pick a vertex $v_i \; \mathcal{I}$ (one of our deleted edges e_i). We can do this for each $i \in 1$ to $\kappa_1(G) - 1$. Call our set of vertices meeting this requirement $\mathfrak{W}$. Let $G'' := G - \mathfrak{W}$. The following two cases are then exhaustive.

i) G'' is not connected. Then

$$\kappa(G) < \kappa_1(G) . \tag{35.4}$$

ii) G'' is connected. Then it must contain a vertex whose deletion totally disconnects our graph. I.e. a star-point vertex s, by which case ii) solely involves star graphs S_n. Then

$$\mathfrak{W}_s := \mathfrak{W} \cup \{ s \}$$

constitutes a separating set for G. Furthermore,

$$|\mathfrak{W}_s| := |\mathfrak{W}| + \{ s \} = (\kappa_1(G) - 1) + 1 = \kappa_1(G)$$

Thus

$$\kappa(G) \leq \kappa_1(G) . \tag{35.5}$$

This last equation includes case i)'s conclusion (35.4) as well, thus holding for all graphs with $\kappa_1(G) > 2$.

With cases A) and B) also being special subcases of (35.5), this holds for all graphs unreservedly. □

Exercise 6 By joining two copies of $K_5 - e$, show that both of the above inequalities can be strict. Also find simpler examples to illustrate that both of these inequalities are tight.

35.6 ν-connectivity

Definition 1 A graph G is *ν-connected* if

$$\kappa \geq \nu .$$

This is often referred to as k-connectivity: using k in the role of ν.

Examples 1-3 Trees are 1-connected and cycles are 2-connected. Having a sidechain, or, more generally a *sidetree*, renders a graph 1-connected.

Pointer 1 ν-connectivity is a measure of network robustness. If this is high, more nodes or connecting wires need to fail before the network goes down.

Worked Example 1 Show that K_N is $(N - 1)$-connected What is the smallest 4-connected graph that is not complete?

Answer 1 In K_N, all vertices are connected to all others. So one needs to remove all but 1 vertex in order to separate it. Thus $\kappa = N - 1$. So by the definition of ν-connectivity, K_N is $(N - 1)$-connected.

K_5 is the smallest 4-connected graph, and now forbidden. Let us thus try finding a suitable 6-graph. K_6 is 5-connected, which is more than what we need (and forbidden). But removing 1 edge from this leaves us with a 4-connected graph. This is not however the smallest such. For we nu-Connectivity can remove 2 more edges so as to produce a 4-regular graph whose separating set is of size 4 (Fig 35.2).

Exercise 7 Show that $K_{p,q}$ is q-connected for $q = \min(p, q)$. What is the smallest 3-connected bi-partite graph that is not complete-bi-partite?

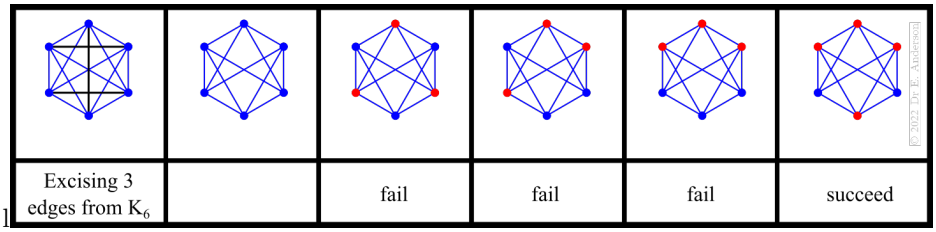


Figure 35.2:

Exercise 8^{+} a) Suppose that

$$d(v) \geq \frac{|\,\mathrm{G}\,| + \nu - 2}{2}$$

for every vertex $v \in \mathfrak{V}(\mathrm{G})$. Then ν-connectivity is limited to the range

$$1 \leq \nu \leq \Delta_{\max}(\mathfrak{G}\mathrm{raph}) = |\,\mathrm{G}\,| - 1 \,.$$

b) Show that Wheel graphs Wheel_n are 3-connected. Deduce that the above sufficiency condition is not a necessary one.

Proposition 2 Excluding 0 from this range, ν-connectedness is guaranteed if for each vertex v, the following holds.

$$d(v) \geq \frac{(N + \nu - 2)}{2} \,. \tag{35.6}$$

Exercise 9 Prove this inequality.

35.7 Connectivity ratios

Structure 1 The naïve *connectivity per vertex* ratio is

$$k := \frac{\kappa}{N} \,.$$

One naïve possibility for the edge analogue of this is edge connectivity per edge,

$$k_1 := \frac{\kappa_1}{E} \,.$$

Definition 1 The *edge-vertex connectivity ratio*

$$\mathcal{K} := \frac{\kappa_1}{\kappa} \,.$$

Proposition 3 Whitney's inequality extends and simplifies to

$$0 \leq k \leq 1 \,,$$

$$0 \leq l \leq 1 \,.$$

Remark 1 This result motivates the following definitions.

Definition 1 The *edge connectivity per minimal degree*

$$m := \frac{\kappa_1}{\delta} \,.$$

Remark 2 The l in the ratio form of Whitney's inequality is then the reciprocal of this,

$$l := \frac{1}{m} \,.$$

Remark 3 Whitney's inequality bounds κ_1 above by δ, which goes at most like N for N large. Thus normalizing κ_1 by maximum edge number is not appropriate. There are moreover subarenas of graphs for which κ and κ_1 are both sub-extensive, including these two differing in sub-extensiveness. $\mathcal{K}$ *characterizes* these, while Structure 1's preliminaries often fall by the wayside.

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Definition 5 The *extent-of-connectivity fraction* is

$$n \ := \ \frac{\nu}{\nu(\max)} \ = \ \frac{\nu}{\Delta(N)} \ = \ \frac{\nu}{N \ - \ 1} \ .$$

Remark 4 This normalization entails a $\mathfrak{G}$raph rather than G ratio.

Exercise 10$^-$ Formulating a ratio version of the bound on ν-connectivity motivates the following notion. The *n-connectivity adjusted degree fraction*

$$\widetilde{d} \ := \ \frac{2\,d}{|\,\mathrm{G}\,| \ + \ n \ - \ 2} \ .$$

Show that all of k, l and $\widetilde{d}$ run over the standard range.

Exercise 11 Investigate how maximal and average connectivity per unit size scales for $N \ \leq \ 5$.

[This is a small toy model of building a robust network on limited wire length or road length constraints on resources.]

35.8 Connectivity in arrow-graphs

Definition 1 An ori-, di- or mixed-edge-type graph is *strongly connected* if it possesses a directed path from each of its vertices to each other vertex.

Definition 2 An arrow-graph is *weakly connected* if its underlying graph skeleton is connected.

Remark 1 Strong connectivity requires directed paths between vertices in both directions. It is thus possible to formulate the following middling-strength notions (though what these are called may vary between authors).

Definition 3 An arrow-graph is *unilaterally connected* if there is at least one of a path $u \to w$ or $w \to u$ for every vertex pair $u \ \neq \ w$. It is *properly unilaterally connected* if each vertex pair u, w supports precisely one of a directed path $u \to w$ or $w \to u$.

Exercise 12 What are the smallest nontrivial ori-graphs that are each of strongly connected, weakly connected, unilaterally connected and properly unilaterally connected?

Definition 4 An arrow-graph is *double-connected* if it is strongly connected in both orientations at once.

Definition 5 A mixed-edge-type graph is *plain-connected* if it is connected by an unordered path.

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Definition 6 A simple graph is *strongly orientable* if it admits arrows rendering it strongly connected.

Exercise 13 a) Which oriented path graphs on 2 to 4 vertices are each of strongly and weakly connected? What is the smallest graph admitting multiple strong orientations?

Remark 2 See e.g. [176] for Robbins' and Roberts' Theorems about strongly-orientable graphs. In both cases, the necessary and sufficient condition is that the graph be connected and bridgeless.

Remark 3 Binary relations' strong connectivity is arrow-graphs' weak connectivity. Whether the region $\mathfrak{R} \subseteq \mathfrak{X} \times \mathfrak{X}$ is connected is an unrelated matter.

Pointer 2 See e.g. [183, 156, 193] for more about notions of connectivity for graphs.

Pointer 3 Notions of path connectivity are much more general than just for graphs (Fig 35.3.a). They are in particular topological space level notions. In this setting, connectivity is itself a distinct notion, as follows.

A topological space $\mathfrak{X}$ is *connected* if it cannot be decomposed into two open sets $\mathfrak{U}, \mathfrak{V}$ that nontrivially partition $\mathfrak{X}$ (Fig 35.3.b). I.e. such that

$$\mathfrak{U}, \mathfrak{V} \neq \emptyset\,, \quad \mathfrak{U} \bigcap \mathfrak{V} = \emptyset\,, \quad \mathfrak{X} = \mathfrak{U} \bigcup \mathfrak{V}\,. \tag{35.7}$$

While for graphs these two notions of connectivity coincide, more generally they do not. (Path) components and *cut sets* (generalizing cut vertices and cut edges) are correspondingly generally meaningful.

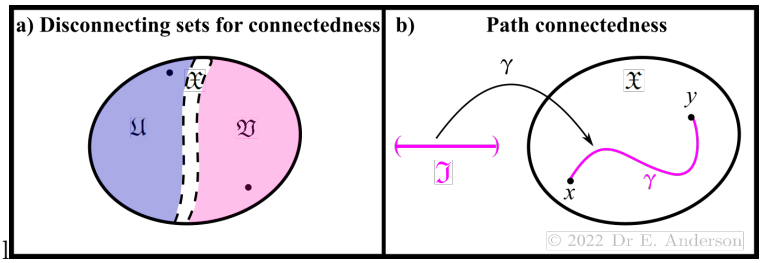


Figure 35.3:

Chapter 36

Tree graphs

36.1 The trees encountered so far, and the starlike trees

Recall the trees on $N = 1$ to 5 vertices from Chapter 29 and those on 6 and 7 vertices from table 29.4.

Exercise 0 Find all the trees on 8 vertices (see Fig 36.2 overleaf for the answer).

Exercise 1 and Naming Remark 1 A large proportion of small trees are included among the *starlike* alias *generalized star* graphs $\mathrm{S}(r_1, \ldots, r_p)$ with p radiant paths of excess lengths r_1 to r_p emanating from a single star point.

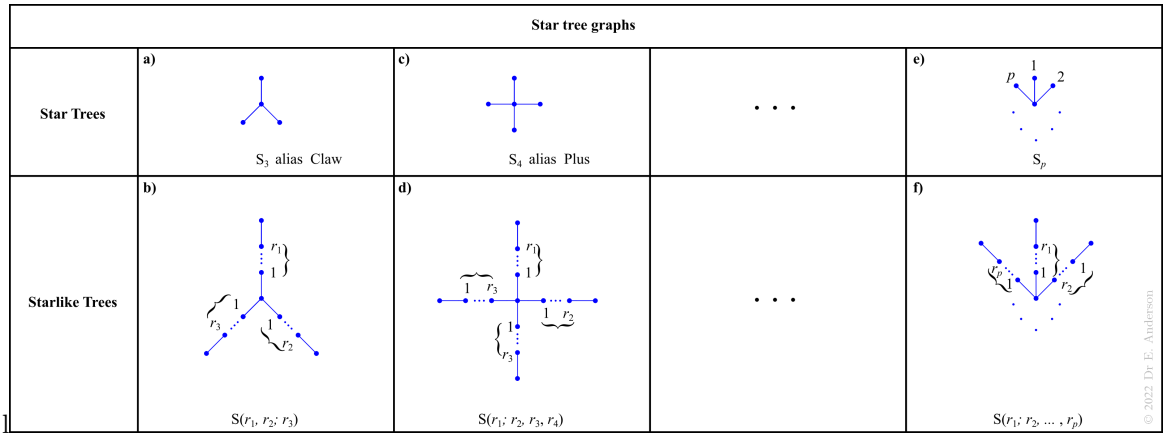


Figure 36.1:

a) Which trees are the $\mathrm{S}(a, b)$? For these, and these alone, which enumerative ambiguity is present?

b) Find all the generalized stars on $N \leq 6$ vertices. Which tree of order 6 is not a generalized star?

c) What are the generalized star names of all the trees on 7 and 8 vertices that admit such a description?

d) How many generalized stars are there on N vertices?

e) What regularity condition do starlike graphs obey?

Pointer 1 This, alongside streamlining of its parameters into partition notation for $n = N - 1$ into p parts, can be seen as a start on a more efficient naming system for small trees, to which we return in Sec 38.6.

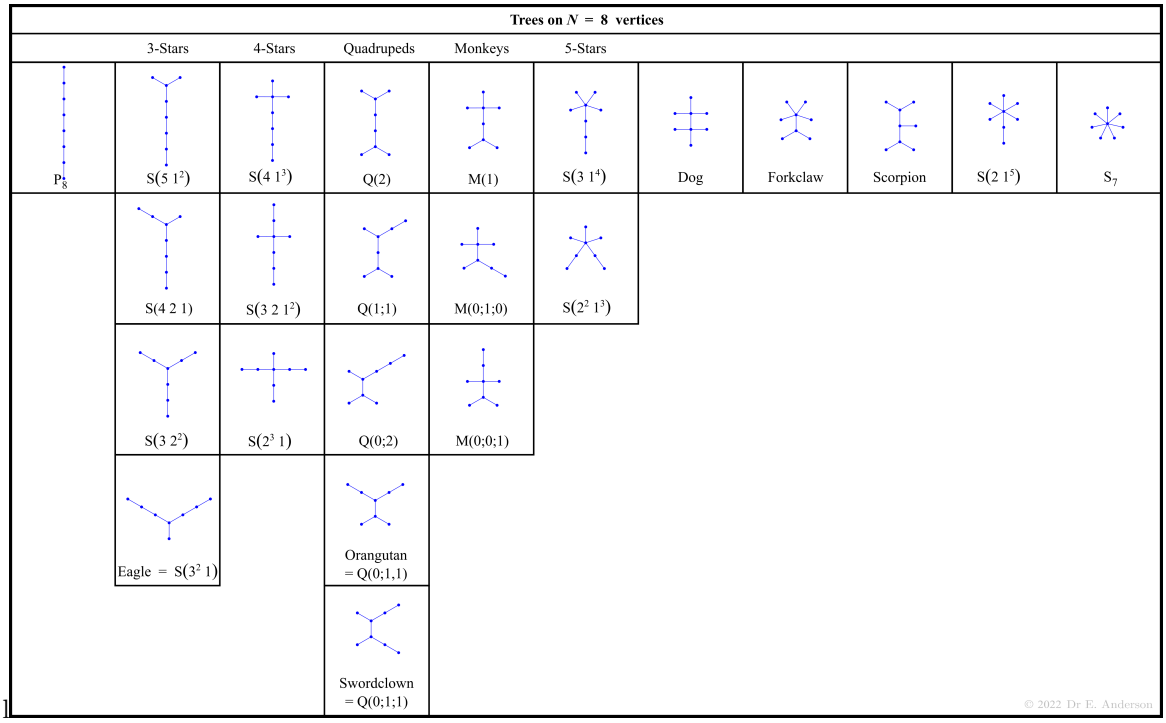


Figure 36.2:

36.2 Rooted trees

Definition 1 A *rooted tree* is a tree in which some particular vertex – called the *root* – has been assigned a distinguished role.

Notation Remark 1 A rooted tree is usually depicted vertically, with the root as the sole element at the bottom, or, actually more often, at the top ('reflected-tree presentation').

Remark 1 The root confers a direction on the rest of the tree, by which rooted trees may furthermore be viewed as ori-graphs. The order down the tree is usually phrased as v_1 *precedes* (or *antecedes*) v_2 if there is a path from v_1 to v_2.

Remark 2 See Fig 36.3 for the rooted trees ≤ 5 vertices.

36.3 Leaves

Definition 1 A vertex v in a graph is a *leaf* if it is terminal, as characterized by being of degree one: $\mathrm{d}(v) = 1$.

Remark 1 In an ori-graph, it is possible to make distinction between initial and final leaves. In the case of a rooted tree, whether or not it is also treated as a ori-graph, one vertex – which may or may not itself be a leaf – is assigned to be the root. Whichever other leaves are present could then

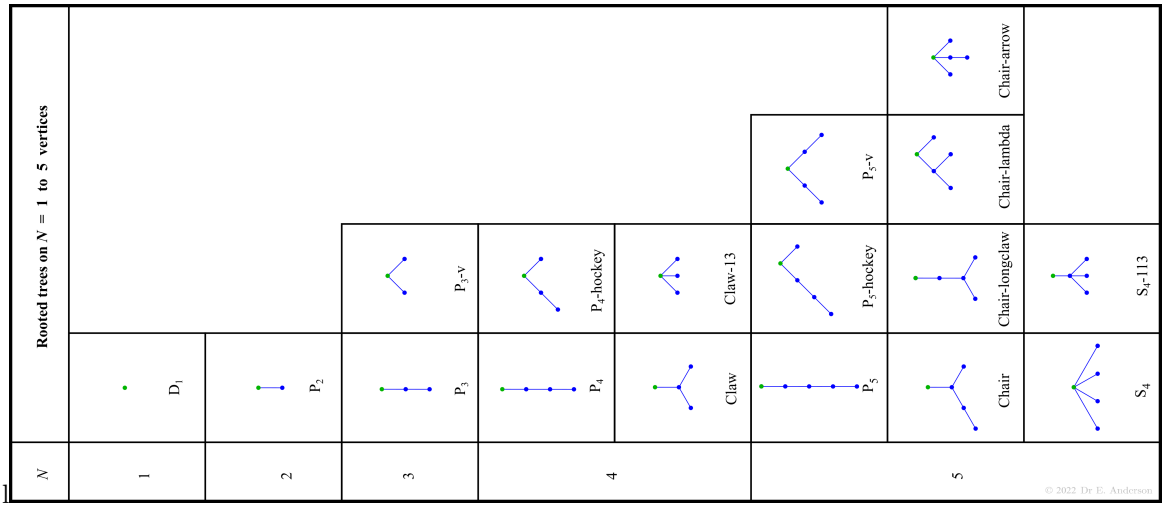


Figure 36.3:

be called *final leaves*, so as to distinguish this situation from that in which all terminal vertices are leaves.

Definition 2 The *leaf number* of a graph is the number of leaves that it possesses:

$$L \ := \ \#(\,\text{terminal vertices}\,) \ . \tag{36.1}$$

Di-graphs support two leaf numbers, $L_{\pm}$, with - for initial leaves and + for final leaves.

Remark 2 For an ori-graph,

$$L_{-} \ + \ L_{+} \ = \ L \ : \tag{36.2}$$

the leaf number for the underlying graph. For a rooted tree, L_{-} is either 1 or 0, depending on whether the root is or is not a leaf.

36.4 Relational leaf variables

Definition 1 *Leaf fraction*

$$l \ := \ \frac{L}{|\,\mathrm{G}\,|} \ . \tag{36.3}$$

Definition 2 The '*in-out index*' of an ori-graph is

$$\Delta L \ = \ L_{+} \ - \ L_{-} \ . \tag{36.4}$$

Remark 1 The *in-out proportion* is

$$\frac{\Delta L}{L} \ = \ \frac{L_{+} \ - \ L_{-}}{L_{+} \ + \ L_{-}} \ = \ \frac{1 \ - \ \mathcal{L}}{1 \ + \ \mathcal{L}}$$

for

$$\mathcal{L} \ := \ \frac{L_{-}}{L_{+}}$$

the *out-in* ratio.

36.5 Branching point number and n-arity

Definition 1 A graph's *branching point number* Br is the number of branching points that it contains.

Definition 2 A branching point is *n-ary* if it has n branches leading out from it. This amounts to having $n + 1$ edges in total, the extra one being the parent branch.

Remark 1 Looking back to Sec 31.11, this has excess branching point strength $e = n - 3$.

Remark 2 Also introduce I counting the *interior points* of a branch (for trees) or of a branch or cycle (for general graphs). I.e. neither branching points nor terminal points (the leaves). Then we have the following partition of vertices by their degree values.

$$L + I + Br = |\,\mathrm{G}\,| \,.$$

Remark 3 Branching points can furthermore be more finely split up according to their particular n-arities. This gives the counts Br_i, $i = 3$ to Δ. This in effect returns us to the degree sum split of formula 32.2.

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Definition 4 The *branching-point* and *interior fractions* are respectively

$$br := \frac{Br}{|\,\mathrm{G}\,|}$$

and

$$i := \frac{I}{|\,\mathrm{G}\,|} \,.$$

Remark 4 These obey

$$l + i + br = 1 \,.$$

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Remark 5 Such as leaf number, branching point number and branching point n-arity constitute $\mathbb{N}$-valued information about graphs. On the one hand, this 'cardinal information' is easy to compute, and to compare between graphs. One can imagine that some structures' variety could be simple enough to be uniquely characterizable by some finite string of cardinal numbers. For instance, point clouds are, since these can be characterized by size alone. On the other hand, some structures are complicated enough that discerning between them using cardinal functions can be hard.

Exercise 2 a) Show that every tree of order ≥ 2 must contain ≥ 2 leaves.

b) i) What are the smallest trees that are not distinguished by just their order N and leaf number L?

ii) By N, Br, I and L?

iii) By these and Br_i?

Remark 6 Degree alias valency is more useful than other branching numbers upon generalizing from trees to arbitrary graphs.

36.6 Binary trees

Definition 1 A *binary tree* is defined recursively as follows. It is a set of vertices which is either empty or contains a root r plus other vertices forming two disjoint binary trees.

Nomenclature The two disjoint binary trees are called *left* and *right* subtrees. Then each has its own root called *left successor* and *right successor* (or *left child* and *right child*) of the preceding root, called the *parent*. This model has single parents, 'reproducing by binary fission' like many single-celled organisms do, to give precisely 2 children: a pair of *siblings*. Rooted trees also involve single parents though siblings may now come in multiplicities other than 2. Every vertex w below a given vertex v with a path $v - w$ is a *descendant* of v. If w is a descendent of v, then conversely v is an ancestor of w.

Remark 1 From developments below, you are welcome to patch together a purely graph-theoretic equivalent alternative to the above recursive definition of binary tree.



36.7 Forest and Tree Characterization Theorems

Remark 1 Some further characterizations of forests and trees are as follows. These can be useful in systematically figuring out whether a given graph is a forest or even a tree. Some context for this is that computers cannot just see pictures. Nor is the human eye much use either if e.g. vertex number, edge number, and the number of removable edge crossings are large. What a computer would usually be handling is some fairly large *adjacency matrix* corresponding to the graph (see Volume 1 for more).

Theorem 1 (Forest Characterization Theorem) A graph is a forest iff $\exists \quad \geq 1$ path between any vertex pair [178].

Proof ($\Leftarrow$) Suppose that our graph is not a forest. We defined forests to be acyclic graphs (Chapter 29). Thus our graph must contain at least one cycle $(v_1, \ldots v_k)$ (Fig 36.4.a). But then $v_1 \ldots v_k$ gives the path from v_1 to v_k as marked in red in Fig 36.4.b), while v_1-v_k is also a path from v_1 to v_k, as marked in black in this Figure. These are not the same since being a cycle requires $m \geq 3$. Then apply the contrapositive.

($\Rightarrow$) Suppose that our graph contains 2 vertices y_0 and y_m which have 2 distinct paths

$$P_1 = y_0\, y_1 \ldots y_m \text{ and } P_2 = y_0\, z_1 \ldots z_l\, y_m \ .$$

Being distinct points, $\exists$ some minimal index $i + 1$ for which $y_{i+1} \neq z_{i+1}$. Also in order for our two paths to end at the common point y_m, $\exists$ some minimal index $j + 1$ with $j \geq i$ such that z_{j+1} is some vertex of P_1. Let us call this y_{k+1}. This is enough to obtain a cycle (Fig 36.4.c),

$$(y_i, y_{i+1}, \cdots, y_k, z_{j+1}, z_j \cdots, z_{i+1}) \ .$$

So 2 paths between a vertex pair $\Rightarrow$ a cycle $\Rightarrow$ the graph is not a forest. Then apply the contrapositive. □.

Remark 2 If some vertex pairs have no path between them, then they belong to different trees in the forest. Banning such pairs from ocurring gives us one way of characterizing trees.

Theorem 2 (Path Characterization Theorem for Trees) A graph is a tree iff there is precisely 1 path between any of its vertex pairs.

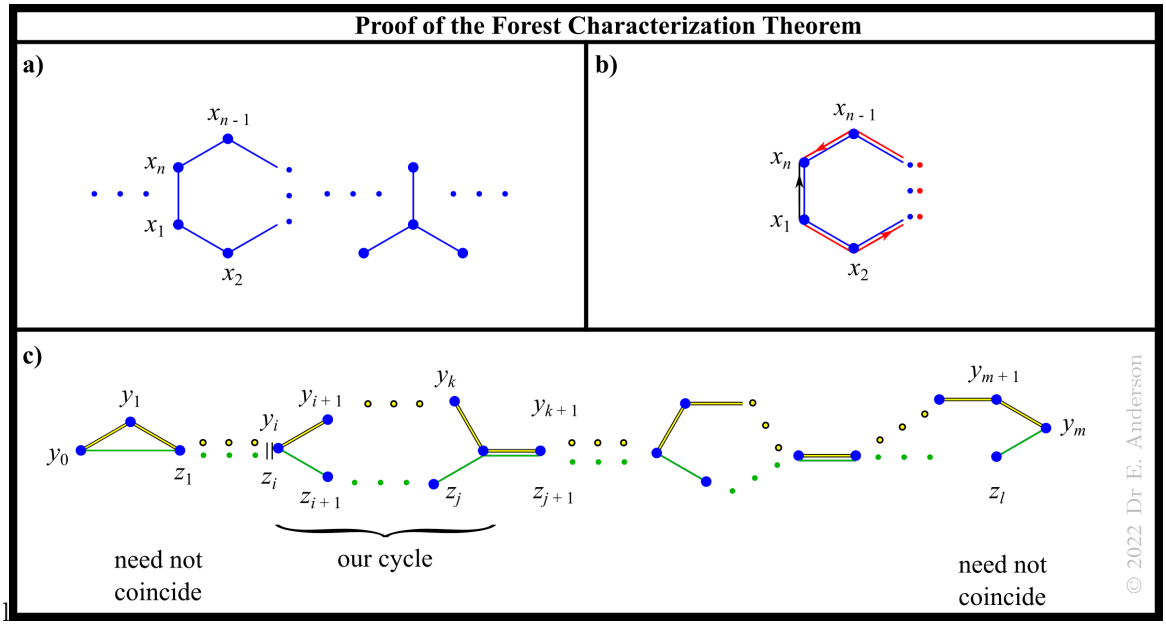


Figure 36.4:

Proof The previous Theorem's proof straightforwardly adapts to this tree case. □

Theorem 3 (All-Bridges Theorem) A graph is a tree iff it is connected and every edge in it is a bridge.

Proof ($\Rightarrow$) Let the graph G be a tree. Then by definition it is connected. By Theorem 2, $\exists$! path between every vertex pair in G. Thus $\exists$! edge joining adjacent vertices u, v without any possibility of getting there elsewise. So removing the edge u-v leaves us with no path from u to v, thus disconnecting the graph. So this edge meets the definition of bridge.

Finally, we picked an arbitrary edge to examine. Thus every edge of a tree is a bridge.

($\Rightarrow$) Let G be a connected graph with all edges being bridges, and yet which is not a tree. So it must contain at least 1 cycle, so we can pick a cycle C. Let H be a subgraph of G obtained by deleting one of C's edges, e. This amounts to removing a bridge, so

$$\text{H cannot possibly be connected } (*) .$$

Let x and y be two vertices in G. G being (path-)connected $\Rightarrow$ $\exists$ a path P between x and y. There are 2 separate cases to consider.

Either $e \notin$ P, so P is also a path between x and y.

Or $e \notin$ P. Then consider subpaths $x \to u$, and $w \to y$, which we refer to as P_1 and P_2 respectively. Let Q be the ! path in the cycle that is free of e. Finally let

$$\mathrm{R} = \mathrm{P}_1 \cup \mathrm{Q} \cup \mathrm{P}_2 ;$$

this is a path $x \to y$ in H.

By our two cases combined, $\exists$ path between any vertex pair in H. But $\mathfrak{H}$ is not (path-)connected by statement (*). Thus G is forced to be a tree. □

Lemma 1 Any tree T possesses precisely $|\,\mathrm{T}\,| - 1$ edges.

Proof Induct on $|\,\mathrm{T}\,| \;=\; n$. For $n \;=\; 2$, the P_2 path has $2 \;-\; 1 \;=\; 1$ edge.

Induction Hypothesis: assume this result is true up to $n \;=\; p$. Theorem 3 gives that every edge is a bridge. So $\mathrm{H} \;=\; \mathrm{G} \;-\; e$ has 2 components.

1) G_1 with n_1 vertices.

2) G_2 with n_2 vertices.

These obey

$$n_1 \;+\; n_2 \;=\; n \;+\; 1\;. \tag{36.5}$$

Our induction hypothesis applies to each component. Thus

$$e(\mathrm{H}) \;=\; (\,n_1 \;-\; 1\,) \;+\; (\,n_2 \;-\; 1\,) \;=\; n_1 \;+\; n_2 \;-\; 2 \;=\; n \;+\; 1 \;-\; 2 \;=\; n \;-\; 1\;. \tag{36.6}$$

The third step here makes use of (36.5). Then

$$e(\mathrm{G}) \;=\; e(\mathrm{H}) \;+\; 1 \;=\; n \;-\; 1 \;+\; 1 \;=\; n\;.$$

The second step here uses (36.6). Thus our induction hypothesis is true $\forall\;\; n \;\geq\; 2$. □

Corollary 1 A forest of order n that contains t trees is of size $n \;-\; t$.

Corollary 2 Any tree of order $\;>\; 2$ contains $\;\geq\; 2$ leaves.

Exercise 3 Prove these Corollaries.

Exercise 4 Prove the following two extremal properties of trees.

a) A graph is a tree iff it is a maximally acyclic graph (one in which adding any further edge forces a cycle).

b) A graph is a tree iff it is a minimally connected graph (removing any edge disconnects the graph).



36.8 Small counts for trees

Lemma 2 Only graphs of odd order can be binary trees.

Proof Starting with the trivial binary tree – just a root vertex – other binary trees arise by adding left-right sibling pairs. All possible binary trees arise in this way, so all must have an odd number of vertices. □

Remark 1 We can thus narrow down our search for small examples of binary trees to the trees of odd order.

Example 0 $N \;=\; 1$ admits just D_1, which is trivial in the sense of just consisting of an initial = final node.

Remark 2 Henceforth, we can also discard all trees with any vertices with degrees other than 1, 2 or 3. I.e. apart from D_1, all binary trees must be (1, 2, 3)-regular. Furthermore, only one vertex of degree 2 can be supported: the one in 'root position'.

Example 1 $N = 3$ admits just P_3, bent into the '2 cherries' or 'smallest V' as per Fig 60.5.a.ii). This is still trivial, in the milder sense of having no branching points.

Example 2 $N = 5$ is minimal for a binary tree to admit branching points. This case admits just the Chair graph, configured as per Fig 60.5.a.iii).

Example 3 $N = 7$ is minimal for supporting two distinct unlabelled binary trees: SegmentedQuadruped and Swordfighter in the configurations given in Fig 60.5.a.iv). In fact this configuration gives the first of these a nicer and commonly-used alias: Grandparents, as in a family tree that goes back to one's grandparents. [The modelling here simplifies out the possibility that some of the parents or the child might have siblings.]

♠ ◇ ♣

Exercise 5^-

a) How many unlabelled binary trees are there on 9 vertices?

b) Construct also the 6 unlabelled binary trees on 11 vertices.

c) How many leaves does the general binary tree possess?

Lemma 3 Suppose that a binary tree has N nodes. Then its leaf number is precisely

$$\mathrm{L} = \frac{N + 1}{2} .$$

Proof For 1 vertex, there is 1 leaf. Suppose that the result holds for $N = p$ vertices. Then for $N = p + 1$ vertices, we have

$$\frac{p + 1}{2} - 1 + 2$$

leaves. For one leaf is converted to new branching point but in the process two new leaves arise. The above expression readily rearranges to

$$\frac{(p + 1) + 1}{2} .$$

Thus our result holds by induction. □

Corollary 1 For $N > 1$, binary trees have the following combination of degrees.

$$\left(\frac{N + 1}{2} , \ 1 , \ \frac{N - 3}{2} \right) = (L, I, Br_2) \text{ i.e. of degree } (1, 2, 3) . \tag{36.7}$$

Exercise 6 a) Prove Corollary 1.

b) Redo Exercise 5 using this extra piece of knowledge.

c) Count and draw out all the binary trees on 13 vertices.

36.9 n-ary trees and their counts

Example 1 A *ternary tree* is defined recursively as follows. It is a set of vertices which is either empty or contains a root r plus other vertices forming three disjoint ternary trees.

Nomenclature 1 In place of left-right distinctions for binary trees, we now have left–middle–right distinctions. Siblings now always occur in threes.

Remark 1 Fig 60.7 presents the first three ternary trees that have no binary counterpart.

Example 2 *Quad trees* are the counterpart with four disjoint trees at each vertex. One application in which these arise is in refining area samples by quartering them; this occurs in zooming into a map. This setting gives the names top-right, top-left, bottom-right, bottom-left names to the four descendents at each stage.

Example 3 *Oct trees* refine volume samples similarly; this occurs in zooming into a 3-d spatial map. 2^p trees refine p-volumes similarly.

Definition 1 An *n-ary tree* is defined recursively as follows. It is a set of vertices which is either empty or contains a root r plus other nodes forming n disjoint n-ary trees.

Remark 2 n-ary trees are almost regular, in that all but their root and terminal points have degree $n + 1$. They are furthermore precisly $(1, n, n + 1)$-regular, the unique root being the sole degree-n vertex.

♠ ◇ ♣

Exercise 7 a) How does our odd-vertex number result for binary trees generalize to n-ary trees?

b) How many leaves and branching points must an n-ary tree possess?

c) Prove that for each n, the corresponding tri-regular graph degree count is sufficient as well as necessary, thus characterizing n-ary trees.

Exercise 8 a) The minimal nontrivial binary tree is Chair. This is in the sense of being the first to contain an actual binary branching point of degree 3, as opposed to the root which is just of degree 2. Draw the minimal nontrivial ternary and quad trees. What names did we use to previously refer to their underlying graph skeletons?

b) What are the jointly-smallest trees that are not n-ary for any n?

Theorem 4 (At-Most Binary-Tree Representation Theorem) The general tree can be represented by an at-most binary tree.

Exercise 9 Prove this.

36.10 Further tree counts

Exercise 10 How many unlabelled rooted trees are there on $N = 0$ to 7 vertices?

Definition 1 A tree is *at most binary* if every vertex has 0, 1 or 2 descendents.

Exercise 11 An *at-most binary tree* is a tree in which each vertex has one, both or neither of a left child and a right child. How many at-most binary trees are there on $N = 0$ to 5 vertices if left and right siblings are taken to be distinguishable in each case?

Example 2 As regards counting unlabelled trees, tables 29.4 and 36.2 go as far as $N = 8$.

Exercise 12 How many labelled trees are there on 0 to 5 vertices?

36.11 Counting the number of labelled trees: another Theorem of Cayley's

This case enjoys a straightforward derivation of a simple closed formula, as follows.

Theorem 5 (Cayley)

$$\#(\text{ labelled trees on } \ n \ \text{ vertices}) \ = \ n^{n \ - \ 2} \ . \tag{36.8}$$

Proof Let us first establish a bijection between labelled trees on n vertices and $(n \ - \ 2)$-vectors.

Let T be any tree with distinctly-labelled vertices 1 to n. Arrange its leaves L_i in increasing label order. Let T_1 be the label of the only vertex adjacent to the leaf with the lowest label, L_1. It is always possible to make this choice because trees must have $\geq$ 2 leaves by Exercise 2.a).

Delete L_1 to obtain a new tree $\mathrm{T}^{(1)}$. Arrange its leaves $L_i^{(1)}$ in increasing label order. Continue until we have obtained $T_{n \ - \ 3}$.

We are then left with a P_2 graph. Finally let $T_{n \ - \ 2}$ be the smaller label value on this.

Conversely, take v_1 to be the following. The first element $\in \ \mathfrak{N} \ = \ \{ 1 \ \text{ to } \ n \}$ that $\notin \ \{ T_1, \ ..., \ T_{n \ - \ 2} \}$. v_2 to be the first element $\in \ \mathfrak{N} \ - \ \{ v_1 \}$ that $\notin \ \{ T_2, \ ..., \ T_{n \ - \ 2} \}$. Keep on going until we have reached $v_{n \ - \ 2}$.

Label the last two vertices as w and x. This leaves us with an n-vertex graph with $n \ - \ 1$ edges and no cycles. I.e. an n-tree.

Finally, it is straightforward to count the number of vectors. I.e. there are n independent possibilities for each of the $n \ - \ 2$ components. We thus arrive at an overall count of $n^{n \ - \ 2}$. □

Exercise 13$^+$ a) An *arborescence* rooted at a vertex u is a directed tree with the following property. It has a unique vertex u with $d_-(u) \ = \ 0$ while $d_-(v) \ = \ 1$ for all other vertices v. Show that

$$\#(\text{ labelled arborescences on } \ n \ \text{ vertices }) \ = \ n^{n \ - \ 1} \ .$$

b) An *edge-labelled tree* on n vertices is a tree in which each edge is assigned a unique integer $\in \ \{1, \ ..., \ n \ - \ 1\}$. Show that

$$\#(\text{ edge-labelled trees on } n \text{ vertices}) \ = \ n^{n \ - \ 3} \ .$$

Remark 1 The unlabelled case is a much harder problem (see Sec 62.3 for further discussion).

36.12 Counting the at-most binary trees with left–right distinction gives the Catalan numbers

This is another exactly-calculable case. It moreover gives rise to a structure that is widely useful in prima facie unrelated problems.

Worked Example 1 With left–right distinction, what are the at-most binary trees on $N \ = \ 0$ to 4?

Answer 1 See Fig 36.5.

Worked Example 2 What is the number of at-most binary trees with left-right distinction?

Answer 2 Given a tree, removing its root splits it into 2 subtrees of size k and $n - k - 1$. (These sum to $n - 1$, as needed.) This can be carried out for any k. We thus obtain the following recurrence relation.

$$C_n = \sum_{k=0}^{n-1} C_k \, C_{n-1-k} \, . \tag{36.9}$$

Remark 1 This is quadratic.

Remark 2 More specifically, it is a discrete convolution product.

We can straightforwardly solve this by the generating series method. Substitute

$$C(z) = \sum_{n \in \mathbb{N}} C_n \, z^n \tag{36.10}$$

in (36.9).

We also know from Worked Example 2 that

$$C_0 = C(0) = 1 \, .$$

Making further use of Worked Example 2, we obtain

$$C(z)^2 = \frac{C(z) + 1}{z} \, .$$

This rearranges to

$$z \, C(z)^2 - C(z) + 1 = 0 \, .$$

Treating this as a quadratic equation in $C(z)$, we obtain

$$C(z) = \frac{1 \pm \sqrt{1 - 4\,z}}{2\,z} \, . \tag{36.11}$$

By the initial condition, we must discard the plus case and keep the minus case.

Furthermore, by the Binomial Theorem,

$$\sqrt{1 - 4\,z} = \sum_{n \in \mathbb{N}} \binom{\frac{1}{2}}{n} (-4\,z)^{n+1} \, .$$

Thus equating coefficients

$$C_n = -\frac{1}{2} \binom{\frac{1}{2}}{n} (-4)^{n+1} = 2^n \frac{(2\,n)!}{2^n \, n! \, (n+1)!} = \frac{1}{n+1} \frac{(2\,n)!}{(n!)^2} = \frac{1}{n+1} \binom{2\,n}{n} \, .$$

C_n is the nth *Catalan number*.

Naming Remark 2 Their most succinctly expressible feature being their recurrence relation, we elect to call Catalan numbers *convolution numbers*. We take it as implicit that, being discrete objects, the convolution in question is discrete.

Exercise 14 a) What are C_6 and C_7?

b) Show that the convolution numbers can also be expressed using combination symbols alone, as $\binom{2\,n}{n} - \binom{2\,n}{n+1}$.

c) Find an approximate formula for C_n when n is large.

Pointer 2 If interested in these numbers, or in any more of the many problems solved by them, see in particular [49, 71].

36.13 Relational variables for the RL-distinguished at-most binary trees

These configurations exhibit an overall mirror symmetry.

A suitable relative variable here is the '*chirality index*'

$$I \ := \ C_+ \pm C_- \ := \ R \ - \ L \tag{36.12}$$

for R the number of right children and L the number of left children. We also have straightforwardly the relation

$$S \ := \ R \ + \ L \ = \ N \ - \ 1 \ .$$

R and L can here be envisaged as mutually distinguishable and yet meaningless labels. This corresponds to a notion of [RL-at-most-binary-trees]. I has critical value 0. This value is necessary, and yet not sufficient, for an RL-tree to be reflection-invariant. This is already clear from the P_3 source-sink pair in Fig 33.2.

2) Characterizing the [RL-at-most-binary-trees] themselves can be done in terms of the '*unsigned chirality index*'

$$UI \ := \ |\, R \ - \ L \,| \ . \tag{36.13}$$

3) The *right–left fraction* is

$$\rho\lambda \ = \ \frac{R}{L} \ .$$

Some dilatational variables then are the *right* and *left fractions*

$$r \ := \ \frac{R}{S} \ = \ \frac{R}{R \ + \ L} \ ,$$

and

$$l \ := \ \frac{L}{S} \ = \ \frac{L}{R \ + \ L} \ ,$$

subject to

$$r \ + \ l \ = \ 1 \ .$$

4)

$$i \ := \ \frac{I}{S} \ = \ \frac{R \ - \ L}{R \ + \ L} \ = \ \frac{\rho\lambda \ - \ 1}{\rho\lambda \ + \ 1} \ .$$

5) Finally,

$$ui \ := \ \frac{UI}{S} \ = \ \frac{|\, R \ - \ L \,|}{R \ + \ L} \ = \ \frac{|\, \rho\lambda \ - \ 1 \,|}{\rho\lambda \ + \ 1} \ . \tag{36.14}$$

Remark 1 See Fig 36.5 for the rooted trees on 0 to 4 vertices.



36.14 Spanning trees

Remark 1 This follows on from Sec 34.1. It gives a way in which one or more trees appear in the study of every other graph.

Exercise 15 Find the possible distinct spanning trees for the cube graph of Fig 31.1.b).

Exercise 16 a) Prove that any finite connected graph admits a spanning tree.

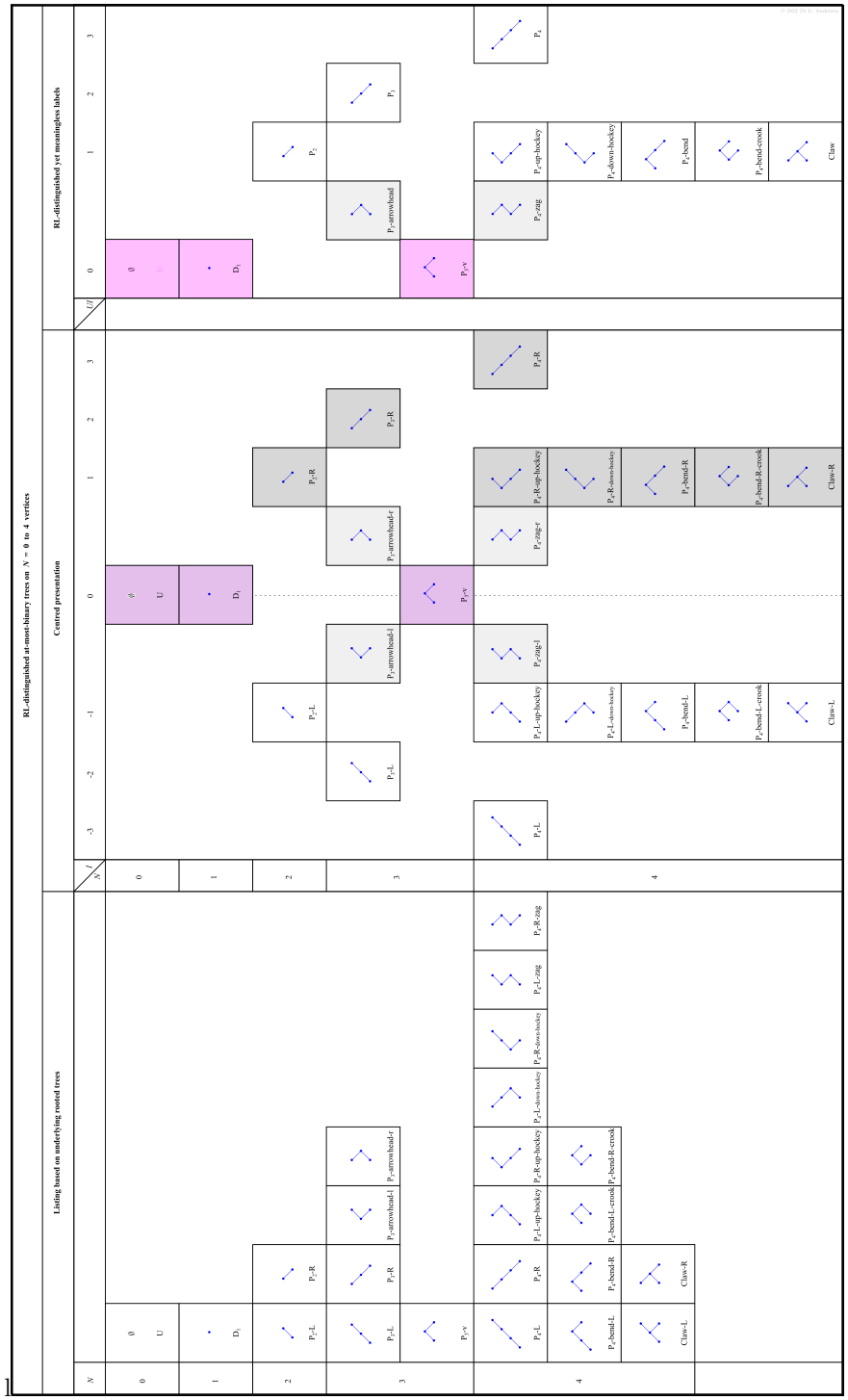


Figure 36.5:

b) What is the analogous result for the general graph?

Remark 2 The number of distinct spanning trees that a connected graph contains provides a further cardinal function $\#ST$ for graphs.

Exercise 17$^-$ a) What is $\#ST$ for each connected 4-graph, 5-graph and for the three smallest platonic graphs?

b) Propose an analogous cardinal function for the general graph.

36.15 Computer Science and Information Theory applications of trees

Pointer 3 The $N - 2$ vector in the proof that we gave for Theorem 5 is the so-called Prüfer code [129, 67, 49]. This gives, by the bijection established in that proof, a unique specification of each labelled tree. A truer name for it is *labelled-tree-vector*. Thereby, computers can fully encode labelled trees as just $N - 2$ vectors. This is more efficient than encoding graphs in general as $(N - 1) \times (N - 1)$ adjacency matrices, which we detail in Volume 1.

Exercise 18 a)$^-$ Starting our notation with (_) for P_2. (1), (2) and (3) for the 3 labellings supported by P_3. And (1, 2) for a first labelling of P_4. Find the Prüfer codes for all labelled trees on ≤ 5 vertices. Count out how many labellings each unlabelled tree supports.

b) Arrive at these counts by considering the symmetry groups of each of these unlabelled trees and performing an orbit-stabilizer analysis. '
c) By what distinctions between Prüfer codes can one uniquely pick out the various *unlabelled* trees on 5 vertices?

d)$^+$ Produce a counter-example for higher N to these distinctions' capacity to uniquely pick out the types of unlabelled tree.

Pointer 4 Rooted trees fully model, and can thus encode, and store all the possible forms that a sequence of events can take. This can be encoded by the following *universal address system*.

Firstly, multi-label the vertices as follows. Place a 0 label on the root vertex. Label its children 1 to A from left to right (say). Label the grandchildren via the leftmost child as 1.1 to $1.B_1$. Proceed similarly for its siblings: 2.1 to $2.B_2$ through to $A.1$ to $A.B_A$. And then for each of the great-grandchildren using a third label.

Secondly, declare that the following lexicographic order holds on these labellings. Order first by the leftmost label, second by the second-leftmost label (if any) and so on.

Pointer 5 Binary trees also model multi-step decision processes with yes/no answers. For instance, some Biological identification keys are of this form.

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Pointer 6 See e.g. [36] for tree models of *sorting* in Computer Science .

Exercise 19 How many ways can a string of n objects have brackets inserted into it?

Pointer 7 Binary trees model *searching* within a database [36]. In a search tree,
(all values in right subtree) > (sought value) > (all values in left subtree) .

Pointer 8 Spanning trees often suffice to detect high-degree nodes, which in turn are one of the key features in large networks (e.g. 'internet hubs').

Pointer 9 See Sec 38.6 for the notion of most economical spanning tree: an Optimization problem of much practical value.

Chapter 37

Independence, cover, domination... graph cardinals

37.1 Vertex independence and cover concepts

Definition 1 A vertex set $\mathfrak{I} \subseteq \mathfrak{V}(\mathrm{G})$ is *independent*, alias *internally stable* [183] in G if no pair v, w both in $\mathfrak{I}$ are adjacent.

Naming Remark 1 This is 'internal' because v and w are both to belong to $\mathfrak{I}$ itself. These vertices are independent in the sense of not being adjacent. It is too early on in this Series to discuss notions of stability.

Definition 2 The *set of independent sets* supported by a graph G is

$$\mathfrak{I}\mathrm{ndep}(\mathrm{G}) \ . \tag{37.1}$$

Definition 3 A vertex set $\mathfrak{C} \subseteq \mathfrak{V}(\mathrm{G})$ is a *vertex cover* if every $e \in \mathfrak{e}(\mathrm{G})$ is incident to at least one vertex of $\mathfrak{C}$.

Naming Remark 2 This is indeed a covering concept since this definition immediately means that all the edges of our graph G are included. With reference to Chapter 9, this is thus is a vertex cover in the sense of consisting of vertices that cover the edge set.

Definition 4 The *set of vertex covers* supported by a graph G is

$$\mathfrak{C}\mathrm{over}(\mathrm{G}) \ .$$

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Remark 1 The following gives a first reason to introduce these two notions as a pair.

Independence–Cover Complementation Lemma $\mathfrak{I} = \overline{\mathfrak{C}}$ is an independent set iff $\overline{\mathfrak{I}} = \mathfrak{C}$ is a vertex cover.

Proof ($\Rightarrow$) Let $\mathfrak{C}$ be a vertex cover. Then by definition

$$(\text{every } e \in \mathfrak{e}(\mathrm{G}))\ \mathfrak{r}\ (\text{some vertex } v \in \mathfrak{V}(\mathrm{G})) \ .$$

So no edge $\mathfrak{r}$ to each vertex $w \in \overline{\mathfrak{C}}$. Hence no pair of vertices $\in \overline{\mathfrak{C}}$ are adjacent. $\overline{\mathfrak{C}}$ thus meets the definition of independent set.

($\Leftarrow$) Let $\mathfrak{I}$ be an independent set. Then by definition every pair of vertices joined by an edge has $\geq$ 1 edge $\in \overline{\mathfrak{E}}$. So (every edge) $\mathfrak{x}$ some $v \in \overline{\mathfrak{I}}$. $\overline{\mathfrak{I}}$ thus meets the definition of vertex cover. □

Remark 3 E.g. Sec 37.5's results, König's Theorem [176, 441] and Hall's Pairs-Matching Theorem (Sec 37.6) give further reasons for introducing these notions side by side.

Exercise 1$^-$ Find independent sets and proper covers for all the connected graphs of order 5. Check that these and their complements satisfy the Complementation Lemma. ['Proper' here is to mean that it is not to cover all the edges simply by including all the vertices.]

37.2 Independence and cover numbers

Definition 1 An independent set $\mathfrak{I} \subseteq \mathrm{G}$ is *maximum* if $\nexists$ an independent set $\mathfrak{I}'$ such that $|\,\mathfrak{I}'\,| \geq |\,\mathfrak{I}\,|$.

Definition 2 A vertex cover $\mathfrak{C} \subseteq \mathrm{G}$ is *minimum* if $\nexists$ a vertex cover $\mathfrak{C}'$ such that $|\,\mathfrak{C}'\,| \leq |\,\mathfrak{C}\,|$.

Remark 1 This amounts to bringing in a pinch of Order Theory, which is simple enough to also be a pinch of Optimization.

Observe furthermore that this ascribes a type of *minimax* structure to the previous section's pair of concepts. The two parts of this are furthermore ordered such that, firstly, covers are as small as possible: a matter of efficiency. This is also the more interesting extremum, given that the full vertex set always covers the corresponding graph. Secondly, such that independent sets are as large as possible: a matter of interesting structure. The opposite extremum is now uninteresting since the empty set constitutes an independent set for any graph.

Involving opposite extrema is furthermore consistent with the Complementation Lemma.

Definition 3 The *independence number* alias *internal stability number*

$$\alpha(\mathrm{G}) := \#(\text{ vertices in a maximum independent set in G }) = \mathfrak{I} \in \max_{\mathfrak{I}\mathrm{ndep}(\mathrm{G})} (\,|\,\mathfrak{I}\,|\,) \,. \quad (37.2)$$

Definition 4 The *vertex covering number*

$$\beta(\mathrm{G}) := \#(\text{ vertices in a minimum vertex cover in G }) = \mathfrak{C} \in \min_{\mathfrak{C}\mathrm{over}(\mathrm{G})} (\,|\,\mathfrak{C}\,|\,) \,. \quad (37.3)$$

Remark 2 The above definitions amount to extracting a cardinal number quantifier from our Order-or-Optimization development. This Combinatorially captures what turns out to be a small amount of Topological and structural information of each graph.

37.3 Matchings and edge covers

Definition 1 A set $\mathfrak{M} \subseteq \mathfrak{V}(\mathrm{G})$ is an *independent edge set*, alias *matching* in G if the following holds. No pair e, f both in $\mathfrak{M}$ are incident on a common vertex.

Naming Remark 3 These pairs are what is getting matched up. The nonexistence of a vertex incident on both is the manner in which a notion of independence is being expressed here.

Definition 2 The *set of matchings* supported by a graph G is

$$\mathfrak{Match}(\mathrm{G}) = \mathfrak{Indep}_1(\mathrm{G}) \,.$$

Definition 3 A set $\mathfrak{C}_1 \subseteq \mathfrak{V}(\mathrm{G})$ is an *edge cover* if the following holds. Every $v \in \mathfrak{V}(\mathrm{G})$ such that $d(v) > 0$ meets at least one edge of $\mathfrak{C}_1$.

Definition 4 The *set of edge covers* supported by a graph G is

$$\mathfrak{Cover}_1(\mathrm{G}) \ .$$

Exercise 2 Show that edge covers and matchings also obey a Complementation Lemma (**Lemma 1**$_1$).

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Structure 1 Using the alias edge independence number for matchings, these Complementation Lemmas furthermore align as follows.

$$\overline{\mathfrak{C}}_i \ = \ \mathfrak{M}_i \text{ and } \ \overline{\mathfrak{M}}_i \ = \ \mathfrak{C}_i \ ,$$

where i runs 0-or-blank for vertices and 1 for edges. i is here the *democracy indexing variable* over the types of simplex realized in a graph: 0-simplex vertices and 1-simplex edges.

Naming Remark 4 I.e. by recognizing the 'Matching–Covering Complementation Lemma' as the 'Edge Independence–Cover Complementation Lemma', it can be jointly-summarized with the 'Vertex Independence–Cover Complementation Lemma' as the 'Democratic Independence-Cover Complementation Lemma'. We number this Lemma 1.d (with 'd' for democratic).

Remark 1 *Democracy invariance* for graphs involves structures and results for which uses of the words vertex and edge can be swapped. I.e. they are related by an $S_2 = C_2$ exchange symmetry.

Remark 2 Let us caution that only some Graph Theory structures can be profitably packaged into democratic pairs. Such profitability can furthermore depend on the context, e.g. further results are capable of breaking democracy invariance.

37.4 Matching number and edge covering number

Definition 1 A matching $\mathfrak{M} \subseteq \mathrm{G}$ is *maximal* if $\nexists$ a matching $\mathfrak{M}'$ such that $\left|\, \mathfrak{M}' \,\right| > |\, \mathfrak{M} \,|$.

Definition 2 An edge cover $\mathfrak{C}_1 \subseteq \mathrm{G}$ is *minimal* if $\nexists$ an edge cover $\mathfrak{C}_1'$ such that $|\, \mathfrak{C}_1' \,| < |\, \mathfrak{C}_1 \,|$.

Definition 3 The *edge-independence number* alias *matching number*

$$\alpha_1 \ := \ \#(\text{ edges in a maximum matching in } \mathrm{G}) \ = \ \mathfrak{M} \in \overset{\min}{\mathfrak{m}_{\mathfrak{atch}(\mathrm{G})}} |\, \mathfrak{M} \,| \ . \tag{37.4}$$

Definition 4 The *edge-covering number*

$$\beta_1 \ := \ \#(\text{ vertices in a minimum edge cover in } \mathrm{G}) \ = \ \mathfrak{C}_1 \in \overset{\max}{\mathfrak{c}_{\mathfrak{over}_1(\mathrm{G})}} |\, \mathfrak{C}_1 \,| \ . \tag{37.5}$$

37.5 Inter-relation between independence and cover counts

Lemma 2.d [183, 176]

$$\alpha_i \ + \ \beta_i \ = \ |\, \mathrm{G} \,| \ . \tag{37.6}$$

where i is the democratic indexing variable.

Naming Remark 5 The 1-version of i) is often called *Gallai's Theorem* i) as stated is then furthermore the 'Democratic Gallai Theorem'. A truer name for this is *Democratic Independence–Cover*

Minimax–Count–Complementation Lemma. This embodies our above chain of thought, as the result of cardinalizing the minimax structure that our original Complementation Lemma enables.

Proof . For $\mathfrak{I}_i \subseteq$ G a maximum independent set,

$$|\,\mathfrak{I}_i\,| \;=\; \alpha_i(\mathrm{G})\ ,$$

so

$$|\,\overline{\mathfrak{I}}_i\,| \;=\; |\,\mathrm{G}\,| \;-\; \alpha_i(\mathrm{G})\ .$$

But Lemma 1 $\Rightarrow$ $\overline{\mathfrak{I}_i}$ is a cover, so

$$|\,\overline{\mathfrak{I}}_i\,| \;\leq\; \beta_i(\mathrm{G})\ .$$

Combining these two equations,

$$\alpha_i \;+\; \beta_i \;\leq\; |\,\mathrm{G}\,|\ . \tag{37.7}$$

For $\mathfrak{C}_i \subseteq$ G a vertex cover,

$$|\,\mathfrak{C}_i\,| \;=\; \beta_i(\mathrm{G})\ .$$

So

$$|\,\overline{\mathfrak{C}}_i\,| \;=\; |\,\mathrm{G}\,| \;-\; \beta_i(\mathrm{G})$$

But Lemma 1 $\Rightarrow$ $\overline{\mathfrak{C}}_i$ is an independent set, so

$$|\,\overline{\mathfrak{C}}\,| \;\leq\; \alpha(\mathrm{G})\ .$$

Combining these two equations,

$$\alpha_i \;+\; \beta_i \;\geq\; |\,\mathrm{G}\,|\ . \tag{37.8}$$

Together, (37.7, 37.8) give (37.6). □

37.6 Why Pairs-Matching Theorem?

Naming Remark 6 Let $\mathfrak{D}$ be a set of Don Quixotes and $\mathfrak{S}$ be a set of Sancho Panzas. Suppose that the Sancho Panzas are only willing to squire for Don Quixotes that they know (and vice versa). So for each Sancho Panza, one can consider a $\mathfrak{D}_i\mathfrak{S}$ubset$\mathfrak{D}$ of Don Quixotes that they know. Then it is possible for all the Sancho Panzas to pair up with Don Quixotes that they know iff any k Sancho Panzas know between them at least k Don Quixotes.

Exercise 3 The previous two sentences are not manifestly symmetric under exchange of the distinguishable labels 'Don Quixote' and 'Sancho Panza'. Show however that the label-exchanged version of these sentences is equivalent. Also show that Naming Remark 6's claim is true by figuring out how it is an equivalent statement to Hall's Pairs-Matching Theorem. [1]

Exercise 4$^+$ Consequently find an alternative Graph-Theoretical proof for the Theorem.



[1] Indeed, this provides the reason why we term it 'pairs-matching'. This is becoming rather more convenient to justify than the term 'Marriage Theorem', which remains more common in the literature.

37.7 Vertex domination

Definition 1 A subset $\mathfrak{D}$ of vertices in a graph G is a *vertex dominating set* alias *vertex external stability number* if every vertex not therein is adjacent to at least one which is.

Naming Remark 7 This is external in the sense that each internal vertex is now considered in relation to each external vertex. I.e. it is mutual internal–external. In contrast, internal stability is self-internal: its two input vertices are both internal. It is dominating in the sense that if occupying its vertices, one is adjacent to all other vertices. So one could move to anywhere else in the graph in just a single edge move.

Definition 2 The *set of vertex dominating sets* supported by a graph G is

$$\mathfrak{Domina}(\mathrm{G}) \ .$$

Definition 3 A dominating vertex set $\mathfrak{D}$ is *minimum* if $\nexists$ any dominating vertex set $\mathfrak{D}'$ such that $\left| \mathfrak{D}' \right| \ < \ |\mathfrak{D}|$.

Definition 4 The *vertex domination number* alias *vertex external stability number* is

$$\gamma(\mathrm{G}) \ := \ \#(\text{ vertices in a minimum dominating vertex set in } \ \mathrm{G}\,) \ = \ \mathfrak{D} \in \overset{\min}{\mathfrak{Domina}(\mathrm{G})} \ (\,|\,\mathfrak{D}\,|\,) \ .$$

37.8 Edge domination

Definition 1 A subset $\mathfrak{D}_1$ of edges in a graph G is a *dominating edge-set* alias *external-stability edge-set* if every edge not therein shares a vertex with an edge which is.

Definition 2 The *set of dominating edge-sets* supported by a graph G is

$$\mathfrak{Domina}_1(\mathrm{G}) \ .$$

Definition 3 A dominating edge set is *minimum* if $\nexists$ any dominating edge-set $\mathfrak{D}_1'$ such that $\left| \mathfrak{D}_1' \right| \ < \ |\mathfrak{D}_1|$.

Definition 4 The *edge-domination number* alias *external-stability edge number* is

$$\gamma(\mathrm{G}) \ := \ \#(\text{ edges in a minimum dominating edge-set in } \ \mathrm{G}\,) \ = \ \mathfrak{D}_1 \in \overset{\min}{\mathfrak{Domina}_1(\mathrm{G})} \ (\,|\,\mathfrak{D}_1\,|\,) \ . \tag{37.9}$$

Lemma 3

$$\gamma_i \ \leq \ \alpha_i \ . \tag{37.10}$$

Naming Remark 8 This is the Democratic Independence–on–Domination Bound. It is also the Democratic Internal-on-External Stability Bound. (It and its vertex and edge subcases are not usually given any name).

Proof Proceed via a convenient intermediary: any subset $\mathfrak{a}_i$ which is both dominating and independent. Then

$$\gamma \ := \ |\,\mathfrak{D}_i\,| \ \leq \ |\,\mathfrak{a}_i\,| \ \leq \ |\,\mathfrak{I}_i\,| \ := \ \alpha \ .$$

The first inequality calls upon $\mathfrak{a}_i$ being a dominating set and thus minimized by $\mathfrak{D}_i$. The second inequality calls instead upon $\mathfrak{a}_i$ being an independent set and thus maximized by $\mathfrak{I}_i$. □

Exercise 5 a) Find the democratic 'independence and domination' numbers for all the connected graphs of order $\leq$ 4. Also write down the corresponding cover numbers. What is the smallest

order for which γ_i does not suffice to distinguish between connected graphs? For α_i not to? For both together not to?

b) For a given G, while maximum independent sets are to be of a unique size for each of edges and vertices they need not be unique. Show that this is the case for the graph in Fig 37.1. What is the smallest graph exhibiting this ambiguity? [Excluding merely isomorphic non-uniqueness.] How many of the graphs with the same V but larger E also exhibit it?

c) Introduce α_i' and γ_i' as the number of distinct unlabelled realizations of the maximum independent set and the minimal domination set. Does this improve the last answer to a)?

Our overall conclusion is that we need further independent cardinals for graphs.

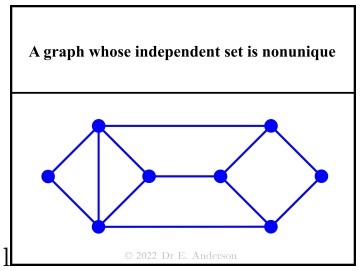


Figure 37.1:



37.9 Clique and intersection numbers

Definition 1 The *clique number* $\omega(\mathrm{G})$ is the size of the largest clique (K_n subgraph) in a graph.

Definition 2 The *intersection number* $\iota(\mathrm{G})$ is the minimum number of elements in a set $\mathfrak{X}$ such that G is a graph representing the intersections of a family of subsets of $\mathfrak{X}$.

Remark 1 These are presented together here because intersection number is conceptually a *clique cover* number.

Exercise 6 Convince yourself that this is a valid point of view.



37.10 Independence, cover, domination, clique and intersection fractions

Let $|\,\mathrm{G}\,| \geq 1$.

Definition 1 The democratic *independence fraction* alias *internal stability fraction*

$$a_i := \frac{\alpha_i}{N} .$$

Definition 2 The democratic *covering fraction*

$$b_i \ := \ \frac{\beta_i}{N} \ .$$

Definition 3 The democratic *domination fraction* alias *external stability fraction*

$$c_i \ := \ \frac{\gamma_i}{N} \ .$$

Notation Remark 1 Why do we use γ and c for domination? δ and d have already been taken up by commonly encountered – and quite often juxtaposed – notions of degree.

We also need the following non-independent ratio.

Definition 4

$$\mathcal{C}_i \ := \ \frac{c_i}{a_i}$$

is the democratic *external-to-internal stability ratio*.

♠ ◇ ♣

Proposition 1 i) a_i, b_i and c_i run over the standard range.

ii)

$$a_i \ + \ b_i \ = \ 1 \ . \tag{37.11}$$

ii)

$$\mathcal{C}_i \ \leq \ 1 \ . \tag{37.12}$$

Proof The ungraph being excluded, i) follows from the definitions of these quantities. Also ii) and iii) follow from Lemmas 1 and 2 respectively upon dividing by $|\,\mathrm{G}\,|$ to form fractional quantities. □

Remark 1 i) signifies that a_i and b_i are collectively-exhaustive and thus partial variables (see Appendix A).

Remark 2 We can thus rewrite

$$a_i \ = \quad = \ \frac{\alpha_i}{\alpha_i \ + \ \beta_i}$$

and

$$b_i \ = \quad = \ \frac{\beta_i}{\alpha_i \ + \ \beta_i} \ .$$

Remark 3 Using the ratio form requires explicit exclusion of the trivial case of the ungraph, since dividing by the order of the graph would then entail division by 0.

Naming Remark 9 We call this form Democratic Independence-Cover Fraction-Exhaustion Lemma, complementation reducing to exhaustion in the case of fractions.

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Definition 1 The *clique fraction*

$$w \ := \ \frac{\omega}{N} \ .$$

Definition 2 The *intersection fraction*

$$i \ := \ \frac{\iota}{E} \ .$$

Remark 3 For G connected, this takes the standard range

See e.g. [183, 195] for further consideration of this Chapter's topics.

Aside 1 Arrow-graphs come with separate notions of in- and out- dominating.

Chapter 38

Notions of distance on graphs

38.1 Path length for graphs

Definition 0 The *length* $l(\mathrm{P})$ *of a path* P (including when realized as a subgraph) is the number of edges that it contains. The *length* $l(\mathrm{C})$ *of a cycle* C is defined likewise.

Definition 1 The *distance* between vertices x and y in a graph G [160, 157, 183] is given by the following.

$$\mathrm{d}(x, y) \;=\; \min_{\text{paths P}\,:\;x \longrightarrow y} \left(l(\mathrm{P}) \right) \;. \tag{38.1}$$

[For x and y not path-connected, the standard convention is that $\mathrm{d}(x, y) \;:=\; \infty$.]

38.2 Notions of extent of centrality

Definition 1 The *eccentricity* $e(v)$ of a vertex v [160, 157, 183] is given by the following.

$$ecc(v) \;:=\; \max_{w \,\in\, \mathfrak{V}(\mathrm{G})} \left(d(v, w) \right) \;. \tag{38.2}$$

Remark 1 Concatenating (38.1) and (38.2),

$$ecc(v) \;=\; \max_{w \,\in\, \mathfrak{V}(\mathrm{G})} \left(\min_{\text{paths P}\,:\;x \longrightarrow y} \left(l(\mathrm{P}) \right) \right) \;.$$

Definition 2 The *radius* $r(\mathrm{G})$ of a graph G [160, 157, 183] is given by the following.

$$rad(\mathrm{G}) \;:=\; \min_{v \,\in\, \mathfrak{V}(\mathrm{G})} \left(ecc(v) \right) \;. \tag{38.3}$$

Remark 2 Concatenating (38.2) and (38.3),

$$rad(\mathrm{G}) \;=\; \min_{v \,\in\, \mathfrak{V}(\mathrm{G})} \left(\max_{w \,\in\, \mathfrak{V}(\mathrm{G})} \left(\min_{\text{paths P}\,:\;x \longrightarrow y} \left(l(\mathrm{P}) \right) \right) \right) \;. \tag{38.4}$$

Pointer 1 Observe the alternation of the min and max here. This is another common feature of *minimax mathematics*, to Chapter 37's pairing of a min concept with an allied max concept. Minimax mathematics carries further connotations of duality (in fact already visible in floor and ceiling, which are not-quite minimax alternations: Sec 11.3).

Definition 3 A vertex v is *central* [160, 157, 183] if

$$ecc(v) \;=\; rad(\mathrm{G}) \;. \tag{38.5}$$

The *centre* of G is the set of central vertices

$$\mathfrak{C}(\mathrm{G}) \;:=\; \{\, v \;\in\; \mathfrak{V}(\mathrm{G}) \,|\, ecc(v) \;=\; rad(\mathrm{G}) \,\}\;. \tag{38.6}$$

♠ ◇ ♣

Definition 4 The *diameter* $diam(\mathrm{G})$ of a graph G [183] is

$$diam(\mathrm{G}) \;:=\; \max_{v \,\in\, \mathfrak{V}(\mathrm{G})} \left(ecc(v) \right)\;. \tag{38.7}$$

Remark 3 Concatenating (38.2) and (38.7),

$$diam(\mathrm{G}) \;=\; \max_{w,\, v \,\in\, \mathfrak{V}(\mathrm{G})} \left(\min_{\text{paths P} \;:\; x \,\longrightarrow\, y} \left(l(\mathrm{P}) \right) \right)\;.$$

♠ ◇ ♣

Remark 4 Radius and diameter obey

$$rad \;\leq\; diam \;\leq\; 2\, rad\;. \tag{38.8}$$

Remark 5 There is some interest in which graphs saturate either of these inequalities. For instance, P_2, C_3, C_4 and K_4 saturate the first. While P_3, Claw, Paw and Diamond saturate the second. Leaving P_4 as the sole connected graph on $\leq\; 4$ vertices with an intermediate-sized diameter.

Exercise 1^{-} Which 5-graphs have the largest possible diameter, and which have the smallest possible diameter?

Exercise 2 What are the diameters of Wheel(n), Sun(n), Grid(m, n) and Web(n, m)?

♠ ◇ ♣

Definition 5 A vertex v is *peripheral* [160, 157, 183] if

$$ecc(v) \;=\; diam(\mathrm{G})\;. \tag{38.9}$$

Definition 6 The *periphery* of G is the set of peripheral vertices

$$\mathfrak{p}\mathrm{eriph}(\mathrm{G}) \;:=\; \{\, v \;\in\; \mathfrak{V}(\mathrm{G}) \,|\, ecc(v) \;=\; diam(\mathrm{G}) \,\}\;. \tag{38.10}$$

Definition 7 Let us term a vertex *penultimately peripheral* if

$$ecc(v) \;=\; diam(\mathrm{G}) \;-\; 1\;. \tag{38.11}$$

The *penultimate periphery* of G is the set of vertices

$$\mathfrak{p}\mathrm{eriph}_2(\mathrm{G}) \;:=\; \{\, v \;\in\; \mathfrak{V}(\mathrm{G}) \,|\, ecc(v) \;=\; diam(\mathrm{G}) \;-\; 1 \,\}\;. \tag{38.12}$$

Remark 6 One can furthermore partition each graph into *ecentricity shells* running from central to peripheral cases. Ones in between these two extremes could be termed *middling vertices*.

Exercise 3 a) Construct a smallest possible graph that has three layers, i.e. more than just a centre and a periphery.

b) Repeat for four layers.

c) Repeat a) and b) under the further condition that the graph must contain at least two cycles.

[Consider 'smallest' here to refer to lexicographic order, with graph order counting first, and graph size counting second.]

38.3 Distance ratios for graphs

Definition 1 The *diameter per radius*

$$\mathcal{Z} \ := \ \frac{diam}{rad} \ . \tag{38.13}$$

Definition 2 The *acentrality*

$$\overline{\mathcal{C}} \ := \ \frac{rad(\mathrm{G})}{ecc(v)} \ . \tag{38.14}$$

Definition 3 The *aperipherality*

$$\overline{\mathcal{P}} \ := \ \frac{ecc(v)}{rad(\mathrm{G})} \ . \tag{38.15}$$

Remark 1 These last two are measures of departure from centrality and from peripherality respectively.

Remark 2 These are both standard-range, taking value 1 in the central and peripheral cases respectively.

Remark 3 On the other hand,

$$1 \ \leq \ \mathcal{Z} \ \leq \ 2 \tag{38.16}$$

follows from (38.8).

♠ ◇ ♣

Structure 1 If we consider the *diameter-to-radius increment*

$$\zeta \ := \ \frac{diam \ - \ rad}{rad} \ = \ \mathcal{Z} \ - \ 1 \ , \tag{38.17}$$

however, we arrive at a further standard-range variable.

Exercise 4 What ranges of values of $\mathcal{Z}$, $\overline{\mathcal{C}}$ and $\overline{\mathcal{P}}$ can be realized by a) 5-graphs and b) 6-graphs?

38.4 Leaves and twigs as notions of peripherality

Remark 1 The notion of leaves introduced in chapter 36 can also be viewed as an alternative notion of peripherality: the outermost layer of vertices of a graph.

Remark 2 Extending one layer further inward in this sense ('defoliating') we arrive at the following further notions and cardinals.

Definition 1 The *twig* vertices are the leaves of the graph obtained by a single defoliation.

Definition 2 *Twig number*

$$\mathrm{T} \ := \ \#(\text{ penultimately terminal vertices }) \ . \tag{38.18}$$

Definition 3 The *p-twig* vertices are the leaves of the graph obtained after 'serially defoliating' out p layers.
Structure 1 *Serial defoliation* is recursive removal of leaves until a graph with no leaves is obtained. This has the effect of *desidechaining* and, more generally, of *desidetreeing* any finite connected graph down to its core *cycle system*.

This might still be split into disjoint combinations of cycles, the manner that the Bridge graph minimally exhibits. Namely a tree in which some vertices are blown up into cycle subsystems. All

terminal vertices are blown up thus (for if not, there would still be leaves present). Bridge is then P_2 with both terminal vertices blown up to triangles.

Definition 4 *p-Twig number*

$$T := \#(\text{ vertices } p \text{ layers from being terminal }) . \tag{38.19}$$

Remark 3 p-twigs provide a further partition of each given graph. I call this the *hedge sculptor's partition*, since it is based on shearing off external layer after external layer until all vertices are accounted for.

Exercise 5 Give the hedge sculptor's partition for

a)$^-$ all the trees on 7 vertices.
b) A 16-graph for which our shearing off of layers terminates with a 10-graph.

♠ ◇ ♣

Definition 5 The *defoliation number* $F(v)$ of a vertex is the number of defoliations required for it to become a leaf.

Remark 4 So $F(v) = 0$ for leaves, $F(v) = 1$for twigs and $F(v) = p$ for p-twigs.

♠ ◇ ♣

Definition 6 *n-ary leaves* are those emerging in a group of n from certain twigs. *n-ary twigs* are those emerging in a group of n from certain 2-twigs. *n-ary p-twigs* are those emerging in a group of n from certain $p + 1$-twigs.

Definition 7 The *n-leaf number* L_n and *n-twig number* T_n of a graph are the number of n-leaves and n-twigs it contains respectively.

Remark 5 Each n-ary version is a more refined partition of the corresponding plain version.

$$T = \sum_{n=1}^{m} T_n$$

Remark 6 One can furthermore define maximal leaf, twig and p-twig n-arities. However, maximal n-arity of p-twigs for all p at once just returns one short of the maximal degree, $\Delta - 1$.

Remark 7 Leaf and twig number are most useful for trees. This is since these have little structure apart from leaf, twig and branching point related information. leaf and twig number can be defined for any graph. Leaf and twig number can however be defined for every graph.

38.5 Twig fractions

Definition 1 *Twig fraction*

$$t := \frac{T}{|G|} .$$

Non-twig fraction

$$t^c := \frac{T^c}{|G|} .$$

Remark 1 By (8.2), these obey

$$t + t^c = 1 .$$

Definition 2 *Twig significance*

$$\mathcal{T} \;:=\; \frac{\#(\,\text{twig vertices}\,)}{\#(\,\text{non-twig vertices in }\Sigma\,)} \;=\; \frac{\mathrm{T}}{\overline{\overline{\mathrm{T}}}} \;=\; \frac{\mathrm{t}}{\bar{\mathrm{t}}}\;. \tag{38.20}$$

Correspondingly, *n-leaf density*

$$\mathrm{l_n} \;:=\; \frac{\mathrm{L_n}}{|\,\mathrm{G}\,|}$$

and *n-twig density*

$$\mathrm{t_n} \;:=\; \frac{\mathrm{T_n}}{|\,\mathrm{G}\,|}\;. \tag{38.21}$$

Definition 3 The *fraction of leaves that are n-ary* is

$$\mathrm{l}'_\mathrm{n} \;:=\; \frac{\mathrm{L_n}}{\mathrm{L}} \;=\; \frac{\mathrm{l_n}}{\mathrm{l}}\;.$$

The *fraction of twigs that are n-ary* is

$$\mathrm{t}'_\mathrm{n} \;:=\; \frac{\mathrm{T_n}}{\mathrm{T}} \;=\; \frac{\mathrm{t_n}}{\mathrm{t}}\;.$$

So also

$$\mathrm{t} \;=\; \sum_{\mathrm{n}\,=\,1}^{\mathrm{m}} \mathrm{t_n}$$

and

$$\mathrm{l} \;=\; \sum_{\mathrm{n}\,=\,1}^{\mathrm{m}} \mathrm{l_n} \;=\; \sum_{\mathrm{n}\,=\,1}^{\mathrm{m}} \mathrm{n\,t_n}\;.$$

Exercise 6 What range of values can each of the quantities in this section take within the set of all possible 8-trees?

38.6 Studying trees on up to 21 vertices

♠ ◇ ♣

Exercise 7 and Naming Remark 1 The smallest tree that is not starlike is Quadruped (Fig 38.1.a) [answering Exercise 36.1.b)]. Let us now define the *generalized quadruped* graph to be

$$\text{Quadruped}\,(\,b;\,a_1,\,a_2;\,l_1,\,l_2\,)$$

for body length b arm lengths a_i and leg lengths l_i (Fig 38.1.b).

a) Which generalized quadrupeds are realized for $N \;\leq\; 9$?

b) What regularity condition do quadruped trees obey?

c) What is the general form for the Prüfer code of a 3-pointed starlike graph? Of an N-pointed starlike graph? Of a generalized quadruped?

Exercise 8 Which trees on 6 and 7 vertices cannot be distinguished using L, T and T_2? Which are not distinguished by L' and T' either? Show that B distinguishes these, but that the combined branching and twig cardinals fail to distinguish one pair of $N \;=\; 8$ trees.

Exercise 9 = Naming Remark 2 A *caterpillar* is a tree consisting of at most a path and some leaves, here viewed as legs or hairs on the caterpillar. A *lobster* is a tree for which leaf removal

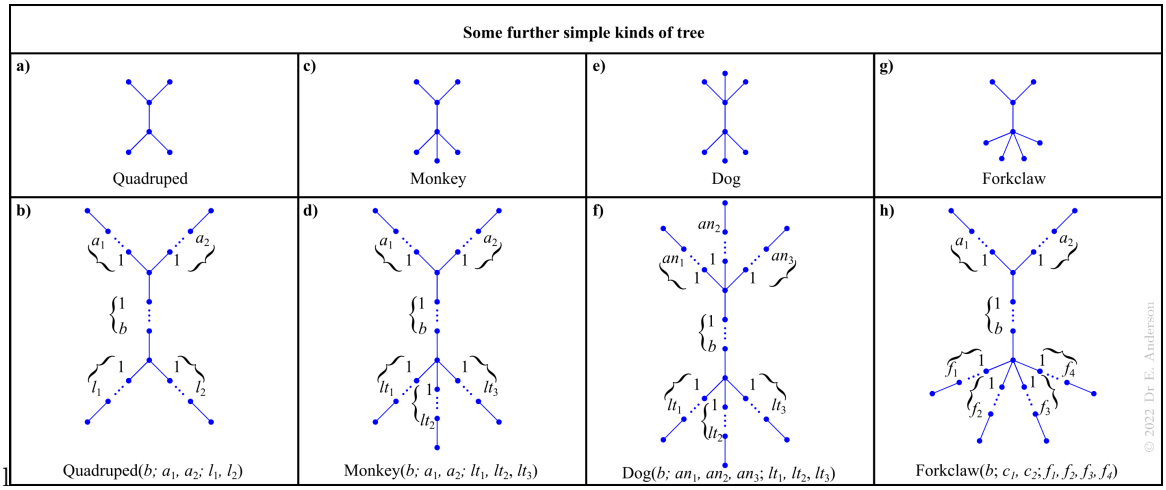


Figure 38.1:

yields a caterpillar. It may or may not itself be a caterpillar. Let us use *nontrivial caterpillar* for caterpillars that are not just paths. And *nontrivial lobster* for lobsters that are not just caterpillars.

Construct:

a) The eight smallest nontrivial caterpillars.

b) The smallest nontrivial lobster.

c) The smallest non-lobster.

d) The smallest four caterpillars that are not starlike.

e) The smallest nontrivial lobster that is not starlike.

f) The smallest two non-lobsters that are not starlike.

g) The smallest asymmetric tree.

h) The smallest asymmetric lobster.

i) The smallest asymmetric non-lobster.

Exercise 10 Find at least two of the smallest nontrivial asymmetric graphs.

Exercise 11 = Naming Remark 3 Show that the only tree on 7 vertices which is not a generalized star or quadruped is Monkey (Fig 38.1.c). Define the *generalized monkey* graph to be

$$\text{Monkey}\,(b;\, a_1;\, a_2;\, lt_1,\, lt_2,\, lt_3)\ ,$$

where the tl_j are leg-tails (i.e. either legs or tails, treated as indistinguishable).

a) What are the generalized monkey graphs on up to 9 vertices?

b) What regularity condition do monkey trees obey?

c) What is the general form for the Prüfer code of a generalized monkey graph?

Exercise 12^+ = Naming Remark 4 Show that the only trees on 8 vertices that are not generalized stars, quadrupeds or monkeys are Dog (Fig 38.1.e), Forkclaw (Fig 38.1.g), and Scorpion (Fig 38.2.a).

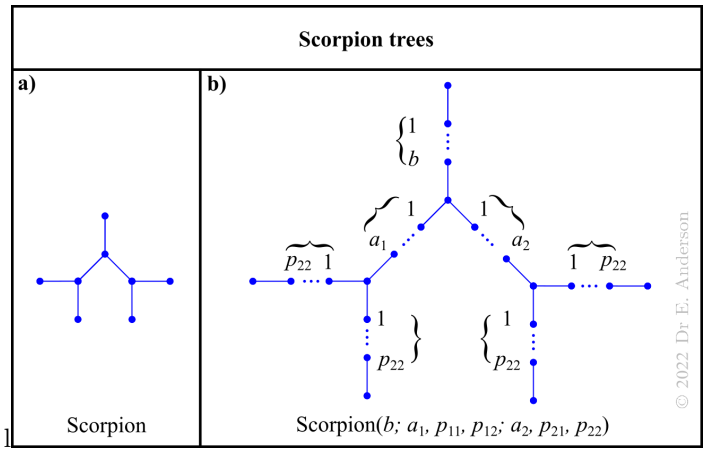


Figure 38.2:

a) Define the *generalized dog* graphs (Fig 38.1.f) to be

$$\text{Dog}\,(b;\, an_1,\, an_2,\, an_3;\, lt_1,\, lt_2,\, lt_3)$$

where the *an* are arm-noses. What are the generalized dog graphs on up to 11 vertices?

b) Define the *generalized Forkclaw* graphs (Fig 38.1.e) to be as

$$\text{Forkclaw}\,(b;\, c_1,\, c_2;\, f_1,\, f_2,\, f_3,\, f_4)$$

for claw nail lengths c_i and fork prong lengths f_k. What are the generalized Forkclaws on up to 10 vertices?

c) Define the *generalized scorpion* graphs to be

$$\text{Scorpion}\,(t\,|\,a_1;\, p_{11},\, p_{12}\,|\,a_2;\, p_{21},\, p_{22})$$

for tail length t, and pincer lengths p_{ij} belonging to the corresponding arms of length a_i (Fig 38.2;b). What are the generalized Scorpions on up to 10 vertices?

d) What are the regularity conditions for each of these families of tree?

e) What is the general form for the Prüfer code for each of these families of tree?

♠ ◇ ♣

Remark 1 Generalized star, quadruped and monkey graphs are examples of *metric parametrizations* of families of graphs. How many distinct such there are for a given N is partly determined by considering partitions. Here the 'body parts' that each parameter measures the length of one of play the role of boxes.

Worked Example 1 Show that this analysis is sufficient for starlike graphs, thus giving a general formula for the number of starlike graphs with R radiances on N vertices.

Answer 1 This analysis is sufficient for the starlike graphs, because there all the 'body parts' are radiances from a single centre and are thus equivalent. Thus the number of these is the number of ways of partitioning the $n := N - 1$ non-nexus vertices between the R radiances,

$$\text{Star}(N, R) = p(n, R) = \left\{ \begin{array}{c} n \\ R \end{array} \right\} .$$

The last equation here following from Chapter 16. This is in the sense that e.g. quadruped has indistinguishable legs, indistinguishable arms and arm–leg pair exchange symmetry. Monkey has indistinguishable tail-legs and indistinguishable arms; but with these being a triple and a pair respectively, there is not a further arm–tail-leg exchange symmetry.

Remark 2 The above case's simplicity is rooted on the stated equivalence. Which in turn follows from the underlying star graph enjoying the full S_n group of symmetries acting as exchange of radiences. More generally, only a proper subgroup of S_n will be realized.

Worked Example 2 Illustrate this by the smallest N for which partitions do not suffice among the generalized quadrupeds.

Answer 1 The minimal counterexample occurs for $N = 7$:

$$\text{Grandparents} \overset{\text{graph}}{\cong} \text{Swordfighter} .$$

Now if all the parameters were inequivalent, there would be 5 cases. But 4 of them – lengthening 1 arm or 1 leg – produce the same case: Swordfighter. The other - lengthening the body instead – gives Grandparents. So overall there are just 2 cases. A systematic way of analyzing such outcomes is as follows.

Firstly observe that

$$Sym(\text{Quadruped}) = C_2 \times C_2 \times C_2 .$$

For the arms are exchangeable, the legs are exchangeable, and the labels 'arms' and 'legs' are themselves meaningless and so can be exchanged. And each of these exchanges independently contributes its own $S_2 \overset{\text{group}}{\cong} C_2$ factor.

Secondly, observe that

$$Sym(\text{Swordfighter}) = C_2 ,$$

corresponding to exchange of the legs (still both of length 1). Then viewing this as a 'little group' subgroup of Sym(Quadruped), i.e. a stabilizer, the corresponding orbit must be of size 4. For Grandparents, however, the full Sym(Quadruped) is retained. Thus the corresponding orbit must be of size 1.

Exercise 13$^-$ Perform a corresponding analysis for the $N = 8$ and 9 quadrupeds.

Exercise 14 [Long] For each of the generalized a) Monkey, b) Dog, c) Forkclaw and d) Scorpion families, identify the following. i) The symmetry group of the defining graph.
ii) All symmetry breakings that its generalized family can manifest.

iii) The count of such graphs for $N \leq 12$.

iv)$^+$ The minimal examples for all cases of symmetry breaking, whether or not $N \geq 13$ is necessary.

♠ ◇ ♣

Exercise 15 a) Find all the ternary trees up to $N = 16$.

b) Find all the quad trees up to $N = 21$.

Exercise 16 Find the number of non-isomorphic spanning trees $\#ST$ supported by the icosahaedron and dodecahaedron graphs.

Exercise 17$^+$ Find a way of constructing spanning trees which preserves the underlying graph's radius.

Exercise 18 Suppose that edge x-y of a graph costs $c(x, y)$ to build. Then an *economical spanning tree* is one that minimizes $\sum_{x\text{-}y \in \mathfrak{C}} c(x, y)$. $c(x, y)$ can be viewed as supplanting the graph's natural metric $d(x, y)$.

a) Show that for

$$c(x, y) = d(x, y) ,$$

all of a graph's spanning trees are equally economical.

Practical reasons to use a nontrivial $c(x, y)$ include that road cost roughly scales with the metric length of the road. That between cities at a given distance as the crow flies from each other, mountain roads will be more expensive per unit crow-metric distance. For they will have to wind about to avoid or accessibly climb the mountains, or perhaps tunnels would have to be drilled through mountains. And that river-traversing roads would be more expensive as well, due to needing bridges or tunnels.

b) The country of Nova Icosia consists of an island in a lake and the shores surrounding the lake, as per Fig 38.3.a). The Nova Icosians wish to expand some of their roads to form a highway interlinking all 20 of their cities. Expanding roads between the island and the mainland will require replacing the existing bridges with larger bridges. To first approximation, these parts of the road will cost 110 million icos each, while expanding each other section of road will cost 20 million icos. Find a most economical road. How non-unique is it? [Treat Nova Icosia's cities as distinguishable vertices!]

c)$^+$ Having realized that there are quite a lot of possibilities to that level of approximation, the Nova Icosians decide to carry out a more detailed survey. This reveals the cost estimates in Fig 38.3.b) (in millions of icos). This reflects that Nova Icosia's outer, middle and inner cycles of roads are each shorter than the previous, while their outer radial roads are shortest. That Nova Icosia is slightly hilly around the edges. And that in some parts, land adjacent to the existing road would not need to be purchased. Show that there is now a unique cheapest road.

d)$^+$ Is this uniqueness a fluke?

38.7 Depth of a rooted tree

Remark 1 In a rooted tree ${}^r\mathrm{T}$, the presence of a distinguished vertex – the root r – points to the following absolute notion of distance.

Definition 1 The *depth* of a vertex v in a rooted tree ${}^r\mathrm{T}$ is

$$depth(v) := dist(r, v) . \tag{38.22}$$

Remark 2 Within a tree, distance between two vertices is unambiguous without having to specify a minimization. For if there were two ways to get between a vertex pair, then a cycle would be present, by which the graph would not be a tree.

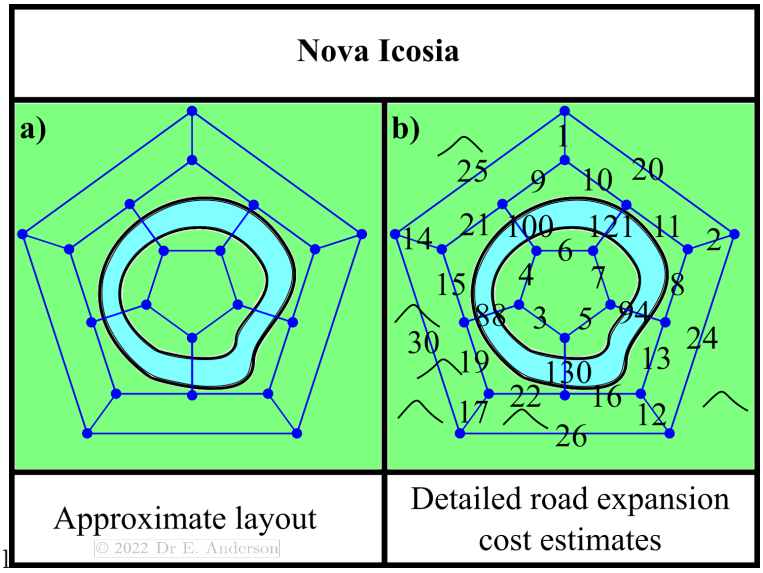


Figure 38.3:

Definition 2 The *depth* of a rooted tree is

$$depth(^r\mathrm{T}) \;=\; \overset{\max}{v \in \mathrm{T}} \left(depth(v) \right) \; . \tag{38.23}$$

Remark 3 Concatenating,

$$depth(^r\mathrm{T}) \;=\; \overset{\max}{v \in \mathrm{T}} \left(dist(r, v) \right) \; . \tag{38.24}$$

Naming Remark 5 If the root is positioned at the bottom, depth becomes *height*.

Example 1 In Grandparents with the child viewed as the root, the parents are of depth 1, and the grandparents of depth 2. This rooted tree is thus of depth 2.

Exercise 19 a) What are the depths of Monkey and Scorpion, in each case as rooted by the tail?

b) For trees, let us also introduce

$$\Delta\mathrm{epth}(\mathrm{T}) \;:=\; \overset{\max}{r \in \mathfrak{V}(\mathrm{T})} \left(depth\left(^r\mathrm{T}\right) \right) \;=\; \overset{\max}{r \in \mathfrak{V}(\mathrm{T})} \left(\overset{\max}{v \in \mathrm{T}} \left(dist(r, v) \right) \right) \; .$$

And the minimax

$$\delta\mathrm{epth}(\mathrm{T}) \;:=\; \overset{\min}{r \in \mathfrak{V}(\mathrm{T})} depth\left(^r\mathrm{T}\right) \;=\; \overset{\min}{r \in \mathfrak{V}(\mathrm{T})} \overset{\max}{v \in \mathrm{T}} \left(dist(r, v) \right) \; .$$

What are Δepth and δepth for Quadruped, Monkey, Scorpion, Dog and Lobster?

c) Which notion already encountered are Δepth and δepth subcases of?



38.8 Cycle cardinals

Definition 1 The *cycle length* $\mathrm{l}(S)$ of a cycle S in a graph G is the number of edges that form the cycle.

Definition 2 The *girth* of a graph G is

$$g \;:=\; \min_{\text{cycles } S \;\in\; \mathrm{G}} (\,\mathrm{l}(c)\,)\;. \tag{38.25}$$

Example 1

$$g(\mathrm{C}_n) \;=\; n \;=\; g(n\text{-Pan})\;.$$

Example 2

$$g(\mathrm{K}_n) \;=\; 3\;.$$

Exercise 20 Find $g(n$-Fan), $g(n$-Wheel), g(Petersen) g(Chvátal) and g(Heawood) (see Fig 38.4 for these last three graphs).

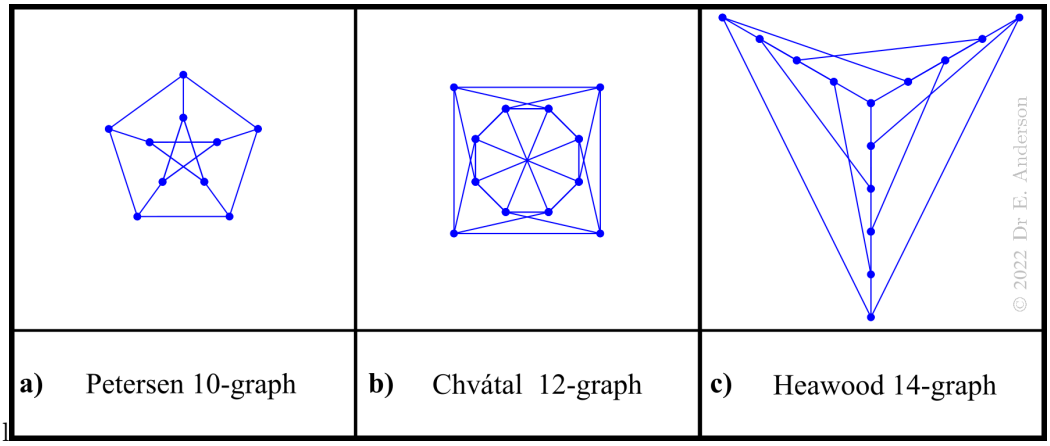


Figure 38.4:

Exercise 21$^+$ and Pointer 2 a) Show that the Petersen graph is moreover a *cage*: a smallest cubic graph to have a given girth (here 5).

[It is somewhat harder to show that the Heawood graph is also a cage; finding cages remains an active area of research [199].]

b) What are the cubic cages of girth 3 and 4? Can you generalize to the $g \;-\; k \;=\; 0$ and 1 graphs of girth g and regularity k?

♠ ◇ ♣

Definition 3 The *circumference* of a graph G is

$$C \;:=\; \max_{\text{cycles } S \;\in\; \mathrm{G}} (\,\mathrm{l}(S)\,)\;. \tag{38.26}$$

Lemma 1

$$3 \;\leq\; g \;\leq\; l \;\leq\; C \;\leq\; |\,\mathrm{G}\,|\;. \tag{38.27}$$

Proof The minimal cycle length in a simple graph is 3. g is of a cycle, so its length is bounded by this. g is the minimal cycle length manifested in G. So it bounds from below the length l of any

other cycle in G. C is the maximal cycle length manifested in G, and so bounds l from above. The longest cycle any of the graphs of order $|\,\mathrm{G}\,|$ can possess is of length $|\,\mathrm{G}\,|$. □

Exercise 22 Find graphs for which $g = C$ and $C = \mathrm{G}$. Specialize to the case in which all three of these are equal and those in which precisely 2 are equal.

Definition 4 A graph G is *Hamiltonian* if it possesses at least one cycle of the maximal possible length $|\,\mathrm{G}\,|$.

Pointer 3 Chapter 42 is about these graphs.

♠ ◇ ♣

Definition 5 The cycle-number $\#C$ of a graph, is the number of independet cycles it contains. This refers to a cover by the smallest cycles available. So e.g. Diamond counts as two triangles, rather than these alongside a square

38.9 Relational cycle information

Definition 1 The *cycle range* manifested by a given graph is

$$\Delta c \;=\; C \;-\; g\;. \tag{38.28}$$

Definition 2 The *lower-shifted length* of a cycle S within a given graph is

$$l_{\mathrm{l}}(S) \;=\; l(S) \;-\; g\;. \tag{38.29}$$

Definition 3 The *standardized cycle fraction*

$$\sigma \;:=\; \frac{l_{l}(S)}{\Delta S} \;=\; \frac{1 \;-\; g}{C \;-\; g}\;.$$

Remark 1 This is only defined for $C > g$.

Definition 4 *Circumference density*

$$c \;:=\; \frac{C}{|\,\mathrm{G}\,|}\;.$$

Definition 5 *Girth fraction of circumference*

$$\mathcal{G} \;:=\; \frac{g}{C}\;.$$

♠ ◇ ♣

Lemma 2 σ is standard-range and

$$c \;\leq\; 1\;.$$

$$c \;=\; 1$$

encodes Hamiltonianness at the level of ratios.

Definition 2 Let us introduce the *anHamiltonianness*

$$\overline{\mathcal{H}} \;:=\; 1 \;-\; c$$

as a measure of departure from Hamiltonianness.

Lemma 3

$$\mathcal{J} \ := \ \frac{|\,\mathrm{G}\,|\,\mathcal{G} \ - \ 3}{3 \ - \ |\,\mathrm{G}\,|}$$

is standard-range.

Exercise 23 Prove Lemmas 2 and 3.



38.10 Ori-graph versions

Definition 1 The *oriented distance* between vertices x and y in a graph G [183] is given by the following.

$$\vec{\mathrm{d}}(x,\, y) \ = \ \min_{\text{oriented paths P} \ : \ x \longrightarrow y} (\, l(\mathrm{P})\,) \ . \tag{38.30}$$

[For x and y not strongly-connected, $\mathrm{d}(x,\, y) \ := \ \infty$.]

Remark 1 This is a 'one way traffic system notion of distance'...

Exercise 24 and Pointer 4 A vertex w in a tournament T is called a *k-king* if the following holds.

$$d(w, v) \ \leq \ k \ \ \forall \ \ v \ \in \ \mathfrak{V}(\mathrm{T}) \ , \ \ v \ \neq \ w \ .$$

a) What simpler conceptualizations are possible for 1-kings?

b) By a)'s reduction to simpler concepts, it is the 2-king that is the minimal nontrivial notion of k-king, and is thus often just referred to as king. Which tournaments on up to 5 vertices possess a king?

c)$^{+}$ Prove that every round-robin tournament must contain at least one king.

d) Show that a k-king is also a $k \ + \ 1$-king but not vice-versa.

e) Construct a tournament that contains a nontrivial 3-king (a 3-king that is not already a 2-king), and likewise for a nontrivial 4-king.

Remark 2 Cycle concepts also clearly have corresponding ambiguities of this kind. E.g. *oriented girth* $\vec{g}$ and *oriented circumference* $\vec{C}$.

38.11 Further path, trail, walk, cycle, circuit and tour cardinals

Structure 1 One can similarly assign maximal and minimal numbers for the lengths of each of circuits and tours present in a graph. And radii and diameters based on using each of trails and walks in place of paths. Thus we obtain 8 further conceptually-simple cardinals: g_{ci}, C_{ci}, g_{t}, C_{t}, rad_{tr}, $diam_{\mathrm{tr}}$, rad_{w} and $diam_{\mathrm{w}}$.

Structure 2 One can furthermore count the total number of each of paths, trails, walks, cycles, circuits and tours within a given graph, giving 6 more: $\#(p)$, $\#(tr)$, $\#(w)$, $\#(c)$, $\#(ci)$, and $\#(t)$.

Structure 3 One can finally refine these counts into how many of each of these there are of each length. These are functions of the admissible lengths.

Remark 1 As far as the Author is aware, most of these cardinals are however of somewhat limited use.

Remark 2 One of the more commonly encountered such cardinals is the number of Hamiltonian cycles that a given graph realizes, $\#H = \#(c(|\, \mathrm{G} \,|))$. See Sec 42.8 for a bit more about this.



Chapter 39

Theory of multi-graphs and loop-graphs

39.1 Introduction

Remark 1 Recollect the definitions of multi-graph, loop-graph and loop-multi-graph from Sec 29.14.

Remark 2 $\geq$ 2 edges between a pair of distinct vertices are called *multiple edges* alias *parallel edges*.

Definition 1 An *m-multi-graph* is a graph in which each pair of distinct vertices can have $\leq m$ edges between them.

Remark 3 For $m = 2$, we have at most *double edges* (e.g. double bonds in Chemical models).

For $m = 3$, we additionally have *triple edges* (e.g. triple bonds in Chemical models).

[Chemical models do not support loops.]

Definition 2 An *l-looped* graph is a graph in which each vertex can support $\leq l$ loops.

Definition 3 An (l, m)-*looped* multi-graph is a loop-graph in which each vertex can support $\leq l$ loops and each pair of distinct vertices can have $\leq m$ edges between them.

♠ ◇ ♣

Naming Remark 1 Some authors use *pseudographs* instead of loop-graphs, or *multi-graphs* for what we are calling loop-multi-graphs. So always check definitions in each source you are using.

Naming Remark 2 Another ambiguity is covered by our qualifying as *nontrivial* for loop-graphs that actually contain loops, for multi-graphs that actually contain multiple edges, and for loop-multi-graphs that actually contain both. Some other sources might refer to these unqualifiedly, rather than as nontrivial. We make this choice so as to be able to model the arena of each of these as an extension of a more familiar arena.

Motivation 1 While (loop-)(multi-)graphs are not often mentioned in other introductory texts, widespread applications motivate them. E.g. Chemistry requires multi-graphs, while all of binary relations, Feynman diagrams and rubber band shadows require loop-multi-graphs.

Remark 4 The l-loop- and m-multi- conditions serve as *cutoffs*. Many modelling situations permit these, including with unique natural values for the application in hand. For Chemistry, 3-multi-graphs suffice. For both Feynman diagrams and rubber band shadows, 1-loop-3-multi-graphs suffice with 2 exceptions (pushing each of these up by 1). Once such cutoffs are in place, various Graph-Theoretic notions like $E_{\max}$, E_{crit}, edge density and complementarity become well-defined. Getting this notion of complementarity to work out vindicates making the inclusions in Fig 62.5.

Remark 5 The above modelling can be combined with arrowed edges.

♠ ◇ ♣

Definition 4 The vertex v is *self-adjacent* if there is at least one loop at v.

Remark 6 In the directed case, out-self-adjacency is present wherever in-self-adjacency is.

Remark 7 It is straightforward to extend adjacency relations $\mathcal{A}$ unadorned or whichever combination of 1-subscripted and $\pm$ superscripted.

39.2 The smallest 1-loop graphs

See Fig 39.1 for these on $N = 0$ to 3 vertices. We only pin names on this when a new component appears.

1) L_1 is the looped counterpart of $\mathrm{D}_1 = \{\,\mathrm{pt}\,\}$. This notation extends to $\mathrm{L}_n = {\mathrm{L}_1}^n$ as the looped counterpart of $\mathrm{D}_n = {\mathrm{D}^1}^n$. Mixed unions of L_1's and D_1's are also possible.

2) The P_2 simple graph skeleton carrying one loop is Lollipop, while the version carrying one loop on each vertex is Cuffs. As a fully-looped graph, another name for Cuffs is FP_2. We use the F prefix more widely to denote fully looped. We also use LP_2 to denote 1-Pan, standing for 2-path with one loop.

3) The P_3 graph skeleton offers two new features. Firstly, distinction between central 1-loop and end 1-loop graphs. Also the smallest multiply- and yet not fully-looped graphs Longcuffs and Scooter $= \mathrm{L}_2\mathrm{P}_2$.

4) For the Triangle graph skeleton, loop number suffices to characterize all possibilities. This distincion rests upon P_3 having S_2 symmetry while Triangle has the full S_3 symmetry.

Exercise 1 a) Show that the P_4 graph skeleton supports two homocomplementary 1-loop-graphs. While the C_5 and Bull graph skeletons support none.

b) For which N are homocomplementary 1-loop-graphs prohibited?

♠ ◇ ♣

Definition 1 A *minimal-order simple graph model* M(L) of a loop-graph L is the simple-graph obtained by blowing up each loop to form a C_3 subgraph.

Naming Remark 3 This provides a further naming system, lM(L) for L, where l stands for contraction of M to a loop.

Worked Example 1 and Naming Remark 4 What looped graphs are supported on 0 to 3 vertices? Develop a nomenclature for the nontrivially-looped connected such.

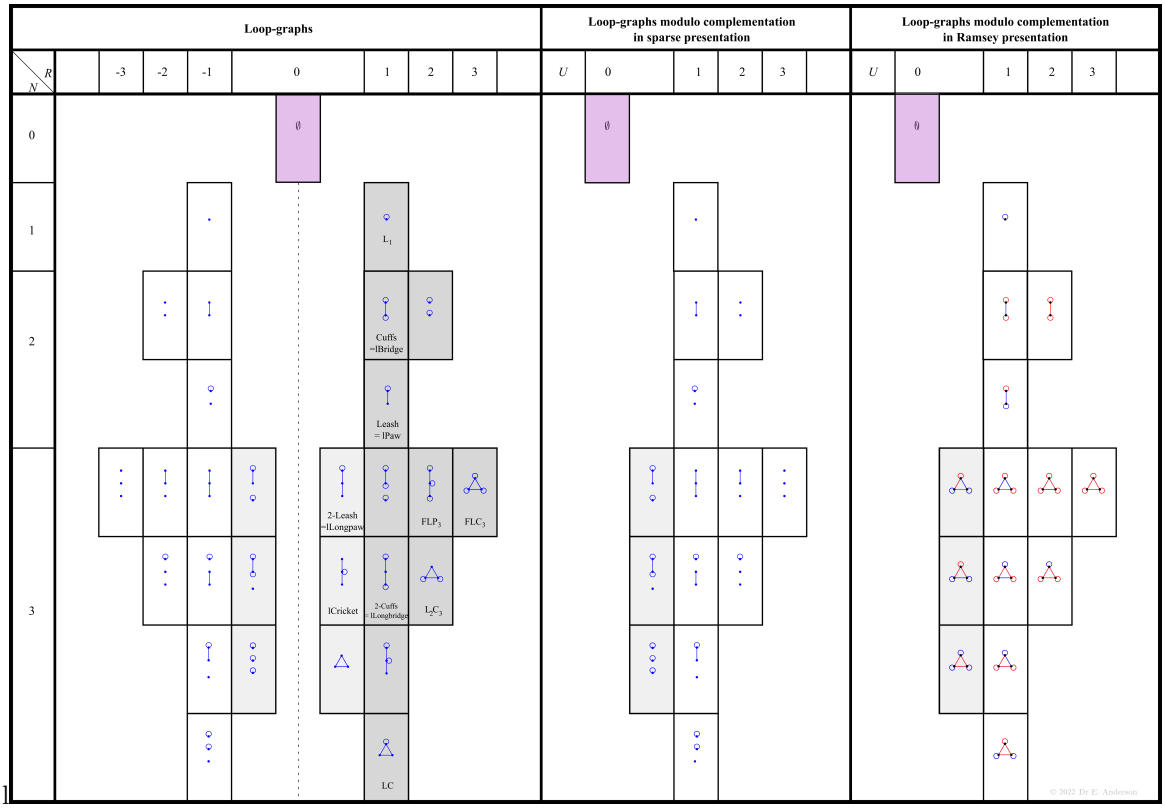


Figure 39.1:

Answer 1 M(Lollipop) = 3-Pan = Paw (Fig 1-Looped-Graph-Spaces-1-3). So another name for Lollipop is lPaw. In this case, the name 1-Pan is also clear, loop-graphs permitting this in contrast to simple graphs' Pans starting with the 3-Pan.

M(Cuffs) = Bridge (Fig 1-Looped-Graph-Spaces-1-3). So another name for Cuffs is lBridge.

M(P_3 with end loop) = 3-Longpan = Longpaw (Fig 39.1). This provides the names lLongpaw and 1-Longpaw.

M(P_3 with central loop) = *Cricket* (Fig 39.1). We can thus call this loop-graph lCricket.

While an ambiguity arises in the form of loop-graphs that also contain 3-cycles, the context of our subsequent Chapter (44) precludes this.

Exercise 2$^-$ What are M(Longcuffs) and M(Scooter)?

Exercise 3$^-$ **and Naming Remark 5** The *p-bouquet* B_p consists of p circles emanating from a single common point. The first two cases of these are the 1-loop $B_1 = L_1$ and the figure-of-8 graph, $B_2 = 8$: the exception to just needing 1-loop-graphs for rubber band shadows and Feynman diagrams. The 3-bouquet B_3 is thus the first new such; see row 1 of Fig 39.2 for the 'n-petalled flower' presentation of this. Some Topology books depict these in the alternative 'n-hoop Hawaiian ear-ring' presentation (row 2 Fig 39.2.b), taking particular interest in the infinite bouquet B_∞ alias infinite Hawaiian ear-ring. What is the minimal simple graph presentation of the p-bouquet?

[$^{++}$ If you know some Topology and yet have not seen this example before, then you might be

interested in investigating which notions of connectivity do and do not hold for the infinite bouquet.]

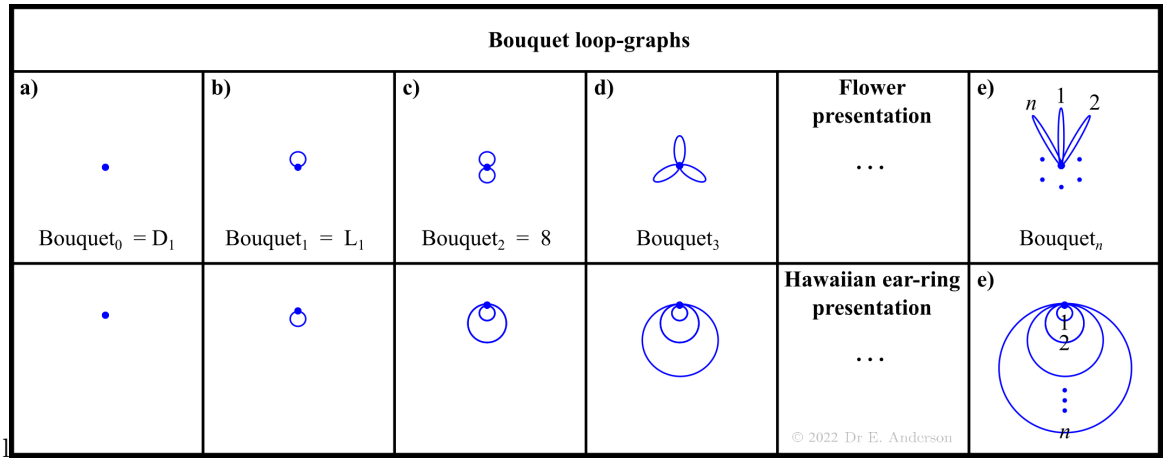


Figure 39.2:

39.3 The smallest 2-multi-graphs

See Fig 39.3 for these on $N = 0$ to 3 vertices.

Naming Remark 5 We now use a T prefix for totally 2-multi-edged if every edge present has a parallel edge. Hence the name $TK_2 = TP_2$ for the 'ethene carbon skeleton'. Though this case alone makes no distinction between being totally multi-edged and just being multi-edged. P_3 is the smallest simple graph skeleton to exhibit this distinction: between TP_3 and DP_3 with its single double edge. To Chemists, these are the propene and allene carbon skeletons respectively.

The K_3 simple graph skeleton supports three manifestations of double edges: the single DK_3, the double D_2K_3 and the full FK_3.

For N any larger than this, ambiguities appear in where a non-full set of double edges are placed. For instance, for the P_4 simple graph skeleton, there is a peripheral DP_4 and an interior DP_4. To Chemists, these are the but-1-ene and but-2-ene graph skeletons. There is also an adjacent D_2P_4 and an alternating D_2P_4, with reference to the placings of the double edges. In this case, equivalent descriptions are a peripheral single edge and an interior single edge respectively.

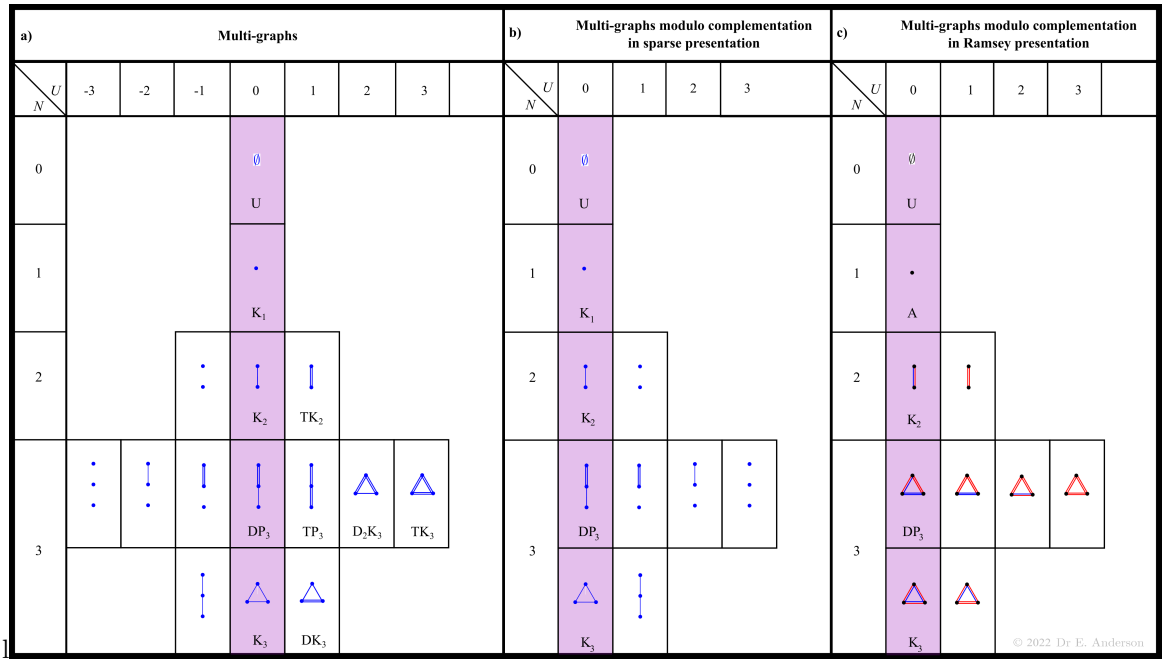


Figure 39.3:

Exercise 4 Find minimal-order simple graph models for each remaining graph in Figs 39.1 and 39.3.

Exercise 5 Find the smallest 3-multi-graphs: on 0 to 3 vertices.

Exercise 6 How many 2- and 3-multi-graphs are there on the K_4 and Diamond simple graph skeletons?

Exercise 7$^-$ and Naming Remark 5 The *n-dipole* Dipole_n consists of n 'field lines' running between a pair of common points (the poles): Fig 39.4. [In some other books, these are denoted by D_n, though we reserve this shorter notation for totally disconnected graphs.] a) What is the minimal simple graph presentation of the n-dipole?

b) Show that minimal simple graph presentations of loop-multi-graphs need not be unique. Show furthermore that this applies even to just multi-graphs.

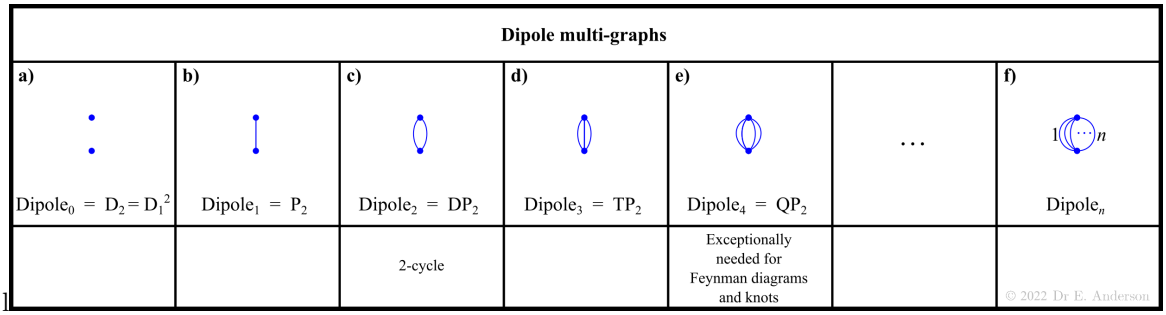


Figure 39.4:

39.4 The smallest 1-loop-2-multi-graphs

See Fig 39.5 for these on $N = 0$ to 3 vertices.

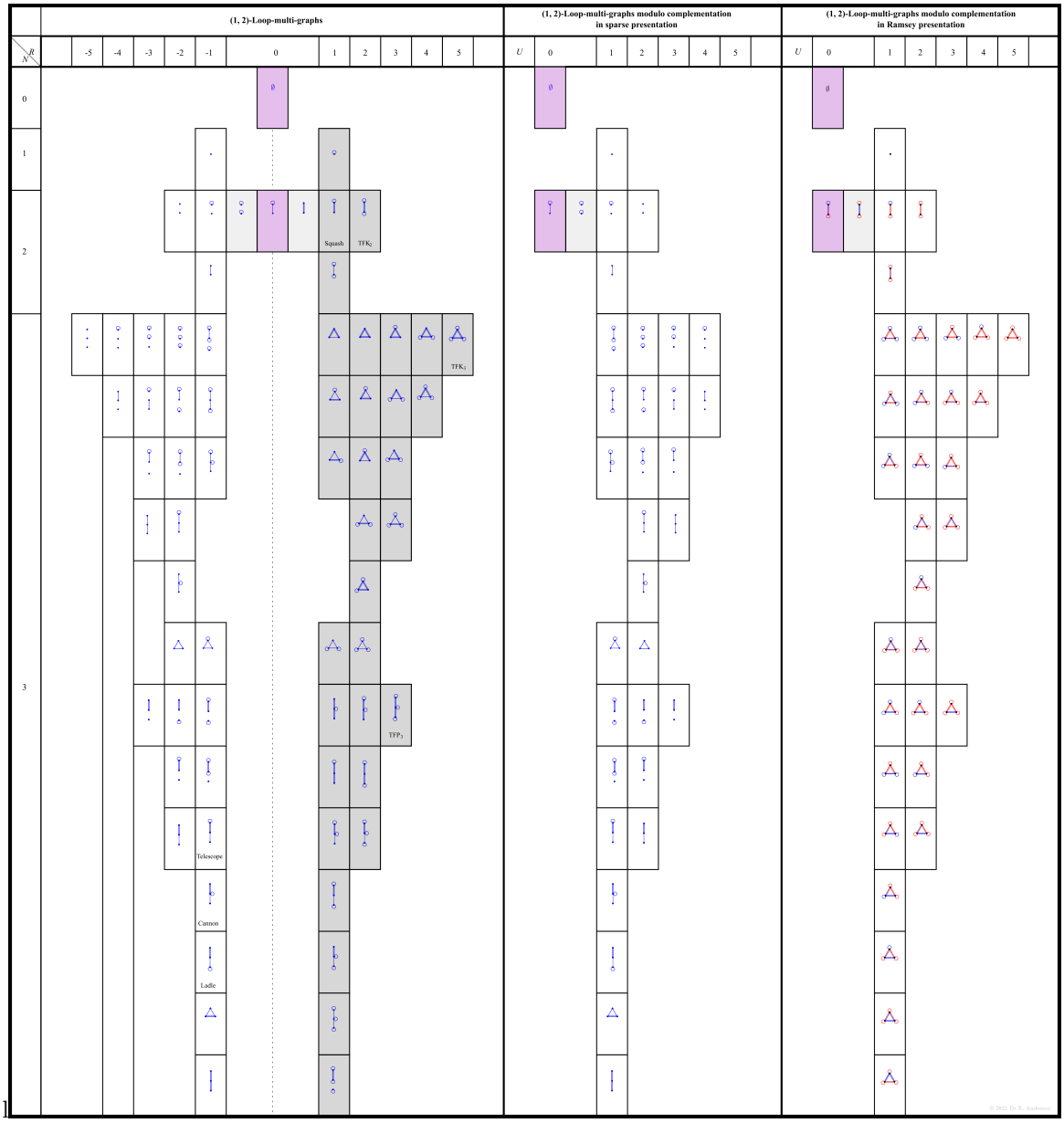


Figure 39.5:

Naming Remark 6 We begin by leaning on the previous two sections' namings. We now additionally have to deal with graphs concurrently exhibiting looping and double-edge features. We name the smallest such Squash, since it both looks a bit like the vegetable, and it is a thicker-handled squash racket to the 1-Pan's alternative interpretation as a thinner-handled badminton racket. See Fig 39.6 for further namings.

Exercise 8 Find minimal-order simple graph models for each remaining graph in Figs 39.1 and 39.3.

Exercise 9 Find the homocomplementary 2- and 3-multi-graphs on up to 5 vertices.

Exercise 10 Find the (1, 3)-loop-multi-graphs on 0 to 3 vertices. Form your own version of Fig 39.5 for these.

39.5 The smallest loop-multi-arrow-graphs

Exercise 11 Find all the arrowed (1, 2)-loop-multi-graphs for $N \leq 2$. What happens to the counts of these if vertex labelling by fully-distinguishable but mathematically unstructured labels is introduced?



39.6 Maximal and critical edge numbers for (l, m)-multi-graphs

Remark 1 The maximal m-multi-graphs on N vertices are the m-fold edge-covered complete graphs $\mathrm{M}_m\mathrm{K}_n$. The maximal l-loop-graphs on N vertices are the everywhere l-eared complete graphs $\mathrm{L}_l\mathrm{K}_n$. Composing, the maximal (l, m)-loop-multi-graphs on N vertices are the everywhere-l-eared m-fold-edge-covered complete graphs $\mathrm{L}_l\mathrm{M}_m\mathrm{K}_n$.

Structure 1 Thus

$$E_{\mathrm{max}} \;=\; E_{\mathrm{max}}(N;\, l,\, m) \;=\; \frac{N\,(\,m\,N \;+\; 2\,l \;-\; m\,)}{2}\;.$$

And then

$$E_{\mathrm{crit}} \;=\; E_{\mathrm{crit}}(N;\, l,\, m) \;=\; \frac{N\,(\,m\,N \;+\; 2\,l \;-\; m\,)}{4}\;.$$

Having this edge number is necessary but not sufficient for (l, m)-loop-multi-graphs to be homocomplementary.

Exercise 12 Find all the homocomplementary (1, 2)-loop-multi-graphs on up to 5 vertices.

Remark 2 For (looped) (multi-)graphs, the dense regime has

$$E(N,\, l,\, m) \;\simeq\; \frac{N\,(\,m\,N \;+\; 2\,l \;-\; m\,)}{2} \;\simeq\; \frac{m\,N^2}{2}\;.$$

Remark 3 The sparse regime has

$$E(N,\, l,\, m) \;\ll\; \frac{N\,(\,m\,N \;+\; 2\,l \;-\; m\,)}{2} \;\simeq\; \frac{m\,N^2}{2}\;.$$

Remark 4 The critically-dense regime has

$$E(N,\, l,\, m) \;\simeq\; \frac{N\,(\,m\,N \;+\; 2\,l \;-\; m\,)}{4} \;\simeq\; \frac{m\,N^2}{4}\;.$$

39.7 Ratio and dilatational variables for (l, m)-multi-graphs

Structure 1 The edge density ratio variable carries over, but now with computational value

$$\rho \ = \ \frac{E(\mathrm{G}(N))}{E_{\max}(N,\, l,\, m)} \ = \ \frac{2\, E(\mathrm{G}(N))}{N\,(\,m\,N \ + \ 2\,l \ - \ m\,)} \ .$$

Structure 2 The Ramsey density ratio variable also carries over, for sure now with a new computational value

$$\sigma \ = \ \frac{E(\mathrm{G}(N))}{E_{\max}(N, l, m)} \ = \ \frac{4\, E(\mathrm{G}(N))}{N\,(\,m\,N \ + \ 2\,l \ - \ m\,)} \ .$$

Remark 1 In terms of density variables, the conditions for dense, sparse and critically-dense regimes now remain unchanged from those in Sec 33.7.

Structure 3 The relational Ramsey variable now takes the form

$$\mathcal{R} \ := \ \frac{E_{\mathrm{crit}}(N;\, l,\, m) \ - \ E(\mathrm{G}(N))}{E_{\mathrm{crit}}(N;\, l,\, m)} \ = \ 1 \ - \ \sigma \ = \ 1 \ - \ \frac{4\, E(\mathrm{G}(N))}{N\,(\,m\,N \ + \ 2\,l \ - \ m\,)}$$

The homocomplementary graphs lie within the zero value of this, as highlighted in yellow in Figs 30.1 and 30.3

Exercise 13$^+$ Critical density still requires division by 4. So a necessary but not sufficient (mod 4)-condition for homocomplementarity arises. Work this out for all l and m. What further necessary but not sufficient (mod 2)-condition can l impose on homocomplementary graphs?

[Wider hint: even if you can only do part of the current Exercise, if you were stuck on while attempting the previous 4 Exercises you may now have a better chance of success with those, so now try again if needs be.]

39.8 Degrees for loop-multi-graphs

Structure 1 In the plain case, the degree of a vertex v now takes the form $d(v) \ := \ p \ + \ 2\,q$ at a vertex with p adjacent edges and q loops.

Structure 2 In the oriented case, in-degree and out-degree of a vertex now take the following form. $d_{\pm}(v) \ := \ p_{\pm} \ + \ q$ for a vertex with $p_{\pm}$ in/out edges and q loops.

Remark 1 These continue to count the numbers of unadorned, in and out edges respectively in the neighbourhood of v.

39.9 Combinatorics of loop-multi-graphs

Exercise 14 Generalize Euler's degree-sum Theorem to loop-multi-graphs and to loop-multi-ori-graphs.



loop blues \ cycle blues	S_3 0	S_2 1	S_2 2		S_3 [cycles] 3	S_1	S_2 4	
0	 $M_3(1,2)$	 P_2	 DP_2	 P_2	 C_3	 Propene	 TP_3 = Max Noncon	 Cyclo propene
1	 L D2	 P L	 DP L	 Bughead	 Shuttlecock	 Ladle		
		 Spoon D	 Squash D	 Longspoon D		 Cannon		
						 Telescope		
2		 Cuffs	 Dumbell	 Pincenez				
	 L2 D	 Spoon L	 Squash L	 Scooter				
3	 L3	 Cuffs L						

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Figure 39.6:

39.10 Notions of connectivity for (looped)(multi-)graphs

Remark 1 Loops do not register among standardly-defined notions of connectivity for graphs. We can however modify the definition of connectedness to include counting self-connectedness via a loop. While loops contribute +2 to vertex degree, they only count as one edge each. Overall, let us term this *self-and-mutual connectedness*.

Remark 2 Adding parallel edges does not change standardly-defined connectivity, yet it can change ν-connectivity. One can also define a notion of $\kappa^{(k)}$ connectivity, meaning connectivity specifically via a path of k-fold parallel edges.

Exercise 15$^-$ What is the smallest nontrivial multi-graph that is connected and yet not $\kappa^{(2)}$ connected? What is the smallest connected multi-graph containing a $\kappa^{(2)}$ cycle which is not however $\kappa^{(2)}$ connected?

Exercise 16$^+$ What happens to Whitney's inequality (35.3) for (l, m)-loop-multi-graphs?

Exercise 17 Give a conceptual analysis of how notions of independence, cover and domination generalize to (l, m)-loop-multi-graphs. Do (37.6) and (37.10) generalize?

Remark 3 One can also conceive of multi-edge independence and multi-edge domination, in each case by considering one or more types of multi-edges only.

Chapter 40

Applications of (loop-)multi-graphs

A. Chemical applications

40.1 Chemical use of 2- and 3-multi-graphs

Chemistry makes use of up to 3-multi-graphs, since double and triple bonds are possible in structural formulae. So Readers interested in Theoretical Chemistry should do the previous section's purer exercises on these as well.

Exercise 1 [Organic Chemistry] a) Following on from Exercise 1 of Chapter 32, find the corresponding molecular formulae for i) *alkenes* (hydrocarbons with no cycles but now with one double bond) and for *cycloalkenes* (one cycle and one multiple bond).

b) Find the corresponding molecular formulae for hydrocarbons with c cycles and bond multiplicity m.

c) Give the structural formulae for all

i) alkenes containing up to 8 carbon atoms.

ii) Cycloalkenes containing up to 8 carbon atoms.

iii)$^+$ Hydrocarbons containing up to 8 carbon atoms.

To first approximation, just work at the Graph-Theoretic level. To second approximation, take into account the following.

i) Double bonds are rigid as opposed to permissive of rotational freedom, so that the *cis-trans ambiguity* in Fig 40.1.a) leads to

ii) Alternating double bonds, for which buta-1,3-diene (Fig 40.1.b) is a minimal example, furthermore have their electrons delocalized along the carbon-skeleton path: a so-called *conjugate bond system*. This is a feature that confers particular stability to molecules. This points to a limitation to using 2-multi-graphs to represent chemical compounds. It also has the knock-on effect that such as Fig 40.1.c)'s two distinct multi-graphs represent one and the same chemical compound: (1 , 2)-dimethylbenzene.

If you practise or study Chemistry, be welcome to consider further eliminating cases with chemically-prohibited bond strains.

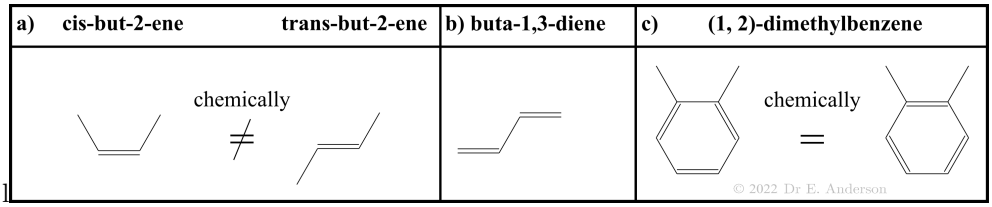


Figure 40.1:

Remark 1 Conjugate bonding along a ring acts so as to keep that ring within some plane. In considering larger rings (somewhat larger than 6-rings), the instantaneous configuration will however show increasing tendency to 'flop' rather than to lie in the plane. This is of some relevance as regards finding examples realizing larger C_n, D_n and Dic_n molecular symmetry groups.

Remark 2 Throughout this Volume, assume integer molecular masses; if you need these for the below question, just read these off a periodic table online.

Exercise 2 [Biochemistry] a) Which of the following can be hydrocarbon molecular masses? i) 72, ii) 78, iii) 46, iv) any odd number, v) each even number $\leq$ 60.

b) If carbon and hydrogen fail to match a given molecular mass, one common ansatz is to consider whether oxygen O (of valency 2) is present. Show that this ansatz resolves only one of the impasses among i) to iv).

c) Consider the larger 'biochemically reasonable' ansatz that only H, C, O, nitrogen N, phosphorus P and sulphur S are present in small biomolecules. Show that odd molecular mass implies the presence of an odd number of N atoms.

[N has valency 3 or occasionally 5. P has valency 5 or 3. S has valency 6, 4 or 2.]

Within this assumption, quickly deduce the structures of all possible biomolecules of molecular mass 61. Also determine which *common* biomolecule has molecular mass 121.

d) If you are studying Chemistry, determine which of i) to v)'s *graphically allowed* hydrocarbon and C, H, O molecules are *chemically allowed* as well.

40.2 Polymeric structures in Chemistry and Biochemistry

Polymers are formed by piecing together individual unit molecules (monomers), perhaps slightly internally altering the constituent units in the process. Infinite-length is an idealization in the Chemical context. Smaller polymers are on some occasions referred to as *oligomers*. 'Mer' means pieces, so monomer, oligomer, polymer mean one piece, a few pieces, and many pieces.

Example 1 *Polythene*, from polymerizing ethene, is just a long-chain alkane (Fig 40.2.a).

Example 2 *Polystyrene* is an example with a nontrivial sidechain. The monomer here is styrene: a benzene ring with an ethene sidechain (Fig 40.2.b). This gives an alkane chain with a benzene *sidegroup* every two molecules.

Exercise 3 What are the symmetry groups of the following cyclic polymers?

a) *Metaldehyde* (Fig 40.2.c).

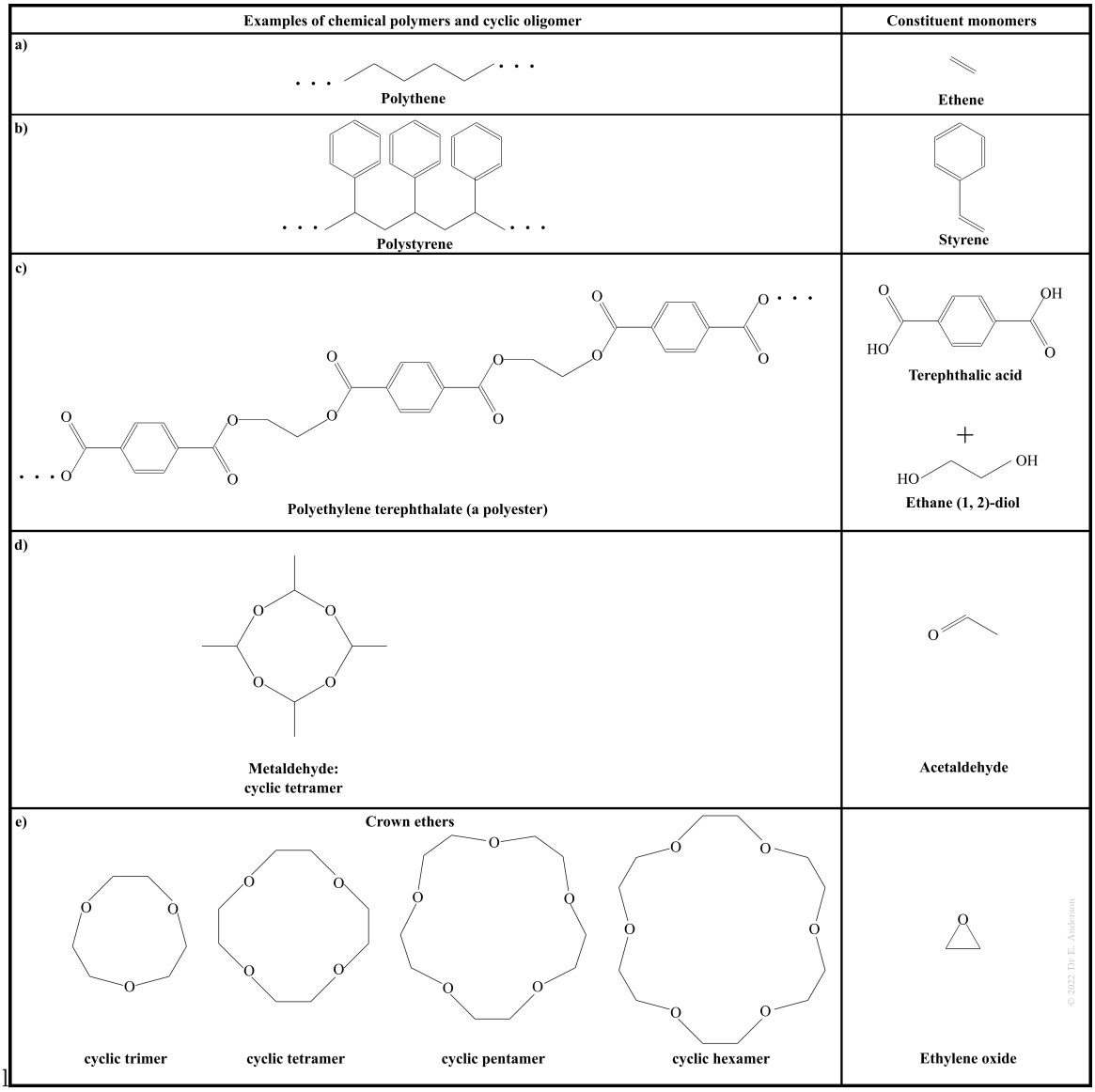


Figure 40.2:

b) The first few, and nth, *crown ethers* (Figs 40.2.d-g).

Exercise 4 A *copolymer* (joint polymer) is a polymer formed from $\geq$ 2 types of monomer, which typically play an interchangeable role in the structure. Suppose that two copolymers are involved and the polymer is a chain. Then in how many orders can a 10-unit *oligomer* be made from 5 copies of each constituent molecule? What is the total number of distinct possible 10-unit oligomers?

Pointer 1 Many of the practical applications of polymers concern however their material properties; see e.g. [413].

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Exercise 5 [Biochemistry] Each strand of a DNA molecule can be viewed as a copolymer of 4 constituent base molecules into a chain. For future reference, these are adenosine (A), thymine (T), guanosine (G) and cytosine (C). How many different possible 10-base strands are there? DNA

furthermore has two strands which are complementary to each other, A complementing T and C complementing G. This gives some indication of mechanism by which further copies of a DNA molecule are made within cells: each strand serving as template to form a copy of its complement. Viewing a strand and its complement as containing the same information, how many possible messages can a 10-base strand carry?

Exercise 6$^-$ [Biochemistry] Proteins are, to first approximation, joint polymers of 20 different kinds of amino acid. The smallest and largest known proteins contain around 20 and 20000 amino acids respectively. Estimate how many 20-amino acid proteins and how many 20000 amino-acid proteins are possible. Also estimate the molecular masses of proteins of these two extreme sizes [Consult the internet as regards the molecular masses of the 20 amino acids.]

Exercise 7 [Biochemistry] For comparison, a human chromosome's DNA molecule contains around 100 million base pairs. What is its approximate molecular mass? [Look up the molecular masses of A, T, C and G on the internet.] Approximately how many combinations can these base pairs encode? Assuming that the DNA base codes for the different amino acids are reasonably efficient, around how many distinct kinds of 600-amino acid protein could be encoded by a human chromosome?

40.3 (Loop-)(multi-)graph polymers

Structure 1 Given a sequence of graphs, these can be rooted on successive vertices of a path graph base to form a *comb*. Our sequence, and so matchingly our path, could be finite, semi-infinite like $\mathbb{N}$, or infinite like $\mathbb{Z}$. Each of our original graphs can now be viewed as a side-graph or 'tooth' of the comb. Some examples of this have a closed formula for the Nth graph in the sequence, and for its rooting, and thus for the Nth sidechain.

Example 1 The *finite 2-toothed comb* (Fig 40.3.a) strings together n P_2 to form a P_n subgraph 'spine'.

Example 2 The *infinite equal-toothed comb* (Fig 40.3.b) strings together a leaf from each of a $\mathbb{Z}$ of P_n.

Example 3 The *semi-infinite linear comb* (Fig 40.3.c) strings together a leaf of P_1, P_2, P_3... in this order.

Remark 1 There are also systematic ways of stringing together a sequence of graphs by more than one point each.

Example 4 *Ladder graphs* (already encountered in Fig 34.3.d) can be viewed in this way: stringing together some quantity of P_2 via both ends.

Example 5 *grid graphs* (already encountered in Fig 34.3.f) can be viewed in this way as well: stringing together some quantity of P_n via every vertex with parallel sets of edges.

Example 6 *Bunting graphs* (Fig 40.3.b) can be viewed as attaching D_1 to a base path by two points each. Or as polymerizing C_3 using two distinct attachment points on each.

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Remark 2 Cyclic polymers can also be formed from graphs.

Example 7 The *sunlet graphs* Sunlet(n) are cyclic polymers of P_2 (Fig 40.3.d).

Example 8 The *sun graphs* Sun(n) are cyclic polymers of points, each of which is attached to two consecutive vertices on a cycle (Fig 40.3.c).

Example 9 The prism graphs (Fig 34.6.a) are the cyclic polymer counterpart of the ladder graphs.

Example 10 The Prism(n, m) graphs (Fig 34.6.d) are the cyclic polymer counterpart of the grid graphs.

Exercise 8 a) How else can Examples 9 and 10 be mathematically characterized?

b) What are the automorphism groups of Examples 1 to 10? [Include infinite cases other than where explicitly stated.]

Pointer 2 Our Volume on Networks [450] shall further consider graph polymers. As well as considering higher levels of structure: filaments, textures, coiling, pairing, liquid crystal structure, defects, protein folding, protein cross-bridges...

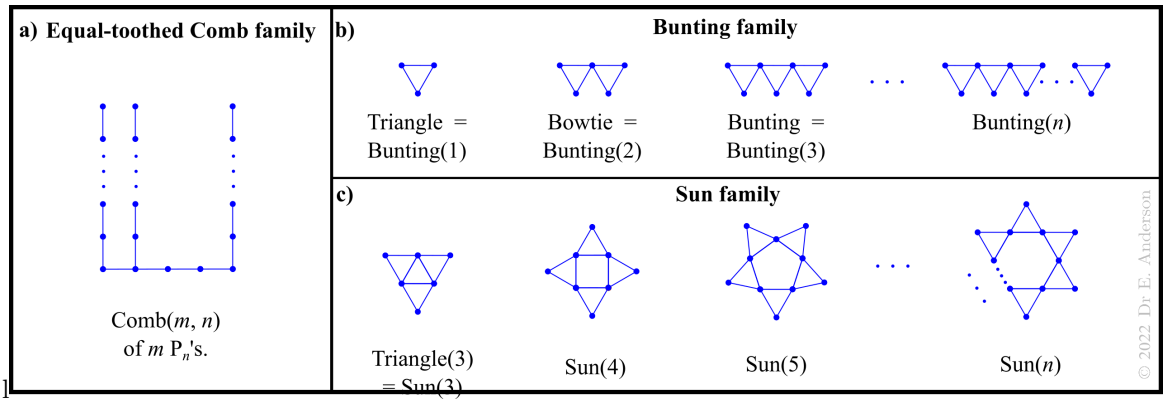


Figure 40.3:

40.4 Buckyballs

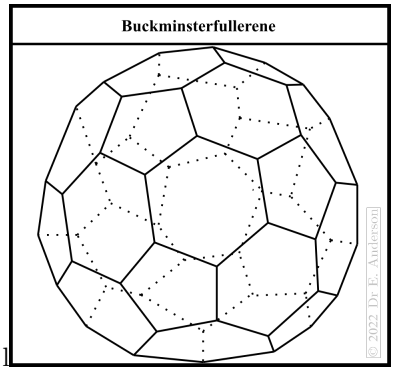


Figure 40.4:

A Chemical application of the polytope modelling regime is as follows. This can also be viewed as a spherical analogue of cyclic polymers.

Example 1 Consider carbon with one electron per atom delocalized once again, but now closed up to form a shell.

Exercise 9 a) Show that such molecules must contain an even number of carbon atoms.

b) Check that containing 20 hexagonal and 12 pentagonal faces is consistent for $V = 60$. This gives the C_{60} molecule Buckminsterfullerene (Fig 40.4.a). What is its symmetry group?

[Observe furthermore that this is identical to the patching pattern on a common model of soccer ball.]

c) Check that it is consistent to mathematically assemble a C_{70} by adding 10 vertices to a C_{60} in a way that solely adds 5 hexagons. If these form an equatorial belt, what is the ensuing molecule's symmetry group?

Pointer 3 It is however *nanotubes* [430] – cylindrical counterparts of the graphite sheet – that presently have many more applications than buckyballs do. This is however again for reasons concerning material properties, for which we are not ready at this stage in our Series.

Pointer 4 Most of Chemical Graph Theory [193] is in fact spectral, though considering this area of Graph Theory should await some of Volume 1's structures.

B. Binary Relations

40.5 Representing binary relations by loop-multi-graphs

Structure 1 Let us here represent the binary relation $\mathcal{R}$ by arc adjacency $\vec{\mathcal{A}}$. (1,2)-loop-multi-graphs, all of whose edges are arrowed, suffice for this purpose. The 2 here corresponds to the two possible orders for heterogeneous inputs, $a \mathcal{R} b$ and $b \mathcal{R} a$. While the 1 suffices to cover the case of homogeneous inputs, $a \mathcal{R} a$.

Worked Example 1 Interpret reflexive, symmetric, transitive and equivalence relations in terms of arrowed (1, 2)-loop-multi-graphs.

Answer 1 By the definition of reflexive, every vertex must carry a 1-loop. The arrow-graph for a reflexive relation is consequently fully-looped.

By the definition of symmetric, wherever there is arc, its reversed counterpart must feature as well. The arrow-graph for a reflexive relation is consequently a pure di-graph.

By the definition of transitive, any oriented 3-path $a \to b \to c$ must be accompanied by an arc $a \to c$. By which an acyclicly-oriented triangle bewteen these three elements is included (Fig 40.5.a). In the case of an oriented m-path, this argument extends by a straightforward induction to (some orientation of) a $\vec{\mathrm{K}}_n$ subgraph being present.
An equivalence relation is defined to be all three of the previous. It is thus represented by a fully 1-looped pure di-graph. Fully-looped $\overleftrightarrow{\mathrm{K}}_N$ is then a possibility. Equivalence relations are however capable of internally relating multiple mutually-unrelated clusters of set elements. This gives the most general possibility,

$$\coprod_{\mathrm{i}\ =\ 1}^{k} (\text{ fully-looped } \overleftrightarrow{\mathrm{K}}_{n_i}) \ ,$$

subject to

$$\sum_{i\ =\ 1}^{k} n_i \ = \ N \ .$$

I.e. the multiple-component version of the previous.

Remark 1 In a proof that n statements are equivalent, it moreover suffices to prove along a spanning tree digraph ($2\,n\ -\ 2$ subproofs). Or, distinctly, and more efficiently for $n \ \geq \ 3$, around an oriented Hamiltonian cycle (n subproofs). The other edges of the $\overleftrightarrow{\mathrm{K}}_n$ then follow by transitivity, while the loops require no proving. This indicates some ways of representing equivalence relations using less edges (and with arrow directions suppressed as well).

Exercise 10 Interpret a-reflexive, irreflexive, a-symmetric, unsymmetric, a-transitive and intransitive relations likewise.

Worked Example 2 Interpret antisymmetric and partially-ordered relations likewise.

Answer 2 The definition of antisymmetry translates to saying that vertex pair interlinked by digraph edges would collapse to a single looped vertex. This leaves us with a partly 1-looped origraph. Including as subcases both of 1-loopedness' extreme cases: unlooped and fully looped.

The stronger definition of partial ordering – reflexive, antisymmetric and transitive – then picks out the fully-looped subcase. There is no bar to there being multiple components. Components are however able to be more general than the fully-looped $\vec{\mathrm{K}}_n$. A minimal counterexample to this is in Fig 40.5.b), whose arrows extend to no further arrows by transitivity.

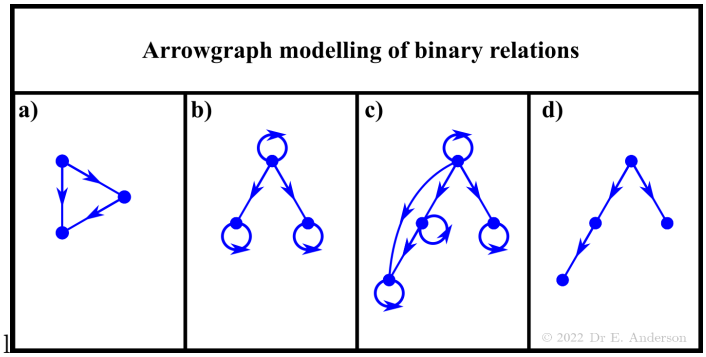


Figure 40.5:

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Remark 2 It is more convenient to represent partial order relations without inclusion of any of the ubiquitous loops implied by reflexivity. Or of the arcs that can be inferred from transitivity. So e.g. the poset in Fig 40.5.c) is more conveniently represented as 40.5.c). See Chapter 49 for further streamlining of this into the final 'cover diagram' notation for posets.

Worked Example 3 Interpret strongly-connected = total and totally-ordered relations likewise.

Answer 3 The definition of antisymmetry translates to saying that each vertex pair must be related by at least one arc. This firstly forces the arrow-graph to be weakly connected (in the arrow-graph sense). The possibility of multiple components has thus been eliminated. This secondly forces a $\vec{K}_n$ subgraph to be present. Thirdly, it forces every vertex to carry a loop.

The definition of totally-ordered – partially-ordered and total – additionally forces us, firstly, to have no two-way arrows, so it is a pure ori-graph. Secondly, to be fully 1-looped. For an N-element set and thus on N vertices, this leaves us with a fully-looped $\vec{K}_N$

Remark 3 The presentational simplification of Remark 2 cuts this down to a directed N-path: the *chain presentation* of our finite total order.

Exercise 11 Interpret connected relations likewise. Deduce that total orderings can be re-axiomatized as connected partial orderings, which is more efficient from the point of view of stipulating looping. And as the total, antisymmetric and transitive combination.

C. Physical Applications: Feynman diagrams

40.6 Introducing Feynman diagrams

Structure 1 Feynman diagrams are pictorial representations of the terms in some Particle Physics type quantum process' expansion. Each of these terms is of the schematic form

$$S \int_{x \in \mathfrak{M}} \mathrm{d}^4 x \, P \, V$$

for constituent propagators P, constituent interaction vertices V and symmetry prefactor S. The integral is over $\mathfrak{M}$: (usually flat) spacetime.

The interaction terms correspond to internal vertices, at which particles decay, or are created or annihilated outright. To be interactions, these must be of degree ≥ 3. For physical reasons, these must be of degree ≤ 4, so 3 and 4 are the only possibilities (at most bi-regularity).
Some processes involve particles coming or going from an (idealized) spatial infinity. These starting points are modelled by leaf vertices: of degree 1; Particle Physicists call leaves' edges *legs*.
The absence of degree-2 vertices in Feynman diagrams is interesting since Part V characterizes these as topologically inessential. By which Feynman diagrams are one of the very few examples of graph models which are a priori purely topological rather than having extra metric content as well.

Pointer 6 In computing the values of Feynman diagrams, propagators are furthermore typically ratios of polynomials, possibly multiplied by exponential functions. They can be thought of as Green's functions (Volume 1) corresponding to certain Representation-Theoretically fundamental classical field equations (Volume 5).

Remark 1 For now, we just look to understand the vertex types, classify the smallest Feynman diagrams, and indicate how the combinatorial factors outside the integral are calculated. These combinatorial factors simplify the overall calculation of the expansion, by indicating how many terms contribute the same integral.

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Example 1 For all Quantum Field Theory models the propagator (Fig 40.6.a) by itself corresponds to a non-interacting particle moving into our region of study and then just exiting it without interaction.

Example 2 Two such can be set up in three different combinations, as per Fig 40.6.b). From now on, we will only study connected Feynman diagrams.

Example 3 Let us now consider more specifically the ϕ^4 scalar field theory: one of the simplest models of interacting Field Theory. The simplest interacting Feynman diagram here is the *4-point interaction* of Fig 40.6.c); its graph is $+ \quad = \mathrm{S}_4$.

Example 4 So far, every example considered is *tree-level* in the usual graph-Theoretic sense. A first example which is not is the loop riding a propagator (Fig 40.6.d). This can be thought of as arising by joining two of $+$'s legs together.

Example 5 Further joining together the other two legs gives the *vacuum bubble* of Fig 40.6.e). This double-loop '8' is one of the two exceptions to Feynman diagrams being (1, 2)-loop-multi-graphs.

Example 6 Feynman diagrams with 2 and 3 internal vertices are given in Figs 40.6.f,g,h).

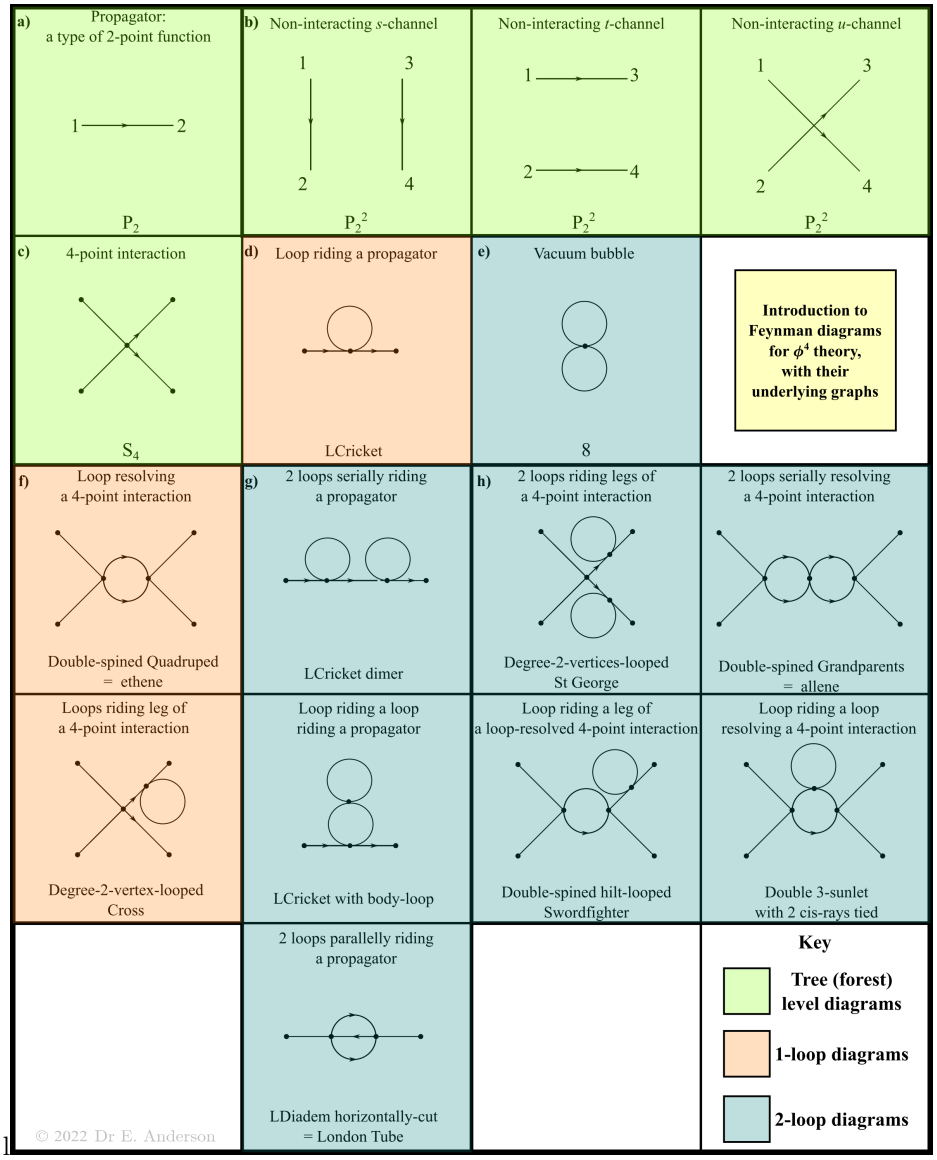


Figure 40.6:

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Remark 1 Let us next re-organize these examples using the numbers of internal vertices, legs and loops as critical variables. Loops are not defined here as previously in this Volume, now covering both multi-edges and loops.

Exercise 12 a) Extend this classification to connected Feynman diagrams with concurrently ≤ 4 legs and ≤ 4 internal vertices.

b) Count how many diagrams there are if their leaves are distinguishable.

Exercise 13 a) Redo Exercise 12 for ϕ^3-Theory, whose internal vertices are instead of degree 3.

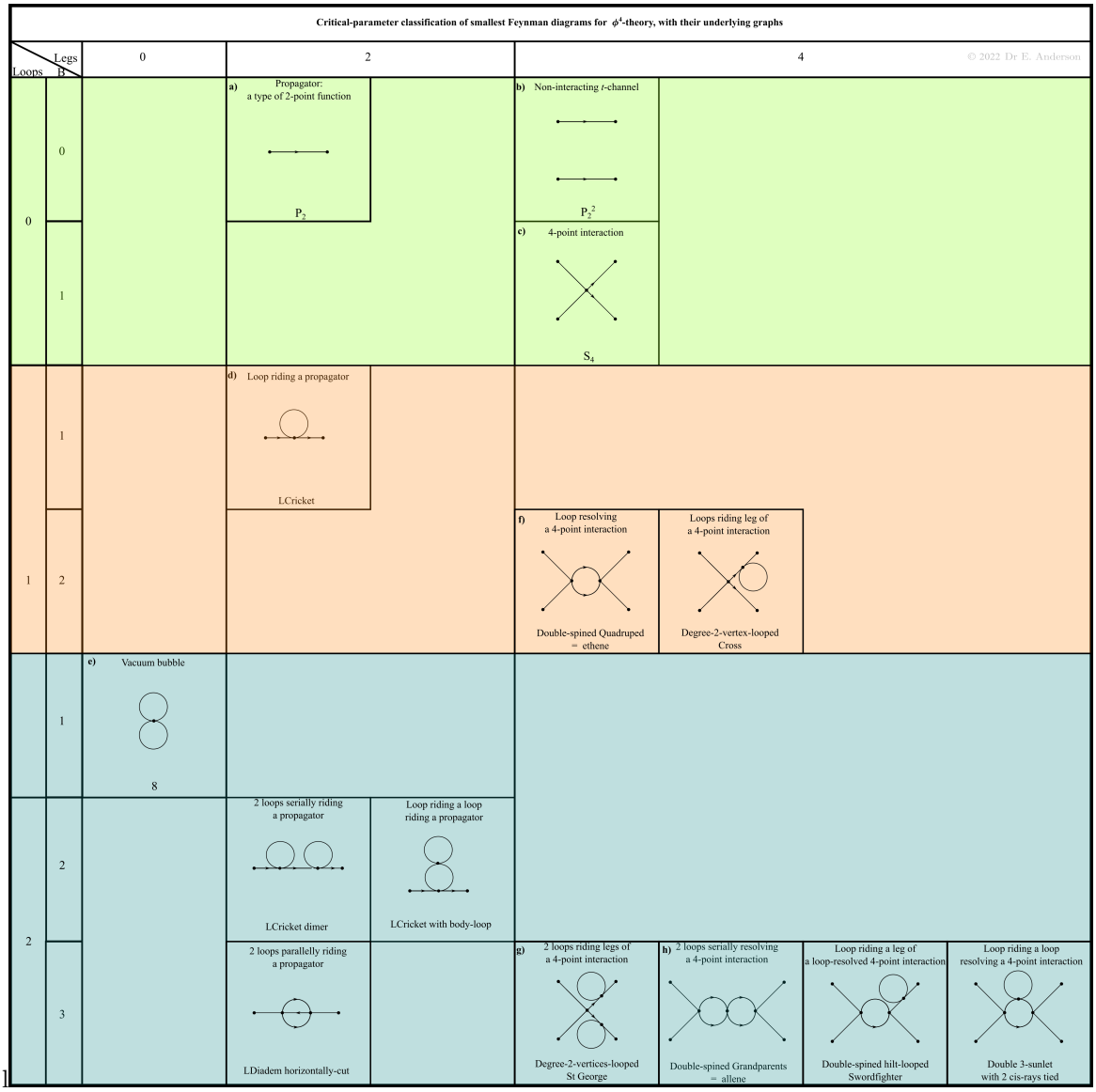


Figure 40.7:

b) How many graphs are supported if furthermore the edges at each internal vertex have to carry precisely one in-arrow and precisely one out-arrow?

Exercise 14 Repeat Exercise 12 for $\phi^4 + \phi^3$ Theory: with internal vertices of degree 3 and 4.

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Remark 2 By *Wick's Theorem* [386], each way of pairing up external legs contributes equally. So one only needs to evaluate one integral, which then comes with a *combinatorial symmetry prefactor*. Namely,

$$S := \#(\text{ways of interchanging the internal legs without changing the underlying graph}) .$$

This keeps count of the number of contributions.

Example 0 For a propagator, there are no internal lines, so $S = 0! = 1$.

Example 1 For the same with a loop riding thereupon, the two ends can be bolted on either way around, $S = 2! = 2$.

Example 2 For the same with two loops riding in series, $S = 2! \times 2! = 4$.

Example 3 For the tube sign, $S = 3!$.

Exercise 15 Evaluate S for every other Feynman diagram in Fig 40.7 and Ex's 12 to 14.

40.7 The Standard Model's Feynman diagrams

Structure 1 See Fig 40.8 or the possible interactions here, while noting that the Yukawa interaction is a physically-illicit precursor. Also the weak force's bosons interact in a manner loosely paralleling the gluons.

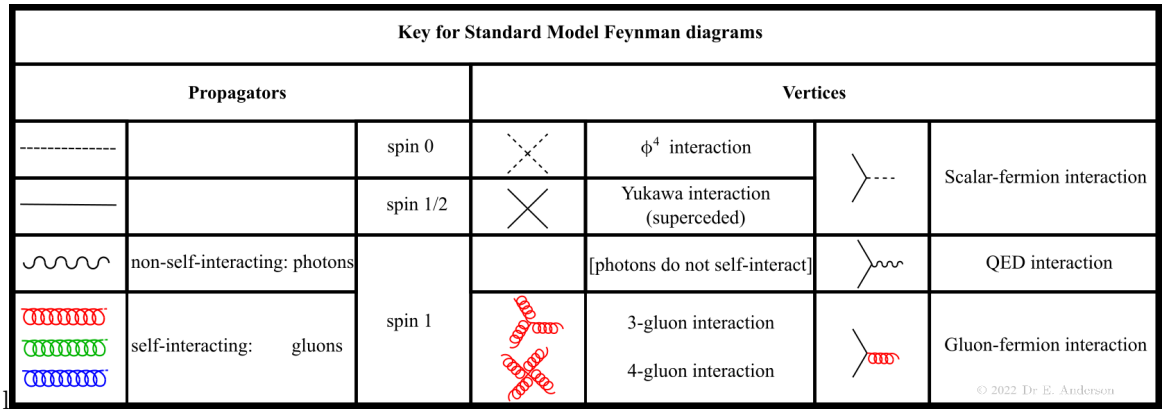


Figure 40.8:

Remark 1 The simplest part of the Standard Model of Particle Physics is *Quantum Electrodynamics (QED)*. I.e. the interaction between Electromagnetism's mediator particles – photons – and charged fermions, such as electrons, positrons protons, and antiprotons. This has just the one interaction term, involving three particles (marked 'QED' in Fig 40.8). The two fermions at each interaction vertex carry 1 in-arrow and 1 out-arrow precisely. I.e. their in-out index ΔL must be zero, as a matter of implementing a physical conservation law.

Feynman diagrams for this concurrently on up to 4 legs and 4 interaction vertices are in Fig 40.9. This Figure also names various well-known and relatively basic Particle Physics processes.

Exercise 16 a) The connected Feynman diagrams on 2 arrowed legs represent a process with 1 input fermion and 1 output fermion. What are its correction terms with $\leq$ 3 loops?

b) Likewise expand the 4 arrowed-leg diagrams up to 3 loops.

And the connected vacuum bubbles on $\leq$ 5 loops.

c) Also evaluate the symmetry prefactors for each new diagram.

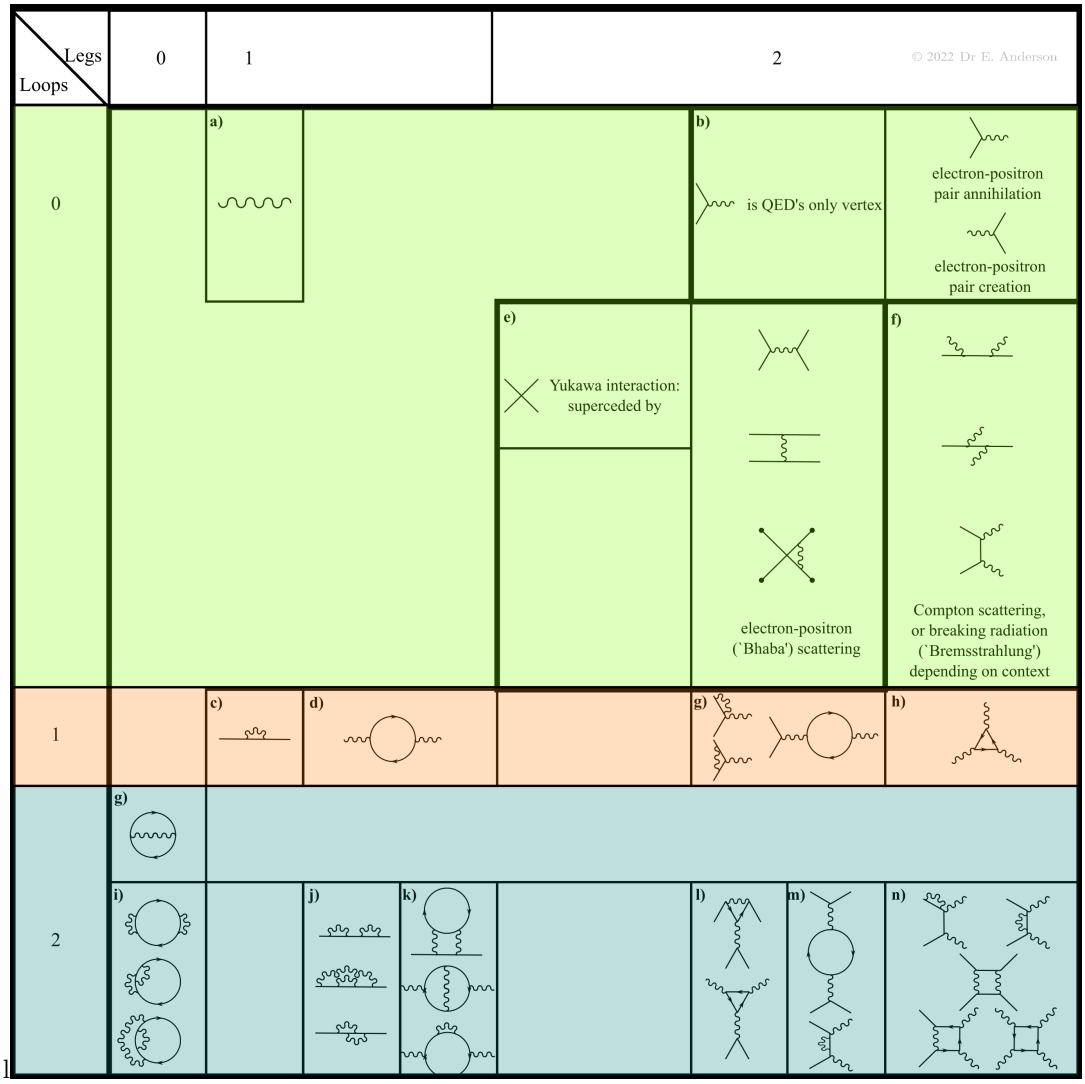


Figure 40.9:

40.8 Pointers about Feynman diagrams

Exercise 17 Classify QCD's Feynman diagrams (involving precisely Fig 40.8's coloured subfigures) on up to 4 internal vertices, and QCD's connected vacuum bubbles on $\leq$ 5 loops. Also evaluate the combinatorial factors for each new diagram.

Pointer 7 Feynman diagrams only make sense if the expansion that they form converges, so that its nth order terms are not swamped by subsequent n's terms. This is tied to why Feynman diagrams are limited to being 4-sub-regular.

Pointer 8 See [386, 387] for more on Feynman diagrams (though these will only be accessible if you have already done two courses on QM and one on Specially-Relativistic Electrodynamics).

Pointer 9 Another large area of Physics in which Combinatorics has started to be significantly applied is Statistical Mechanics.

Part VI

A first installment of Graph Theory

Chapter 41

Eulerian graphs

41.1 Introduction

Remark 1 This Chapter and the next concern *notions of traversability* for graphs. We shall further detail this conceptual type at the end.

Definition 1 A graph is *Eulerian* if it contains a circuit that uses each of the graph's edges precisely once. We also call such a circuit an *Eulerian circuit*.

Motivation 1 Euler considered this in addressing the 'bridges of Königsberg' walking game. In this, people attempted to walk around the city, going precisely once over each of its bridges. They found that they never seemed to be able to do this, no matter in which order they tried to cross the bridges.

Motivation 2 Another widely-motivated problem concerns a postal worker who needs to go down every street (modelled by a graph edge) so as to deliver mail to the houses there. Given some street plan of the part of town that they need to service, is it possible for them to carry this out in a rather efficient way: involving only ever walking along each street once?

Exercise 1 Let us first suppose that this part of town has at most 5 cross-roads. It is thus modelled by a connected graph on $N \leq 5$ vertices, all instances of which can be found within Figs 30.1 and 30.3. Which of these is Eulerian?

41.2 The Eulerian Even-Degrees Theorem

Remark 1 Euler resolved the general such problem as follows.

Theorem 1 (Euler–Hierholzer) A nontrivial connected graph is Eulerian iff each of its vertices is of even degree. I.e.

$$d(v) \; = \; 2\,k \;\; \forall \;\; v \;\in\; \mathrm{G}\;.$$

Proof [1]

($\Rightarrow$) Each Eulerian graph contains some Eulerian circuit. For this circuit to indeed close up, it must enter each vertex the same number of times that it exits that vertex. The number of edges going through each vertex must thus be even. So applying the definition of degree, each degree must be even.

[1] This is a common style of proof; see also e.g. [196, 178].

($\Leftarrow$) This is the more involved half of the proof. We proceed by induction on the size E of the graph G. G being nontrivial and connected,

$$\delta(\mathrm{G}) \;=\; \min_{v \,\in\, \mathfrak{v}(\mathrm{G})} \left(d(v) \right) \;=\; \min_{v \,\in\, \mathfrak{v}(\mathrm{G})} \left(2\,k \right) \;\geq\; 2 \; .$$

Recollect that (Exercise 36.2.a) tree graphs have at least 2 leaves: vertices of degree 1. (41.2) renders this impossible, so our connected graph cannot possibly be a tree. Thus by the definition of tree, it cannot be acyclic, i.e. it must contain at least one cycle.

We next introduce a maximal object that is suitable to probe whether our graph is Eulerian. I.e.

$$\mathrm{M} \;:=\; \{\, \text{circuit} \;\subseteq\; \mathrm{G} \,|\, \text{number of edges covered, E(M), is maximal} \,\} \; .$$

Suppose that M is not Eulerian, i.e.

$$E(\mathrm{M}) \;<\; E \; .$$

Then connectivity ensures that there is some vertex

$$w \;\subseteq\; (\,\text{some nontrivial component} \;\; \mathrm{K} \;\; \text{of G} \;-\; \mathfrak{C}(\mathrm{M})\,) \; .$$

By assumption, each vertex $u \;\subset\; \mathrm{K}$ has even degree in K, i.e.

$$d_{\mathrm{K}}(u) \;=\; 2\,k(u) \; , \quad k(u) \;\in\; \mathbb{N} \; .$$

We now use our induction hypothesis: that K is Eulerian, and thus contains some Euler circuit $\mathrm{C_K}$. Observe that the circuits M and $\mathrm{C_K}$ must have the following properties.

i) They must be edge-disjoint, by the complementation in (41.2).

ii) They must share a common vertex, by u itself having this property.

We can therefore concatenate M and $\mathrm{C_K}$ to form a circuit N with

$$E(\mathrm{N}) \;>\; E(\mathrm{M}) \; .$$

Contradiction! [Because we set up M to have maximal E(M).] We are thus forced to conclude that M must be Eulerian. □

Naming Remark 1 This Eulerian or *Even-Degrees* Theorem succeeds in giving a *full local characterization of Eulerianness*.

Remark 2 This is moreover a useful diagnostic, in particular enabling quick refutation or somewhat less quick confirmation of the Eulerianness of some given graph that is not too large. At least for finite and sufficiently small graphs, and for sufficiently regularly-structured graphs. For this is a general solution in principle, while in practice it may not be possible to check the degrees of the vertices.

Remark 3 Mentioning nontriviality in the statement of this Theorem does not do too much, covering the case of $\{\mathrm{pt}\} \;=\; \mathrm{D}_1$. This has *no* edges to traverse. On the other hand, it is both connected and of even degree: 0.

41.3 The Even-Degrees Cycle-Partition Theorem

Theorem 2 (Eulerian Cycle-Partition Theorem) A nontrivial connected graph is Eulerian iff it can be partitioned into cycles.

Remark 1 Instead of proving this as stated, we recharacterize Eulerianness by its even-degree diagnostic as per Theorem 1. Theorem 2 can thus be viewed as a 1-line proof Corollary of the following.

Theorem 3 (Veblen 1912 [127]). The edge set of a graph G can be partitioned into cycles iff all vertices are of even degree. I.e. (Fig 41.1.a)

$$\mathfrak{e}(\mathrm{G}) \;=\; \coprod_{j\,=\,1}^{l} \mathrm{C}_l \quad\Leftrightarrow\quad d(v) \;=\; 0 \pmod 2 \;\;\forall\;\; v \;\in\; \mathfrak{V}(\mathrm{G}) \;.$$

Proof [2] ($\Rightarrow$) Suppose that a graph is the union of some edge-disjoint cycles (possibly including 0-cycles: isolated vertices). Then a vertex belonging to k cycles has an 'in' and an 'out' edge on each cycle. These add up to degree $2\,k$.

($\Leftarrow$) Suppose that $d(v)$ is even for every $v \;\in\; \mathfrak{V}(\mathrm{G})$. Additionally suppose that the graph contains $\geq\; 1$ edge.

1) A cycle in G can be found as follows. Pick a maximal-length path in G,

$$v_0 \,...\, v_m \;.$$

Thus

$$v_0\, v_1 \;\in\; \mathfrak{e}(\mathrm{G}) \;,$$

and so

$$d(v_0) \;\geq\; 1 \;, \qquad \text{and thus } d(v_0) \;\geq\; 2$$

by the hypothesized evenness. v_0 thus has a further neighbour, call it u.

This u must also be a v_i for some $i \;\in\; [\,2,\, m\,]$. For suppose not. Then

$$u\, v_0\, v_1 \,...\, v_m$$

would form a path of length $m \;+\; 1$ (Fig 41.1.b), contradicting maximality. We have thus found a cycle

$$\mathrm{C}_1 \;=\; (\, v_0,\, v_1,\, ...\,,\, v_m \,) \;.$$

2) We next apply this procedure recursively, as follows. Let $\mathrm{G} \;=\; \mathrm{G}_1$, for which we have found a cycle $\mathrm{C}_1 \;\in\; \mathrm{G}_1$. Next let

$$\mathrm{G}_2 \;:=\; \mathrm{G}_1 \;-\; \mathfrak{e}(\mathrm{C}_1) \;,$$

i.e. remove the edges that have already been used. This G_2 itself inherits even degrees for all vertices. For G_1 has this property, and deleting the cycle preserves this evenness. This is since its effect is to subtract 2 from the degree of every vertex that it touches.

There are now two cases to consider.

a) Either G_2 has no edges, in which case our procedure terminates.

b) Or G_2 contains a cycle C_2 by re-using 1)'s argument.

Proceeding in this manner, we obtain disjoint cycles C_k for $k \;=\; 1$ to c obeying the following.

$$\mathfrak{e}(\mathrm{G}) \;=\; \coprod_{j\,=\,1}^{c} \mathfrak{e}(\mathrm{C}_j) \;.$$

I.e. a partition into cycles. □

Remark 2 Corollary 1 clearly gives a second diagnostic for Eulerianness among connected graphs. It is harder to use since partitioning into cycles is less immediate than counting degrees until if and when an odd degree is found.

[2] This is also a common style of proof; see also e.g. [176, 178]

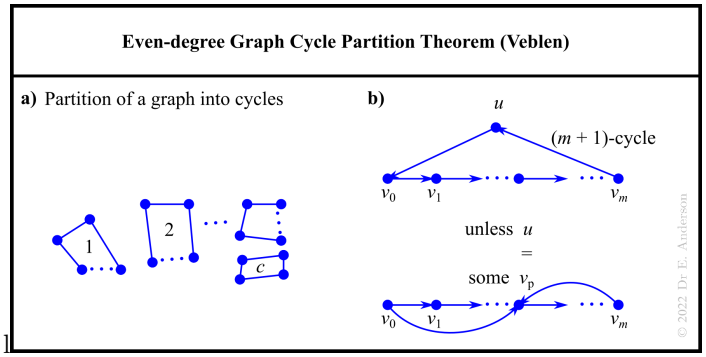


Figure 41.1:

Remark 3 Non-connected graphs clearly cannot be Eulerian, giving a third simple diagnostic.

Naming Remark 2 The truer-names 'Even-Degrees Cycle-Partition Theorem' and 'Eulerian Cycle-Partition Theorem' are new to this Volume.

41.4 The Eulerian No-Bridge Theorem

Theorem 4 If a graph contains a bridge, then it is not Eulerian.

Proof Suppose that the graph G contains a bridge. This runs between some subgraph H_1 and some subgraph H_2 (Fig 41.2.a).

We are to probe our graph with a candidate Euler circuit C, marked in lime-green. W.l.o.g. (relabelling 1 and 2 if needs be), C is to start at some vertex $v \in H_1$. If C is to be Euler, it must go through all the edges of G. Thus at some stage, it must go over the bridge edge itself, to H_2 (Fig 41.2.b).

But then by the definition of bridge, there is no unused path back to H_1. So either C re-uses the bridge, in which case it fails to be Eulerian [Fig 41.2.c), using magenta for non-Eulerian circuits]. Or C does not re-use the bridge, and so does not close up and so fails to even be a circuit (already depicted in Fig 41.2.b). These two cases being exhaustive, C always fails to be an Eulerian circuit. Thus bridged graphs fail to be Eulerian □

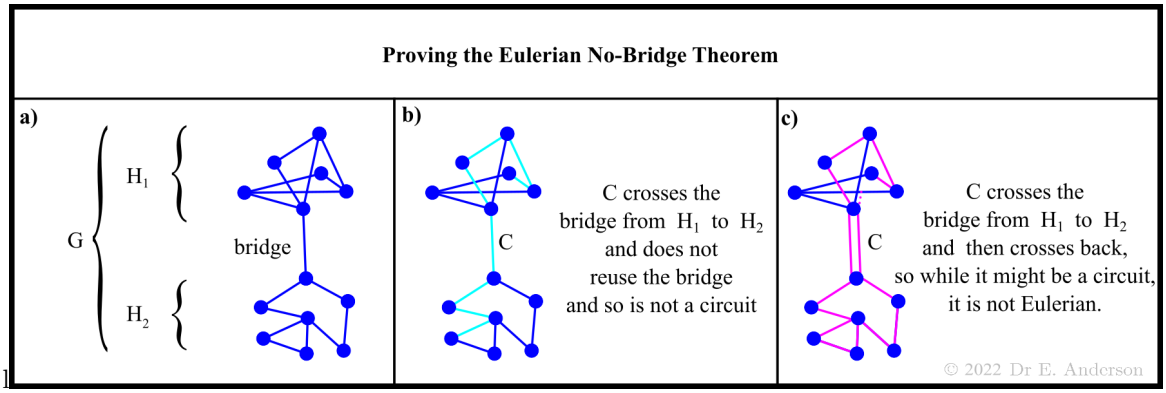


Figure 41.2:

Remark 1 The Eulerian No-Bridge Theorem provides a diagnostic to discard examples of graphs containing bridges from being Eulerian, (at least for graphs small enough to be visually handleable).

41.5 Exercises

Exercise 2 a) Spot which graphs in Fig 41.3 are Eulerian.

b) Which 7 other graphs on 6 vertices are Eulerian?

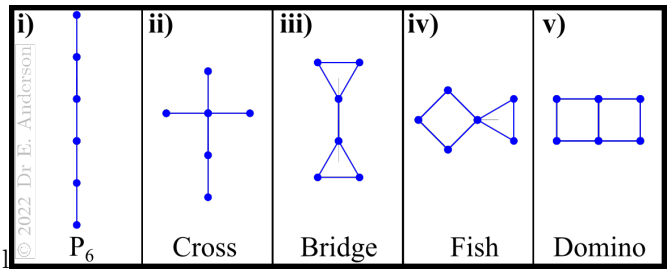


Figure 41.3:

Exercise 3 Which of the graphs in Fig 41.4 are Eulerian? Exhibit at least one Euler circuit for each of these that is Eulerian.

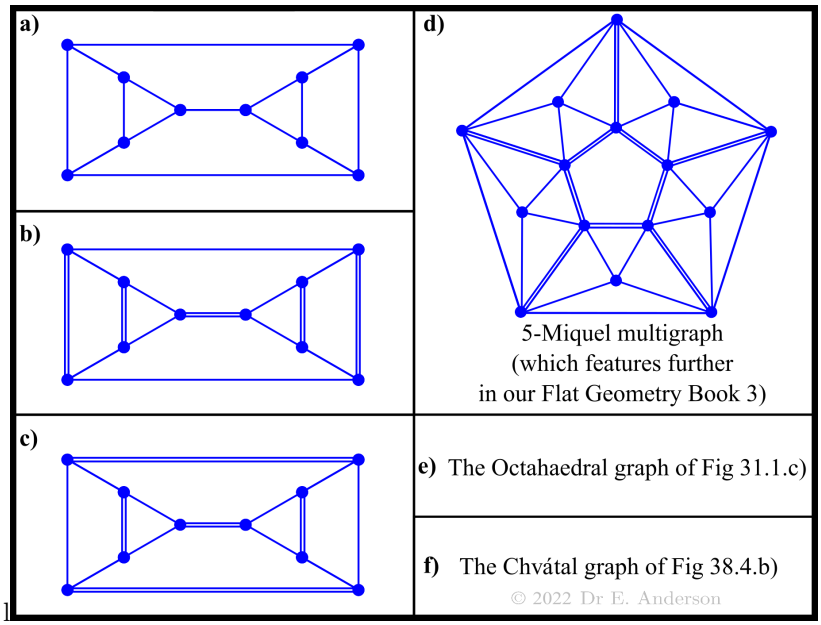


Figure 41.4:

Exercise 4 Show as succinctly as possible that the 19- and 37-Rose graphs (Fig 46.10) are not Eulerian, while the CompassRose 20-graph derived in Fig 55.3 is. Is the C_2 quotient of the last of these, derived in Fig 55.4, Eulerian?

Exercise 5$^+$ Extend the Eulerian definition to (loop-)(multi-)graphs. Include proof of Theorems 1 to 3 in these more general settings.

♠ ◇ ♣

Exercise 6 Show that Königsberg is not Eulerian, using the models in Fig 41.5. Answering this should include attaining an understanding of why these models are equivalent.

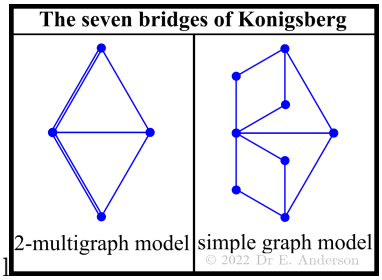


Figure 41.5:

Exercise 7$^+$ Use the technique that we used to prove Theorem 3 to obtain an alternative proof of Theorem 1.

Exercise 8$^+$ An *Euler trail* is a trail that makes use of all the edges in a graph. A graph containing an Euler trail is known as *semi-Eulerian*.

a) Show that semi-Eulerianness is a weaker notion of traversability than Eulerianness is.

b) Which graphs on up to 5 vertices are properly semi-Eulerian (i.e. semi-Eulerian and yet not Eulerian?

c) Generalize the Eulerian Even-Degrees Theorem to this setting.

Exercise 9$^+$ Extend the Eulerian Even-Degrees Theorem to ori-graphs.

41.6 Applications

Pointer 1 We shall see in Volume 2 that Eulerian considerations enter design of optimal circuits in Electronics and de Bruijn sequences [141, 175] in Information Theory .

Pointer 2 Volume 1 – chiefly on Linear Algebra – is also a treatise on notions of independence. One sizeable abstraction of which is Whitney's matroids [279, 284], which possess their own analogue of Eulerianness.

Pointer 3 One place where Euler circuits enter Topology is in the study of knots; see Sec 48.9 for a start on this and mention of its applications.

Chapter 42

Hamiltonian graphs

42.1 Introduction

Suppose that a travelling salesperson needs to visit n cities. We model these as vertices. Unlike the previous Chapter's postal worker, there is no need for the salesperson to drive along all of the roads between these cities (modelled by edges). One simple way toward minimizing travel is to try to find a cycle passing through each city once. Circuits that are not cycles would be inefficient in passing though some vertices multiple times. The corresponding problem of which graphs support such a cycle though all of their vertices is the Hamiltonian cycle problem. This turns out to be a global problem, and much less straightforward to answer in general than the Eulerian circuit problem.

Definition 1 A graph is *Hamiltonian* if it contains a cycle going through all of its vertices.

Remark 1 These are harder to handle than Eulerian graphs. For the simple theorems of Eulerian graphs give systematic tests, but Hamiltonian graph theorems – simple or otherwise – only really cover some particular cases.

Remark 2 Whether small graphs are Hamiltonian is straightforward to determine by inspection, however! Some very structurally simple families of graphs are also straightforward to understand from a Hamiltonian point of view.

♠ ◇ ♣

Exercise 1 a) Which graphs on 0 to 4 vertices are Hamiltonian?

Prove that:

b) can be Hamiltonian.

c) No tree is Hamiltonian.

d) No member of the n-Pan family is Hamiltonian

e) No graph containing any sidetrees can be Hamiltonian.

f) K_n is Hamiltonian for $n \geq 3$.

Worked Example 1 Show that all the platonic solids' graphs (Fig 31.1) are Hamiltonian.

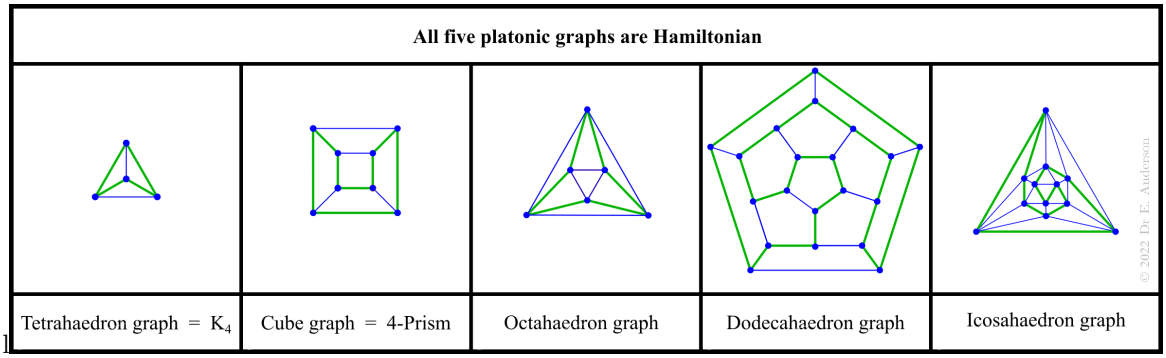


Figure 42.1:

Answer 1 Fig 42.1 places a Hamiltonian cycle on each of these, by 'burrowing inwards' from the outermost layer. [These graphs having a well-defined notion of radial layers of faces.]

Exercise 2 Which graphs in Fig 42.2 are Hamiltonian?

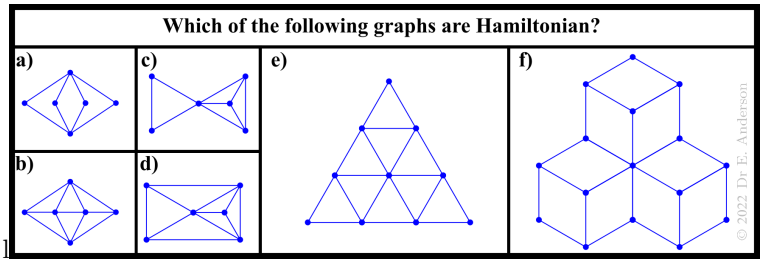


Figure 42.2:

Exercise 3 The Wall graphs are depicted in Fig 42.3. What is the smallest $n \geq 1$ for which Wall(n) is Hamiltonian?

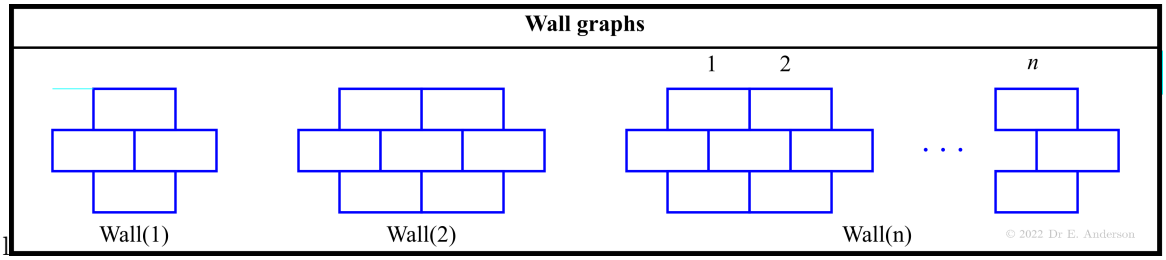


Figure 42.3:

Exercise 4 Show that the Petersen graph (Fig 38.4.a) and the Heawood graph (Fig 38.4.b) are Hamiltonian.

Exercise 5 Show that if a spanning subgraph of a graph is Hamiltonian, then the graph itself is as well.

Exercise 6 Where in the current Book have you already encountered a slightly disguised travelling salesperson problem?

Exercise 7 a) Take a knight for a Hamiltonian cycle around a 5×5 chessboard.

b) Show that this is not possible for 3×3 and 4×4 chessboards.

c)$^+$ Take a knight around a Hamiltonian cycle on a full-sized chessboard.

42.2 Hamiltonian presentation of graphs

Definition 1 A *Hamiltonian presentation* of a graph is a presentation in which a maximal and thus Hamiltonian cycle is manifestly exhibited.

Remark 1 This usually takes the form of a regular N-a-gon passing through the graph's N vertices.

Remark 2 At least for graphs of moderate size, such presentations can be useful in demonstrating planarity or extent of nonplanarity. Furthermore, even if a graph fails to be Hamiltonian, it may have some other large cycle (relative to N) which serves this purpose: more general *large-cycle presentations*. Finally, these applications continue to be useful even if the primary graph has obviously non-Hamiltonian features such as side-trees or multiple components. For one can still consider whether each component's cycle system – arising by serially defoliating it – has a Hamiltonian cycle or other large cycle.

Exercise 8 Put Fig 38.4.c)'s Heawood graph into a Hamiltonian presentation.

42.3 Dirac's Theorem

Theorem (Dirac 1952 [351]). A simple graph G with $N \geq 3$ vertices is Hamiltonian if the following holds.

$$d(v_i) \geq \frac{N}{2} \text{ for every vertex } v_i \in \mathfrak{V}(\mathrm{G}) \, . \tag{42.1}$$

Proof (We here follow a first-principles approach along the lines of that e.g. in [183].)

We can rephrase inequality (42.1) as

$$\delta(v_i) \geq \frac{N}{2} \, . \tag{42.2}$$

Suppose that this is not true. I.e. that $\exists$ some non-Hamiltonian graph for which this inequality holds. Adding edges cannot cause the minimal degree δ to decrease. We can thus narrow down consideration to *maximal non-Hamiltonian graphs*. I.e. those for which addition of a single further edge would force the graph to become Hamiltonian.

Let $u \not\mathcal{A} w$ both be $\in \mathfrak{V}(\mathrm{G})$. Then our maximality $\Rightarrow$ G has a spanning path

$$u = v_1, v_2, \ldots, v_N = w \, .$$

Every spanning cycle in G $+$ $u\,w$ contains the added edge $u\,w$. [Each of these spanning cycles is furthermore a Hamiltonian cycle.]

The idea is then to modify our cycle so as to avoid using the edge $u\,w$. This would give us a spanning cycle in G. Suppose that one of u's neighbours directly follows one of w's neighbours on our path, E.g.

$$u \mathcal{A} v_i \, ,$$

while

$$w \mathcal{A} v_{i-1} \, .$$

Then

$$(u, v_{i+1}, v_{i+2}, \ldots, v\ v_i, v_{i-1}, \ldots, v_2)$$

is a spanning cycle (Fig 42.4.a).

It remains to establish that $\exists$ such a cycle. We proceed by showing that the following *neighbourhood indexing sets* share at least one index.

$$\mathfrak{U} := \{ i \,|\, u \not\mathcal{A} v_i \}$$

and

$$\mathfrak{W} := \{ i \,|, w \not\mathcal{A} v_{i-1} \} .$$

PIE gives that

$$|\mathfrak{U} \cup \mathfrak{W}| + |\mathfrak{U} \cap \mathfrak{W}| = |\mathfrak{U}| + |\mathfrak{W}| .$$

But by (31.5), the RHS is also

$$= d(u) + d(v) .$$

Then using (42.2) twice, this

$$\geq \frac{N}{2} + \frac{N}{2} = N .$$

However, neither $\mathfrak{U}$ nor $\mathfrak{V}$ contain the index value N. Thus

$$|\mathfrak{U} \cup \mathfrak{W}| < N .$$

Also from the existence of the spanning cycle,

$$|\mathfrak{U} \cap \mathfrak{W}| \geq 1$$

We thus arrive at the contradiction $N \geq N - 1$.

Thus there is no maximal non-Hamiltonian graph obeying our hypothesis. □

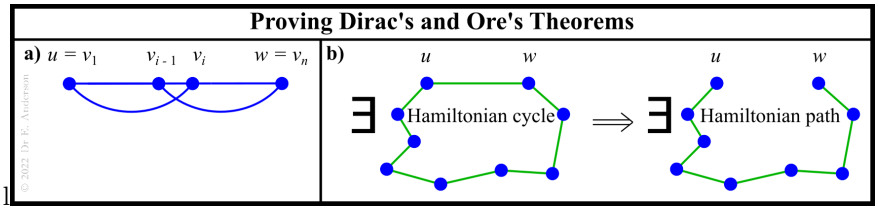


Figure 42.4:

42.4 Ore's Theorem

Theorem (Ore 1960 [151]). A simple graph with $N \geq 3$ is Hamiltonian if the following holds.

$$d(u) + d(w) > |\mathrm{G}| = N \text{ for every } u, w \in \mathfrak{V}(\mathrm{G}) \text{ such that } u \not\mathcal{A} w . \tag{42.3}$$

Proof Suppose again that this is not true, so that $\exists$ some non-Hamiltonian graph, for which (42.3) This is subgraph of K_n with fewer edges holds. Keep on adding edges by joining hitherto nonadjacent vertices, until a maximal non-Hamiltonian graph G' is reached. Let $x \not\mathcal{A} y$ in G'. Then since $\mathfrak{E}(\mathrm{G}) \subset \mathfrak{E}(\mathrm{G}')$, $x \not\mathcal{A} y$ in G as well.

Now

$$d_G(u) + d_G(w) \geq N$$

$\Rightarrow$

$$d_{G'}(u) \ + \ d_{G'}(w) \ \geq \ N \ .$$

Joining u and w, the resulting graph $\mathrm{G}' \ + \ u\,w$ is Hamiltonian by maximality of G'. Thus $\exists$ a Hamiltonian path between u and w (Fig 42.4.b). Finally joining u and w, $\mathrm{G}' \ + \ u\,w$ is also Hamiltonian.

Write our Hamiltonian path as

$$u \ = \ v_1, \, v_2, \, \ldots \, , \, v_{n \, - \, 1}, \, v_n \ = \ w \ .$$

Suppose that there is an edge in G' between v_1 and some non-terminal v_i. Then there is an edge $v_{i \, - \, 1}\, v_n$. Thus G' is Hamiltonian. So

$$v_1 \not\sim v_i \quad \Rightarrow \quad v_n \not\sim v_{i \, - \, 1}$$

[Fig 42.4.a) again].

This argument applies to each non-terminal v_i, i.e. for $i \ = \ 2$ to $N \ - \ 1$. Thus

$$d_{G'}(v_n) \ \leq \ (\text{ a priori maximum number of edges leading to } \ v_n \,) \ -$$

$$(\text{ edges forbidden by the number of vertices adjacent to } \ v_1 \,) \ = \ n \ - \ 1 \ - \ d_{G'}(v_1) \ .$$

So

$$d_{G'}(v_n) \ + \ d_{G'}(v_1) \ \leq \ n \ - \ 1 \ .$$

Contradiction!

Thus any graph satisfying (42.4) is Hamiltonian. $\square$

42.5 Exercises

Exercise 9 a) By looking at small graphs, show that the condition $N \ \geq \ 3$ is necessary for each of the above two theorems.

b) What does Ore's Theorem imply about the Hamiltonianness of the connected graphs on 5 vertices?

Exercise 10 a) Which decorations of the hexagon by diagonals have restrictions placed on them by each of Dirac and Ore's Theorems?

b) What about the graphs in Fig 42.5? Also establish from first principles that these are all Hamiltonian.

c)$^+$ Find minimal examples that establish that both Dirac's and Ore's bounds are sharp.

Exercise 11 Show that Ore's Theorem implies Dirac's Theorem.

Exercise 12 Find a 5-graph counterexample that suffices to establish that the the converses of both of these Theorems are false. Also show that this counterexample is minimal.

Remark 1 So we have two necessary conditions – one generalizing the other – but neither is sufficient.

Exercise 13$^-$ What are Dirac and Ore's sufficiency condition for Hamiltonianness in terms of ratio quantities?

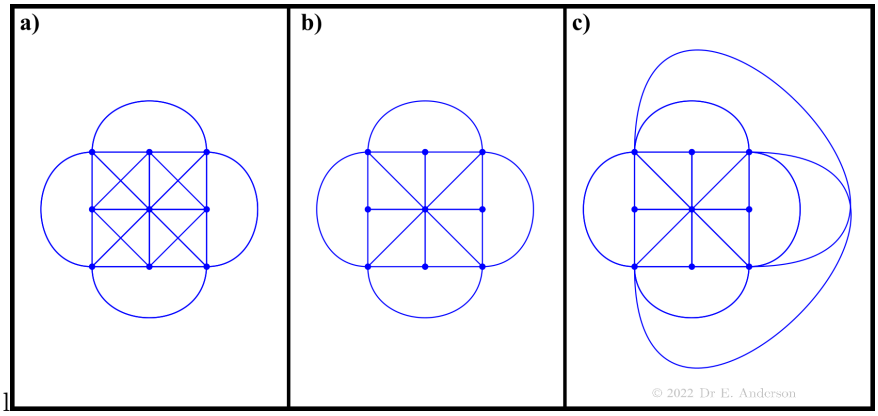


Figure 42.5:

42.6 Conceptual analysis of notions of traversability for graphs and beyond

Remark 1 Hamiltonianness is a recurrent 0-traversability: the existence of a cycle through all 0-simplices. Eulerianness is a recurrent 1-traversability: the existence of a circuit through all 1-simplices.

Remark 2 Let us next complete the conceptual square so far comprised of Eulerianness, semi-Eulerianness and Hamiltonianness.

Definition 1 A *Hamiltonian path* is a path in a graph that visits each of its vertices precisely once. A graph is *semi-Hamiltonian* if it contains a Hamiltonian path. It is *properly* semi-Hamiltonian if it contains a Hamiltonian path and yet does not contain a Hamiltonian cycle.

Remark 3 Proper semi-Eulerianness is a non-recurrent 1-traversability. Namely, the existence of a trail through all 1-simplices that is not a circuit. While proper semi-Hamiltonianness is a non-recurrent 0-traversability. Namely, the existence of a path through all 0-simplices that is not a cycle.

Remark 4 Including loops makes no difference to either Eulerianness or Hamiltonianness. For on the one hand, going around loops would spoil most Hamiltonian cycles, one can opt to never do so. On the other hand, a candidate Euler circuit in a loop-graph would necessarily go around every loop. Yet whether it is a successful candidate is entirely determined by whether the underlying unlooped graph is Eulerian.

Remark 5 Including parallel edges does not affect our basic Eulerian Theorems. Yet underlying simple graph skeletons can differ from multi-graphs thereupon as regards Eulerianness (see Fig 42.6.a,b). As regards Hamiltonianness, P_2 is exceptionally rendered Hamiltonian by a second edge in forming DP_2 (Fig 42.6.a), And yet Paw and other graphs are not (Fig 42.6.b). Hamiltonianness is thus an underlying simple graph skeleton property (for $N \geq 3$), while Eulerianness is an unlooped multi-graph property.

Remark 6 This distinction is itself rooted in Hamiltonianness being a 0-traversability to Eulerianness being a 1-traversability. 1-traversability's change in passing from a multi-graph to its underlying simple graph skeleton then perturbs the latter but not the former. The double-edged P_2 is an exception because it is the 2-cycle C_2. While loops affect the circuit realized, they cannot affect its Eulerianness, since each loop adds 2 to its vertex's degree. This is thus incapable of altering the

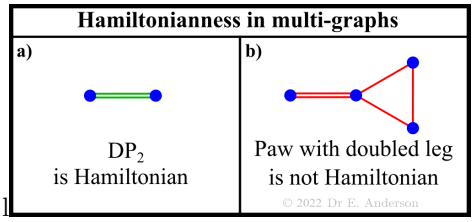


Figure 42.6:

parity of vertex degrees.

♠ ◇ ♣

Notions of traversability can be found in many other areas of Mathematics, including the following.

Example 1 Transitive group actions (Sec 20.2) are also a notion of traversability, corresponding to all configurations lying on a single group orbit.

Exercise 14 a) What are $Aut(K_5)$ and Aut(Petersen)?

A graph G is *vertex-transitive* if its vertices form a single orbit under Aut(G). It is *edge-transitive* if its edges form a single orbit under Aut(G).

b) Show that both K_5 and Petersen have both of these properties.

♠ ◇ ♣

Pointer 1 *Design Theory* [188] affords a large generalization of traversability in Combinatorics. A well-known motivational problem for this branch of Combinatorics can be phrased as follows.

Question 1$^+$ A 15-person marching band is to march in 3 columns of 5 every day for a week. On each day, everybody's position in the march is fixed. How can the marches be arranged so that no two people march side by side on more than one day?

[You are welcome to try to solve this now; the current Volume does not however contain the right material to solve such problems *systematically*. As such, if you cannot solve this at all, or you cannot prove that there are no more solutions, or you suspect your answer was not obtained with anything close to optimal efficiency. Then you are also welcome to try again after reading Volume 1 [441].]

Remark 7 Some other problems which benefit from a Design Theory point of view include the following.

2.a) *Magic squares* and *sudoku grids*, which are commonly found as a form of entertainment in newspapers and magazines.

2.b) *Experimental design*, accounting for the historical reason behind the name 'Design Theory'.

2.c) Some approaches to Finite Geometry, which we shall outline in Volume 3 [443].

♠ ◇ ♣

Pointer 2 'Functions that must take every value at least once' appear in Complex Analysis [363].

Pointer 3 *Ergodicity* [366] in part involves every point in a configuration space being visited by a dynamical system.

Remark 8 These examples point to notions of traversability being a widely valuable class to consider.

42.7 Discussion of basic Graph Traversability Theorems presented

Naming Remark 1 Let us first identify the above two Hamiltonian Graph Theorems as *local overcrowding* Theorems. In this context, this translates to large enough (pairs of nonadjacent) degrees, degrees being Graph Theory's neighbourhoods and thus local notions of size. The underlying idea here is that once these are large enough, then a Hamiltonian cycle becomes inevitable. It is then interesting for universal bounding estimates to exist. For all that these have various shortcomings, one of which is their being much larger than some applications' graphs' degrees. Check e.g. the Dodecahaedron (Fig 31.1.d), Heawood (Fig 38.4.c) and Icosahaedron (Fig 31.1.e) graphs for simple examples of this.

Naming Remark 2 Dirac's Theorem (Sec 42.3) is the 'δ-overcrowding $\Rightarrow$ Hamiltonian Graph' Theorem. While Ore's Theorem (Sec 42.4) is the 'nonadjacent-degree-pair-overcrowding $\Rightarrow$ Hamiltonian Graph' Theorem.[1]

Remark 1 In contrast, our basic Eulerian Graph Theorems make no allusion to overcrowding. All the degrees being even is a *structural* restriction. And yet it applies just as well if all the degrees are

$$d(v) \; = \; 2$$

(or

$$d(v) \; = \; 2\,k \;\; \text{for} \;\; k \; \ll \; N \;)$$

rather than very close to $N/2$. Partitioning into cycles is likewise highly independent of the magnitudes of the degrees. The No-Bridge Theorem moreover does not make any reference to local structure, since bridge number $Bri \; \neq \; 0$ is a global property. this).

Remark 2 All the Theorems we presented are moreover *global* in some sense. The δ condition is global by involving a global minimum. The vertex-pair condition is global since it makes use of *every* nonadjacent pair of vertices. The even-degree condition is global since all vertices' degrees need to be inspected. The cycle-partition is global by being of the whole graph. Such globality is inevitable since Eulerianness and Hamiltonianness are themselves global by involving *all* edges and vertices respectively. And yet, at least at this level of study, simple Combinatorics and inequalities suffices to prove the results, without any significant Topological machinery being required.

Remark 3 Our first two Eulerian Theorems are strong in a way that points to a further shortcoming for our Hamiltonian Theorems. For they are both necessary and sufficient, by which they provide universally useable diagnostics, while our Hamiltonian Theorems are sufficient but not necessary. This one-wayness has already been built into our names for the two basic Overcrowding Theorems. Combining this shortcoming with our overcrowding comment, it is apparent that our Hamiltonian Theorems are useful in some 'homogeneously large' edge-density regime. While saying nothing about whichever of low and inhomogeneous edge-density regimes. Use Sec 31.10's degree variance to quantify inhomogeneity here, while largeness is relative to ρ_{crit}, not $\rho_{\max}$.

[1] Not using the name 'Dirac's Theorem' is particularly clarifying. The conventional literature disambiguates our Theorem out as "Dirac's Theorem on Hamiltonian graphs" 'Dirac's Graph Theorem' would not do, due to other Theorems elsewhere in G.A. Dirac's work. While 'Dirac's Hamiltonian Theorem' would not do either, because *P.A.M.* Dirac considerably extended Hamiltonian *Mechanics* [351].

End-Remark 1 All in all, traversability in graphs is not simplex-democratic. For Remark 3 in particular has started to unveil that notions of 0-traversability – Hamiltonian-type – are much harder to study than notions of 1-traversability: Eulerian-type. This leaves Hamiltonian Graph Theory open to needing further investigation. See the next section's Pointers, while noting that this remains an unfinished and research-active topic.

42.8 Pointers to further topics on Hamiltonian graphs

Pointer 4 Over the two decades following Ore's work, further Local-Overcrowding Hamiltonian Theorems were established by Pósa, Chvátal, and Bondy–Chvátal.

Pointer 5 There are also many Hamiltonian Overcrowding Theorems for ori-graphs, including Ghouila-Houri's, Woodall's, and Meyniel's.

Exercise 15 For instance, Woodall's Theorem (1972) is that a nontrivial ori-graph on N vertices is Hamiltonian if the following holds.

$$d_-(u) \ + \ d+_(\, w) \ \geq \ n$$

whenever there is no arc from vertex u to vertex w.

a) Which simple graph theorem is this a close analogue of?

b) Propose a conceptual name for this Theorem.

c) Check that this is specific enough to not be confuseable with the other two Theorems mentioned above.

d)$^{++}$ Prove this Theorem.

Pointer 6 Planar graphs will turn out to be very common throughout the current Series. As we shall see in Chapter 45, these are mostly part of the low edge-density regime, and thus are seldom well-served by Overcrowding Theorems. We can however give an elementarily-provable Theorem due to Grinberg [159, 183, 176] in Sec 46.5 that is useful for planar graphs.

Pointer 7 4-connectivity suffices for the reasonably straightforward Hamiltonian Theorems of Whitney and of Tutte, also for planar graphs.

Pointer 8 Tournaments facilitate some guarantees of Hamiltonian paths or cycles being present.

Exercise 16 a)$^{+}$ Prove that every tournament must contain a Hamiltonian path.

b)$^{++}$ To have the stronger feature of Hamiltonianness itself – a Hamiltonian cycle – prove that the tournament being strongly connected is both a necessary and a sufficient condition.

Pointer 9 The *Hamiltonian cycle number* $\#H$ of Hamiltonian cycles present within a given graph can be taken to be a further graph cardinal. Next imagining regimes with a large enough $\#H$ to cover our graph. Then our graph is *Hamiltonian-connected* if every vertex pair therein is connected by a Hamiltonian cycle. Our graph is *strongly Hamiltonian* if every edge therein belongs to some Hamiltonian cycle.

Exercise 17 What is $\#H$ for each of K_5 and Petersen? Are these Hamiltonian-connected? Strongly connected?

See [176] for a few further Examples and results about Pointer 9).

Pointer 10 Finally, the stage has been set for considering notions of traversability involving higher simplexes.

End-Remark 2 For further reading about Hamiltonian graphs, see e.g. [176, 167, 191, 183, 193].

Chapter 43

Examples of forbidden and forced subgraphs

43.1 Some simple preliminary examples

Example 1 Cycles are forbidden in forests.

Limited Example 2 Multiple components and cycles are jointly forbidden in trees. Having multiple components does not however allude to any *specific type of* subgraph.

Example 3 Paths cannot contain branching points; this can be rephrased as Claw subgraphs being forbidden.

Example 4 Starlike trees cannot contain ≥ 2 branching points. Since, in a tree, two branching points have no choice but to be homeomorphically side by side, this can be rephrased as 2-paths of branching points – quadruped, monkey, dog... – being forbidden.

Exercise 1 Characterize in terms of forbidden subgraphs the generalized Quadruped, Monkey, Dog and Scorpion graphs.

43.2 Cycles forced by overcrowding

Proposition 1 Suppose that in a finite simple G all the degrees ≥ 2. Then it must contain a cycle.

Proof Pick any vertex $v_1 \in$ G. Since $\mathrm{d}(v_1) \geq 2$, we can also pick an edge leaving v_1, leading us to a v_2. As $\mathrm{d}(v_2) \geq 2$, we can furthermore pick a distinct edge leaving v_2. We can keep on going until it is not possible to thus pick a new vertex. Since G is finite, this must happen within a finite number of steps. Then the sequence of edges between the two occurrences of our first repeated vertex constitutes a cycle. □

43.3 Odd-cycle-free graphs = bi-partite graphs

Proposition 2 A graph is bi-partite iff it is free of odd cycles.

Proof ($\Rightarrow$) Suppose that G is bi-partite, with corresponding 2-piece partition into vertex sets W and W'. Let $(v_1 ... v_k)$ be a cycle in G. W.l.o.g. (reversing which is labelled prime if needs be)

$v_1 \in W$. Bi-partiteness then forces the alternation $v_2 \in W'$, $v_3 \in W$... Since C is a cycle, it contains an edge v_1-v_k. Bi-partiteness then forces $v_k \in W'$, which in turn forces k to be even.

($\Leftarrow$) Suppose that G is odd-cycle-free. We established in Exercise 34.2 that bi-partiteness is hereditary. So in particular the component subgraphs of a bi-partite graph are themselves bi-partite. Thus we can w.l.o.g. consider G to be connected.

Pick a vertex $u \in \mathfrak{V}$. Define the vertex subspace

$$\mathfrak{U} := \{ x \,|\, d(x, u) \text{ is odd} \} .$$

We cannot have any edge joining two points in $\mathfrak{U}$ or two points in its set complement $\mathfrak{U}^c$. For if there were a such, an odd cycle would be realized (Fig 43.1). This forces $\mathfrak{U}$ and $\mathfrak{U}^c$ to be disjoint. They also cover $\mathfrak{V}$ by the definition of set complementation. Thus they constitute a 2-part partition of $\mathfrak{V}$, and we have deduced that G is bi-partite. □

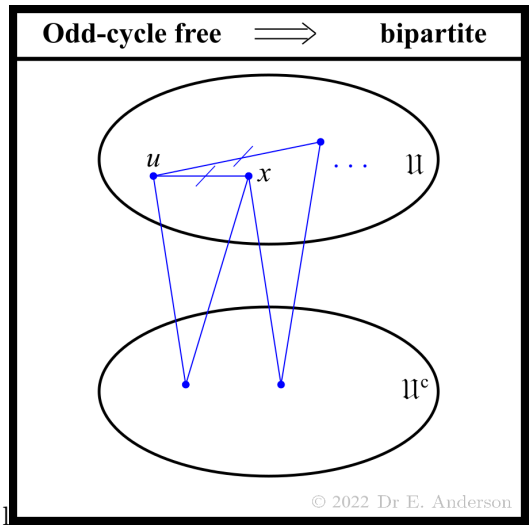


Figure 43.1:

43.4 Triangle-free graphs

Overcrowding Forced-Triangle Theorem (Mantel 1907). Let G be a graph of order N and size

$$E \geq \left\lfloor \frac{N^2}{4} \right\rfloor . \tag{43.1}$$

Then G contains a triangle.

Proof Suppose that G is an order-n triangle-free graph.

1) Consider an edge $u\,w$ therein. Then the punctured neighbourhoods of u and w are disjoint. For suppose not. Then there would be a vertex y adjacent to both u and w. But then u, w, y would constitute a triangle. Contradiction!

It then follows that our graph's adjacent-vertex degree sums

$$d(u) + d(w) \leq N . \tag{43.2}$$

For each vertex y can be adjacent to at most one of u or w by the preceding paragraph. So it can contribute to at most one of $d(u)$ or $d(w)$. Repeat this argument for each $y \neq u, w$. We thus obtain a bound of $N - 2$ on our adjacent-vertex degree sum. Finally u adds 1 to $d(w)$ while w adds 1 to $d(u)$. So we arrive at an overall bound of

$$(N - 2) + 1 + 1 = N + (- 2 + 2) = N .$$

Next sum (43.2) over all of G's edges to obtain the following.

$$\sum_{u\,w \in \mathfrak{E}(\mathrm{G})} (d(u) + d(w)) \leq \sum_{u\,w \in \mathfrak{E}(\mathrm{G})} N = N \sum_{u\,w \in \mathfrak{E}(\mathrm{G})} 1 = N\,E(\mathrm{G}) .$$

The second step here just pulls out a constant factor, while the third uses the definition of size = edge number.

This sum can furthermore be rearranged to run instead over our graph's vertex set. For it picks up a $d(u)$ contribution from the u end of the edge $u\,w$. And a $d(w)$ contribution from the w end. Thus each vertex v contributes $d(v)$ per edge with one end at v, of which there are precisely $d(v)$. The overall contribution is thus $d(v)^2$. We have thereby recast our inequality as

$$\sum_{v \in \mathfrak{V}(\mathrm{G})} d(v)^2 \leq N\,E(\mathrm{G}) . \tag{43.3}$$

2) The following inequality ensues from squaring Euler's degree-sum formula.

$$4\,E(\mathrm{G})^2 = (2\,E(\mathrm{G}))^2 = \left(\sum_{v \in \mathfrak{V}(\mathrm{G})} d(v) \right)^2 \leq \sum_{v \in \mathfrak{V}(\mathrm{G})} d(v)^2 \leq N \sum_{v \in \mathfrak{V}(\mathrm{G})} d(v)^2 .$$

The first inequality here is Cauchy–Schwarz's. This should be familiar to the Reader from a basic course on Vector Space Algebra (if not, see e.g. Volume 1 [441]). The second inequality is just an example of a *throw-away inequality*. In this case, it is all the cross-terms which are thrown away. This is licit since degree ≥ 0 follows from its definition as an edge count.

Next substitute 1)'s inequality in 2)'s to obtain

$$4\,E(\mathrm{G})^2 \geq N\,(N\,E(\mathrm{G})) = N^2\,E(\mathrm{G}) .$$

Both sides can now be divided by $4\,E(\mathrm{G})$, assuming that our graph contains ≥ 1 edge. Thus

$$E(\mathrm{G}) \leq \frac{N^2}{4} \quad \Rightarrow$$

$$E(\mathrm{G}) \leq \left\lfloor \frac{N^2}{4} \right\rfloor ,$$

since $E(\mathrm{G})$ is $\mathbb{N}_0$-valued.

Our above assumption does not affect the outcome since

$$0 = E(\mathrm{G}) \not> \left\lfloor \frac{N^2}{4} \right\rfloor .$$

So the Theorem does not apply. Indeed, a Theorem about graphs containing triangles of edges will not be undone by edgeless point cloud cases.

Finally apply the contrapositive. Namely that if

$$E(\mathrm{G}) \; > \; \left\lfloor \frac{N^2}{4} \right\rfloor$$

then it is not triangle-free, i.e. it must contain a triangle. □

Remark 1 Viewed as an edge density result, the critical value is

$$\rho_{\triangle} \; = \; \frac{1}{2} \, \frac{N}{N \, - \, 1} \quad \text{for} \quad N \quad \text{even}$$

and

$$\rho_{\triangle} \; = \; \frac{1}{2} \, \frac{N \, - \, 1}{N} \quad \text{for} \quad N \quad \text{odd} \quad .$$

So the result is that once approximately half the possible edges have been realized, a triangle is forced.

Remark 2 This Theorem is moreover a *guarantor of included substructure*. In setting about generalizing this Theorem, one should take into account that Triangle is both C_3 and K_3. It then turns out that a generalization to K_n is readily available: the *Overcrowding Forced-Clique Theorem* (Turán 1941). Mantel's Theorem is thus part of a larger Theorem about overcrowding guaranteeing the presence of cliques, rather than of N-a-gons.

Remark 3 Mantel's Theorem is moreover also an *extremal result*. Its bound is tight, by the complete bi-partite graph

$$\mathrm{K}_{\lfloor \frac{N}{2} \rfloor, \lceil \frac{N}{2} \rceil}$$

containing no triangle. For it is bi-partite and thus cannot contain any odd cycles, of which triangles are the smallest case.

♠ ◇ ♣

Exercise 2 Show that the above is the maximally large bi-partite graph. By which forbidding all odd cycles is no more stringent on graph size than forbidding the smallest odd cycles: triangles.

Application 1 We shall see in Chapter 49 that Triangle is forbidden in graphs that underlie posets.

Exercise 3 What are the connected triangle-free graphs on a) 4 vertices? b) 5 vertices? c)$^+$ 6 vertices?

Exercise 4$^+$ a) Show that a graph with

$$E \; > \; \left\lfloor \frac{N \, + \, 3}{2} \right\rfloor$$

must contain $\geq$ 3 vertices of degree $\geq$ 2.

b) Show that any graph whose degrees are all $\geq$ 2 must contain a 3-cycle.

c) Why is this Exercise less interesting than Mantel's Theorem?

Exercise 5^{++} and Pointer 1 Prove the *Overcrowding Forced-Clique Theorem* "Every graph of order N and size

$$E \; \geq \; \left(1 \, - \, \frac{1}{r} \right) \; {}^{n}C_2 \; = \; \frac{r \, - \, 1}{r} \, \frac{n \, (\, n \, - \, 1 \,)}{2}$$

must contain an $(r \, + \, 1)$-clique."

[This result is beyond the scope of the current Volume. It is however included for Readers who are both very keen and yet relatively inexperienced with Graph Theory, and so have not yet proven this result elsewhere.]

Exercise 6 Find the largest complete tripartite graph on N vertices. Does this saturate the $r = 3$ bound in Turán's Theorem?

43.5 Further Pointers

Pointer 2 More famously, we shall see in Chapter 45 that subgraphs containing K_5 or $K_{3,3}$ are forbidden if one is to have a planar graph. Also that subgraphs containing K_4 or $K_{3,2}$ are forbidden if one is to have an outer-planar graph.

Pointer 3 We shall see in Sec 52.2 that a certain ordering of pentagons C_5 are additionally forbidden in graphs that underlie modular lattices.

Pointer 4 We shall additionally see in Sec 52.7 that $K_{3,2}$ subgraphs are additionally forbidden in graphs that underlie a fortiori distributive lattices.

Chapter 44

Graph Homeomorphs

44.1 Homeomorphs

Definition 1 Two graphs G_1, G_2 are *homeomorphs* if one of them can be formed from the other by inserting vertices into some of its edges. Let us denote this binary relation by

$$G_1 \ \mathcal{H} \ G_2 \ .$$

Definition 2 They are *homeomorphic* if both are homeomorphs of some common graph G_P. Let us denote this by

$$G_1 \overset{\text{homeo}}{\sim} G_2 \ .$$

Remark 1 I.e. $G_1 \overset{\text{homeo}}{\sim} G_2$ iff $G_1 \ \mathcal{H} \ G_P$ and $G_2 \ \mathcal{H} \ G_P$.

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Remark 2 This need not mean that they are homeomorphs of each other.

Counterexample 1 See Fig 44.1.b).

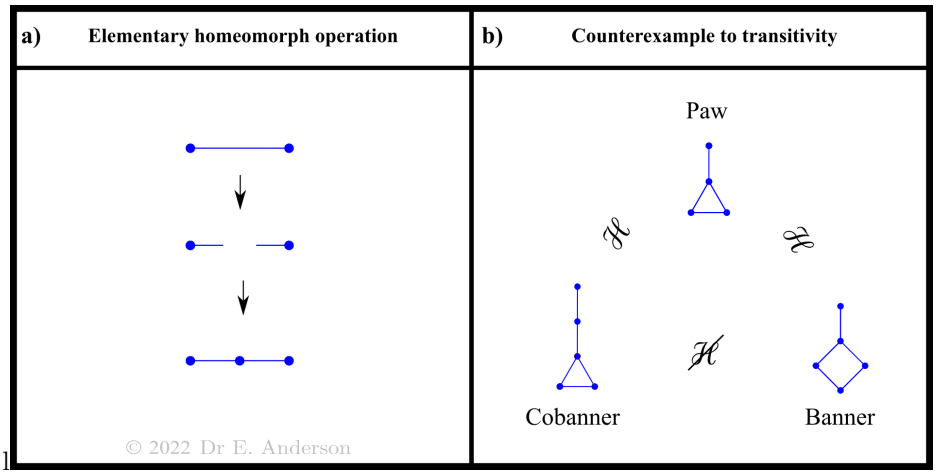


Figure 44.1:

Remark 3 So on the one hand, $\mathcal{H}$ fails to be transitive; for that matter, its definition is not symmetric either. It is reflexive by the trivial operation of not inserting any vertices.

Exercise 1$^-$ Show that, on the other hand, $\overset{\text{homeo}}{\sim}$ is an equivalence relation.

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Remark 4 The reverse process can also be used to define homeomorphs, as follows.

Definition 1′ Two graphs are *homeomorphs* if one of them is formed from the other by multiple iterations of the following process.

a) Removal of a vertex.

b) Each individual removal is followed by unique identification of edges thereby left incomplete.

c) Each such removal is licit provided that the corresponding identification is possible, nontrivial and unambiguous. [This forces the vertices being removed to be of degree 2, and also not to be in 3-cycles if the outcome is to be a simple graph.]

Remark 5 For now, we observe that forming a homeomorph is a type of stretching (or, in reverse, a type of contracting). *Metric* properties of the graph change, while various *Topological* features remain invariant. This includes not adding or removing components or branching points.

44.2 Small examples

Worked Example 1 Which 5, 6 and 7 trees are homeomorphs of Claw?

Answer 1 Claw is alias S_3, so its homeomorphs are the starlike graphs with a degree-3 branching point. Recollect that we argued in Chapter 38 that starlike graphs are just a standard matter in terms of partitions. Our particular problem translates to that of finding the partitions of the $n = N - 1$ non-branching vertices into the 3 boxes made available by the radiances.

Inserting one vertex into an edge gives $S\left(2\,1^2\right)$ = Chair. Inserting two vertices into the same edge gives $S\left(3\,1^2\right)$ = Longchair. Inserting one vertex into two distinct edges gives $S\left(2^2\,1\right)$ = Bird. Inserting three vertices into the same edge gives $S\left(4\,1^2\right)$ = 2-Longchair. Inserting two vertices into one edge and one into another gives $S(3\,2\,1)$ = SAT. Finally, inserting one into all three gives $S\left(2^3\right)$ = Lobster.

See Fig 44.2.a) for these in the above order. This Volume uses emerald green to pick out homeomorph irreducibles. The arrows indicate the sequential ways in which these arise by composition of single-vertex insertion homeomorphs. We shall be developing these arrows somewhat further in Part VIII using Part VII's Order Theory.

Exercise 2$^-$ Which 7- and 8-trees are homeomorphs of Quadruped?

Worked Example 2 What are the homeomorphs of Diamond on 5 to 7 vertices?

Answer 2 Diamond has a distinguished central edge. This splits the other edges into two equivalent pairs, which we refer to here as the upper pair and lower pair.

Inserting one vertex can thus occur in two places, giving $K_{3,2}$ on the central edge and House elsewhere. Inserting two vertices can involve these being together on the central edge. Or split between one on the central edge and one elsewhere. Or split between the edge pairs, forming Domino. Or together on a non-central edge.

Three vertices can be inserted on the central edge. Or two on the central edge and one elsewhere. Or one on each of the central edge, the upper pair and the lower pair. Or one on the central edge and two in the same pair. Or one into one pair and two into the other. Or all three into the same pair. See Fig 44.2.b) for these in the above order.

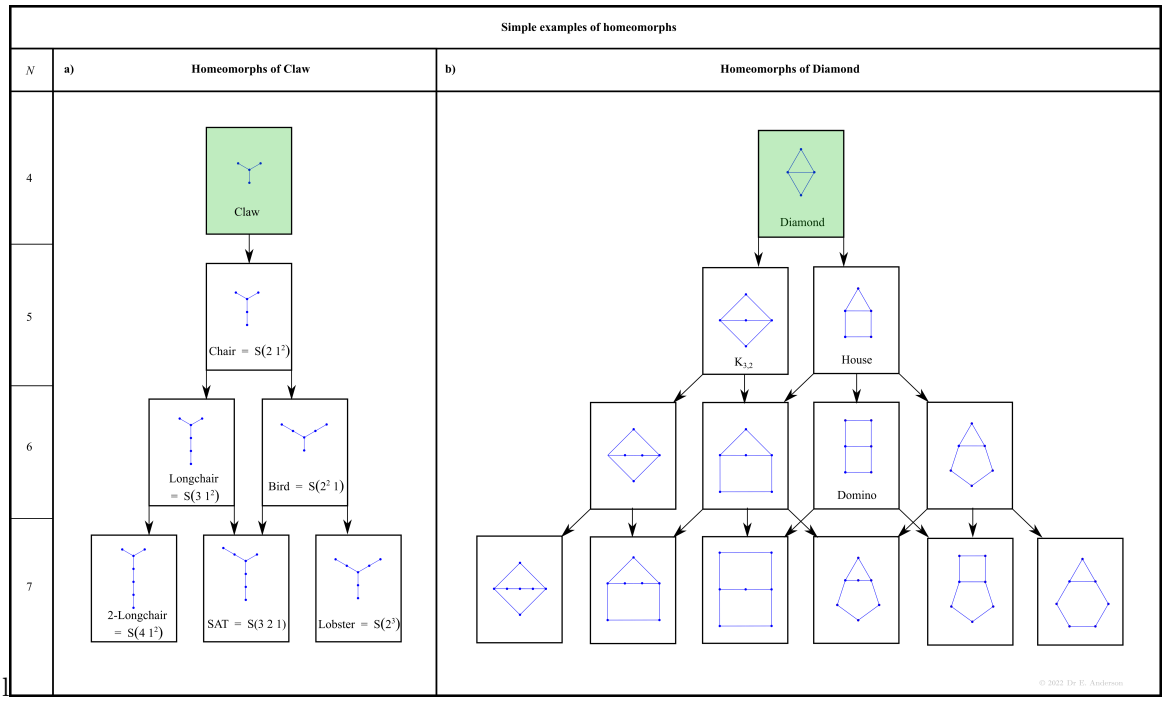


Figure 44.2:

Exercise 3 a)$^-$ What are the homeomorphs of K_2 and of K_3?

b) What are the homeomorphs of K_4 which have 5 to 7 vertices?

Exercise 4 Which 4-graphs are homeomorphs of which 3-graphs? Which 5-graphs are homeomorphs of which 4-graphs?

Pointer 1 See e.g. [187, 193] for more about graph homeomorphs.

Pointer 2 and Exercise 5 Find (or even recollect, from earlier on in this Book) a minimalist way of 'patching over' any vertices of degree 4 present in an approximate model for a network. Extend this to a method for 'patching over' any vertex of degree ≥ 4. Combined with defoliation and homeomorphing, this method gives a way of converting any graph to a cubic graph (or the point).

44.3 Homeomorphism irreducibles

Definition 1 A '*homeomorphism irreducible*' is a graph that is not the homeomorph of any smaller graph.

Remark 1 This is a minimality property.

Naming Remark 1 I use 'irreducible' instead of 'prime' here because of the apparent lack of a decomposition of the general graph into multiple such. The general graph is, rather, stripped down

by reverse homeomorph moves until an individual homeomorphism irreducible is arrived at.

44.4 Small examples of homeomorphism-irreducible trees

In the examples below, we only consider connected graphs. From these, unconnected graphs can be built up componentwise. Also to start off with, let us just consider trees.

Example 0 D_1 is a homeomorphism irreducible that no other graphs are homeomorphs of. This is because it has *no* edges in which to insert further vertices.

Example 1 P_2 is a homeomorphism irreducible whose homeomorphs are the path graphs P_n, $n \geq 2$.

Example 2 Claw, and more generally each S_n are homeomorphism irreducibles.

Examples 3-7 Quadruped, Monkey, Forkclaw, Dog and Scorpion are homeomorph irreducibles (see Exercises 1, 4 and 5 of Sec 38).

Remark 1 See Fig 44.3 for the first few homeomorphs of the first few homeomorph-irreducible trees.

Exercise 6 a) What are the regularities of each tree whose homeomorph irreducible has a 2-path of branching points?

b) What about the 3-path counterpart?

c) What are the smallest trees that manifests the generic answer to b)?

44.5 Homeomorph-irreducible connectedness graphs on up to 5 vertices

See Fig 44.4 for the non-tree connected homeomorph irreducibles on ≤ 5 vertices.

44.6 A few larger examples

Proposition 1 (Local structural diagnostic for homeomorph irreducibles). A graph is a homeomorph irreducible iff it possesses no vertices of degree 2 anywhere except possibly on vertices that form part of 3-cycles.

Remark 1 This means that we can always homeomorph out degree-2 vertices from trees, or from sidetrees on other graphs. We cannot necessarily homeomorph out degree-2 vertices that form part of cycles. For instance, we can homeomorph any C_n graph down to the 3-cycle C_3 but not any further. This is the only connected homeomorph irreducible graph all of whose vertices are of degree 2.

Remark 2 We can homeomorph n-Pan down to the homeomorph irreducible graph Paw, which contains 2 degree-2 vertices on its sole 3 cycle. We can similarly homeomorph 'the pans with handles at 2 different points' down to the homeomorph irreducible graph Bull, which contains 1 degree-2 vertex on its sole 3 cycle. A simple example justifying the word 'possibly' is 3-Sunlet, which contains a 3-cycle all of whose vertices are of degree 3 and thus not 2.

Remark 3 It is certainly possible for a vertex on a 3-cycle to also form part of a larger cycle. If such a vertex is of degree 2, then it must be isolated (not adjacent to other vertices of degree 2).

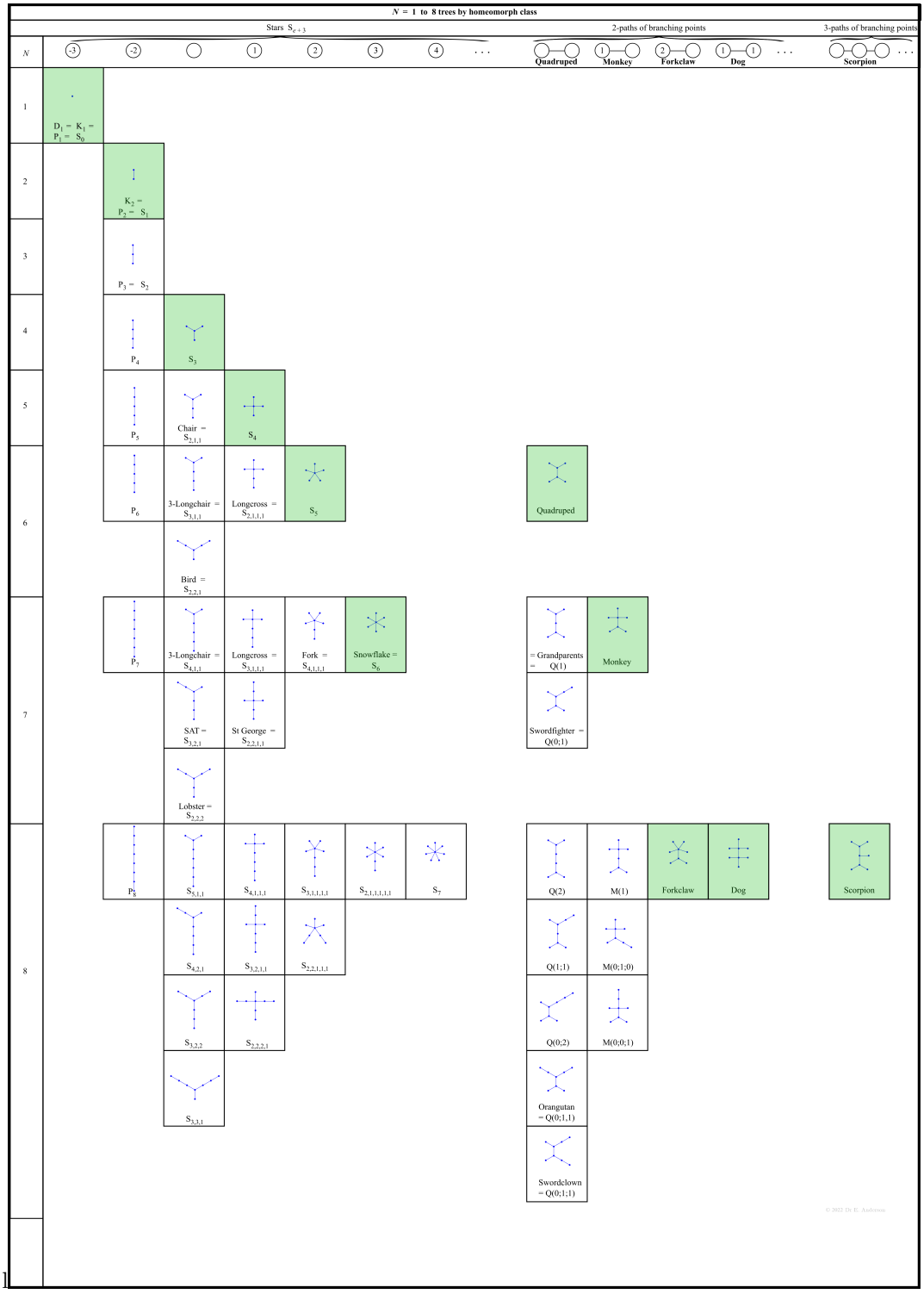


Figure 44.3:

For suppose that two such were adjacent. Then any path extending the edge through them must then cross itself at the third vertex of the 3-cycle. This leaves no room for such a path to form a larger cycle.

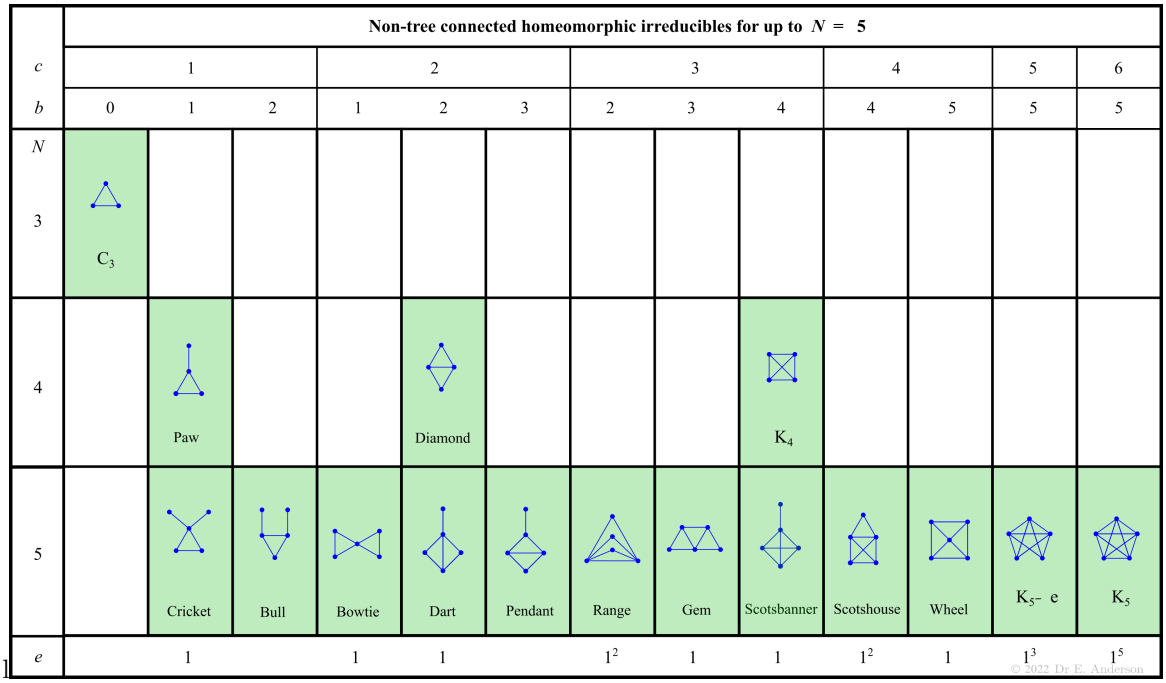


Figure 44.4:

Remark 4 That isolated degree-2 vertices on 3-cycles are possible is size-minimally illustrated by the homeomorph irreducible graph Diamond. Since our vertex is of degree 2, it has just the one path entering and leaving it. Because our vertex is on a 3-cycle, the two adjacent vertices reached by our path must thus have an edge joining them. So our 3-cycle must close up before any longer cycle is reached. So our 3-cycle is an elementary piece ('local triangulation') of our larger cycle.

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Examples 1 to 3 Homeomorph irreducibles *can* include longer non-triangulated cycles, but only if these cycles' vertices are exclusively branching points. The minimal single-cycle example of this is the 4-Sunlet graph on 8 vertices (Fig 44.5.a). Size-minimal, if more edge-dense examples are given in Fig 44.5.b-c).

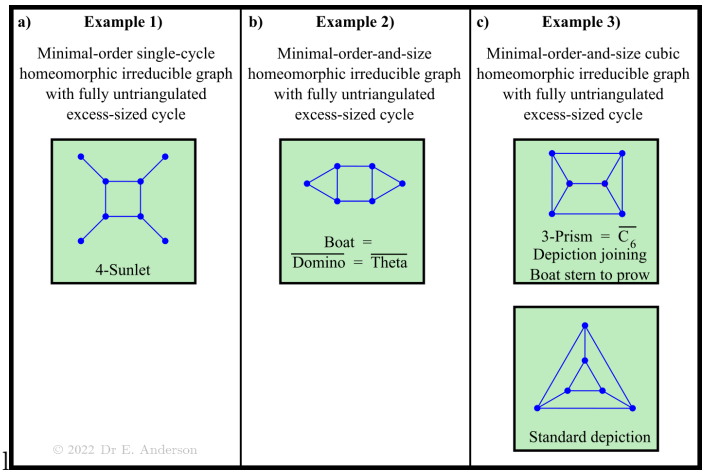


Figure 44.5:

Exercise 7 Show that all of the following are homeomorph irreducible graphs.

i) The K_n and $K_n - e$.

ii) All Wheel, Fan, and Range graphs.

iii) All Sun, Sunlet, and Windmill graphs.

♠ ◇ ♣

Proof (of Proposition 1). ($\Leftarrow$) Suppose that some homeomorph irreducible contains at least one vertex of degree 2 that is not part of a 3-cycle. Pick such a vertex. Then removing this vertex and identifying the two half-formed edges produces a smaller graph homeomorphic to the original graph. Contradiction! (of the minimality of homeomorph irreducibles).

($\Rightarrow$) Suppose that a graph possesses no vertices of degree 2. Then each vertex present has degree 0, 1 or $\geq$ 3. For degree 0, triviality strikes. For degree 1, there are too few edges for such an identification to be possible. For degree $\geq$ 3, ambiguity strikes. Thus our graph meets the minimality of a homeomorph irreducible.

Finally suppose that vertices of degree 2 on 3-cycles are included. These are homeomorphism-stable by the definition of simple graph. I.e. removing a vertex here and then joining the two broken edges would produce a 2-multi-graph version of P_2 as a subgraph, which is forbidden. □

Remark 5 This indicates the sense in which 3-cycles are topologically stable in simple graphs. This situation must change for loop- and multi-graphs; see Sec 44.11.

Remark 6 In the context of our main 13 models, each is regular, multi-regular or sub-regular. With maximal realized degree 3, 4 or 6. So passing to homeomorph irreducibles enforces a somewhat more stringent kind of multi-regularity on these. It is moreover a *structurally conditional* multi-regularity: what degree-2 vertices remain *must lie on a 3-cycle*.

Remark 7 Having a classification and a notation for small to medium-sized trees means also having such for sidetrees, which are now to be taken to be rooted at their attachment point.

Exercise 8 What are the homeomorph irreducibles with a single cycle on up to 7 vertices?

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44.7 Discussion distancing Chemical Graph Theory from this Volume's other applications

Remark 1 Taking homeomorphs merits contrast with serial defoliation, the end-result of which is no vertices of degree 1. Serial defoliation's overall effect is moreover the removal of all sidetrees from a graph. Thus some degree-2 vertices may be wiped out as well: those in sidetrees. Sidetrees can furthermore contain branching points, which then get wiped out as well. All cycles are however preserved. So in general the loss of all degree-1 vertices here is accompanied by partial losses in both degree-2 vertices and branching points.

Difference 1 While taking homeomorphs can shorten some sidetrees, it can never completely eliminate any sidetree. *Sidetree number S* is a homeomorph invariant.

Difference 2 Taking homeomorphs is in general only partially successful at eliminating degree-2 vertices. Serial defoliation is in contrast fully effective at eliminating degree-1 vertices. Yet this is in general at the cost of losing both some degree-2 vertices and some branching points. In contrast, taking homeomorphs entirely respects leaf number L and branching point number Br, which are further homeomorph invariants.

Similarity 1 Both taking homeomorphs and serial defoliation respect *cycle number* $\#C$, which is thus a *joint* homeomorph and defoliation invariant.

Difference 3 *Cycle lengths* are furthermore defoliation invariants, while in general not being homeomorph invariants. It is only *cycle lengths of cycles formed exclusively from branching points* that are homeomorph invariant.

Remark 2 Branching points are a Topological feature because deletion of an individual branching point changes one or both of component number and cycle structure. In particular, branching points in sidetrees are Topological features because their deletion increases component number. This gives one sense in which serial defoliation does not respect Topology. While deletion of degree-2 points in sidetrees alters component number as well, the homeomorphism operation as a whole preserves component number. In contrast, homeomorphisms do not respect metric properties of cycles, while serial defoliation does.

Remark 3 These properties leave serial defoliation well-suited to a subset of chemical reactions, which remove or add ends to chains while preserving cycle structure. Homeomorphic differences are however largely not well-attuned to Chemistry

Remark 4 *Taking* homeomorphs is irrelevant for specifically regular models like pure-carbon, polytopes, and knots. These *already realize homeomorph irreducibles*. As do Feynman diagrams, which are (1, 3, 4)-regular or some subcase thereof. The same argument applies to n-ary trees for $n \geq 3$. For these are (1, n, $n + 1$)-regular, with none of these values being 2. For binary trees, there is precisely one degree-2 vertex present, which plays the key role of root, and thus cannot be homeomorphed out in this application.

Thus our models are split between the majority involving just homeomorph irreducibles. While Chemistry attaches specific significance to metric properties, corresponding to the numbers of atoms in its rings and chains. And binary trees require the minimal deviation from homeomorph irreducibles. This should confer a sense in which homeomorph primes are very significant. Being more quotiented-out objects, however, these are rather harder to study, thus posing a valuable challenge. A deep sense in which Chemical Graph Theory is simpler than that of Physical, Geometrical, Computer-Scientific and Knot-Theoretic Graph Theories has furthermore been pointed out.

44.8 Uniqueness of homeomorph irreducible for a given graph

Proposition 2 Given a finite graph, the corresponding homeomorph irreducible is unique.

Proof Suppose that a graph contains $m \geq 2$ degree-2 vertices that do not lie on 3-cycles. Then we can pick two vertices among these, call them v and w. It makes no difference in which order these two are removed. Inducting on this, removing the finite number m of vertices produces a common end-product. □

Remark 1 Thus our minimality is encoded by a comparable ordering. Meaning that any graph G homeomorphic to a given homeomorph irreducible HIG has $V(\text{G}) > V(\text{HIG})$. This is precludes

the a priori possibility of some distinct graphs having equal order to their homeomorph irreducible graph.

44.9 Systematic notation for caterpillar trees

Remark 1 The homeomorph irreducible trees we have introduced so far are all caterpillars, for which the following two notations are more efficient.

Notation Remark 1 Introduce the *branching point strength*: a shifted-degree relative variable

$$b(v) \;:=\; d(v) \;-\; d(\,\max(\text{non branching point}\,) \;=\; d(v) \;-\; 2\;. \tag{44.1}$$

Then caterpillars can be expressed as path graphs whose vertices are labelled by this variable (which we decree to take lower-case letter values). For instance, the above Examples 0 to 7 of Fig 44.4 are cast in this notation within Fig 44.6.

Exercise 9$^-$ Finish working out the automorphism for the cases in this Figure.

Notation Remark 2 Label the caterpillar's non-leaf vertices with their leaf numbers.

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Exercise 10$^-$ Which caterpillars are minimal for the above two notations to not coincide?

Remark 2 Notation 2 is more attuned to computing the automorphism group of the graph.

Remark 3 Notice the difference between homeomorph irreducibles and displaying metrically-distinct versions thereof. This is now to be done as an *edge labelling* by edge length in internal vertices of in excess of those in the underlying homeomorph irreducible graph.

Exercise 11$^-$ Reinterpret Ex 38.6, 10 and 11 to form tree examples of this.

Exercise 12 i) How many homeomorphs of excess vertex count 1 to 3 vertices does each subfigure of Fig 44.7 possess?

ii) What are the automorphism groups for each of these homeomorph irreducibles?

44.10 Adding metric information to homeomorph irreducibles that contain cycles

Example 1 The homeomorphs of C_3 form the cycle family C_n. These are parametrized by a single metric variable: the *excess cycle length*

$$e_{\rm c} \;:=\; c \;-\; c_{\rm min} \;=\; c \;-\; 3\;.$$

This is a further relative length variable.

Example 2 The homeomorphs of Paw are parametrized by two metric variables: excess cycle length and *excess handle length*,

$$e_{\rm h} \;:=\; l \;-\; l_{\rm min} \;=\; l \;-\; 1\;.$$

For excess handle length 0, the n-Pan family is realized. In general, the $(m,\, n)$-Longpan graphs are thus realized.

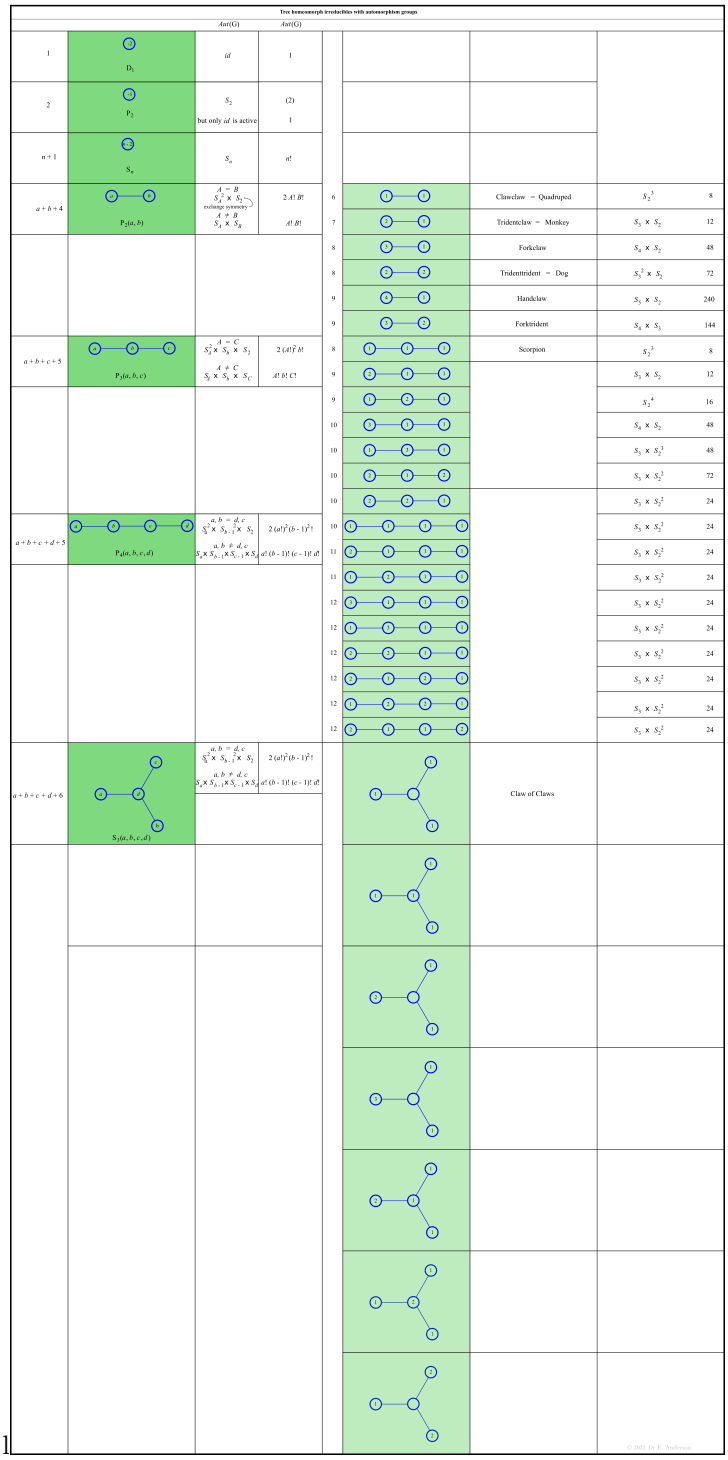


Figure 44.6:

Exercise 13 a) How many homeomorphs do each of the following possess on each of 5 to 7 vertices? Paw, Diamond and K_4.

b) How many homeomorphs does Bowtie possess on each of 6 to 8 vertices?

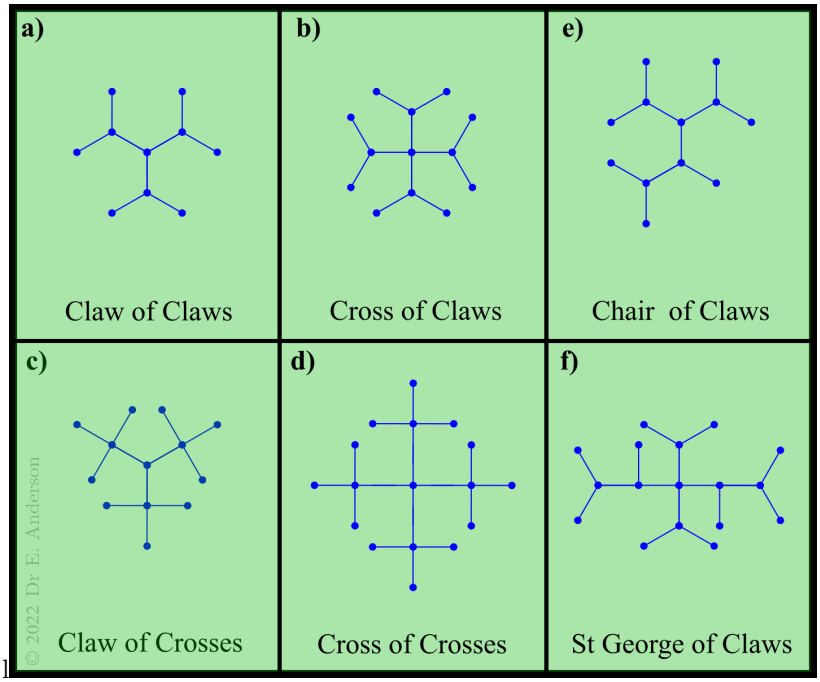


Figure 44.7:

c) How many homeomorphs on 7 to 9 vertices are there of Bowtie's bridged counterpart Bridge (Fig 41.3.iii).

In each case, start by figuring out how to metrically parametrize each family.

Pointer 2 Sec 46.1 considers homeomorphs of K_5 and $K_{3,3}$ since these are crucial to the theory of planar graphs.

Pointer 3 Sec 46.3 considers homeomorphs of $K_{3,2}$, since these and those of K_4 are crucial to the theory of outer-planar graphs.

Pointer 4 Sec 47.11 considers homeomorphs of the Petersen graph, since these are crucial to the theory of graph colouring.

Pointer 5 Spaces of graphs modulo homeomorphic equivalence are the subject of Chapter 62 .

44.11 Homeomorph irreducibles for loop-multi-graphs

These are provided up to $N = 4$ in Fig 44.8.

Exercise 14$^+$ What are the homeomorph irreducibles of the 2-multi-graphs on ≤ 4 vertices?

Exercise 15$^+$ What are the homeomorph irreducibles of the (1, 2)-loop–multi-graphs on ≤ 3 vertices?



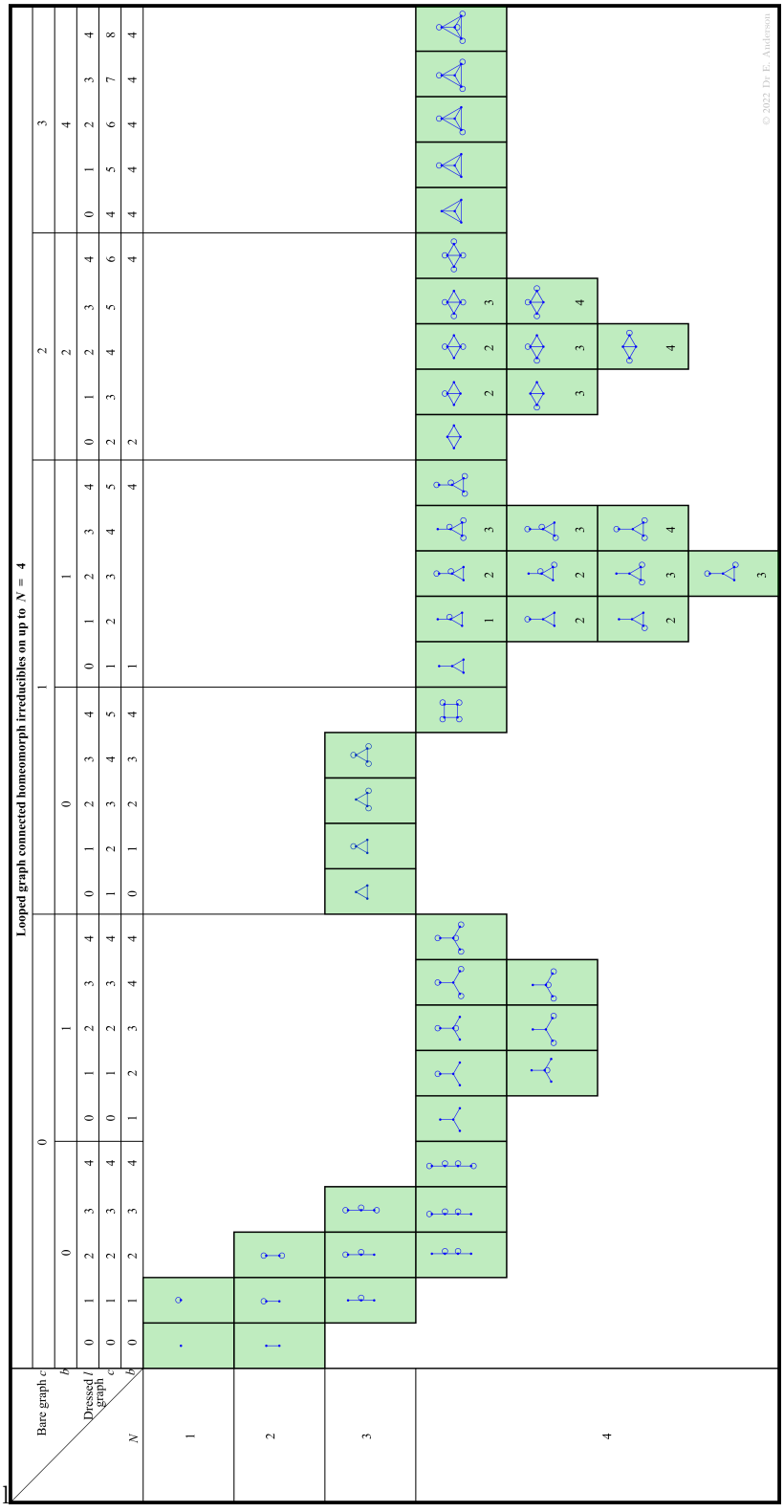

Looped graph connected homeomorph irreducibles on up to N = 4
Bare graph c
Dressed l graph c
N
© 2022 Dr. E. Anderson

Figure 44.8:

Chapter 45

Planar graphs

45.1 Introduction

Definition 1 A *planar graph* is one that can be embedded in the plane without edges crossing.

Remark 1 This means that it can be represented by a drawing in the plane in which no edges cross.

Remark 2 Observe that planarity is a property of the graph itself, and not of how it is embedded in the plane. It is entirely possible to planarly-embed almost any planar graph in a manner which *does* have crossing edges! For instance, P_4 can be embedded so as to have its leaf line-segment edges cross (Fig 45.1.a). For P_3, this is not possible with straight edges, but Fig 45.1.b) shows it done for curved edges. Even P_2 can be represented with edge crossings, provided that we allow self-intersecting curves to represent edges (Fig 45.1.c).

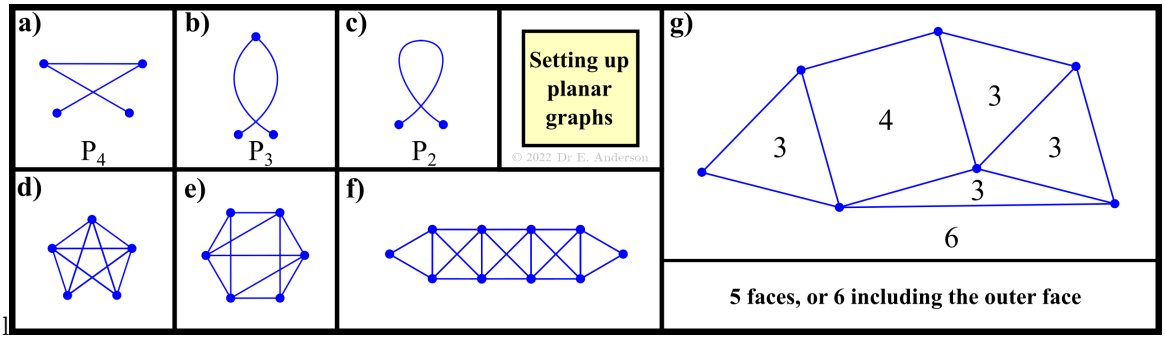


Figure 45.1:

Our aim is however to represent graphs as simply as possible, and one criterion for this is having as few edge crossings as possible. If we can get this down to no crossings at all, then we have succeeded in demonstrating that our graph is planar.

Pointer 1 Eliminating or minimizing crossings is a major motivator as regards effectively depicting graphs [194]. This is not however the only criterion that one's modelling might require. Indeed, in Sec 22.3, we mentioned manifest presentation of automorphisms of a graph. These two criteria moreover often turn out to be incompatible. Which is already apparent from the widespread use of all of the following for K_4.

Presentation 1 The '3-triangles' presentation (Fig 30.2.c). Which is planar. And yet only manifests a $D_3 \cong S_3$ and thus order-6 subgroup of the graph's S_4 and thus order-24 symmetry group.

Presentation 2 The 'square with crossed diagonals' presentation (Fig 30.2.d): non-planar. As interpreted literally, so that the vertices but not the edges enter it on an equal footing. So that now a D_4 subgroup – order 8 – of S_4 is manifest.

Presentation 3 The previous, but now interpreted as depicting a tetrahaedron. By which its edges additionally enter on an equal footing, and so the full S_4 symmetry group is manifest. This is however not even pretending to be a 2-d presentation.

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Remark 3 Trying to minimize, or fully eliminate, edge crossings can be approached simply as a subcase of performing graph isomorphisms. On the long run, performing such 'by eye' is not likely to be the most efficient strategy. Especially not in cases for which it can be proven that there is *no way* of further reducing the number of edge crossings. Yet doing this 'by eye' does quickly lead one to start to get a feel for planar graphs. We thus point back to Exercises 9 to 14 of Chapter 31 as already being Worked Examples in this direction. Thus armed, we invite the Reader to try the following Exercise.

Exercise 1 a) Find a planar presentation for each of the graphs depicted in Figs 45.1.d-f).

It turns out that (Sec 46.2) any planar graph can be drawn not only with no crossings but also concurrently with rectilinear edges only.
b) Find such a no-crossings rectilinear presentation for each of the graphs in Fig 45.1.d-f).

c) What are the symmetry groups of these three graphs? Compare with those of their given presentations and of your answers' presentations. How can your answers' presentations' symmetries be maximized?

Remark 4 Using just straight edges is consequently common. By which P_3 becomes manifestly planar in all licit presentations. It is even more common to forbid self-intersecting edges. By which the above problem that even P_2 manifests goes away.

Exercise 2 How about using just unit-length straight edges?

a) Show that any path, star, starlike, or quadruped graph can be represented in this way.

b) Give 8 examples of infinite families of connected cycle-containing graphs which can be represented in this way without any crossings.

b) To show that this way of drawing graphs is rather too special as regards representing all planar graphs, which 4- and 5-graphs cannot be represented in this way?

45.2 Face count

Structure 1 Let us denote the number of faces supported by a planar graph by F. This is a covering by smallest face-disjoint cycles concept. [Rather than keeping on counting larger cycles supporting multiple smaller cycles' faces.] For many purposes, including to become a graph-isomorphism invariant notion count is furthermore to include the outer face covering the rest of the plane.

Pointer 2 In more complicated cases, it may not be straightforward to see which is the outer face. And yet which face this is remains well-defined by the famous *Jordan Curve Theorem* of Topology. By which every simple (i.e. non self-intersecting) closed curve in the plane must have a well-defined interior and exterior [260].

45.3 Euler's formula

Theorem 1 For a planar connected graph,

$$V - E + F = 2 . \tag{45.1}$$

Remark 1

$$\chi := V - E + F$$

is a kind of relative variable, though it does not have equal sums of each sign.

Remark 2 It is an *alternating sum*, whose indexing variable is a democracy indexing variable, i.e. running over dimensionality: 0 for vertices, 1 for edges and 2 for faces.

Remark 3 It is profoundly surprising that this must take a fixed value for connected planar graphs.

Remark 4 It is less surprising that this quantity is free from the heterogeneous extensiveness that it would exhibit for a generic graph. This is because the planar graphs manage to be a linearly-sparse regime.

Remark 5 Let us now rephrase the Theorem as

$$\chi = \chi_{\mathbb{R}^2} = 2 .$$

Naming Remark 1 The standard name for χ is the *Euler characteristic*. (45.1) is known as *Euler's formula*.

Remark 6 (45.1), equivalent statements and generalizations enter many different subjects at a basic and yet foundational level. Multiple different styles of proof ensue, using ideas from each of many of the subjects that it touches upon. For now, we give a Graph-Theoretic proof. Yet in Volume 3 we shall give an alternative geometrical proof. And Series 1 shall contain further proofs of it and its generalizations using various more specifically topological types of idea. Indeed, one of the reasons to seek multiple conceptually-distinct proofs for an already-profound Theorem is that some of these proofs may point to further or distinct generalizations.

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Proof Induct on F.

Base step: When $F = 1$, there is just room for the obligatory outer face. This leaves our connected graph bereft of cycles. Thus forcing it to be a tree.

Strong induction step: Suppose that the Theorem is true $\forall\ F < P$ for some $P > 1$. Then consider $F = P + 1$. Now $F > 1 + 1$, so the obligatory outer face must be accompanied by our graph having ≥ 1 cycle. It is thus meaningful to pick some edge in some cycle. Delete that edge. Then the two pieces of the plane separated by that cycle conflate to form a new face. This operation furthermore leaves all other faces invariant.

What is the 'cost' of our deletion in terms of faces and edges? Let the resulting subgraph be G'. Then

$$V' = V ,$$

for, in forming G', no vertices have been deleted. While

$$E' = E - 1 ,$$

since one edge has certainly been deleted. Finally,

$$F' \;=\; F \;-\; 1$$

by the above conflation.

Substitute the above three equations in $\chi(G)$ to obtain

$$\chi(G) \;=\; V \;-\; E \;+\; F \;=\; V' \;-\; E' \;-\; 1 \;+\; F' \;+\; 1 \;=\; V' \;-\; E' \;+\; F' \;=\; \chi(G') \;=\; 2\,. \quad (45.2)$$

The second step here is a simple cancellation, the third identifies the $\chi(G')$ combination, and the last makes use of our induction hypothesis. □

Corollary 1 For a k-component graph that can be embedded in the plane,

$$\chi \;=\; \chi_{k\text{-component planar}} \;=\; k \;+\; 1\,.$$

Exercise 3 Prove this.

Exercise 4 The undecorated graph truncation of this sum is then $\chi_1 := V - E$ (already mentioned in Sec 30.2). This does have equal sums of each sign. Investigate χ for trees, C_n, n-Pan, -Fan, Windmill(n) and Grid(m, n). Interpret these examples in terms of the number of holes that they realize in the form of cycles. Prove your conjecture for the general connected case.

b) What happens to Corollary 1 in the absence of faces? And of edges as well?

Pointer 3 For connected graphs embedded on spheres, χ shares planar graphs' fixed value of 2:

$$\chi \;=\; \chi_{\mathbb{S}^2} \;=\; 2\,.$$

This can be envisaged by beating the sphere flat (one effect of which is to select one face to play the outside role).

Pointer 4 χ takes a fixed value for connected graphs embedded on tori as well, albeit now with a different fixed value:

$$\chi \;=\; \chi_{\mathbb{T}^2} \;=\; 2\,.$$

Excluding the outer face in the plane is topologically equivalent to working on the disc:

$$\chi \;=\; \chi_{\mathbb{R}^2\text{inner only}} \;=\; \chi_{\mathbb{D}^2} \;=\; 1\,.$$

And so on for surfaces with progressively more holes.

Euler's formula is in fact capturing this very information about the number of holes. This is usually quantified by *topological genus* g: g = 0 holes for spheres, g = 1 holes for tori... This is however just a slight repackaging of Euler's characteristic. Indeed, the two are linearly-related by

$$\chi \;=\; 2\,(\,1 \;-\; \mathrm{g}\,)\,.$$

Which inverts to

$$\mathrm{g} \;=\; 1 \;-\; \frac{\chi}{2}\,.$$

45.4 Face degrees

Definition 1 The *face-degree* fd_i of a face F_i is the number of edges in its bounding cycle.

Remark 1 In fact, this definition extends to [183] *region-degrees* rd_i of *regions* R_i. Meaning partitions of the plane by cycles of edges from our graph. Regions are a more general concept than faces, in that some regions could consist of more than one face. The partition into faces itself being the finest-graining possible among the partitions into regions.

Lemma 1 (Face-Euler Degree Theorem) For connected bridgeless planar graphs,

$$\sum_i fd_i \;=\; 2\,E\;.$$

Exercise 5 Given a planar graph, define its *dual graph* as follows. Let each face be a vertex. Then place edges between vertices corresponding to faces that are adjacent across one or more common edges. Show that the usual Euler Degree Sum Theorem for the dual graph implies the Face–Euler Theorem. Why is the bridgeless condition necessary?

Structure 1 In parallel to (32.2), we can decompose our sum into

$$2\,E \;=\; \sum_i i\,f(i)\;.$$

Where the $f(i)$ count the faces of size i, so that

$$\sum_i f(i) \;=\; F\;.$$

Observe that we have the slight asymmetry with (32.2) that nontrivial face degrees start at 3 to ordinary degrees starting at 1.

45.5 Planar density

Remark 1 The theory of planar graphs involves the following key number of edges.

Definition 1 The *maximum planar edge number*

$$E_{\text{max-planar}} \;:=\; 3\,(\,N\;-\;2\,)\;=\;3\,N\;-\;6\;. \tag{45.3}$$

Exercise 6 Prove that this is the maximum amount.

Structure 1 (45.3) in turn supports the following ratio variable. The *planar edge density*

$$\rho_{\text{planar}} \;:=\; \frac{E}{E_{\text{max-planar}}} \;=\; \frac{E}{3\,(\,N\;-\;2\,)}\;.$$

Remark 2 This has standardized range provided that we are computing it for a planar graph. I.e. when computed restricted to the planar graphs, it can run from 0 for 0 edges to 1 when $E_{\text{max-planar}}$ is realized,

Structure 2 The *first planar fraction* for a given N is

$$\mathcal{P} \;=\; \frac{6\,(\,N\;-\;2\,)}{N\,(\,N\;-\;1\,)}\;.$$

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Exercise 7 What happens to planar density as N increases from 0 to 6?

Structure 3 (48.1) also supports the following shifted difference variable. The '*nonplanator*'

$$N_{\text{-planar}} := E - E_{\text{max-planar}} .$$

This gives the following diagnostic.

$$N_{\text{max-planar}} > 0 : \quad \text{cannot be planar} .$$

$$N_{\text{max-planar}} = 0 \text{ and } N_{\text{max-planar}} < 0$$

are however inconclusive: these might be critically-planar and planar respectively, but in each case they might also not. This will be further clarified in Sec 46.1.

Exercise 8 a) For a planar graph, show that

$$V \geq \frac{E}{3} + 2 ,$$

$$F \leq 2(V - 2)$$

and

$$E \geq \frac{3}{2} F .$$

b) For a polytope, the dual graph is also planar, so

$$E \leq 3(F - 2)$$

Deduce that

$$\frac{3}{2} V \leq E \leq 3(V - 2) ,$$

$$\frac{F}{2} + 2 \leq V \leq 2(F - 2)$$

and

$$\frac{E}{3} + 2 \leq F \leq \frac{2}{3} E .$$

Exercise 9 [Long] Bi-partite planar graphs exhibit the following parallel of this Section's structure.

a) Prove that they can have maximum edge number

$$E_{\text{max-bi-partite-planar}} := 2(N - 2) .$$

b) This in turn supports the following ratio variable, which is conceptually a type of density. The *bi-partite planar edge density*

$$\rho_{\text{bi-partite-planar}} := \frac{E}{E_{\text{max-bi-partite-planar}}} = \frac{E}{2(N - 2)} .$$

Check that this has standardized range provided that we are computing it for a bi-partite graph.

c) Define a 'non-complete-bi-partator' $N_{\text{max-bi-partite-planar}}$, and clarify what it sharply diagnoses and what it leaves inconclusive.

d) The *bi-partite fraction* of planar graphs for a given N is

$$\mathcal{BP} \ := \ \frac{4\,(\,N\ -\ 2\,)}{N\,(\,N\ -\ 1\,)}\ .$$

Show that

$$\frac{E_{\text{max-planar}}}{E_{\text{max-bi-partite-planar}}} \ = \ \frac{\mathcal{P}(\mathrm{G})}{\mathcal{BP}(\mathrm{G})} \ = \ \frac{3}{2}\ :$$

a fixed N-independent value.

[We write explicit G dependence here since Sec 59's second such are $\mathfrak{G}$raph entities, rather.]

e) Which of the graphs in Figures 36.2, 41.3 and 31.5 are bi-partite?

f)$^+$ How many connected bi-partite graphs are there on 0 to 7 vertices? How many of these are planar?

45.6 Simple proofs that K_5 and $\mathrm{K}_{3,3}$ are nonplanar

Worked Example 1 Show that K_5 is nonplanar.

Answer 1 Here

$$N_{\text{planar}} \ = \ \frac{N\,(\,N\ -\ 1\,)}{2} \ -\ 3\,N \ +\ 6 \ = \ \frac{5\ \times\ 4}{2} \ -\ 3\ \times\ 5\ +\ 6 \ =\ 10\ -\ 15\ +\ 6\ =\ 1\ >\ 0\ .$$

So nonplanarity is forced.

Worked Example 2 Show that Utilities $=$ $\mathrm{K}_{3,3}$ is nonplanar.

Answer 2 Here

$$N_{\text{bi-partite-planar}} \ :=\ E\ -\ E_{\text{bi-partite-planar}} \ =\ p^2\ -\ 2\,N\ +\ 4$$

$$=\ 3^2\ -\ 2\ \times\ 6\ +\ 4\ =\ 9\ -\ 12\ +\ 4\ =\ 1\ >\ 0\ .$$

So nonplanarity is forced.

Exercise 10 a) Show that using $N_{\text{-planar}}$ instead fails to prove that Utilities is planar.

b) Show directly that K_6, $\mathrm{K}_{4,3}$ and $\mathrm{K}_{4,4}$ are nonplanar.

c) Show that neither of the above arguments manage to be conclusive about K_4, $\mathrm{K}_{3,2}$ or $\mathrm{K}_{4,2}$.

d) Find the smallest $(p,\,q)$ such that either argument applies to $\mathrm{K}_{p,q}$ if i) $p\ =\ q$. ii) $q\ <\ p$ and smallest (p,q) is judged lexicographically. [This suggests that $N_{\text{bi-partite-planar}}$ may have limited shelf life.]



45.7 Inequalities involving girth, and corresponding ratio variables

Lemma 2 (Edge-Face-girth inequality) [183].

$$2\,E \;\geq\; F\,g\;. \tag{45.4}$$

Proof By face-Euler

$$2\,E \;=\; \sum_{i\,=\,1}^{F} f d_i \;\geq\; \sum_{i\,=\,1}^{F} g \;=\; g\,\sum_{i\,=\,1}^{F} 1 \;=\; g\,F\;.$$

The second step here uses, firstly the definition of girth g as a lower bound on all cycles in the graph. Secondly, that this therefore applies in particular to the boundaries of each face. Thirdly, the basic defining inequality that

$$\min_{x\,\in\,\mathfrak{X}}\,(x) \;\leq\; x$$

for each x in the totally-ordered set $\mathfrak{X}$. The third step then pulls g outside the sum. For minima of a set are independent of the indexing of the set, which is the summation index. The final step then just sums up the summand's sole remaining factor of 1. □

Lemma 3 (Edge-Vertex-Girth Inequalities) i)

$$E \;\leq\; (\,N \;-\; 2\,)\,\frac{g}{g \;-\; 2} \tag{45.5}$$

ii)

$$\frac{F\,g}{E} \;\;\leq\;\; \max\left(\,N \;-\; 2\,)\,\frac{g}{g \;-\; 2}\;,\;\; N \;-\; 1\right). \tag{45.6}$$

Exercise 11 a) Prove this Lemma.

b) Find an example illustrating that the second inequality in ii) can manage to be more stringent in practice than i).

[Combining inequalities with the same subject and pointing in the same direction is a more widely useful technical trick.]

Remark 1 The above suggests using the following ratio variables. The *edges-per-face ratio variable*

$$\epsilon \;:=\; \frac{E}{F}\;. \tag{45.7}$$

And the *edges-per-calibrated-vertex ratio variable*

$$\eta \;:=\; \frac{E}{V \;-\; 2}\;. \tag{45.8}$$

Remark 2 Then

$$2\,\epsilon \;\geq\; g\;. \tag{45.9}$$

And

$$\eta \;\leq\; \left(1 \;-\; 2\,g^{-1}\right)^{-1}\;. \tag{45.10}$$

Remark 3 These provide a second example of comparing a ratio of graph numbers to a graph number, here the girth g (or a function thereof).

45.8 Face cardinals

Structure 1 Because the boundaries of faces are cycles, the number of vertices around a face equals the number of edges around the face. Call this face size, which we denote by f.

Correct this to

$$e := f - f_{\text{minimal face}} = f - f_{\text{Triangle}} = f - 3$$

for *excess* face size relative variable. Or to

$$s := f - f_{\text{maximal non-face}} = f - 2$$

for face *strength* relative variable. Triangles contribute nothing to e but still contribute 1 to s. So face strength s registers *something* for each face.

Definition 1 Let us term a graph in which all interior faces have the same f *face regular*.

Structure 2 Some graphs have a well-defined central face and well-defined layers thereabout. In such a case, the face layer number $Flayer$ is well-defined.

Exercise 12 a)$^-$ Show that a convex polyhaedron being both r-regular and k-face-regular forces it to have an even number of vertices.

b) Show that $(4; 3; 3)$, $(6; 3; 4)$, $(8; 4; 3)$, $(12; 3; 5)$ and $(20; 5; 3)$ are permitted combinations of $(V; k; r)$. What are the corresponding values of F and E? Identify one well-known geometrical example for each of these.

c)$^+$ Prove that the above five are the only geometrically-nontrivial solutions.

45.9 Face multi- and sub-regularity

Definition 1 A graph is *face-multi-regular* if it has a limited number of face sizes. More specifically, let the number of face sizes it possesses be p, and these face sizes be the components k_i $i = 1$ to p of the multi-regularity vector $\underline{k}$. Then we call our graph *face-p-regular*, or, in more detail, *face-$\underline{k}$-regular*.

Let us also call a planar graph *fully face-regular* if its exterior face additionally shares the regularity value k; if not, one has a type of bi-regular graph.

Remark 1 W.l.o.g. we can order our k_i by strictly-increasing size. Geometrical nontriviality then requires $k_1 \geq 3$, while for finite graphs k_p provides an upper bound as well.

Definition 2 A graph is *face-k_p-sub-regular* if its face sizes are k_p or less.

Remark 2 So

$$3 \leq k_1 < k_2 < ... < k_p < \infty .$$

45.10 Buckyballs revisited

Exercise 13 a)$^-$ Show that face-bi-regular regular graphs are also forced to have an even number of vertices.

b) A standard Chemical ansatz is that if a ring system cannot contain just 6-rings, then one should look into whether it could contain just 5- and 6-rings. Show that for buckyballs, this ansatz is forced

to contain precisely 12 5-rings. What is the smallest such? And the smallest such that nontrivially contains at lest one 6-ring?

Exercise 14 Show that it is impossible for a buckyball to contain just 6- and 7-rings.

Exercise 15$^+$ Find all face-(6, k)-bi-regular cubic graphs, giving what their V, F_6 and F_k are.

Exercise 16 a) Find all face-(5, 6, 7)-regular cubic graphs.

b)$^+$ Find all face-(a, 6, b)-regular cubic graphs for $a < 6$ and $b > 6$ which are minimally nontrivial. I.e. with > 1 6-ring and > 1 b-ring.

[Exercises 15 and 16 can also be interpreted in terms of buckyballs. In doing the above Exercises, we are only requiring *face counts*, rather than also how the faces are glued relative to one other.]

45.11 Outer-planarity

Exercise 17 A graph is *outer-planar* if it admits an embedding into the plane such that the following holds. Every vertex of the graph belongs to the boundary of one particular region (usually drawn as the exterior region).

a) Show that

$$N_{\text{max-outerplaner}} = E - 2N + 3 > 0$$

forces a graph to be non-outerplaner.

b) Show that K_4 and $K_{3,2}$ are non-outer-planar.



45.12 Triangulations. i. Conceptual outline

Structure 1 A *triangulation* is the partition of a planar region (or, more generally, of a surface) into triangular faces.

0) Suppose that our region or surface comes with no given vertices, Then some vertices are to be sprinkled on it so as to initiate the triangulation process.

1) Adjacent triangles are to meet at most pairwise on the entirety of a common edge (Fig 45.2.a). In contrast, Fig 45.2.b)'s 'dart fin' is not a triangulation by having an edge shared by three faces. While Fig 45.2.c) is to be interpreted as having quadrilaterals for faces, rather.

2) Triangles with missing vertices (Fig 45.2.d) or missing edges (Fig 45.2.e) are also disallowed. This precludes trying to interpret Fig 45.2.c)'s faces as e.g. Fig 45.2.f).

3) At this point, we do not preclude triangulating a surface sufficiently sparsely that some pairs of triangles share two (Fig 45.2.g), or even all three faces (Fig 45.2.h).

Structure 2 Figure 45.2.i) is disallowed because its face overlaps mean that it cannot be a partition. Such overlaps could however feature in a *covering by triangles*.

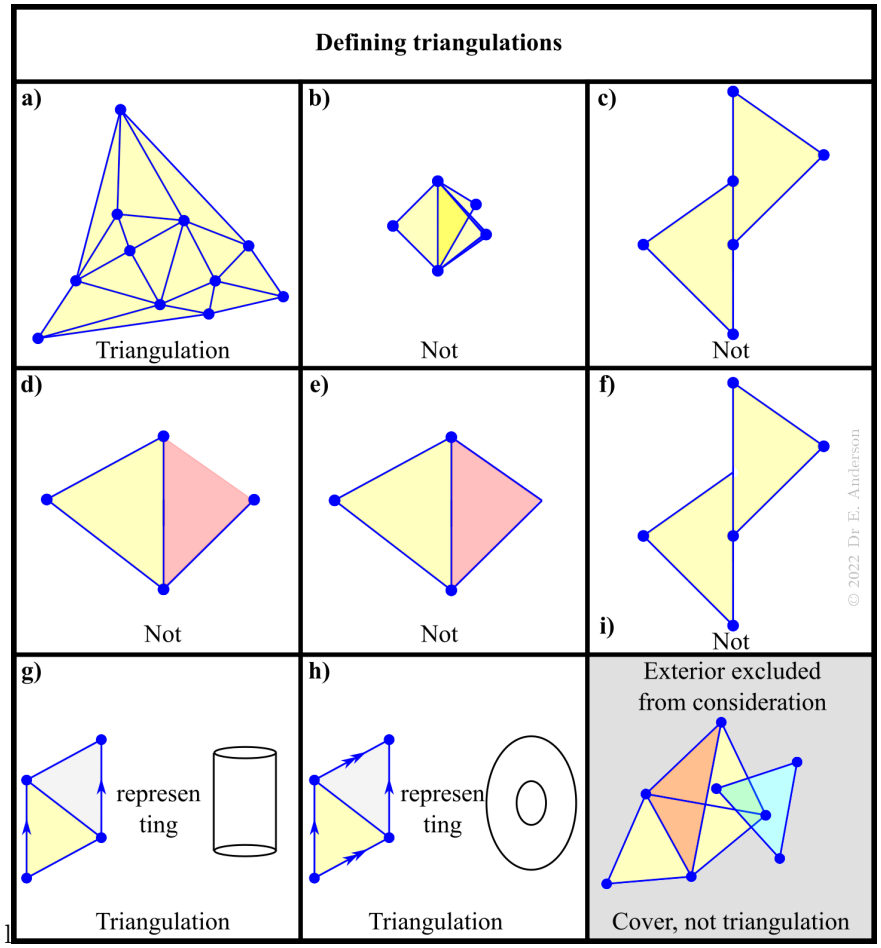


Figure 45.2:

Structure 3 For some applications, we need to insist that the outer face is itself a triangle. One way of phrasing this is as a *triangulation of a triangle*. Though some authors call this a *triangulated graph*. And some even just call it a triangulation, which may lead to confusion between Structure 1 and Structure 2. I call it *a triangulation in the strong sense*.

45.13 ii. Examples and Exercises

Exercise 18 Prove that all triangulations are homeomorph irreducibles, as are all their leavings.

Exercise 19^{+} In how many ways can a (fully distinguishably-and-meaningfully!) labelled regular polygon be triangulated without introduction of further points?

Exercise 20^{+} Answer the previous again, to whatever extent you can, if the polygon is a) unlabelled, b) asymmetric, and c) both unlabelled and asymmetric.

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Worked Example 3 Show that nontrivial triangulations of triangles – for $N \geq 3$ – coincide with the graphs which have exactly $3N - 6$ edges and are thus *maximally-planar graphs*.

Answer 3 Base step: For $N = 3$,

$$3 = 3 \times 3 - 6 .$$

Induction hypothesis: Suppose that our claim holds for some $N = P \geq 3$.

Induction step: Consider a $(P + 1)$-point triangulation of a triangle, TT. Excise a vertex from it, alongside all edges emanating from it (Fig Triangulation-Size.a). This leaves us with a P-vertex graph G', but this is not in general even a triangulation. For the excision has left it with some Q-a-gon face.

We remedy this shortcoming as follows. Pick one of this face's vertices (red in Fig 45.3.c) and spray out $Q - 3$ red diagonals therefrom (Fig 45.3.b). Now we have a triangulation of a triangle, TT' on P vertices. So the induction hypothesis guarantees that

$$E\left(\mathrm{TT}'\right) = 3P - 6 .$$

But then

$$E\left(\mathrm{G}'\right) = E\left(\mathrm{T}'\right) - (Q - 3) = 3P - 6 - Q + 3) = 3P - 3 - Q .$$

Finally,

$$E(\mathrm{TT}^{prime}) = E(\mathrm{G}') + Q = 3P - 3 - Q + Q = 3(P + 1) - 6 .$$

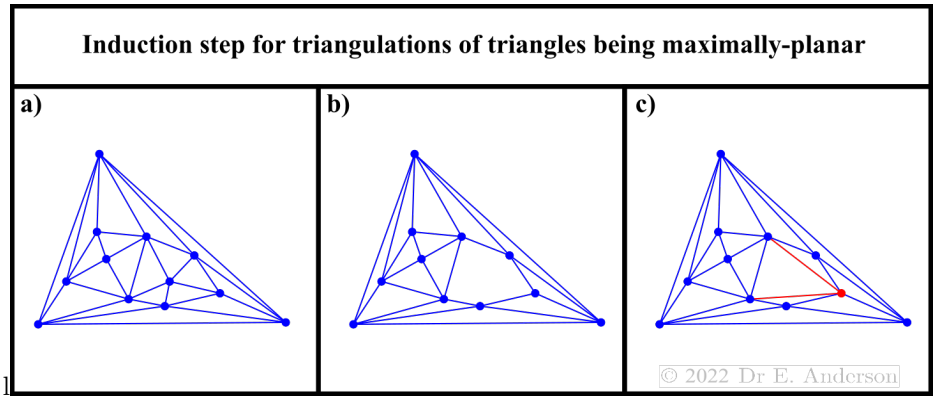


Figure 45.3:

Conversely, if a maximally planar graph has a face bigger than a triangle. Then we could split this up into triangles by addition of edges while maintaining planarity. But this would contradict maximality.

Remark 1 As the zero of

$$3N - 6 = 3(N - 2) ,$$

we could have included $N = 2$ having no edges. The result does not however hold for the even sparser cases of $N = 1 - \mathrm{D}_1 =$ pt – or $N = 0$ – the ungraph or unpoint $\mathfrak{U}$.

Exercise 21 $\mathrm{C}_3 =$ Triangle $= \mathrm{K}_3$ and K_4 are the smallest triangulations of triangles. Which further such are supported on 5 and 6 vertices?

Worked Example 4 Prove that any $\geq$ 4-vertex planar graph has ≥ 4 vertices of degree ≤ 4.

Answer 4 Being planar, our graph G can be extended to a maximally-planar graph $G^{\rm E}$. For this,

$$\sum_{i \leq 3} i\, v_i \;=\; \sum_{i} i\, v_i \;=\; 2\,E \;=\; 2 \;\times\; 3\,(\,N \;-\; 2\,)\;. \tag{45.11}$$

The first step here is by maximal planarity implying triangulation. The second step is the expanded form (32.2) of Euler's Degree-Sum Theorem. And the final step is maximal planarity's edge-number condition.

Next set

$$w \;:=\; v_3 \;+\; v_4 \;+\; v_5$$

and

$$x \;:=\; \sum_{i \geq 6} v_i\;.$$

w and x are not independent, since

$$w \;+\; x \;=\; V\;. \tag{45.12}$$

Next use the following throwaway inequalities in (45.11): 5 and 4 $\geq$ 3 and each of 7 or more $\geq$ 6. Also cancelling off the overall factor of 3 thus obtained, we arrive at

$$w \;+\; 2\,x \;\leq\; 2\,(\,V \;-\; 2\,)\;.$$

Next apply (45.12) to eliminate x:

$$w \;+\; 2\,V \;-\; 2\,w \;\leq\; 2\,V \;-\; 4$$

So, cancelling off the V's,

$$4 \;\leq\; w \;=\; v_3 \;+\; v_4 \;+\; v_5$$

ensues, and we are done. [This cancellation of the V's gives stronger reason for the particular extents of throwaway stated above.]

Exercise 22^{++} Prove that any triangulation must have at least one pair of adjacent vertices whose degrees sum up to $\leq$ 13.

45.14 Simplicial complexes and their cardinals

Pointer 5 Point clouds and graphs can be viewed as the two lowest-dimensional simplicial complexes (0- and 1-d) Dimension here refers to the *maximal* dimension realized by the complex: the 0-d of vertices for point clouds and the 1-d of edges for graphs.

Structure 1 Let us set

$$S_0 \;:=\; V\;,\quad S_1 \;:=\; E \;\text{ and }\; S_2 \;:=\; F\;.$$

so as to have a dimensionally-democratic notation for the vertex, edge and face counting cardinals. Let us furthermore envisage these S_d as *d-simplex-counting cardinals* for $d \;=\; 0$ to 2 and thus have no reason to stop at 2. For 3-simplexes, we also need one for volume chunks, for 4-simplexes, one for 4-hypervolume chunks, and so on.

Exercise 23 Much as $\rm C_3$ = Triangle = $\rm K_3$ is minimal in supporting a face, the tetrahaedron model of $\rm K_4$ is minimal in supporting a volume chunk. What nontrivial 3-simplexes – meaning with at least one volume chunk – are supported by the graphs on 5-vertices?

[Assume that a 3-simplex can only be present when its bounding faces, edges and vertices are realized.

Pointer 6 Graphs are the confluence of the 1-cases of the following.

a) Simplicial complex topology [250].

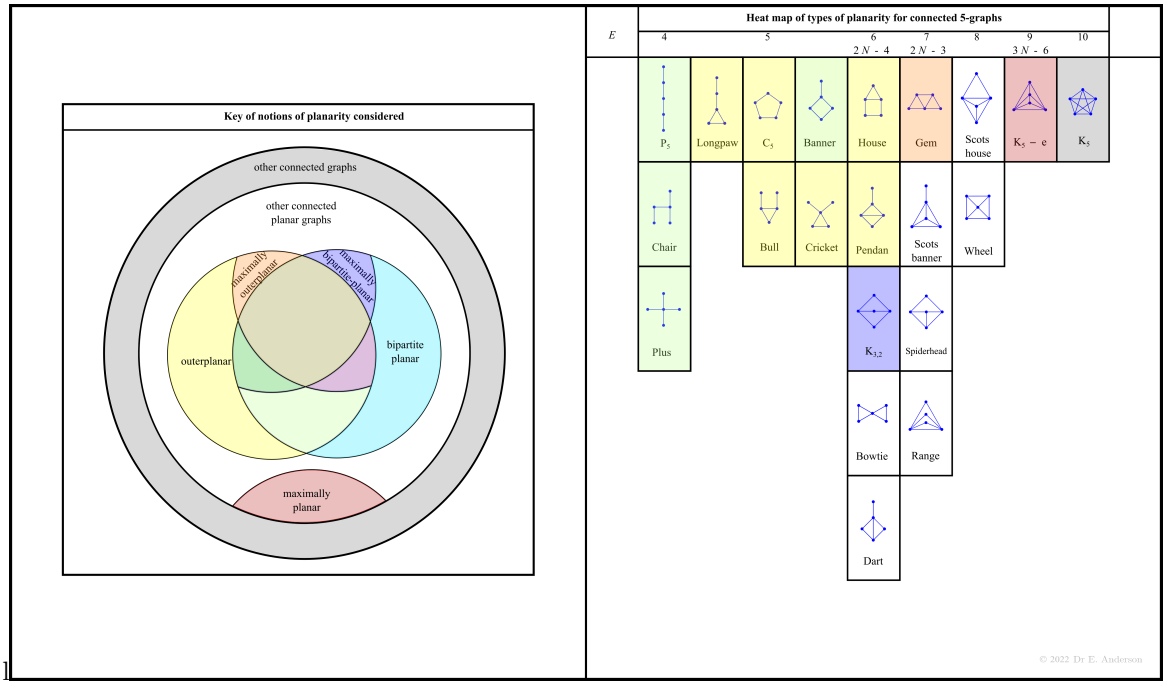


Figure 45.4:

b) *CW topology*, standing for 'closure-finite weak'. -Finite here is as in 'locally-finite; see [261] for further discussion.

These differ as regards further rules of licitness, with smallest possible representations of a space by a complex are often dependent on which modelling assumptions are used.

45.15 Euler characteristic revisited

Naming Remark 2 A truer-name for the Euler characteristic is *dimension-signed simplex count*. This builds in that they are an alternating-sign sum of simplexes, dimension by dimension. This covers both

$$\chi \;=\; \sum_{d\,=\,0}^{2} S_d \;=\; V \;-\; E \;+\; F$$

and its generalization to

$$\chi \;=\; \sum_{d\,\leq\,0}^{2} S_d \;.$$

These will not be the only names these Series shall make use of, due to how this beautiful structure interconnects multiple types of Mathematics.

Pointer 7 Let us subsequently refer to the small portion of Platonia that consists of beautiful objects as *Kallista*, from the Greek $\kappa\alpha\lambda\lambda\iota\sigma\tau\eta$ for 'most beautiful'.

Pointer 8 A more systematic analysis of the number of holes (of whatever dimensionality) that a topological space has yields none other than homology. This *provides the question that* the homology construct outlined in Sec 26.8 *answers*.

Pointer 9 Let us however postpone further discussion of generalizations of Euler's formula to Volume 6 and Series 1.

Pointer 10 One can also imagine face and higher-simplex versions of each type of notion so far considered for vertices and edges. For instance, notions of connectivity, independence, covering, domination, peripherality, of length around a circuit of such simplices... Such will be considered in detail elsewhere.

45.16 Graph Theory as a 'Goldilocks point'

Pointer 11 $d = 0$ is sparse in structure, while $d \geq 2$ is, for now at least, sparse in theorems. $d = 1$ is thus somewhat of a 'Goldilocks balance point' among the complexes. I.e. for $d = 0$, the porridge is too salty to have much mathematics, while for $d \geq 2$, it is too sweet. But for the $n = 1$ sandwiched in between these, it is just right.

45.17 Manifolds, and Hausdorff spaces more generally, are further 'Goldilocks points'

Pointer 12 In subsequent Volumes, we shall encounter various further 'Goldilocks balance points' between too sparse a mathematical structure and too rich a mathematical structure.

Hausdorffness is a such among the topological spaces. A topological space is *Hausdorff* if any two points on it can be 'housed off' by open sets (Fig 45.5.a).

Pointer 13 As are various further qualifications thereof, one of which is the *topological manifolds*. These are the Hausdorff, second-countable locally-Euclidean topological spaces (Fig 45.5.b).

Pointer 14 These and the graphs constitute 'the two ends of' Topology that are all of more straightforward, more mathematically developed and particularly widely applicable. Topological manifolds can moreover *be approximated by* simplicial complexes whose highest dimension matches the manifold's unique dimension.

45.18 Interlude on notions of separation

Independent sets (as seen for graphs in Chapter 37) are additionally notions of separation. For instance, vertex independent sets consist of vertices separated by > 1 edge. Also in the matching conception of edge-independent sets, the vertices are split into two sets. With the matching in question consisting of pairings by mutual edges: each between a vertex in one set and a vertex in the other. This can furthermore be seen as a rather sparse subcase of *bi-partite graph*. Bi-partite graphs indeed also exhibit this same separation into two groups of unrelated vertices.

Viewed in this way, independent sets and multi-partite graphs adjoin with Sec 35.4's notions of vertex-separating set and edge-separating set: as notions of separation. . Since such notions have numerous and significant representatives, further commentary on the diversity of such notions is merited. In fact, Topology and Functional Analysis – rooted on Topological Spaces – have enough such notions that distinction is made there between notions of separation and notions of separability.

Hausdorffness (Sec 45.17) is a notion of separation, as justified by the 'housing off' in its definition. It is one member of a larger family of notions of separation of variable strength. Finite graphs, and discrete objects more generally, are not distinguished between by most of these notions, so they play no further part in the current Volume. Some of the weakest notions of separability, such as

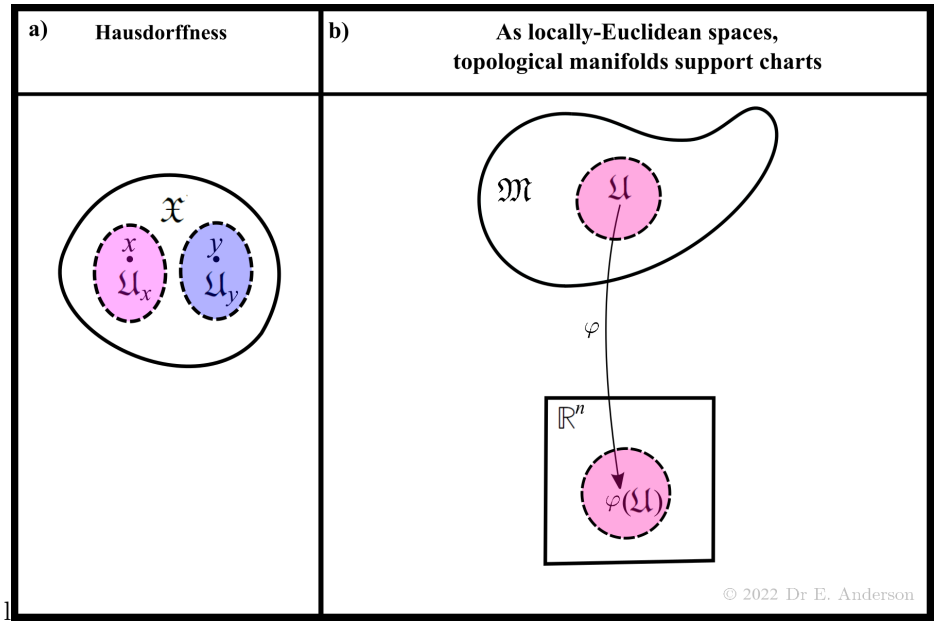


Figure 45.5:

Kolmogorovness, can vary from example to example of discrete structure. These weakest notions are however too advanced to play a role in the current Series; see Volume 4 of Series 1, rather.

Hausdorffness is the second great enabler of conventional Analysis. This is through its guaranteeing that limits of sequences are unique.

Manifolds are then usually defined to be precisely those topological spaces that are Hausdorff, second-countable and locally-Euclidean. This last property enables the use of charts [445]. A prototype for which is locally modelling the surface of the Earth using multiple approximately-correct local charts. This then enables standard $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ Calculus [441] to carry over to the study of differentiable manifolds [332, 445]. It also turns out that the combination of Hausdorffness and second countability implies paracompactness. Which in turn permits differentiable manifolds to have a globally well-defined theory of integration.

Weaker notions of separation permit between much less and essentially no Analysis. Let us finally put together the four topological notions discussed in this Volume that are enablers of Analysis. They are second-countability, Hausdorffness, compactness and connectedness. Compact connected manifolds are thereby particularly well-behaved.

Chapter 46

Planar Graph Theory

A. More advanced Planar Graph Theorems

46.1 Kuratowski's Planar Graph Theorem

Remark 1 A low-density graph can nonetheless be nonplanar if it contains a small dense clique. Re-running this kind of insight in reverse, if a graph is nonplanar, adding vertices and edges to it cannot cure this. So a graph with a K_5 or $K_{3,3}$ subgraph visible in it cannot be planar. This deals with cases in which overall density fails to get close to the critical (bi-partite) planar values. There is therefore such a thing as a linearly-sparse and yet nonplanar graph regime.

Remark 2 In fact, we can say more. For instance, any graph containing a homeomorph of K_5 or $K_{3,3}$ also fails.

Remark 3 This is moreover the only way a graph can fail to be planar, by the following Theorem.

Theorem 1 (Kuratowski's Planarity Theorem) A graph is planar iff it does not contain a homeomorph of the complete graph K_5 or the bi-partite graph $K_{3,3}$.

Exercise 1 What are the homeomorphs of $K_{3,3}$ on $\leq$ 9 vertices?

Exercise 2 What are the homeomorphs of K_5 on $\leq$ 9 vertices?

Exercise 3 a) Which of the graphs in Fig 46.1 are planar?

b) Show that Hypercube(4) is not planar.

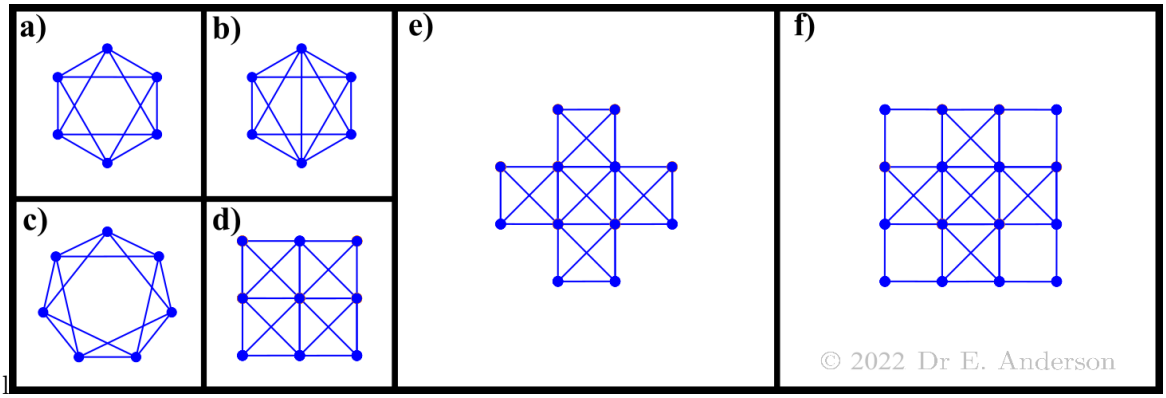


Figure 46.1:

Exercise 4 Prove that the following are nonplanar. The last graph in column 2 of Fig 59.2.c), and the graph derived in Fig 59.5.

46.2 Planar-graph Rectilinear Presentation Theorem

Theorem 2 (Fáry–Stein–Wagner Theorem) Every planar graph admits a rectilinear edge presentation.

Remark 1 We postpone further Exercises on this to Secs 48.5 and 48.7.

46.3 Outer-planarity revisited

Theorem 3 (Chartrand–Harary) [158] Non-outer-planar graphs are precisely those graphs that contain homeomorphs of K_4 or $K_{3,2}$.

Exercise 5 What are the homeomorphs of these on $\leq$ 7 vertices?

Exercise 6$^-$ Which graphs in Fig 46.2 are outer-planar?

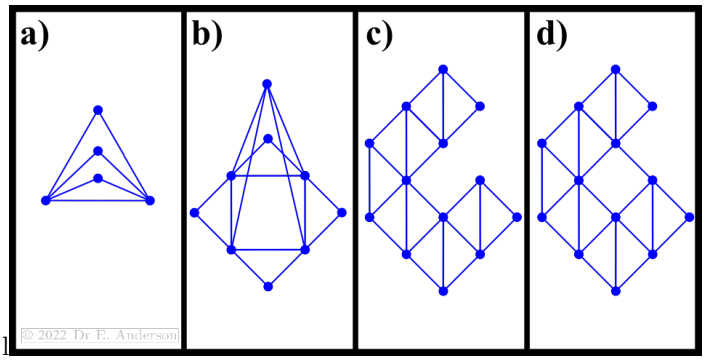


Figure 46.2:

Exercise 7 Find all the connected outer-planar graphs on $\leq$ 6 vertices. Which are maximally outer-planar?

Exercise 8 a), b) See Fig 46.3.

c) Are face-branching-points uniquely characterized by strength?

d) Need they have the same symmetries as their branching-point analogues?

Pointer 1 In the current Series, we do not prove any of the above three Theorems. Volume 1 adds a second Theorem of Kuratowski's to the current Subchapter's list of results relating to planarity.

B. Hamiltonian planar graphs

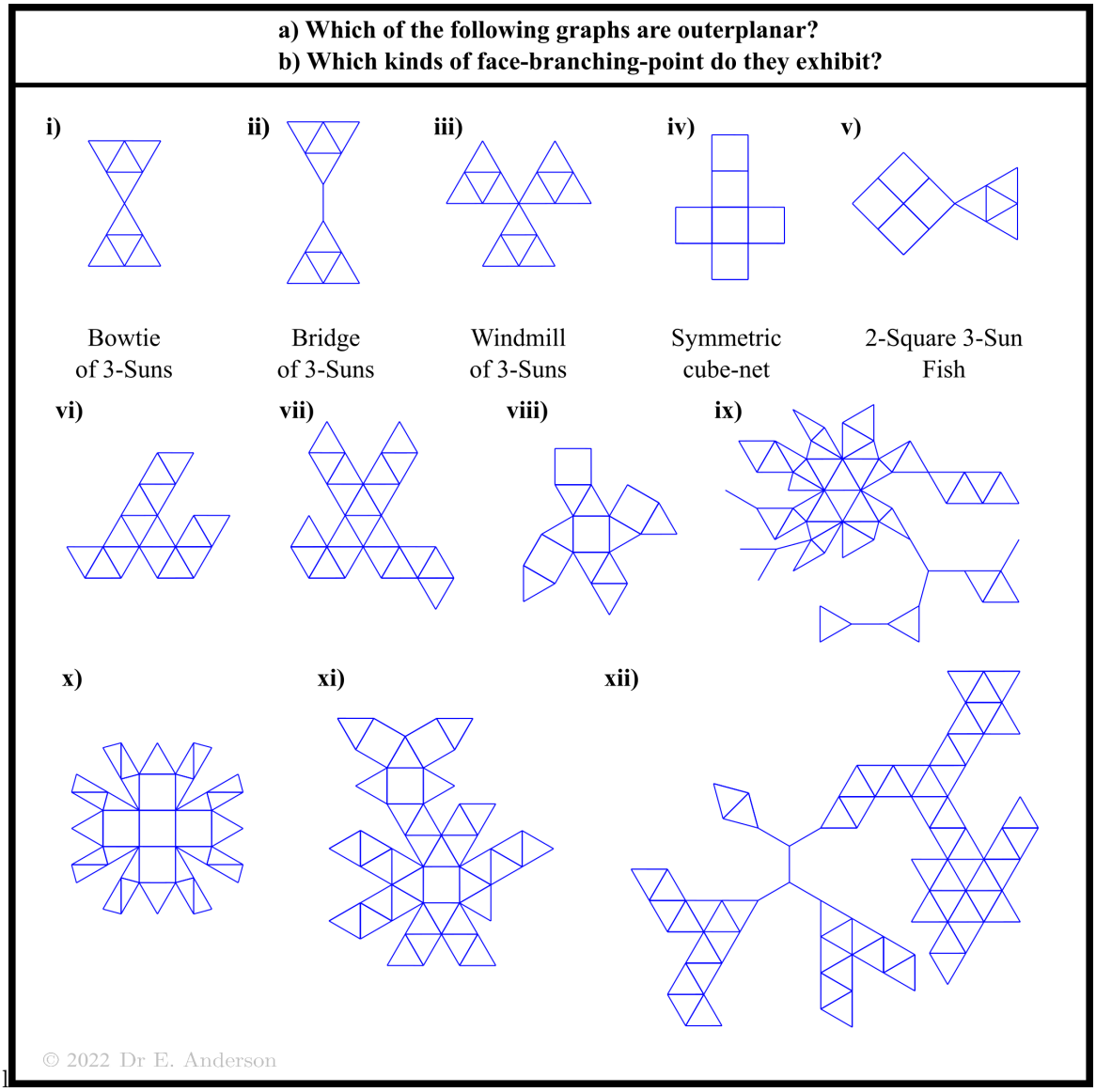


Figure 46.3:

46.4 Cycle system notation

[**Definition 1** A cycle in a graph's cycle system is *elementary* if it cannot be split up into smaller cycles by use of edges of the graph.

Remark 1 These correspond to the internal faces of our planar graph.

Remark 2 Describing a cycle system by the cycle lengths of its constituent cycles is not invariant under graph isomorphisms. For instance, the three graphs in Fig 46.4.a) are isomorphic. And yet the first consists of 3 3-cycles, the second of 2 3-cycles and a 4-cycle and the third of 2 3-cycles and a 5-cycle. The number of faces in our cycle system is however coming out as an invariant.

Exercise 9 Prove that this is an invariant for any planar graph.

Structure 1 A first improvement follows from including the *outer face* in the description. In our first subfigure, this is a 5-cycle, then a 4-cycle and finally a 3-cycle. This illustrates that which cycle is the outer cycle is itself not invariant under graph isomorphisms. This is to be contrasted with Chemistry, where a planarly-representable cycle system does have a unique outer cycle due to molecular rigidity's approximately fixed bond lengths.

Definition 2 Let Y denote the number of cycles present in a cycle system, comprising the internal elementary cycles and the external cycle.

♠ ◇ ♣

Remark 3 Our three descriptions now have moreover another common feature: the sum S of the face sizes – including the outer face – is invariant.

Exercise 10^{-} Check that this is also invariant for Fig 46.4.b)'s isomorphic presentations, and for Fig 46.4.c)'s

Exercise 11 Prove that this is an invariant for any planar graph.

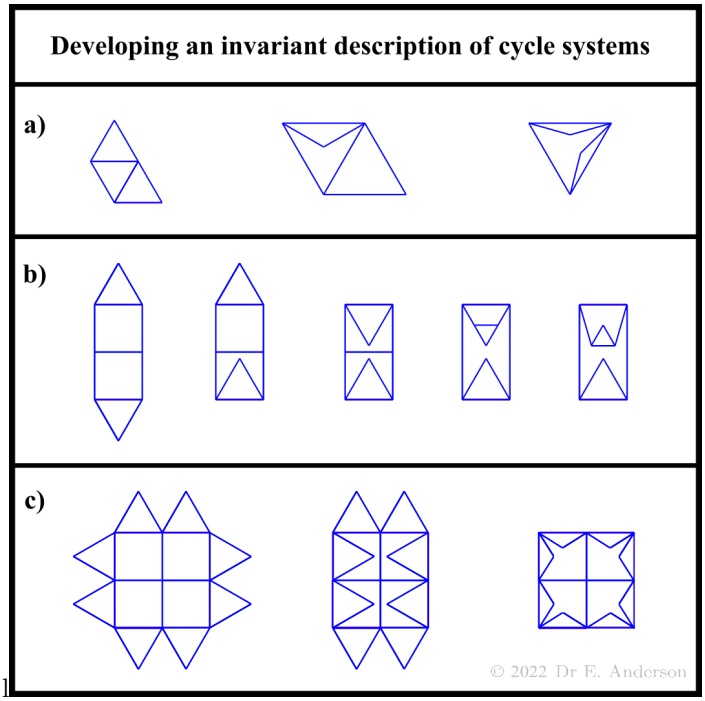


Figure 46.4:

Notation 1 Pictorially, we usually only include vertices if they lie on 2 edges shared by the same 2 cycles, and these edges are drawn mutually-straight. So if the vertex were then not marked with a dot, Readers would miss its presence.

46.5 Grinberg's Theorem for detecting Hamiltonianness in planar graphs

Recollect from Sec 45.8 the face-strength relative variable s to which faces with p sides contribute $p - 2$.

Theorem (Grinberg 1968) Let G be a simple planar graph and H any Hamiltonian cycle therein. Then

$$0 = I := \Delta S = S_+ - S_- = \sum_{F_+ \text{ inside } H} s(F_+) - \sum_{F_- \text{ outside } H} s(F_-) \ . \tag{46.1}$$

Remark 1 This gives a necessary condition rather than a sufficient one. Thus its only direct use is to establish that some graphs are not Hamiltonian.

Remark 2 'Inner' and 'outer' are furthermore well-definedly distinct and yet meaningless labels. This is meaningless in the sense that these notions map into each other under an inversion.

Pointer 2 We shall see in Volume 3 that this operation is, more precisely 1) a projection onto a sphere. 2) Followed by puncturing one of the hitherto-inner faces.

Pointer 3 The well-definedness of which faces group up into each of these halvings follows from *Jordan's Curve Theorem in the Plane*, of Topology [260].

Remark 3 The Theorem has hitherto more usually been phrased as

$$\sum_{F_+ \text{ inside } H} s(F_+) = \sum_{F_- \text{ outside } H} s(F_-) \ .$$

Since the (for Hamiltonian cycles) common value of these two sums shall feature in several dozen of our subsequent calculations, let us temporarily term it the *Grinberg cardinal*, *Grin* of our graph.

46.6 Zero-Index Theorem interpretation

Remark 1 My presentation of this Theorem is, rather, as the enforced zero value of a more generally nonzero relative quantity. Which takes the further conceptual form of an *index*,

$$I := \Delta S = S_+ - S_- \ . \tag{46.2}$$

Naming Remark 1 In a somewhat parallel development, my face strength variable s is quite widely called *face index*. However, while an index *can be formed out of* the inner- and outer-face sums relative to the Hamiltonian cycle H as above, s is not itself an index in this deeper technical sense. So I desist from calling it one.

Pointer 4 What we consider to be an *index* is, a priori, a relative difference of + and − sign objects. Compare (33.1), (36.4), (36.12), and on a second reading, (49.9). The current (46.2) is *face-strength-sum index*. We postpone more commonly-encountered examples of such to Volume 1 and subsequent volumes. Particular technical depth comes in the presence of an *Index Theorem*, by which some such analytically-computable differences additionally coincide with topologically-significant quantities. Such are not claimed in the above four comparisons, though indexes in Volume 1 and further Volumes will often have them. Most Index Theorems are too hard for an introductory book; here detailing a rather more limited *Zero-Index* Theorem is a reasonable compromise.

Remark 2 In a number of simpler settings, 'analytically computable' translates to being a merely-combinatorial object. Along these lines, Grinberg's Theorem is eminently suitable for a basic Book whose primary topic is Combinatorics.

Naming Remark 2 A truer name for this Grinberg(–Kozyrov) Theorem is thus *Zero-Index Planar-Hamiltonian Necessity Theorem ('ZIPHoN')*. With reference to face-strength index. Let us also rename the Grinberg cardinal as the *one-sided face strength sum*, Z.

Remark 3 Clearly

$$Z \; = \; S \; - \; 2\,Y \; .$$

With S and Y being depiction-invariant, Z is as well. Subsquently in these Series, Z supplants S as a cycle system variable, on two counts. Firstly, it is smaller – stripping away some homogeneous absolute information. Secondly, it is the right quantity to directly apply the ZIPHoN Theorem to. The second of these is taken to trump using the even smaller and more relational *excess-cycle-strength sum* $S \; - \; 3\,Y$.

46.7 Proof of the Grinberg = ZIPHoN Theorem

Remark 1 Let us first recollect that just as degree sums can be expanded value by value, so can face degree sums. And thus also relative values of face degree sums, in particular of the currently-posed face-strength sums,

$$S_{\pm} \; = \; \sum_{n \, \in \, \mathbb{N}} n \, s_{n \, \pm} \; . \tag{46.3}$$

Proof By Remark 2 of Sec 46.5, + and − are interchangeable labels Furthermore, changes to one of these two parts of the set of faces does not affect the other. So it suffices to prove that w.l.o.g.

$$\sum_{n \, \in \, \mathbb{N}} n \, s_{n \, +} \; = \; const$$

One of the standard ways of approaching this is as follows (c.f. e.g. [183]; see Exercise 17 for a distinct method of proof). Suppose that there are no + edges. Then our sum is $p \; - \; 2$.

Next suppose that a sum of q terms sum is $p \; - \; 2$. We can then form any graph with $q \; + \; 1 \; +$ edges by adding a single + edge. This must cut some l-face into two pieces, which are say a j-face and a k-face (Fig 46.5). Then

$$j \; + \; k \; = \; l \; + \; 2 \; : \tag{46.4}$$

subtracting off the two copies of the new edge. Every other face and face variable are left invariant by this operation.

Finally observe that the sum of the new face variable values is

$$(j \; - \; 2) \; + \; (k \; - \; 2) \; = \; (j \; + \; k) \; - \; 2 \; - \; 2 \; = \; (l \; + \; 2) \; - \; 2 \; - \; 2$$

$$= \; l \; + \; (2 \; - \; 2) \; - \; 2 \; = \; l \; + \; 0 \; - \; 2 \; = \; (l \; - \; 2) \; .$$

Thus our result holds for $q \; + \; 1$ faces as well. Thus it holds for any number $p \; \in \; \mathbb{N}$ of faces by induction. □

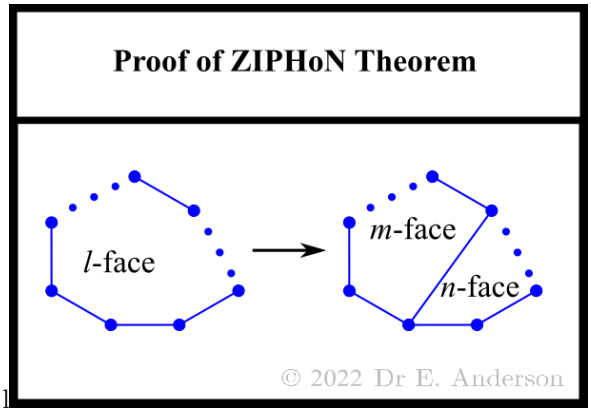


Figure 46.5:

46.8 Simpler Examples

Worked Example 1 What is Z for each of the Platonic graphs? Check that the Hamiltonian cycles in Fig 42.1 comply with the ZIPHoN Theorem.

Answer 1 For the tetrahaedron, the graph is just K_4; all four faces are triangles. So the face strengths are all 1, and we are just required to have 2 + faces and 2 − faces.

For the cube graph, all six faces are quadrilaterals. So the face strengths are all 2, and we just need to have 3 + faces and 3 − faces.

For the octahaedron graph, all eight faces are triangles. So the face strengths are all 1, and we require 4 + faces and 4 − faces.

For the dodecahaedron graph, all twelve faces are pentagons. So the face strengths are all 3, and we need 6 + faces and 6 − faces.

Finally, for the icosahaedron graph, all twenty faces are triangles. So the face strengths are all 1, and we require 10 + faces and 10 − faces.

See Fig 46.6 for Hamiltonian cycles with inner and outer counts. Noting that in highly regular examples, we simplify presentation by omitting writing face strengths' default values. Platonic solid graphs are so regular that their outer faces are also default: 3 triangulations in the strong sense, and 1 quadrangulation and 1 pentangulation in parallel strong senses.

Remark 1 More generally, fully face-regular planar graphs with an even number of faces – outer face included – are not in any way restricted by the zero face-index condition. Such with an odd number of faces with odd face strengths would however be zero-face-index-prohibited from being Hamiltonian. This continues to hold if the face strengths are not all necessarily equal, but all remain odd.

Worked Example 2 Show that the 2×2 and 4×4 grid graphs are not Hamiltonian whereas the 3×3 and 5×5 ones are. Is the 5×3 grid graph Hamiltonian?

Answer 2 All face strengths bar the outer one are 2. The outer face strengths are, respectively, 6, 14, 10, 18 and 14.

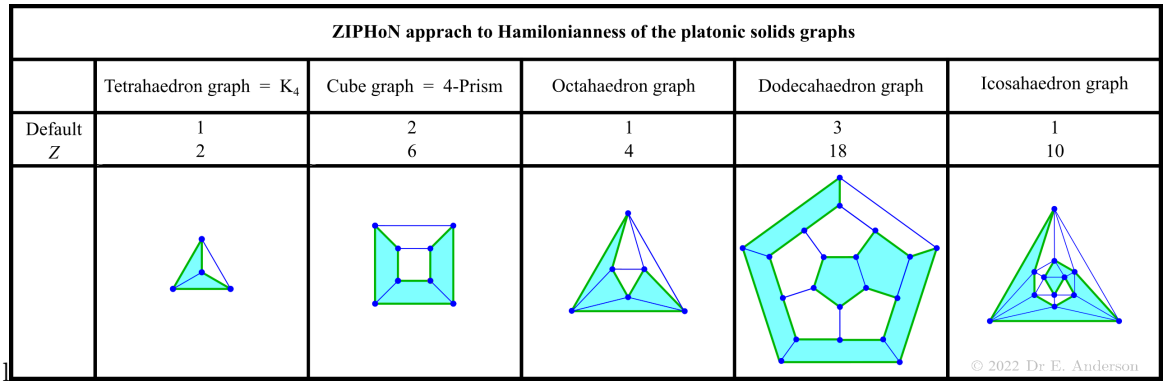


Figure 46.6:

Then for the 2 × 2 case,

$$Z = \frac{4 \times 2 + 6}{2} = 7 .$$

But there is no way of making 7 by summing even terms. So this graph is zero-face-index-prohibited.

Similarly in the 4 × 4 case,

$$Z = \frac{16 \times 2 + 14}{2} = 23 .$$

But there is no way of making 23 by summing even terms.

On the other hand, in the 3 × 3 case,

$$Z = \frac{9 \times 2 + 10}{2} = 14 .$$

Then placing 2 of the squares outside while leaving 7 inside passes our test:

$$2 \times 2 + 10 = 14 = 7 \times 2 .$$

Similarly in the 5 × 5 case,

$$Z = \frac{25 \times 2 + 18}{2} = 34 .$$

Then placing 8 squares outside while leaving 17 inside passes our test:

$$8 \times 2 + 18 = 34 = 17 \times 2 .$$

As to the 5 × 3 grid,

$$Z = \frac{15 \times 2 + 14}{2} = 22 .$$

So this can be attained by placing 4 squares inside and 11 outside:

$$4 \times 2 + 14 = 22 = 11 \times 2 .$$

It then remains to see whether the last three cases are all indeed Hamiltonian rather than just ZIPHoN-allowed. They are, since we can quite readily construct e.g. the Hamiltonian cycles in Fig 46.7. We start this by observing the corners to be of degree 2. So all the edges emanating from these are forced to be part of any Hamiltonian cycle. This *forcing method* [1] also very quickly gives an alternative first-principles proof that the 2 × 2 grid cannot possibly be Hamiltonian. We finally use the above inside and outside face-counts to narrow down the options for which edges the Hamiltonian cycle is to run along.

[1] Some Readers will have already developed this method in solving Exercise 42.3.

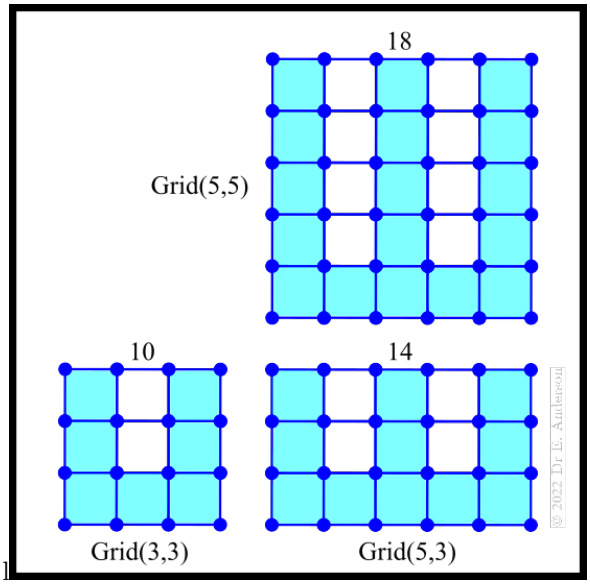


Figure 46.7:

Remark 1 Let us adopt the convention of depicting forced edges as thickened orange line segments. Two ways in which the forcing method can establish non-Hamiltonianness are as follows.

1) If forced edges close up to form a non-maximal and thus non-Hamiltonian cycle, then the presence of a Hamiltonian cycle is precluded (Fig 46.8.a).

2) Forced edges can also further enforce a choice between the remaining vertices. By exhaustively excluding some vertex or other, this ensures that any cycle extending the forced edges cannot be Hamiltonian (Fig 46.8.b).

The forcing method furthermore applies to nonplanar graphs as well as to planar ones.

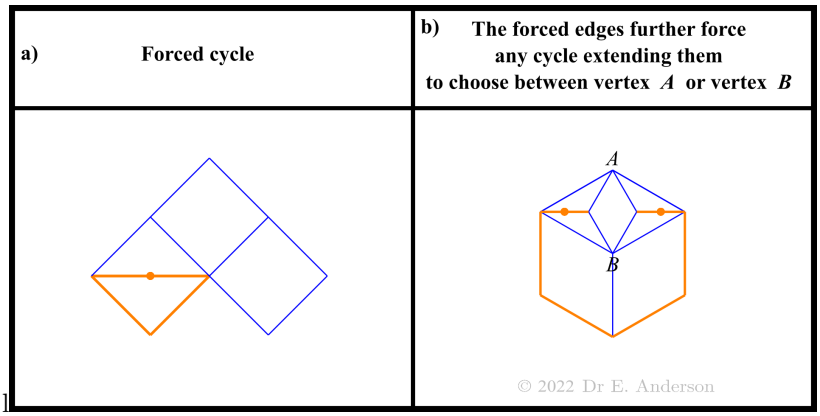


Figure 46.8:

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Exercise 12 Which of the hexagonal tilings in Fig 46.9 are Hamiltonian?

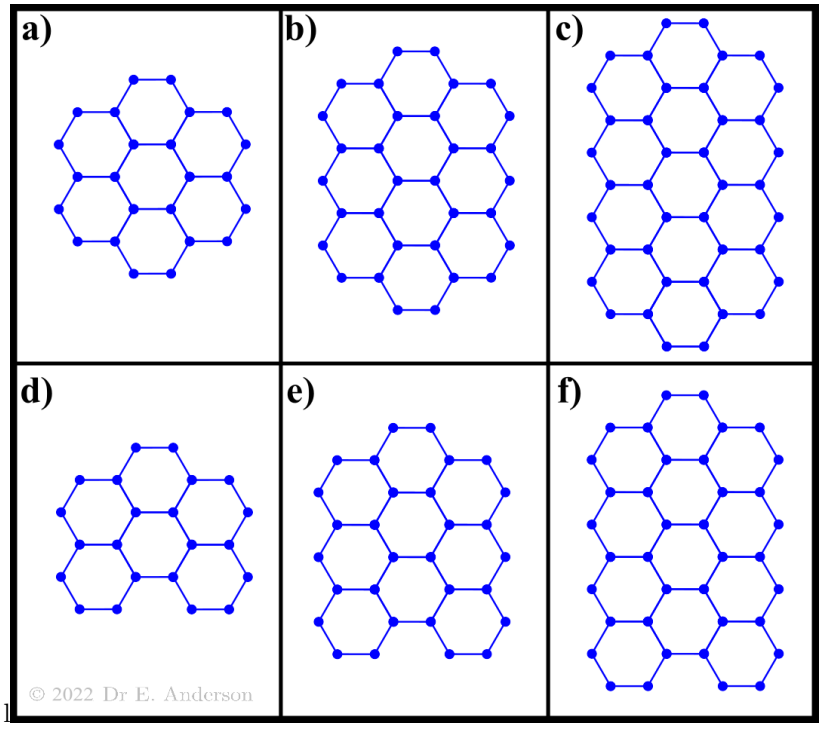


Figure 46.9:

Exercise 13 a) Show that the 6-Wheel graph (Fig 46.10.a) is Hamiltonian.

b) How many distinctly-shaped Hamiltonian cycles can you find for the graph in Fig 46.10.b)?

c) Show that the graph in Fig 46.10.c) is Hamiltonian.

Exercise 14 Is the graph in Fig 46.10.d) Hamiltonian? Where in the current Volume have you seen an isomorphic graph before?

Exercise 15 Show that the family of graphs in Fig 46.11 is not Hamiltonian unless $n = 0$.

Exercise 16 The *Tutte fragment* (Fig 46.12) is a key subgraph in a number of interesting counterexamples to Hamiltonianness. Is its own cycle system Hamiltonian?

♠ ◇ ♣

Exercise 17$^+$ Find an alternative proof for the ZIPHoN Theorem using Euler's formula.

Remark 2 Suppose that we are investigating the Hamiltonianness of a nonplanar graph. Then as a subcase of Exercise 42.5, if a planar spanning subgraph is Hamiltonian, then the graph is as well. Furthermore, for this subcase, the ZIPHoN approach is posed.

Exercise 18 a) Is the CompassRose graph derived in Fig 55.3 Hamiltonian? b) Is its C_2-quotient derived in Fig 55.4?

Exercise 19 After removing sidechains (if any), which of the following graphs are Hamiltonian?

a) $\mathfrak{T}$ree[6] on Fig 60.3.

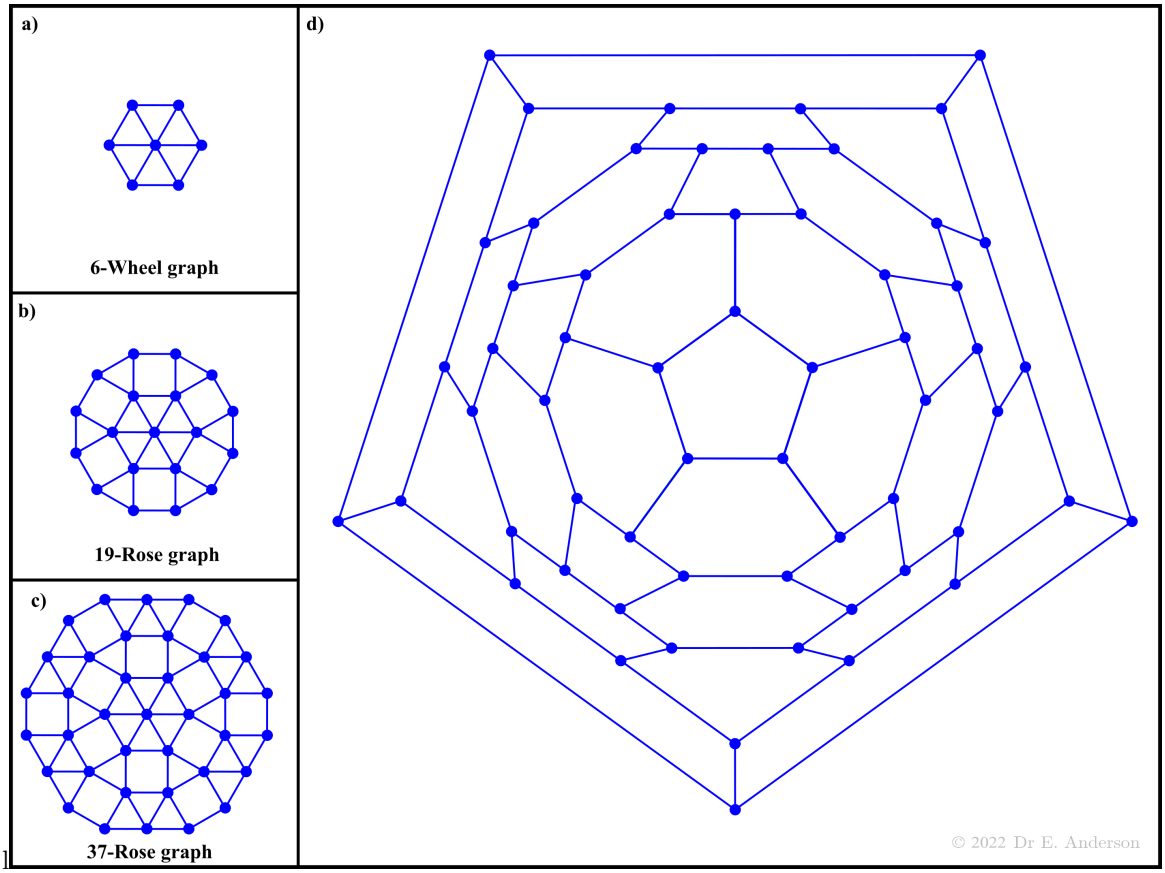


Figure 46.10:

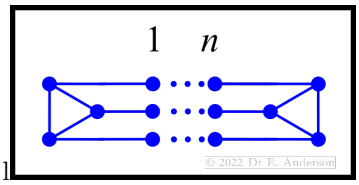


Figure 46.11:

b) $\mathfrak{R}$ooted$\mathfrak{T}$ree[5] on Fig 60.4.

c) The common cycle system of $\mathfrak{B}$in$\mathfrak{T}$ree[11], $\mathfrak{T}$ern$\mathfrak{T}$ree[16] and $\mathfrak{Q}$uad$\mathfrak{T}$ree[21], which can be read off e.g. 60.10.

d) The bottom-left graph in Fig 60.11.

Exercise 20 After removing sidechains (if any), which of the following graphs are Hamiltonian?

a) The last graph in Fig 59.1.

b) The 26-edge graph in Fig 59.2.

c) The graph derived in Fig 62.3.

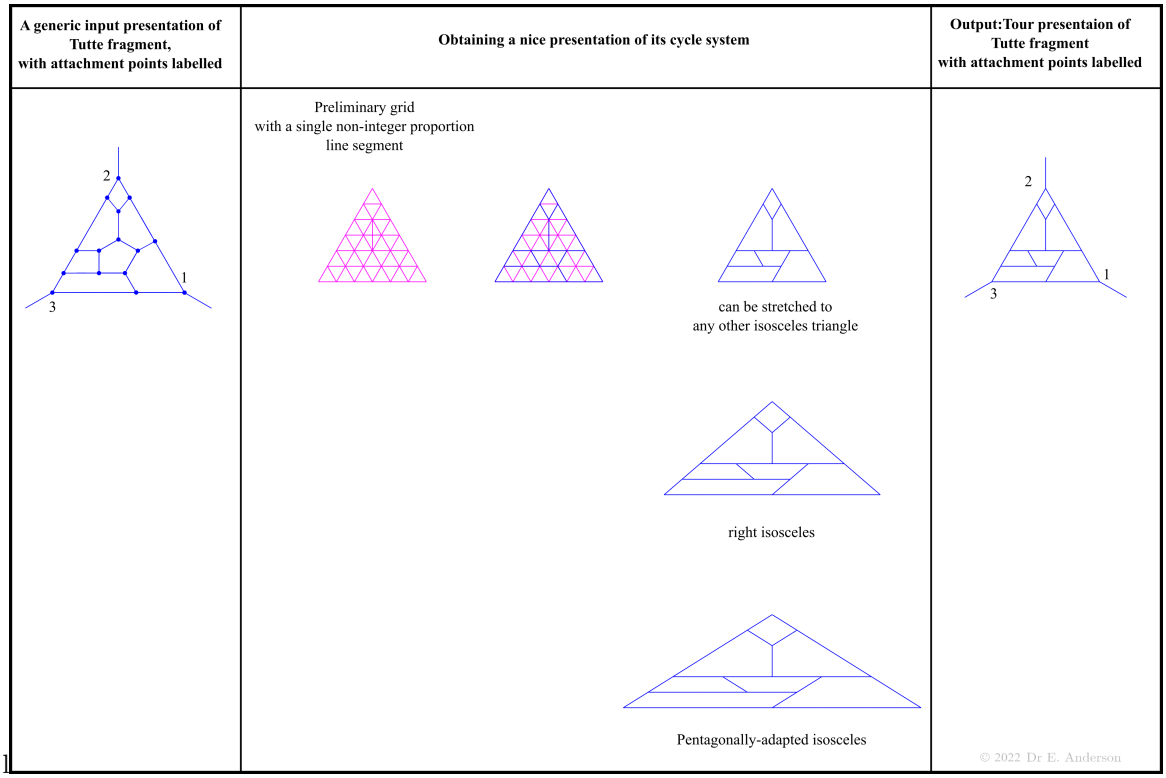


Figure 46.12:

d) The bottom-right graph in Fig 62.6.

e) The bottom-right graph in Fig 63.5.

f) The bottom-right graphs in columns b) and c) 64.1.

46.9 Harder Examples

Worked Example 3 Prove that any graph with only one face index that is not divisible by 3 cannot be Hamiltonian.

[In terms of face sizes, this is a $2 \pmod 3$ condition.]

Answer 3 a) Suppose that our graph is Hamiltonian. Then the side of our face index sum that contains this distinguished face cannot be divisible by 3, while the other side of our face index sum must be. Thus these two terms cannot be equal. But they must be equal by the ZIPHoN Theorem. Contradiction! Thus such a graph cannot possibly be Hamiltonian.

Worked Example 4 Deduce from the ZIPHoN Theorem that Tutte's 46-graph (Fig 46.13.a) is not Hamiltonian.

Answer 4

$$Z \;=\; \frac{7 \;+\; 3(8 \;+\; 5 \;\times\; 3 \;+\; 2 \;\times\; 2)}{2} \;=\; 44\;.$$

Using inner-outer equivalence under inversion, there are 2 cases to consider.

i) That all 3 8's are inner.

ii) That precisely 2 8's are inner.

In Case i), to be Hamiltonian we need a pair of edges through the central vertex. But then this path must extend to a separation of the adjacent 8 from the other 2 8's. For if either had a face-bridge, the outside region would pick up a further component.
In Case ii), our equality reads

$$2\,a \;+\; 3\,b \;+\; 2 \;\times\; 8 \;=\; 2\,c \;+\; 3\,d \;+\; 8 \;+\; 7\;.$$

So introducing relative variables

$$\alpha \;:=\; d \;-\; b$$

and

$$\beta \;:=\; d \;-\; a\;,$$

we obtain the following.[2]

$$2\,\alpha \;+\; 3\beta \;=\; 1\;.$$

Thus

$$2\,\alpha \;\equiv\; 1 \pmod 3\;.$$

Given that this must be solved in integers and

$$|\,\alpha\,| \;\leq\; 6$$

and

$$\beta \;\leq\; 15\;,$$

we are left with

$$\alpha \;=\; 4 \;\text{ or }\; 10\;.$$

But these imply that

$$3\,\beta \;=\; -\,7 \;\text{ or }\; -\,19\;,$$

neither of which are solvable in integers.

Exercise 21 a) Prove that Grinberg's 46-graph is not Hamiltonian.

Exercise 22^{+} Tutte put forward his 46-graph as a first counterexample to polyhaedral (planar cubic 3-connected) graphs being Hamiltonian.

a) Check that this graph is indeed cubic and 3-connected.

b) Since this graph appeared two decades before Grinberg's work, there should be other ways of establishing that it is not Hamiltonian. To this end,

i) Find a Hamiltonian path between 3 and 2 in the Tutte fragment, and another between 1 and 2.

ii) Prove that there is however no Hamiltonian path between 3 and 1.

iii) Deduce that pointing 3 copies of the Tutte fragment's 2's into a common vertex forces non-Hamiltonianness.

[This explains a major way in which the Tutte fragment is special, as well as why Tutte's 46-graph contains precisely 3 copies of the Tutte fragment.]

[2]This works just as well if the 7 is placed on the other side of the equation.

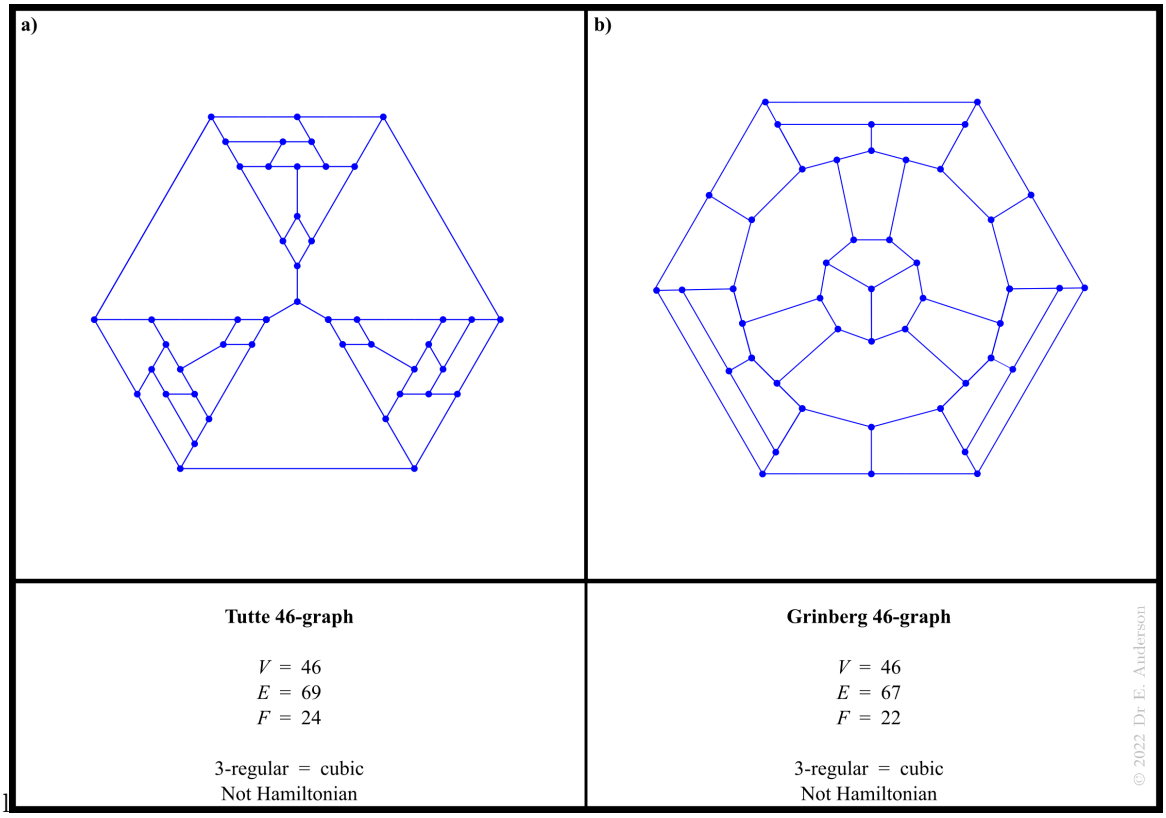


Figure 46.13:

Exercise 23^{+} a)-d) Using the ZIPHoN Theorem, which of the graphs in Fig 46.14 are Hamiltonian?

e) We shall be getting a lot more practice with the ZIPHoN Theorem in Volume 3 on Flat Geometry. Why do you think this might be?

Exercise 24^{+} The fragments in Fig 46.15 are key ingredients in Fig 46.14's graphs.

a) Are their cycle systems Hamiltonian?

b) Do these fragments have analogues of the Tutte fragment's restriction on in and out points?

Exercise 25^{+}+ The graph in Fig 46.16 can be viewed as the outcome of, firstly, ripping out 2 vertices of the 5-prism graph. a Tutte fragment across each resulting triple of loose edges. This jointly attains the 4 objectives of being planar, cubic, 3-connected and not Hamiltonian.

a) Prove that the last three of these properties hold (planarity is manifest).

b) Show that the above ripping and patching strategy cannot yield a non-Hamiltonian graph if carried out on a 3- or 4-prism.

c) For the 5-prism, show that there are precisely 5 other ways of patching on 2 Tutte fragments that attain our 4 objectives.

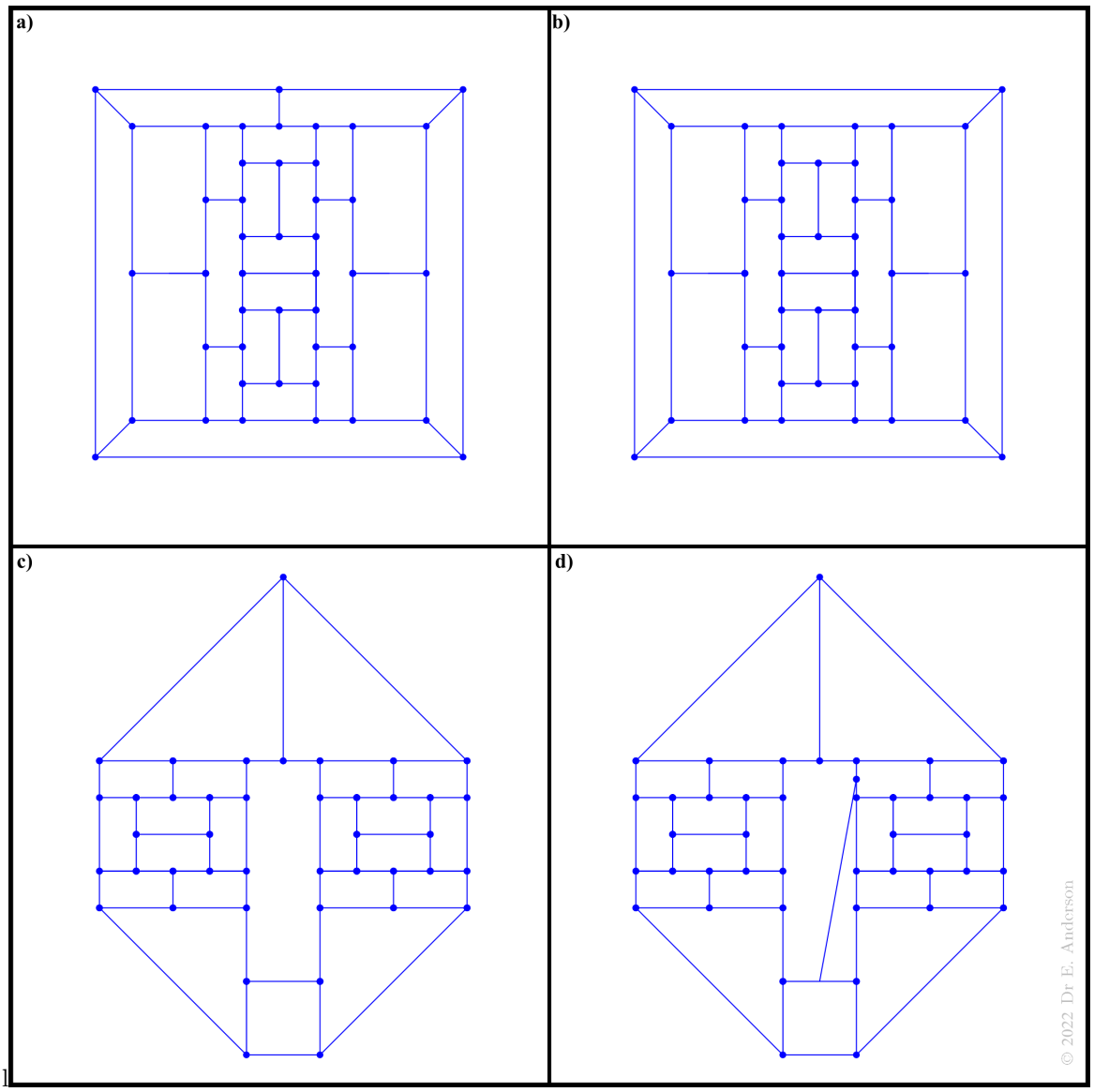


Figure 46.14:

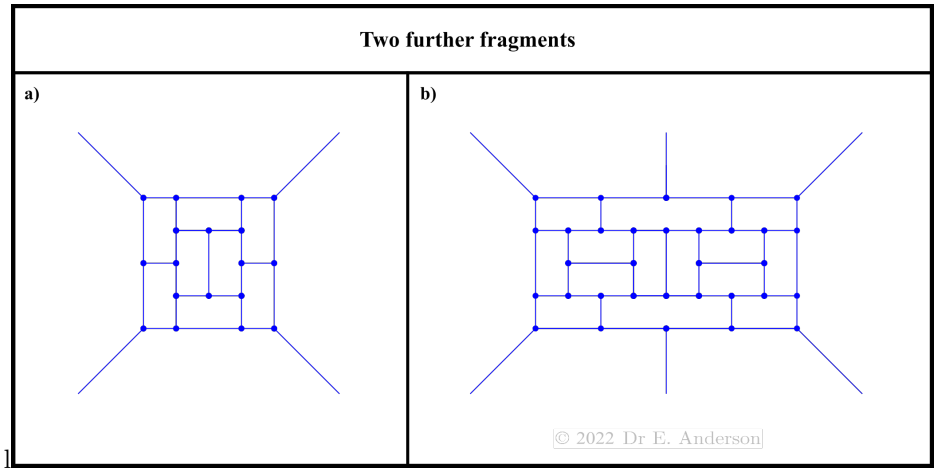


Figure 46.15:

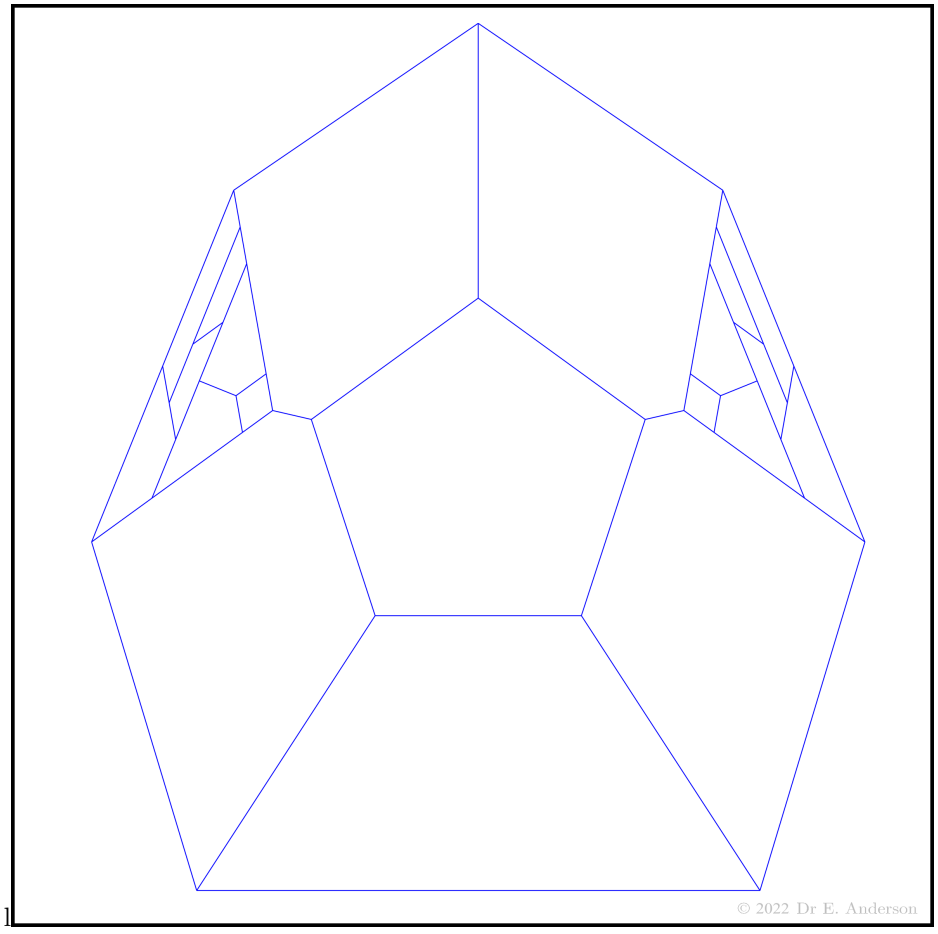


Figure 46.16:

Chapter 47

Graph colouring

47.1 Introduction

The extra structure of colouring vertices with no two adjacent vertices coloured the same, and of colouring edges with no two adjacent edges coloured the same, turns out to be deeply Topological. The second of these is furthermore equivalent to the *map-colouring problem* of what is the least number of colours that can be used to colour in the countries on a map. Over a century after this was posed [122, 163], it was solved by the 4-Colour Theorem in the plane [165, 173]. Colouring has some way of detecting the topology of the plane, differing e.g. for some other underlying surfaces' topologies.

47.2 Vertex colouring

Modelling situation 0 Let us first partition the vertex set $\mathfrak{V}$(G) of our graph G into classes $\mathfrak{V}_i$, $i = 1$ to k such that there are no edges between members of the same class. To visualize this, it is both conventional and useful to imagine that each $\mathfrak{V}_i$ is shaded in a different colour. Our focus is then on what the minimum number of classes k is. We encode this into the following definitions [190, 183, 178].

Definition 1 A *vertex k-colouring* for a graph G is a map

$$C \ : \ \mathfrak{V}(G) \ \longrightarrow \ \{ \, i \ = \ 1 \ \text{ to } \ k \, \} \ ,$$

where each inverse image $C^{-1}(i)$ is an independent vertex set.

Remark 1 k counts the number of colours used; these are usually taken to be *meaningless* colour labels. Each of our k inverse images mathematically encodes the corresponding vertex colouring class. Observe moreover that by vertex colouring classes being vertex independent sets, first contact with Topology has been made.

Definition 2 Let us denote the arena of possible vertex colourings of a graph G by

$$\mathfrak{Col}(\mathrm{G}) \ .$$

[This refers solely to *licit* colourings in the above vertex-independent sense.]

Definition 3 A graph G's *(vertex) chromatic number* is

$$\chi(\mathrm{G}) \ := \ \min_{C \ \in \ \mathfrak{Col}(\mathrm{G})} \ k(C) \ .$$

47.3 Small examples of vertex colourings and vertex chromatic numbers

Example 0

$$\chi(\mathrm{D}_n) \;=\; 1 \;,$$

since the total absence of edges means that all the vertices can be coloured in the same.

Example 1

$$\chi(\mathrm{K}_n) \;=\; n \;,$$

since each vertex has to be coloured differently from the others, by being adjacent to all of them. $\mathrm{K}_2 \;=\; \mathrm{P}_2$ suffices to see that colouring is not a [graph] property.

Example 2

$$\chi(\mathrm{P}_n) \;=\; 2 \;\forall\; n \;\geq\; 2 \;.$$

Exercise 1$^-$ Prove that

$$\chi(\mathrm{C}_n) \;=\; \left\{ \begin{array}{l} 2 \;\; n \;\; \text{even} \\ 3 \;\; n \;\; \text{odd} \end{array} \right. \;.$$

Exercise 2$^-$ What is $\chi(\mathrm{B})$ for B any bi-partite graph.

Exercise 3$^-$ What adjective already encountered in this Volume describes precisely all graphs with $\chi \;=\; 2$?

♠ ◇ ♣

Example 3

$$\chi(n\text{-Pan}) \;=\; \chi(\mathrm{C}_n) \;.$$

Example 4

$$\chi(\mathrm{S}_n) \;=\; 2 \;.$$

Example 5 Fig 47.1 includes all licit vertex colourings of graphs on 0 to 4 vertices. We here treat the colours as distinguishable and yet meaningless labels. The graphs with unmodified names exhibit the vertex chromatic number.

Example 6

$$\chi(\mathrm{Fan}(n)) \;=\; 3 \;=\; \chi(\mathrm{Range}(n)) \;.$$

Example 7

$$\chi(\mathrm{Wheel}(n)) \;=\; \left\{ \begin{array}{l} 3 \;\; n \;\text{even} \\ 4 \;\; n \;\text{odd} \end{array} \right. \;.$$

Exercise 4 Prove Examples 3, 4, 6 and 7's results.

47.4 Edge colouring

Modelling situation 1 So as to democratize, let us next partition [190, 183, 178] the edge set $\mathfrak{e}(\mathrm{G})$ of our graph G correspondingly into edge colouring classes $\mathfrak{e}_i$, $i \;=\; 1$ to k_1. I.e. with no edges between members of the same class. We now visualize the edges in each edge class as painted in a colour exclusive to that class.

Definition 1 An *edge k_1-colouring* for a graph G is a map

$$C_1 \;:\; \mathfrak{e}(\mathrm{G}) \;\longrightarrow\; \{\, i \;=\; 1 \;\text{ to }\; k_1 \,\} \;,$$

where each inverse image $C_1^{-1}(i)$ is an independent edge set alias matching.

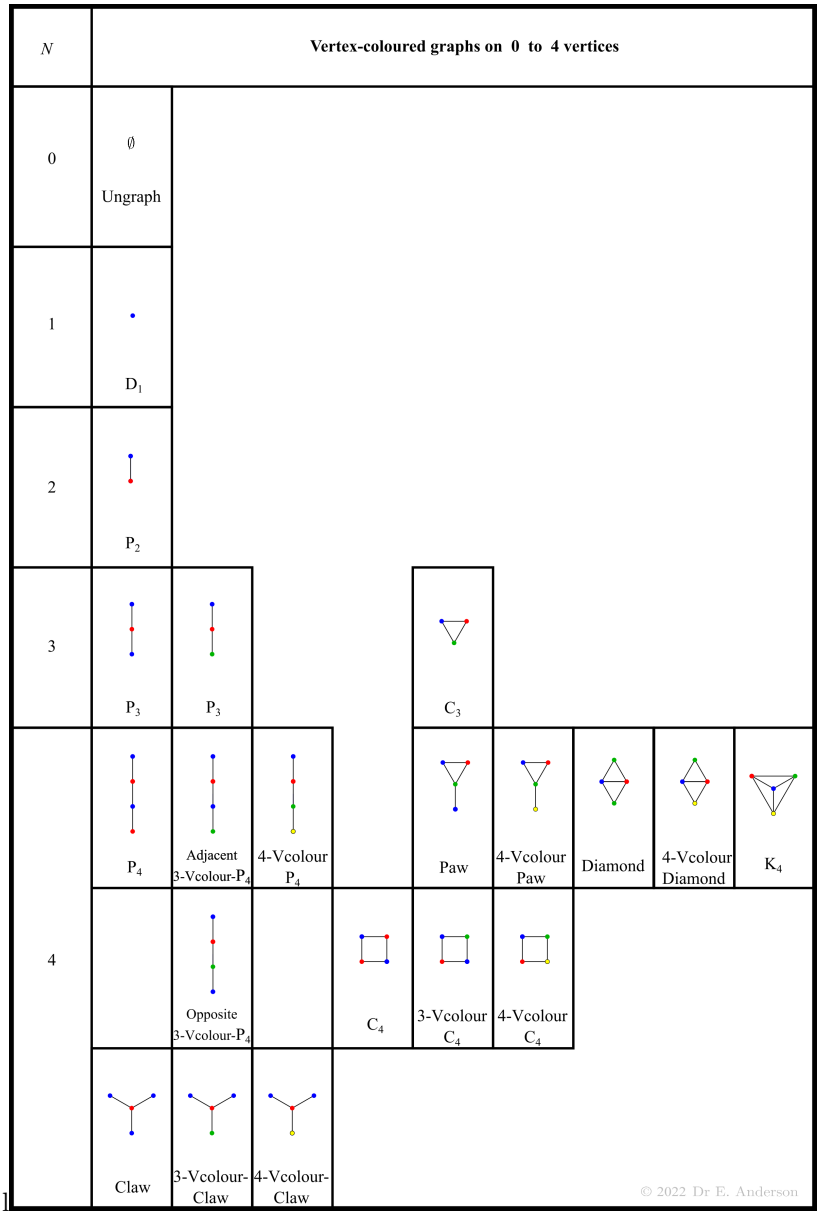


Figure 47.1:

Definition 2 Let us denote the arena of licit edge colourings of a given graph G by

$$\mathfrak{Col}_1(\mathrm{G}) \ .$$

Definition 3 A graph G's *edge chromatic number* is

$$\chi_1(\mathrm{G}) \ := \ \min_{C_1 \ \in \ \mathfrak{Col}_1(\mathrm{G})} k_1\left(C_1\right) \ .$$

47.5 Small examples of edge colourings and edge chromatic numbers

Exercise 5 Which of Sec 47.3's results about vertex numbers carry over to edge chromatic numbers? And which distinct edge number results hold in the other cases?

Example 1 Fig 47.1 includes all licit vertex colourings of graphs on 0 to 4 vertices. The graphs with unmodified names exhibit the edge chromatic number.

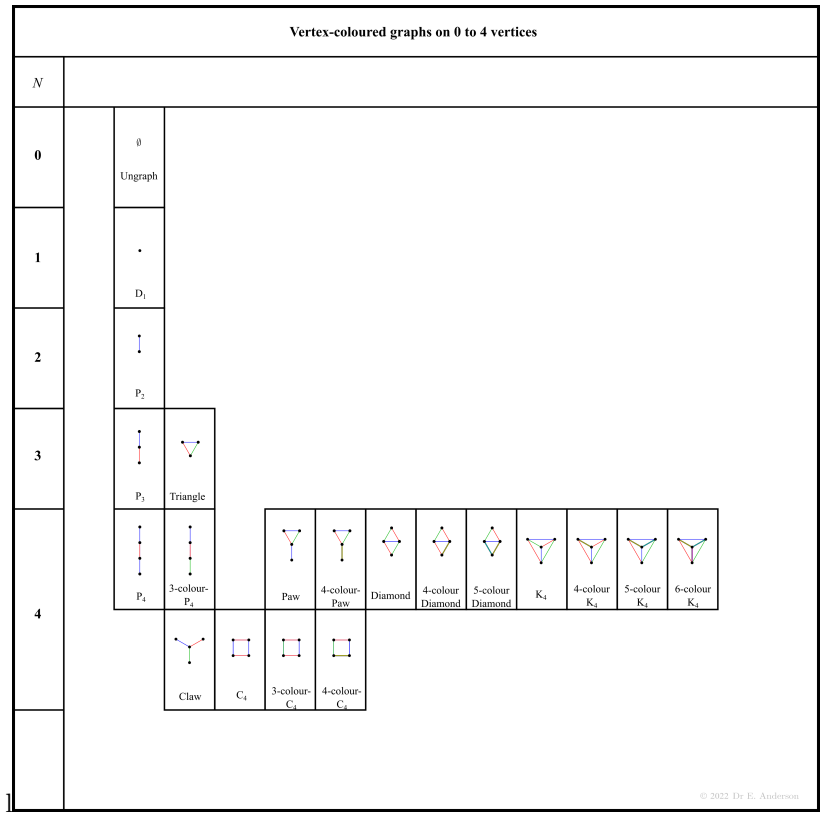


Figure 47.2:

Exercise 6 a) Find χ and χ_1 for all connected graphs on 5 vertices.

b) Which graphs on $\leq$ 5 vertices have $\chi_1 \neq \chi$?

c) P_4 is minimal for exhibiting 2 inequivalent colourings with the same number of colours. Which graphs realize the edge-colouring counterpart of this minimality?

47.6 Edge colouring without contiguity restrictions

We use this (Fig 47.3) as a model in Part VIII. Observe in particular that we now consider full, absent and partial distinguishability of the edge colours.

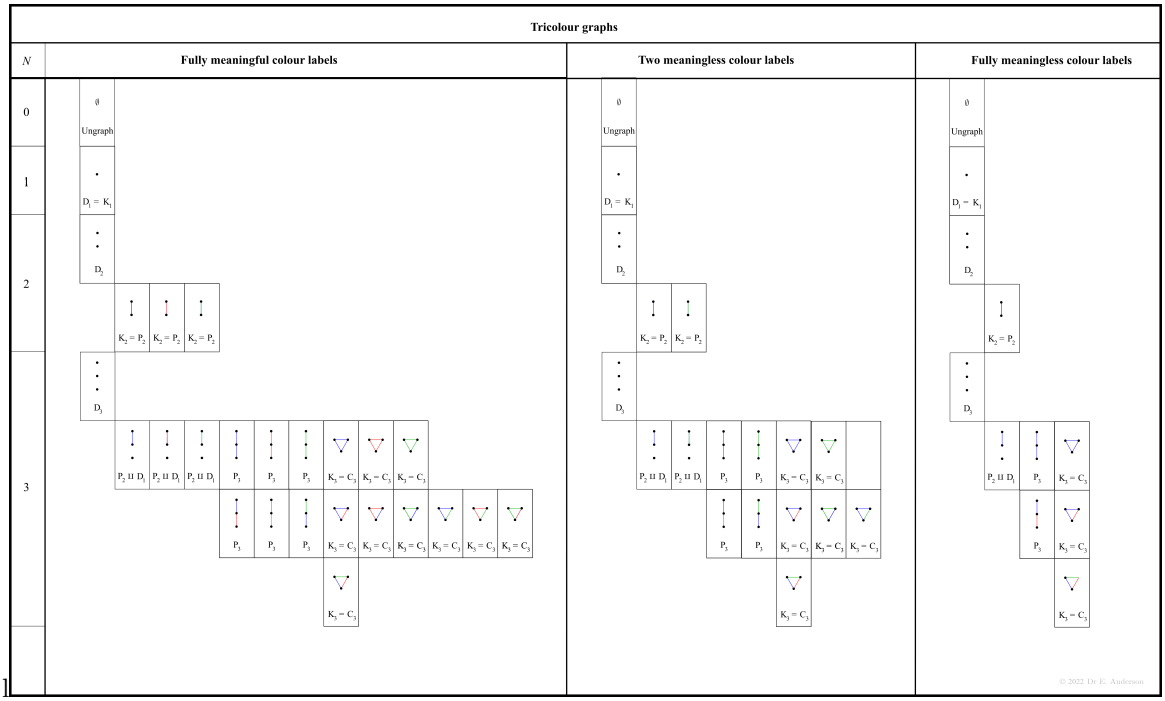


Figure 47.3:

47.7 Bounds on chromatic numbers

Proposition 1

$$\chi \;\geq\; \omega \;. \tag{47.1}$$

Remark 1 I.e. a lower bound on chromatic number is provided by clique number. This is a result by which a local structure – the clique – provides a bound on a global quantity: the chromatic number.

Proof Let graph G contain a clique K_m. Then its m vertices are all adjacent. So $\chi(\mathrm{G}) \;\geq\; m$. Repeat for all cliques that the graph contains. Thus in particular, the best lower bound is given by

$$\chi(\mathrm{G}) \;\geq\; \max\left(\, m \,|\, \mathrm{K}_m \;\in\; G \,\right) \;.$$

This coincides with Sec 37.9's definition of clique number. □

Exercise 7^{-} Show that

$$\chi(\mathrm{G}) \;\leq\; |\,\mathrm{G}\,| \;=\; N \;. \tag{47.2}$$

Proposition 2 For $|\mathrm{G}| \;\neq\; U$,

$$\chi \;\geq\; \frac{|\,\mathrm{G}\,|}{\alpha} \tag{47.3}$$

Proof

Suppose that G's independence number is α. Then no more than α vertices can be coloured in the same. Next use the split count

$$|\,\mathrm{G}\,| \;=\; \sum_{i\,=\,1}^{k} |\,i\text{th colour class}\,| \;\leq\; \chi(\mathrm{G})\,\alpha(\mathrm{G}) \;.$$

Finally divide by α. [This is licit provided that we omit the ungraph.] □

Exercise 8 a) Combine the above three bounds into a single expression. Which of these bounds is least likely to be useful?

b) Find a graph for which each inequality is tight, and another for which each inequality is strict.

♠ ◇ ♣

Theorem 1 (Brooks' inequality) [138]. Let G be a connected graph that is neither complete nor an odd cycle. Then

$$\chi \leq \Delta . \tag{47.4}$$

Exercise 9 a) Check the graphs on 4 and 5 vertices to establish that Brooks' inequality can be both tight and strict. And that each of its conditions is indeed necessary.

b) What are the values of χ that are realized by these graphs?

c) What are the values of χ_1 that are realized by these graphs?

d) Adjust your answer to Exercise 8.a) to take Brooks' Theorem into account.

e) Explain to what extent Brooks' inequality is also a bound by local structural quantities. And Exercise 7's bound? And Proposition 2's bound?

♠ ◇ ♣

Theorem 2 (Vizing) [155, 183, 176].

$$\chi_1 = \Delta \text{ or } \Delta + 1 . \tag{47.5}$$

Definition 3 These two cases are called *Vizing classes* 1 and 2 respectively.

Naming Remark 1 Truer names are Edge-chromatic-number bi-partition Theorem. Alongside 0 and 1 edge-chromatic classes, as developed in the next section.

Remark 2 Proofs of Theorems 1 and 2 are left to harder Exercises at the end of this Chapter.

47.8 Relational presentation of graph colouring

Caveat 1 It becomes clear below that chromatic number clearly does not in general scale with N. This lessens the value of considering

$$\frac{\chi}{N}$$

in comparing between graphs of different orders, or in comparing between distinct graph cardinals. A more general consideration is that while some cardinals are extensive quantities[1]., or well-approximated as such, or asymptotic to such, others may be intensive. Or scaling as some power of e.g. size or length, Or not known to bear any relation to any notion of size or length that the underlying mathematical objects carry. So which particularly meaningful ratios can we form using the chromatic numbers?

Definition 1 *Chromatic number per maximum clique* is

$$x := \frac{\chi}{\omega} .$$

Remark 1 This takes the form of a ratio of two structural quantifiers: global over local, and permits the following rephrasing.

Ratio form of Proposition 1 x is standard-range.

[1] Extensive and intensive are explained in Appendix A

Remark 2 In this way, χ is bounded to scaling by at most a local quantity: a clear sign of non-extensiveness (other than in the complete graphs, in which the clique is everything).

♠ ◇ ♣

Ratio form of Proposition 2

$$a \geq \frac{1}{\chi} .$$

Remark 3 So χ and the at-most extensive a turn out to be compared upside down.

♠ ◇ ♣

Definition 2 *Chromatic number per maximum degree*

$$\mathcal{B} := \frac{\chi}{\Delta} .$$

Remark 4 This is also a normalization by local quantifiers, in that while Δ is a global maximum, it is still the value locally taken by some vertices' degree d. This variable furthermore permits the following rephrasing.

Ratio form of Theorem 1 (Brooks' inequality) The 'Brooks ratio' $\mathcal{B}$ is standard-range.

Remark 5 In this way, χ is also bounded to scaling by a distinct in-a-sense-local quantity: a further sign of non-extensiveness.

♠ ◇ ♣

Definition 3 *Relative edge-chromatic number*

$$V := \chi_1 - \chi_{1\,\min} = \chi_1 - \Delta .$$

Remark 6 This permits the following rephrasing.

Relative form of Theorem 2 (Vizing) The 'Vizing relative difference'

$$V = 0 \text{ or } 1 , \tag{47.6}$$

which can also be phrased as being C_2-valued or parity-valued.

Remark 7 This gives a clear indication that χ_1 is not in general extensive. For Δ is in a sense a local quantity, rather than in general scaling like N.

Caveat 1 continued Graph chromatic numbers thus serve as a first example in which one needs to be more careful in forming relational variables. For these illustrate that the theory of some mathematical objects is capable of internally dictating what constitutes meaningful such variables. It is furthermore interesting that the current Volume swept through *many* objects' relational reformulations before encountering such a less trivial case.

♠ ◇ ♣

Remark 8 χ_1 converts to a relative difference, while χ is amenable to forming ratios. So one should be careful with what to expect of the primitive difference: the *graph democratic colouring difference*

$$d := \chi_1 - \chi .$$

And of the primitive ratio: the *graph democratic colouring ratio*

$$\mathcal{D} := \frac{\chi_1}{\chi} .$$

The first of these is more convenient to analyse. Vizing and Brooks combined give that

$$d \ = \ \chi_1 \ - \ \chi \ \geq \ \Delta(\ + \ 1 \ ) \ - \ \Delta \ \geq \ 0 \ \text{ or } \ 1 \ .$$

Remark 9 Finally introducing the 'Brooks difference'

$$B \ := \ \Delta \ - \ \chi$$

permits us to arrive at the following 'Vizing–Brooks relational variable'.

$$\mathcal{VB} \ := \ \frac{V}{B} \ = \ \frac{0 \ \text{ or } \ 1}{\Delta \ - \ \chi} \ .$$

Though from a computational point of view, this is just a piecewise combination of 0 and the inverse Brooks difference.

47.9 More substantial examples

Worked Example 1 A systematic way of obtaining licit colourings is as follows.

Weak form of Greedy Algorithm

1) Given a finite graph G, order its vertices as v_i, $i \ = \ 1$ to n.

2) Apply colour 1 to vertex v_1.

3) Also apply colour 1 to vertex v_2 if licit, i.e. if $v_2 \not\sim v_1$.

4) Keep on going, with each vertex getting the lowest-number colour possible.

5) A licit colouring must ensue.

Naming Remark 2 This is well-named *greedy*, since the first colour label takes as big a share as the rules permit. *Greedy pecking order* would be even more accurate. For as well as the first chicken gobbling up as many vertex grains of corn as possible, there is then a second chicken that does so conditionally on not confronting the first chicken, and so on.

Remark 1 There is no guarantee that a colouring generated in this way will realize the chromatic number's minimum. Indeed, it will often use a far larger number of colours.

Exercise 10 Find a colouring in which the last chicken nonetheless gets more grain than the first chicken.

♠ ◇ ♣

Strong form of Greedy Algorithm Re-run the weak form for every vertex ordering, keeping only the smallest number of 'colours' (or 'chickens') that any run requires. This exhaustively guarantees that the chromatic number will be found, in principle, given enough running time. We thus have an algorithmic computation of chromatic number for a given finite graph. We make no claim on this being the most efficient algorithm for finding chromatic number.

Exercise 11 Use the Greedy Algorithm to obtain colourings of the wheel graphs W(4) and W(5) as well as House and Domino (find this within Fig 34.3). How many different (meaningless label!) colourings do each of these graphs support? What are their chromatic numbers?

Exercise 12 Find the chromatic number of $K_{5,5} \ -$ the 5 vertical edges by exhaustive use of the Greedy Algorithm.

Exercise 13 A simple algorithm for finding edge colourings is to assign a different colour to each edge at a given vertex, Then likewise at another vertex, and so on until all edges have been licitly coloured.

Find a) χ_1(Octahaedron), b) χ_1(Cube) and c) χ_1(Petersen)?

47.10 N-colour Theorems

Modelling Situation 1 What is the minimum number of colours that can be used to colour in countries on a map? More specifically, the countries are to be connected regions, nor is the map to contain any holes ('lakes'). These connected regions are furthermore to meet at most in threes (another general position condition) at corner points. In each such triple, all three regions must have a different colour.

Modelling Situation 1′ We invite the Reader to work out how the above problem translates to the following Graph Theory problem. Model each corner by a vertex, and adjacency between regions by edges. The general position statement translates to this planar graph being furthermore cubic. Bridgelessness is also required.

Worked Example 2 Prove that every planar graph is 6-colourable [6-Colour Theorem].

Answer 2 Base step. If $|G| \leq 6$, this is trivially true.

Induction step (we induct on $|G| = N$). Take any order-$(N+1)$ planar graph G. Therein, let v be a vertex of degree ≤ 5 (this is always possible, thanks to Worked Example 46.4). By the induction hypothesis, $G - v$ is 6-colourable. But also v is adjacent to ≤ 5 vertices, so we have at least one of our six colours left to use on v. Thus G is also 6-colourable, and we are done. □

Remark 1 We set as an Exercise below to show that every planar graph is 5-colourable. It is vastly harder to prove that the answer to the map-colouring problem in the plane is 4 [190, 173].

4-Colour Theorem Every planar graph is 4-colourable.

47.11 Exercises

Exercise 14 a) Show that the first few steps of the proof of the 6-Colour Theorem carry over to the 5-Colour Theorem. Which step breaks down?

b) Patch over this breakdown by using that at least 2 of the vertices adjacent to a given 5-vertex must be non-adjacent in a planar graph. (Include an explanation of why this works.)

c)$^+$ Merge these 2 non-adjacent vertices to craft a new induction step.

Exercise 15 Show that the Greedy Algorithm implies $\chi \leq \Delta + 1$, i.e. one out from the bound in Brooks' Theorem.

Exercise 16^{++} By splitting consideration between graphs containing a Hamiltonian path and those without, give a short proof by induction of Brooks' Theorem. Your answer is to avoid notions of connectivity (which permit various more usually encountered comparably-short proofs).

Exercise 17 Which of the following are 3-edge colourable?

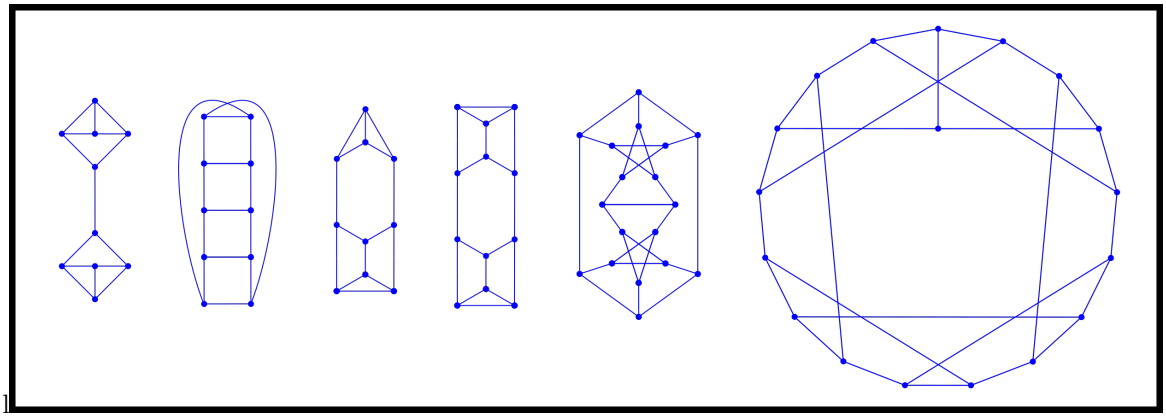


Figure 47.4:

Exercise 18^{++} Prove Vizing's Theorem along the following lines. A colour is *missing* at a vertex v if there is no edge at v that uses that colour. Proceed by constructing a sequence of edges with a missing colour at one end's vertex being present at the other. Show that this sequence is forced to terminate after $\leq \Delta$ terms.

[Hint: the usual way of doing this requires splitting the proof into 2 cases, and then resplitting one of these cases. Figuring out which splits to use accounts for this being one of this Volume's hardest Exercises. If lost, try seeing what variety of such paths is possible in small graphs. Subproofs involving only part of this diversity may suffice, if the parts are well-defined and can be shown to mutually-exhaust.]

Exercise 19 A *snark* is a connected bridgeless cubic graph that is not 3-edge-colourable. These are hard to find.

a) Establish that the Petersen graph is a snark, and that its first homeomorph is as well.

b) It has been announced [181], and yet for now not publicly proven that the Petersen graph is the sole forbidden subgraph for snarks.

Establish that by this definition the graphs in Fig 47.5 are snarks.

c) Where in the current Volume are two more snarks lurking?

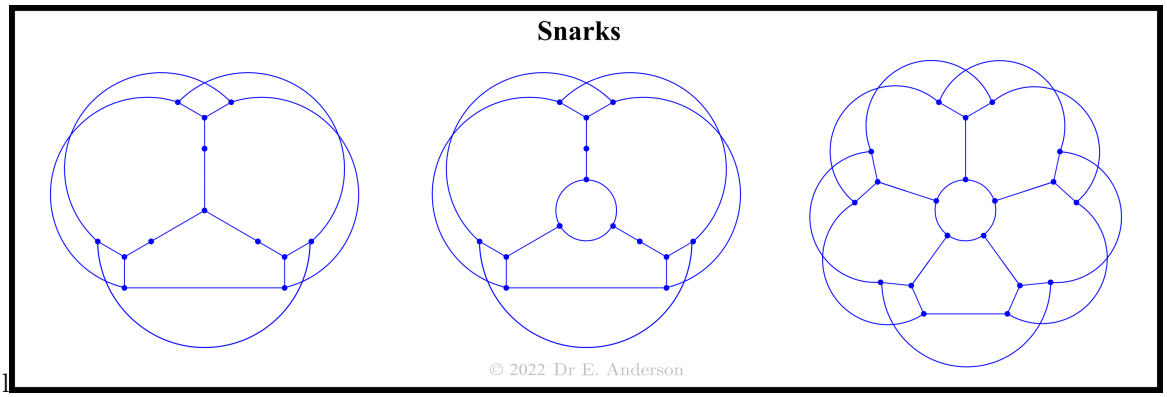


Figure 47.5:

Exercise 20 Find all the 2 and 3 vertex homeomorphs of Petersen.

Exercise 21^{+} Establish equivalence of the map colouring problem and the graph edge colouring problem.

[Attempt this by yourself, and then compare with [190]'s account.]

Exercise 22^{+} What is the face-dual of the conjecture that every snark is nonplanar?

Exercise 23^{+} Find a map on the surface of the 2-torus that is not 4-colourable. Let this serve as a first indication that the map-colouring problem is intricately topological.

Pointer 1 Graph colouring is encoded into *chromatic polynomials* (Volume 1; see also []).

Pointer 2 Graph colouring's sensitivity to the type of surface that a graph is embeddable into is briefly outlined in Volume 6 (see also [178]).

Chapter 48

Describing nonplanarity

48.1 Middle-sized graphs are already quite likely to be non-planar

Remark 1 One can envisage this by e.g. maximal edge count for planar graphs being

$$E_{\text{max-planar}} \;=\; 3\,N \;-\; 6 \;=\; 3\,(\,N \;-\; 2\,)\;. \tag{48.1}$$

while average edge count is

$$E_{\text{crit}} \;=\; \frac{N\,(\,N \;-\; 1\,)}{4}\;.$$

Thus the *maximal-planar to critical edge ratio* is

$$\mathcal{E} \;:=\; \frac{E_{\text{max-planar}}}{E_{\text{crit}}} \;=\; \frac{12\,(\,N \;-\; 2\,)}{N\,(\,N \;-\; 1\,)} \;\simeq\; \frac{12}{N} \;\longrightarrow\; 0 \;\text{ as }\; N \;\longrightarrow\; \infty$$

for N large, which There is thus a need for criteria to measure departure from planarity. We consider two distinct such below.

Remark 1 Also observe that for N large enough, the sparse regime is split in two by whether it can support planarity (Fig 48.1).

Exercise 0 a)$^-$ Bi-partite graphs and outer-planar graphs jointly place an analogous split at $2\,N \;-\; 4 \;=\; 2\,(\,N \;-\; 2\,)$, rather. How does the corresponding ratio

$$\mathcal{E}' \;:=\; \frac{E_{\text{max-planar}}}{E_{\text{crit}}}$$

behave for large N?

b) Check that we need $N \;>\; 11$ for these to become ranked in the Figure's definitive order.

c) What happens to these values for $(l,\,m)$-loop-multi-graphs?

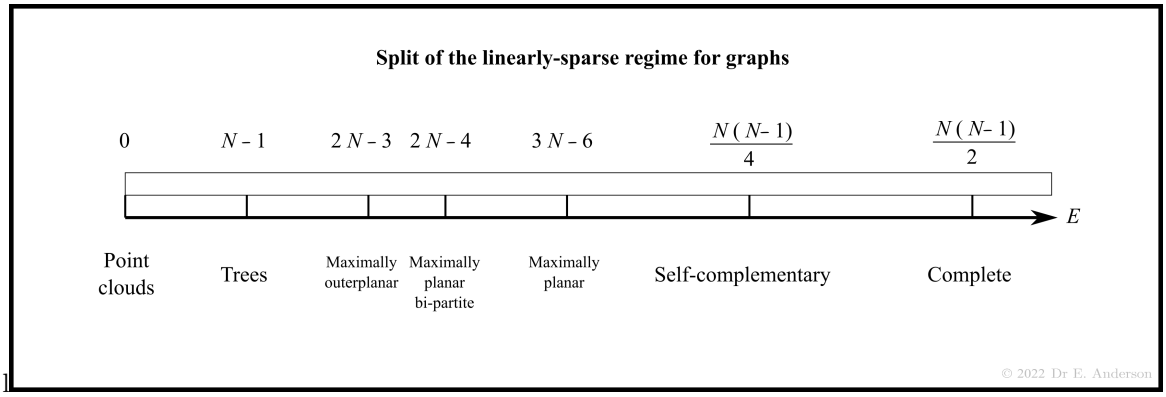


Figure 48.1:

A. Crossing number

48.2 Introducing crossing number, *cr*

Definition 1 A *crossing* in the embedding of a graph is an intersection of edges at other than a vertex.

Remark 1 Making use of a small perturbation if necessary, no further edges are to pass through a crossing point. This is a simple example of general-position argument; such are common in Topology and in somewhat related subjects (Fig 31.3.a).

Definition 2 Let us use

$$\#cr(\mathrm{G},\, \mathfrak{e}) \;:=\; \#\,(\,\text{crossings in embedding}\;\; \mathfrak{e}\;\; \text{of graph}\;\; G\,)\;.$$

Definition 3 The *crossing number* [183, 139, 193, 194] of a graph G is

$$cr(\mathrm{G}) \;:=\; \min_{\mathfrak{e}\;\in\;\mathfrak{e}_{\mathrm{mb}}(\mathrm{G})} \#cr(\mathrm{G},\, \mathfrak{e})\;. \tag{48.2}$$

48.3 Preliminary examples of *cr*

Example 0

$$cr(\mathrm{P}) \;=\; 0$$

for P planar. For these possess planar embeddings in which there are no crossings. This is a first useful check as regards *cr* being a measure of departure from planarity.

Remark 1 A second useful check concerns this quantity being nonzero, and in particular being nonzero for small graphs that are known to be nonplanar.

Example 1

$$cr(\mathrm{K}_5) \;=\; 1\;,$$

by Fig 48.2.a)'s embedding together with Worked Example in Sec 45.6 showing that K_5 is nonplanar.

Example 2

$$cr(\mathrm{K}_{3,3}) \;=\; 1\;,$$

by Fig 48.2.b)'s embedding of $\mathrm{K}_{3,3}$ = Utilities together with Sec 45.6's Worked Example showing that $\mathrm{K}_{3,3}$ is nonplanar.

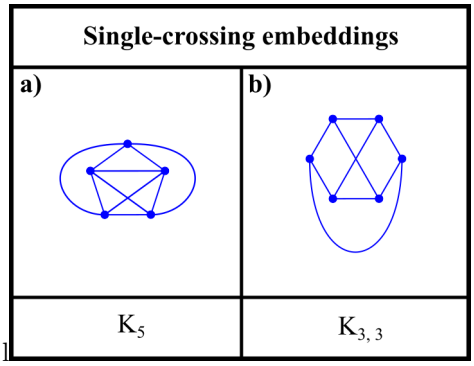


Figure 48.2:

48.4 Crude bounds on *cr*

Remark 1 There is no known algorithm for calculating crossing number [193, 194]. For many more complicated graphs, therefore, all that humanity has for now is bounds on crossing number.

Remark 2 A crude lower bound is provided by

$$cr \geq P := E - E_{\text{planar}} .$$

This rests on splitting out a maximal planar subgraph, and then using that each extra edge must cross at least one edge of our subgraph. For, if not, then that edge could be added to the subgraph to produce a larger-sized planar graph, contradicting maximality.

Remark 3 A crude upper bound is provided by there not possibly being more rectilinear-presentation crossings than there are pairs of edges. I.e.

$$cr(\mathrm{G}) \leq \frac{E(E-1)}{2} . \tag{48.3}$$

This rests on 2 (pieces of distinct) lines at most being able to intersect once.

Remark 4 So *cr* is at least inhomogeneous-linear in E and at most quadratic in E. In particular, for large E, *cr* is between extensive and quadratically super-extensive in E.

Remark 5 For nearly-complete graphs, this can be rephrased as being between at least quadratic in N and at most quartic in N.

More specifically, the lower bound goes as

$$\frac{N(N-1)}{2} - 3(N-2) = \frac{N^2 - 7N + 12}{2} .$$

The upper bound goes as

$$\frac{1}{2}\frac{N(N-1)}{2}\left(\frac{(N(N-1)}{2} - 1\right) = \frac{1}{8}(N+1)N(N-1)(N-2) .$$

Furthermore, the upper bound is much tighter in this highly-dense regime.

48.5 Slightly more substantial examples of *cr*

Worked Example 1 What is the lexicographically smallest graph G with $cr(\mathrm{G}) = 2$?

Answer 1 We know that $N = 5$ supports at most 1 crossing. We can also draw $K_6 - e$ as in Fig 48.4.a) to get down to 2 crossings. Removing just one more edge leaves us with just 1 crossing. Whichever way this is done, one can straightforwardly find a $K_{3,3}$ subgraph therein (Fig 48.4.d-e) so there is no way of presenting away this remaining crossing. So we have found a candidate graph.

$$P = E - E_{\text{planar}} = (E(K_6) - 1) - (3 \times 6 - 6) = (15 - 1) - (18 - 6) = 14 - 12 = 2 .$$

Thus

$$cr(K_6 - e) \geq 2 .$$

But we have just exhibited a 2-crossing presentation, so this holds with equality.

$$cr(K_6 - e) = 2 .$$

Exercise 1 a)$^-$ Take the regular hexagon with diagonals presentation of K_6 and slightly deform the middle part to obtain a general-position presentation. How many crossings are present?

b) Find a presentation of K_6 with just 3 crossings.

c) Prove that

$$cr(K_6) = 3 .$$

Exercise 2 a) What is the crossing number of K_7 with 3 edges emanating from the same vertex deleted?

Exercise 3 Prove that

$$cr(K_{4,3}) = 2 .$$

Worked Example 2 Investigate crossing number for each graph in Fig 48.3.

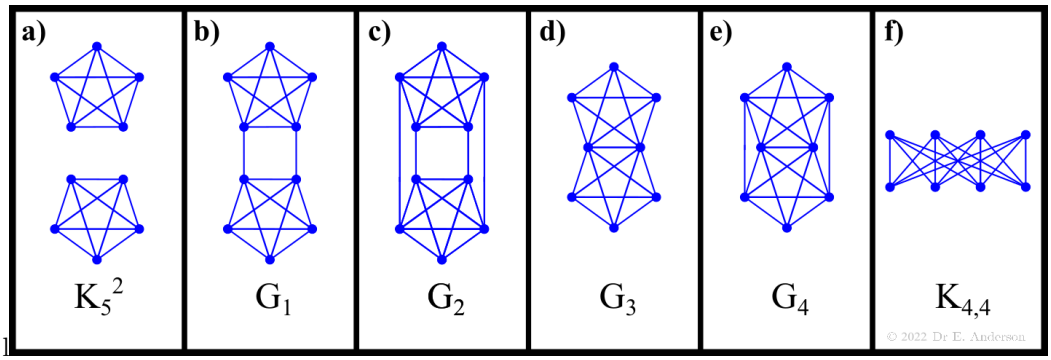


Figure 48.3:

Answer 2 a) For a graph that is not connected, cr is components-wise additive:

$$cr\left(\amalg_{i=1}^{k} C_i\right) = \sum_{i=1}^{k} cr(C_i) .$$

In particular, for all components identical,

$$cr\left(C^k\right) = k\, cr(C) .$$

Thus

$$cr\left(\mathrm{K}_5{}^2\right) \;=\; 2\,cr\left(\mathrm{K}_5\right) \;=\; 2 \;\times\; 1 \;=\; 2\;.$$

b) In this case, our connected graph G_1 admits two entirely independent subgraphs that are nonplanar, with no further crossings. The general result is now

$$cr\left(\;\amalg_{i\,=\,I}^{k}\,\mathrm{I}_i\right) \;\geq\; \sum_{i\,=\,1}^{k} cr(\mathrm{I}_i)\;.$$

Only an inequality is guaranteed in general because parts external to each subgraph might block how that subgraph can in isolation be freed from its spurious crossings.

For our case, however,

$$cr\left(\mathrm{G}_1\right) \;\geq\; 2\,cr\left(\mathrm{K}_5\right) \;=\; 2$$

is saturated, thanks to Fig 48.4.b)'s demonstration that no such blocking occurs.

c) In this case, however, at least the presentation that we used for b) does exhibit blocking. In the absence of further work, all that we can say is that

$$2 \;\leq\; cr\left(\mathrm{G}_3\right) \;\leq\; 4\;.$$

d) This serves to emphasize that we can still add crossing number of nonplanar subgraphs pieces if some vertices and edges therein overlap. Overlap of crossings themselves, however, would be a complication, but this is absent from our G_3. So $cr\left(\mathrm{G}_3\right) \;\geq\; 2$, and this is furthermore saturated thanks to Fig 48.4.c)'s presentation.

e) however exhibits blocking, so all that we can say for now is that

$$2 \;\leq\; cr\left(\mathrm{G}_3\right) \;\leq\; 4\;.$$

f) Let us illustrate the above complication by considering $\mathrm{K}_{4,4}$ in terms of its 16 copies of the $\mathrm{K}_{3,3}$ subgraph. Since each crossing is in precisely 4 of these, we need to correct for quadruple counting by dividing by 4:

$$cr\left(\mathrm{K}_{4,4}\right) \;\geq\; \frac{16}{4}\,cr\left(\mathrm{K}_{3,3}\right) \;=\; 4 \;\times\; 1 \;=\; 4\;.$$

We then provide Fig 48.4.f) to establish that this bound is sharp. In step 1, we split one 4-part into pairs that we place above and below the other 4-part. We then pull out the four 2-crossing edges into the outer face. For this just incurs 1 crossing per edge. Thus sending us overall from 8 to just 4 crossings.

The general result here is as follows (along the lines in [183]). Suppose that a graph G contains M copies of the same nonplanar subgraph H), with each crossing being counted up L times by the totality of these subgraphs. Then

$$cr(\mathrm{G}) \;\geq\; \left\lceil \frac{M}{L} \right\rceil\, cr(\mathrm{H})\;. \tag{48.4}$$

Exercise 4$^-$ Find rectilinear presentations for each graph in Fig 48.3.b), d) and e).

Exercise 5 Show that (48.4) gives a further method for Exercise 3 but not for Exercise 1.

Pointer 1 While we know little about *cr* for general graphs, the matter has been extensively studied for K_n and $\mathrm{K}_{p,q}$. Among other things, indicates that quartic behaviour in N can indeed be attained (with variable constants of proportionality in known bounds).

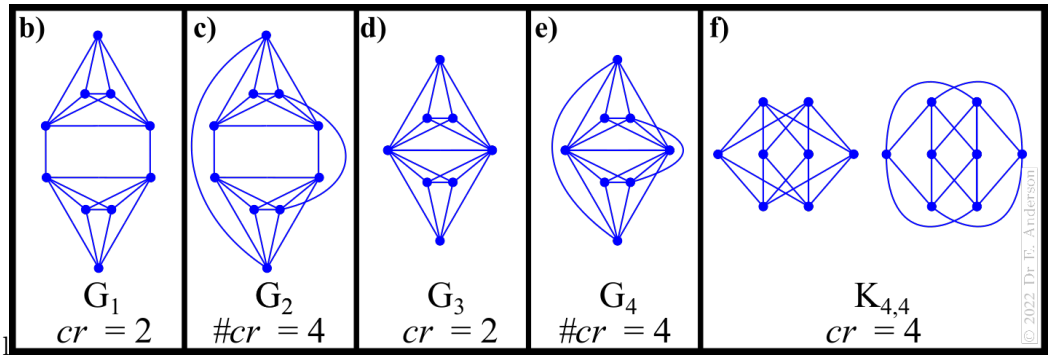


Figure 48.4:

48.6 Relational counterparts of *cr*

Remark 1 Let us first mention the visual relative count of vertices versus crossings in a given presentation of our graph.

'blues minus pinks' =: (Vertex number) − (crossing number) = |G| − cr(G)

$$= N - cr(\mathrm{G}(N)) \ .$$

For small enough edge number for a given N, blues dominate. However, in dense enough graphs for N large enough to support such, pinks dominate. This simple relative variable can thus take both signs. A limitation on this is that the number of crossings can vary considerably between presentations; it is another mismatch of extensivities.

Remark 2 From our bounds, we intuit that, firstly,

$$\frac{cr}{E}$$

is a ratio of interest in comparing sparse nonplanar graphs. If the graphs are linearly-sparse, this translates to

$$\frac{cr}{N}$$

being a ratio of interest. If very close to crititical planarity,

$$\frac{cr}{E - E_{\mathrm{planar}}}$$

is more nuanced. It can also implicity be viewed as a relational quantity:

$$\frac{cr - cr_{\mathrm{planar}}}{E - E_{\mathrm{planar}}} \ ,$$

by virtue of cr_{planar} being another name for zero.

Remark 3 Secondly, that

$$\frac{cr}{E^2} \simeq \frac{cr}{N^4}$$

is of more interest in comparing between dense nonplanar graphs.

Pointer 2 In fact, a more advanced analysis [168, 169] gives a bound on crossing number that is a function of both E and N. Namely, that if $E > 7\,N$ (still linearly sparse, but 7/3rds of the planar limit for N large),

$$cr \geq \frac{E^3}{29\,N^2} \ .$$

Exercise 6 a) Reconcile (48.6) and our previous dense-regime approximation.

b) What bound does (48.6) impose in the linearly-sparse edge regime?

Exercise 7 Prove that the graph in Fig 48.5.a) has crossing number 1.

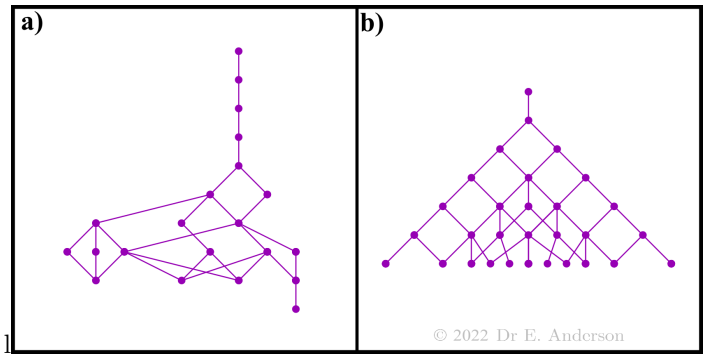


Figure 48.5:

48.7 Harder Exercises and Pointers on *cr*

Exercise 8 a) Show that (48.4) fails to be useful when two copies of $K_{3,3}$ encode two crossings that share a common edge.

This impedes our above method from working for the **F**amily[2] graph: the second subfigure in the last column of Fig 55.1. However, like many other graphs in Part VIII, this graph is of sub-critical-planar density and yet of nontrivial girth. On some occasions, the following alternative method works in this regime.

b)$^+$ By working via a planar spanning subgraph, show that

$$cr \;\geq\; E \;-\; \frac{g}{g \;-\; 2}\,(V \;-\; 2)\;.$$

[Hint: check carefully which variables merely depend on the subgraph, and find a way of eliminating each of these .]

c) Show that this still fails for **F**amily[2]'s most obvious simplified nonplanarity model: the third subfigure in the last column.

And yet, that it works for the furtherly-simplified nonplanarity model with the red background, yielding

$$cr(\textbf{F}\text{amily}[2]) \;\geq\; 2\;.$$

Deduce that this holds with equality.

Exercise 9 Prove that

$$cr(\text{Petersen}) \;=\; 2\;,\quad \text{and}\quad cr(\text{Heawood}) \;=\; 3\;.$$

Exercise 10 a) Count the crossings in Hypercube(4)'s standard presentation in Fig 34.3.

b) Prove that

$$cr(\text{Hypercube}(4)) \;\geq\; 4\;.$$

c)$^+$ Draw a presentation of Hypercube with no more than 12 crossings.

Exercise 11 [Long] a) Show that Exercise 8.b)'s technique fails to produce useful bounds on *cr* for each of the following graphs (even if de-side-treed). i) The bottom graph of Fig 59.2.c) with the top two vertices and consequent edges removed. ii) The graph derived in Fig 59.5. iii) The bottom graph in the first column of Fig 60.2.c). iv) The bottom graph in Fig 61.1.b). v) The graph in Fig 48.5.b). vi) The bottom graph in the second column of Fig 62.3.c).

b) By picking out 2 sufficiently independent $K_{3,3}$ subgraphs on each occasion, deduce that $cr = 2$ for each of v) and vi).

Exercise 12$^+$ a) Prove that any graph can be represented rectilinearly while maintaining a given number of crossings.

b) Find rectilinear presentations for the following (maintaining planarity and exhibited crossing number wherever relevant). i) Fig 64.12.b)'s last graph, ii) Fig 61.1's largest graph, and iii) Fig 59.4's last graph.

Exercise 13^{++} Get the graph at the bottom of Fig 61.1.b) into a presentation with a single-digits number of crossings.

Pointer 3 Further interesting bounds arise once one starts to consider multiple notions of crossing number. [194] provides interesting further reading for both Pointer 1 and Pointer 3.

B. Thickness

48.8 Introducing thickness, th

Definition 1 The *thickness* [160, 154, 190] of a graph G is

$$th \;=\; \min\#\,(\text{ pairwise edge-disjoint planar spanning subgraphs for }\;\mathrm{G}\,)\;. \tag{48.5}$$

Notational Remark 1 While θ is a more common notation elsewhere, we choose th to make a nice pairing with cr.

Remark 1 Thickness is conceptually a notion of cover. More specifically, it is a partitioning of a graph's edge set.

Remark 2 One use for thickness is that planar graphs are rather easier to draw, theoretically understand, and visually parse. One can represent a graph as a 'multi-storey car park' stack of planar graphs. It makes sense for one copy of this to vertically-align those vertices that are common to different floors. With each floor having a manifestly-planar copy next to it to help understand it. As we shall see in the next section, however, decompositions into planar graphs are far from unique. So there is room for further guiding principles in choosing a such for a graph. Thickness itself concerns the principle of using the minimal possible number of planar floors.

48.9 A first few examples of th

Example 0 Thickness is 1 for planar graphs. Thus to form a measure of departure from planarity, let us introduce the shifted *nontrivial thickness variable*

$$nth \;:=\; th \;-\; th(\min) \;=\; th \;-\; th(\text{planar}) \;=\; th \;-\; 1\;.$$

This is to be pronounced 'enth'.

Example 1 The minimal nontrivial value of nth is 1, corresponding to $th \;=\; 2$: the so-called *biplanar graphs*. K_5 and $\mathrm{K}_{3,3}$ exhibit this. For we know these to be nonplanar, and yet in each case we know that lifting out a single edge will suffice to attain planarity.

This is the *minimal-piece presentation*: minimizing the size of one of the pieces of the partition. For K_5 but not for $\mathrm{K}_{3,3}$, it is also the *maximally-planar piece presentation*. For the former's second piece – $\mathrm{K}_5 \;-\; e$ – is $N \;=\; 5$'s unique maximally-planar graph (Fig 48.6.a). The latter's second piece is C_6 with long (1, 2)-diagonals. Or equivalently a first homeomorph of Spiderhead as exhibited in Fig 48.6.d) This does not have the right size to be maximally-outer-planar:

$$8 \;\neq\; 12 \;=\; 3 \;\times\; 6 \;-\; 6\;.$$

The *reductive presentation* sends each member of a family to a smaller member of that family that manages to be planar. For K_5, the corresponding pieces of this are K_4 and Plus (Fig 48.6.b). For K_4, the corresponding pieces for one interpretation of this are $\mathrm{K}_{3,2}$ and Claw (Fig 48.6.d).

Exercise 14$^-$ What are the planar pieces for $\mathrm{K}_{3,3}$ if we interpret the family being reduced as the equal-entries $\mathrm{K}_{p,p}$?

Exercise 15 Demonstrate that the thickness of $\mathrm{K}_{4,4}$, $\mathrm{K}_{5,4}$ and $\mathrm{K}_{5,5}$ is 2. What are their minimal-piece presentations? Do they admit reduced splittings? [With $\mathrm{K}_{4,2}$, $\mathrm{K}_{5,2}$ and $\mathrm{K}_{5,2}$ pieces respectively.]

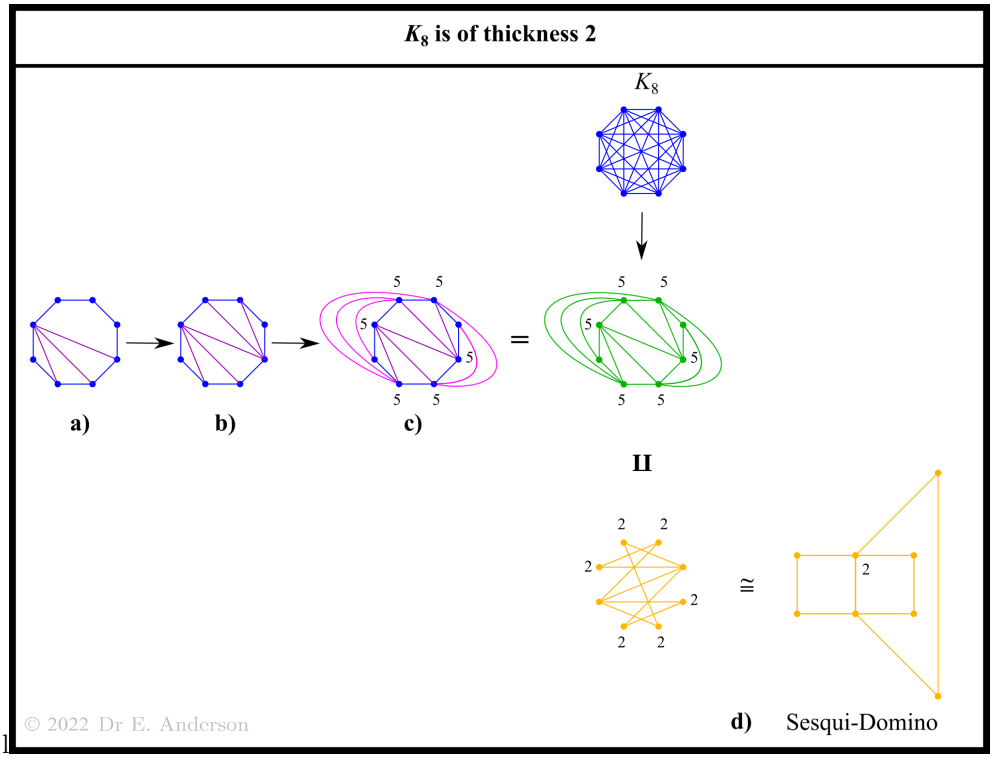


Figure 48.6:

48.10 Bounds on *th*

Proposition 1 For $N \geq 3$,

$$\left\lceil \frac{E}{3(N-2)} \right\rceil \leq th \leq N \,. \tag{48.6}$$

Proof Let G be a graph and P_I index the totality of its planar subgraphs. Then picking a minimal-sized subset of these that partition $\mathfrak{E}(\mathrm{G})$

$$E = E(\mathrm{G}) = \sum_{i=1}^{th} E(P_i) \leq th\, E_{\text{max-planar}} = th\, 3(N-2) \,.$$

The second step here uses additivity of partitions' objects while identifying the above minimal size as *th*. The third is a

$$\sum \leq (\text{number of terms}) \times (\text{maximum value})$$

throwaway inequality. The last uses eq. (45.3).

Since $N \geq 3$, we can furthermore licitly divide by $3(N-2)$. Finally use that thickness is $\mathbb{N}$-valued.

For our other inequality, all trees are planar. We can furthermore decompose each graph into each vertex's neighbourhood tree, yielding N planar stories. □

Remark 1 Requiring $N \geq 3$ is entirely unrestrictive, since $N = 5$ is minimal for nontrivial thickness.

Exercise 16 a) Show that for bi-partite graphs, the lower bound can be improved to

$$th \;\geq\; \left\lceil \frac{E}{2\,(N\;-\;2)} \right\rceil \;. \tag{48.7}$$

b) And for graphs of girth g an alternative lower bound is

$$th \;\geq\; \left\lceil \frac{g\;-\;2}{g}\,\frac{E}{N\;-\;2} \right\rceil \;. \tag{48.8}$$

♠ ◇ ♣

Example 2 Exercise 16's lower bounds mean that

$$th(\mathrm{K}_N) \;\geq\; \left\lceil \frac{N\,(N\;-\;1)/2}{3\,N\;-\;6} \right\rceil \;=\; \left\lceil \frac{N\,(N\;-\;1)}{6\,(N\;-\;2)} \right\rceil \;\simeq\; \frac{N}{6} \quad \text{for} \quad N \quad \text{large} \quad .$$

Example 3 And that

$$th(\mathrm{K}_{P,Q}) \;\geq\; \left\lceil \frac{P\,Q}{2\,(P\;+\;Q\;-\;2)} \right\rceil \;.$$

So for

$$P \;=\; \frac{N}{2} \;=\; Q \;,$$

$$th\left(\mathrm{K}_{\frac{N}{2}\;,\;\frac{N}{2}}\right) \;\geq\; \left\lceil \frac{N^2}{8\,(N\;-\;2)} \right\rceil \;\simeq\; \frac{N}{8} \quad \text{for} \quad N \quad \text{large} \quad .$$

Example 4 For linearly-sparse graphs G, thickness goes at least like

$$th(\mathrm{G}) \;\geq\; \left\lceil \frac{k\,N\;+\;l}{3\,(N\;-\;2)} \right\rceil \;\simeq\; \frac{k}{3} \quad \text{for} \quad N \quad \text{large} \quad .$$

Example 5 For linearly-sparse bi-partite graphs, as

$$th(\mathrm{G}) \;\geq\; \left\lceil \frac{k\,N\;+\;l}{2\,(N\;-\;2)} \right\rceil \;=\; \frac{k}{2} \;.$$

Remark 2 Taking into account the trivial upper bound on thickness as well, it is clear that thickness can be extensive or sub-extensive, but not super-extensive (see Appendix A for definitions).

Worked Example 3 Prove that a)

$$th(\mathrm{K}_{11}) \;\geq\; 3 \;.$$

b)

$$th(\mathrm{K}_{7,7}) \;\geq\; 3 \;.$$

Answer 3 a)

$$E(\mathrm{K}_{11}) \;=\; E_{\max}(11) \;=\; \frac{11\;\times\;10}{2} \;=\; 55 \;.$$

Also

$$E_{\text{planar}} \;=\; 3\;\times\;11\;-\;6\;=\;27 \;.$$

Finally, we know from Proposition 1 that

$$th(\mathrm{K}_{11}) \;\geq\; \frac{E_{\max}}{E_{\text{planar}}} \;.$$

So

$$th(\mathrm{K}_{11}) \;\geq\; \frac{55}{27} \;>\; 2 \;.$$

Thus since the answer must be a natural number, it is ≥ 3.

b)

$$E(\mathrm{K}_{7,7}) = 7 \times 7 = 49 .$$

Also

$$E_{\text{bi-partite}} = 2(7 + 7) - 4 = 24 .$$

Finally, we know that

$$th(\mathrm{K}_{7,7}) \geq \frac{E_{\max}}{E_{\text{bi-partite}}} .$$

So

$$th(\mathrm{K}_{7,7}) \geq \frac{49}{24} > 2 .$$

So since the answer must be an integer, it is ≥ 3.

Exercise 17 a) Check that the above argument gives another proof that K_5 and $\mathrm{K}_{3,3}$ are nonplanar.

b) What are the smallest K_n and the smallest $\mathrm{K}_{m,m}$ for which $th \geq 4$ is guaranteed by the above argument?

48.11 Further examples of th

Exercise 18 By going for a maximally-planar larger piece that leaves the second piece with not enough edges to be nonplanar, give $th = 2$ presentations of K_6 and K_7.

Worked Example 4 Show that the above method is insufficient for $N \geq 8$. But that we can build a larger planar piece which sufficiently loads up enough degrees that no K_5 or $\mathrm{K}_{3,3}$ can occur in the second piece.

Answer 4 K_8 has

$$\frac{8 \times 7}{2} = 28$$

edges. While a maximally-planar 8-graph has

$$3 \times 8 - 6 = 18 .$$

And

$$28 - 18 = 10 > 8 :$$

the maximum edge number not supporting a nonplanar graph.

Build the larger maximally-planar piece as follows. Take the Hamiltonian cycle – C_8 – and load up a vertex with internal diagonals until its degree is 5 (e.g. Fig 48.7.a). This is a key value for 8-graphs, since then the smaller part's corresponding vertex is of degree

$$8 - 5 - 1 = 2 .$$

This is not enough to contribute to any $\mathrm{K}_{3,3}$ or K_5 subgraphs. We still have 2 diagonals left to complete the inner face's triangulation, which we can use to load up a second vertex to degree 5 (Fig 48.7.b).

By internal–external symmetry, the external face also admits 5 edges. With many vertices already of degree ≥ 3, it is easier to load up vertices' degrees up to ≥ 5 with the external edges. For instance, in Fig 48.7.c), we attain five 5's and a 6. The remaining 2 higher-degree vertices are clearly outnumbered out of being able to contribute a $\mathrm{K}_{3,3}$ or a K_5. The second planar graph in this case is exhibited in Fig 48.7.d).

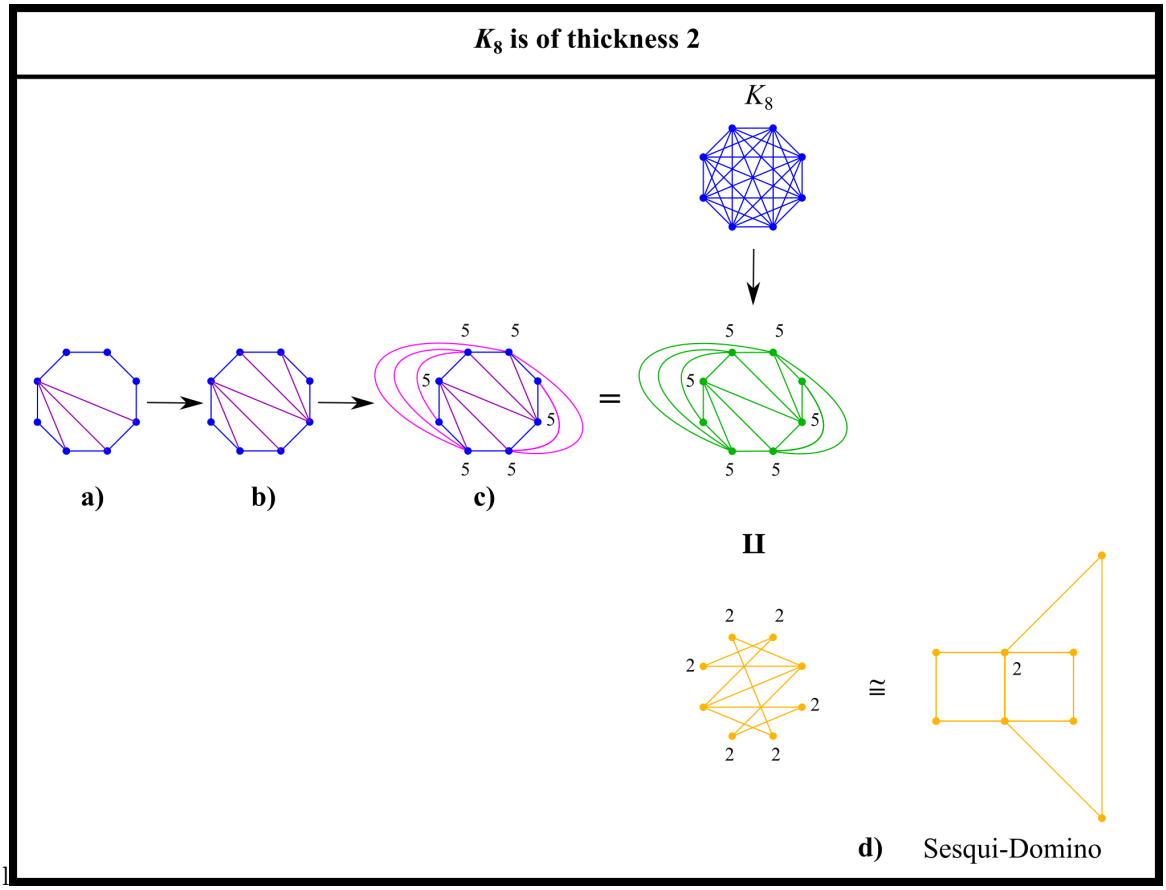


Figure 48.7:

Exercise 19 Show that Worked Example 3's 2 bounds hold strictly. I.e. exhibit a pair of planar graphs that cover a) K_9, b) $^+ K_{10}$ and c) $K_{6,6}$.

Exercise 20 Show that

$$th(K_{4,m}) = 2$$

for a) $m = 6$ to 9, and b)$^+$ $m = 10$ to 12.

Exercise 21$^+$ What is the lexicographically-smallest graph G with $th(G) = 3$?

48.12 Ratio variables for th

Remark 1 One of Proposition 1's inequalities above is comparing thickness with the ratio

$$\frac{E}{E_{\text{planar}}} .$$

For dense graphs, this goes as $N/6$, while for linearly-sparse graphs it is constant. Not-even-linearly sparse graphs can have this bound going as $1/N$, but this is not realized for large enough N. For $th \in \mathbb{N}$, giving another inequality $th \geq 1$, which then becomes a stronger bound than $1/N$.

Structure 1 This suggests that *order-normalized thickness*

$$\frac{th}{N}$$

is of some use in comparing between dense graphs. While linearly-sparse graphs would use just th. And sub-linearly sparse graphs

$$\frac{th}{N^{\alpha}}$$

for some $\alpha \in (0, 1)$.

48.13 cr and th compared

Remark 1 The following weak inequalities hold.

$$rth \leq cr \ .$$

$$th \leq N \ .$$

These combine into

$$rth \leq \min(cr, N - 1) \ .$$

The second of these holds because, until this bound is hit, each crossing can be resolved by putting a vertex involved in it into its own planar graph. While on some occasions, adding a planar will resolve multiple crossings, by which the inequality will not always be tight.

Structure 1 The *naïve crossing-thickness ratio*

$$\frac{cr}{th}$$

is somewhat motivated by the first inequality above.

Remark 2 We have however argued above that crossing number and thickness each go as a different power of N. This points to using instead the following.

Structure 2 The *asymptotically-constant highly-dense graph crossing number to thickness ratio* is

$$\frac{cr}{th^4} \ . \tag{48.9}$$

Remark 3 For linearly-sparse and yet nonplanar graphs, the bound on th is independent of N. Since our bounds on cr are N-dependent, this means that an N-independent ratio of cr and th cannot be formed in this regime. This corresponds to thickness by itself staying fixed, while crossing number continues to increase with N in this regime.

48.14 Further Exercises on th

Lemma 1 a) Graphs with $cr = 1$ must be biplanar.

b) Graphs with all crossings lying on a single edge must be biplanar.

c) Graphs with all crossings lying on vertex-disjoint edges must be biplanar.

Lemma 2 If a graph has $th = k$, then each of its subgraphs has $th \leq k$.

Exercise 22 Prove these Lemmas.

Exercise 23 Calculate (or estimate where needs be) inequality (48.9) for K_n and $\mathrm{K}_{n,n}$ for $n = 10$ to 12.

Exercise 24 [Long] Show that every nonplanar graph produced in Part VIII for which a presentation with a single-digits number of crossings is offered is biplanar.

Exercise 25^{+} And likewise for those for which only presentations are offered with more crossings than that.

[Observe that for girth 4, density $\rho > 4$ is needed for the girth-thickness inequality to carry useful information. The answer to Exercise 30.7's survey of Part VIII's nonplanar graph densities is that all lie between 1 and 2 bar the S_4 subgroup lattice having density slightly over 2. Thus, unlike in evaluating some cr, evoking that many of these graphs are of girth 4 does not help in evaluating th.]

Pointer 4 See e.g. [177] for further diversity of notions of thickness.

C. Towards Knot Theory

48.15 Rubber band shadows

Modelling situation Consider some rubber bands in $\mathbb{R}^3$ casting a shadow onto a plane. This shadow consists of edges and intersection points. We can deform the rubber bands in $\mathbb{R}^3$, and this can change the number of intersection points in the shadows. One cannot necessarily remove all intersection points in this way, however. For the rubber bands might be knotted or linked, so that separating their shadows would involve cutting and rejoining. But such topology-altering moves lie outside the remit of our allowed deformations.

This modelling situation comes with a notion of crossing number,

$$Cr \;=\; \min_{\text{all shadows of all presentations}} (\,\#cr\,)\;.$$

Where $\#cr$ is the number of crossings in a given shadow in Subchapter A's Graph-Theoretic sense.

Remark 1 The examples in Fig 48.8.a-d) illustrate that Cr is not the same as cr.

Remark 2 a) and b) furthermore illustrate that (1, 2)-loop-multi-graphs are necessary.

Remark 3 Among connected rubber band shadows, c) and d) contain the sole exceptional 2-loop-graph and 4-multi-graph. Aside from these, (1, 3)-loop-multi-graphs will suffice, though we will often also have reason to discard the 3's.

Remark 4 These subfigures also all illustrate that we can also now have loops with no vertex on them.

Remark 5 The previous three Remarks collect ways in which our modelling of rubber band shadows is somewhat more general than the (1, 2)-loop-multi-graphs.

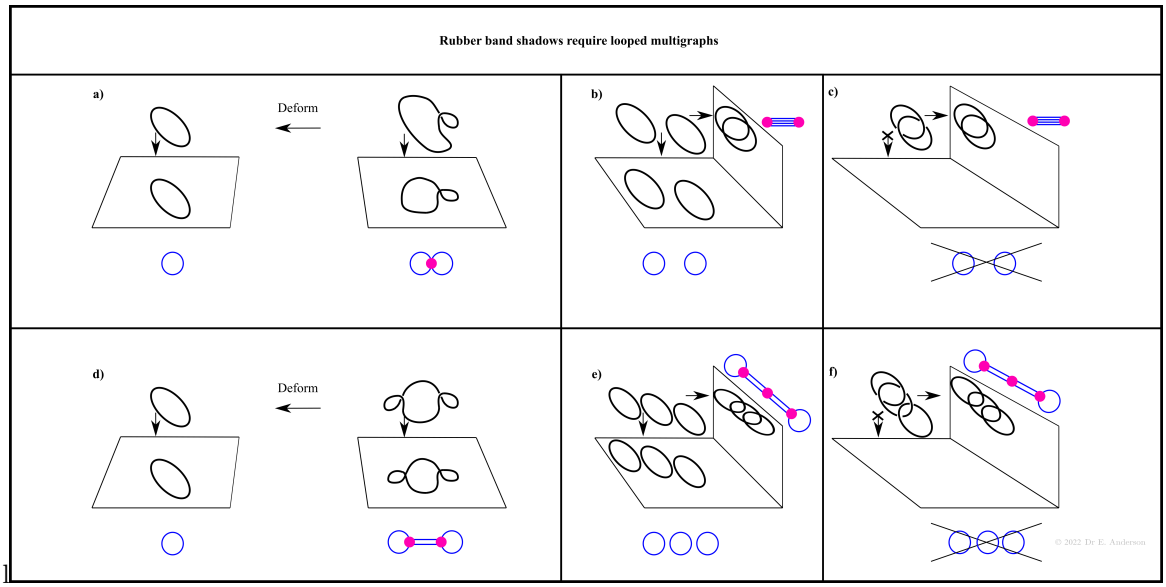


Figure 48.8:

Caveat 1 We also need our shadows to have crossing points in 'general position. Then the ensuing shadow loop-multi-graph with these crossing points as its vertices must be 4-regular.

Structure 2 The difference between a *knot* and a *link* is that a knot is a single closed curve embedded in $\mathbb{R}^3$. While a link is multiple such that cannot be teased apart with deformations. These definitions respectively include however the trivial *unknot* – just a loop with no knotting – among the knots. And the trivial *unlink*: just two loops with no linking. Most discussion in these Series and in the specific literature [254, 250, 247, 255, 277] about knots and links refers to *nontrivial* knots and links: to the exclusion of the unknot and the unlink.

48.16 Smallest examples of rubber band shadows

Worked Example 4 Characterize the possible connected rubber shadows with 0 to 2 crossings. What can they be shadows of?

Answer 4 On 0 crossings, we have just the shadow of a single rubber band, such that its shadow is not twisted. On 1 crossing, we have a single loop with 1 twist, giving the double-looped single-vertex graph, 8 (Fig 48.8.a).

On 2 crossings, we have a single double-twisted band, corresponding to the double-edged P_2 with a loop at either end. I.e. the complete (1, 2)-loop-multi-graph on 2 vertices TFK_2 (Fig 48.8.b). At the level of the shadows, we can finally have Fig 48.8.c)'s QP_2 with its exceptional quadruple edge. This could just represent two separate rubber bands whose shadows happen to overlap. It could however also be the shadow of 2 rubber bands that are linked to each other: the simplest link, occasionally alias *Hopf link* (Fig 48.8.e).

Example 1 Looking through the 3-crossing shadows in Fig 48.10, one of the objects that the last of these could represent is the simplest knot: the *trefoil knot* $\mathfrak{T}$refoil (Fig 48.9.a).

Example 2 Looking through the 4-crossing shadows in Fig 48.10, the last of these can represent the *figure-of-8* knot $\mathfrak{F}$igure-of-8 of Fig 48.9.b).

Example 3 Also the fourth of these can represent the *Solomon link* $\mathfrak{S}$olomon of Fig 48.9.c).

Exercise 26 a) Show that rubber band shadows, and thus knot shadows and link shadows, must in all cases be 4-regular and Eulerian.

b) Which of the above modelling assumptions bring(s) about this 4-regularity?

c) Deduce that Fig 48.8.a) and b)'s knot shadows contain the sole exceptions possible to (1, 3)-loop-multi-graphs among the connected rubber band shadows.

Exercise 27 Connectedness is necessary for the rubber band shadow to be of a knot or link: our main cases of interest. The previous Exercise explains the contents of Fig 48.10. A further organizational point about this Figure is that it lexicographically traverses along the connected simple graphs in the role of underlying graph skeleton.

a) Explain why Claw does not underlie any rubber band shadows.

b) [Long] Extend this lexicographically-traversing, 4-regularity-restricted construction to $N = 5$.

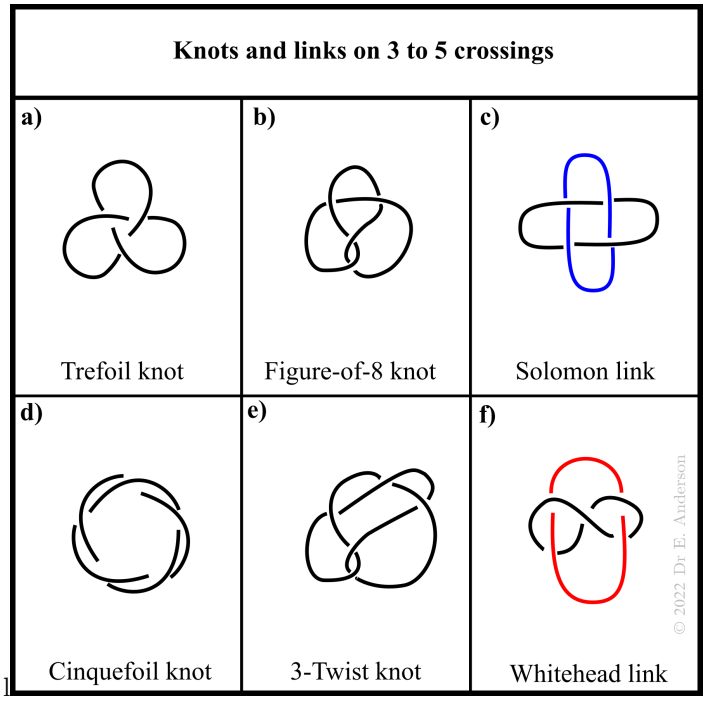


Figure 48.9:

c) Ensure that your extended figure has succeeded to include shadows of the *cinquefoil knot* $\mathfrak{C}$inquefoil, *3-twist knot* 3-$\mathfrak{T}$wist and the *Whitehead link* $\mathfrak{W}$hitehead [Figs 48.9.d-f) respectively]. What are the underlying simple graph skeletons for each of these?

d) Which connected simple 5-graphs do not underlie any rubber band shadows? Can you place any Lemmas restricting which connected simple higher-N graphs can be discarded from consideration?

48.17 Reidemeister moves and knot shadows

Structure 1 One distinction from plain graphs is that a notion of which strand is above which other needs to be retained in order to model knotting and linking information. We do this by drawing all of the upper strand. While leaving gaps around the lower strand near the crossing point so as to visually indicate that this passes underneath. If we include this information, then we pass from a rubber band shadow to an actual model of knotting and linking.

Structure 2 Fig 48.11 exhibits the Combinatorial *Reidemeister moves* under which presentations of knots and links are invariant. The idea is that two knots are equivalent if only differing from each other by these moves and trivial curve deformations involving no crossings at all. And similarly for links.

Exercise 28 a) Explain why knot shadows in presentations that minimize their crossing number should have no loops present.

b) Show that the configurations in Fig 48.12.a) are also $\mathfrak{T}$refoil.

c) Show that the configurations in Fig 48.12.b) are also $\mathfrak{F}$igure-of-8.

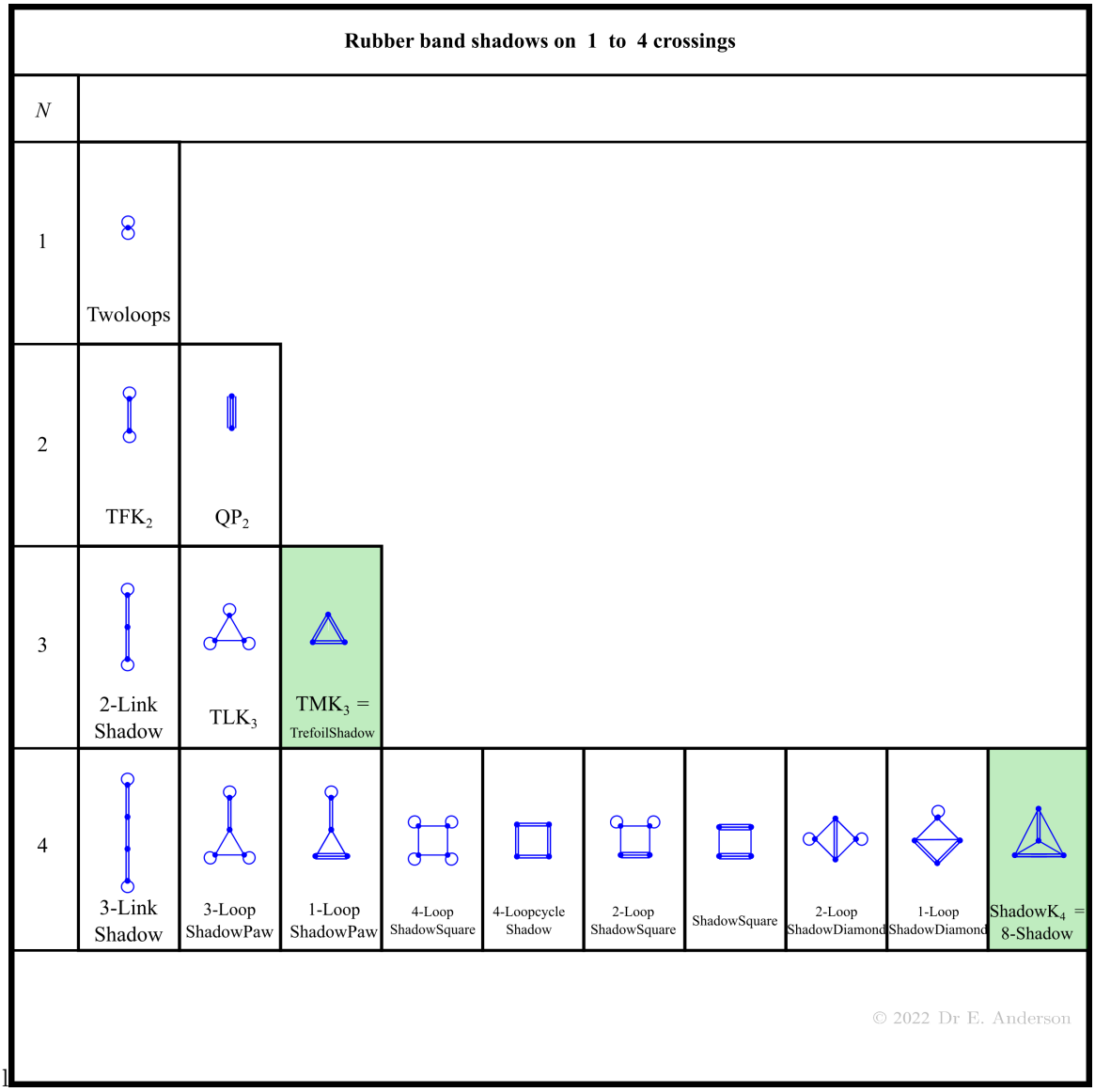


Figure 48.10:

d) Identifying the two ends of a shoelace after it has been tied in the conventional way, which knot is realized?

e) Fig 48.12.c-d) shows how to form the 'standard' and 'Windsor' knots on a tie. Finally identifying the two ends of the tie, which knot is realized in each case?

Exercise 29 Show that the trefoil knot is chiral but the figure-of-8 knot is not.

Remark 1 Connectedness and 4-regularity are necessary, but not sufficient, conditions to have a knot shadow. For starters, it could also be a link, or even multiple unlinked rubber bands whose shadows however overlap.

♠ ◇ ♣

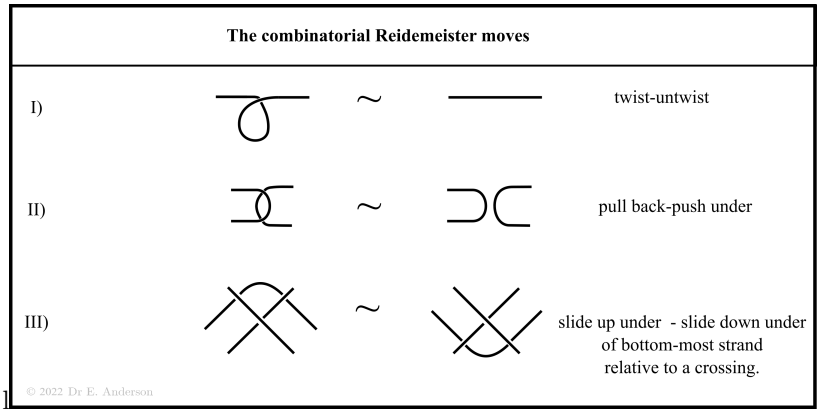


Figure 48.11:

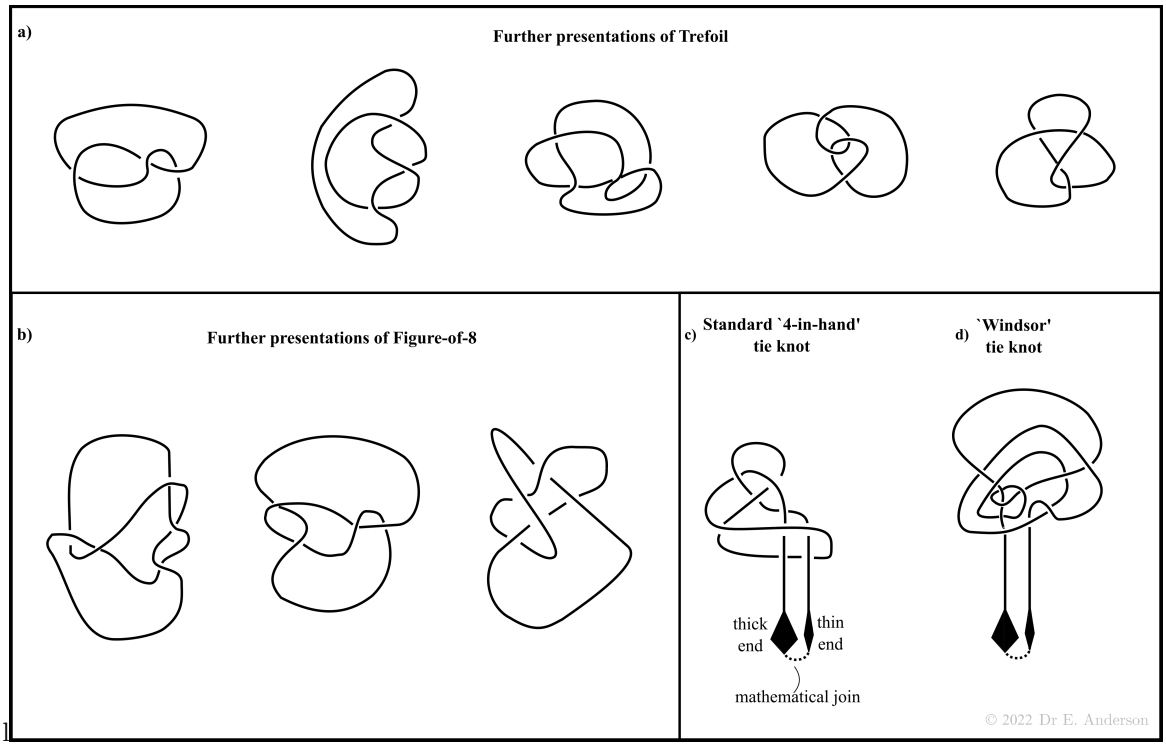


Figure 48.12:

Exercise 30 a) Justify that knot shadows cannot contain triple edges.

b) Justify that knot shadows must be connected leaf-free 4-regular unlooped 2-multi-graphs. Use this to cut down our classification of $\leq$ 5-vertex rubber band shadows.

c) [Long] Find all the connected leaf-free 4-regular unlooped 2-multi-graphs on 6 vertices. Identify among these the shadows of the three prime knots on 6 vertices (Fig 48.13.a-c). Identify also among these the shadows of the first two composite knots: $\mathfrak{S}$quare (Fig 48.13.d) and $\mathfrak{G}$ranny (Fig 48.13.e)

[**Structure 3** A knot is *composite* if we cannot find a 2-sphere that intersects it in exactly 2 points such that the edges on both sides are knotted. We will subsequently reformulate this as a cut-and-paste topological sum.

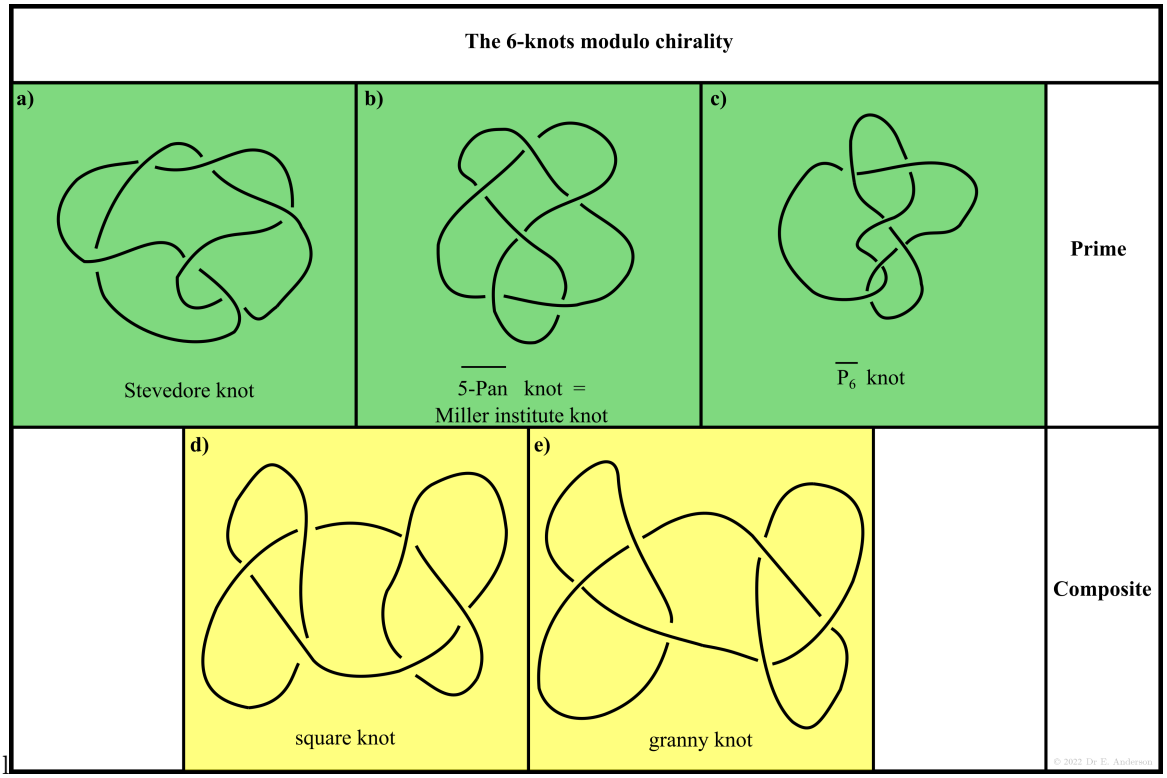


Figure 48.13:

Remark 2 Under this operation in reverse, knots can be decomposed into prime knots. Up to $Cr = 6$, these are in Fig 48.9.a),b),d),e) and in Fig 48.13.a-c).

Exercise 31 Worked Example 4 can be quickly adapted to prove that there are no nontrivial knots on ≤ 2 vertices.
a) Use our classification of rubber band shadows, alongside some understanding of what can be realized in the primary $\mathbb{R}^3$ setting in general position to prove the following.

i) That on 3 crossings, $\mathfrak{T}$refoil is the only possible knot (modulo chirality).

ii) That on 4 crossings, $\mathfrak{F}$igure-of-8 is the only possible knot.

iii)$^+$ That on 5 crossings, $\mathfrak{C}$inquefoil and 3-$\mathfrak{T}$wist are the only possible knots. Determine also whether each of these is chiral.

iv)$^{++}$ That on 6 crossings, Fig 48.13's 5 cases are all the possible knots, stating also the extent to which this is modulo chirality.

b)$^+$ How many links are supported on ≤ 5 crossings?

c)$^{++}$ Find also all the links supported on 6 crossings.

[You may assume that the Reidemeister moves account for all possible deformations of the rubber bands. Proving this, and more efficient ways of finding knots up to a given number of crossings Cr, pertain to Topology and Invariant Theory. Knots are notable moreover for remaining unclassified:

there is no known systematic way of finding all the prime knots, even up to moderate-sized Cr.]

Naming Remark 2 $\mathfrak{T}$refoil and $\mathfrak{C}$inquefoil are the first two members of the infinite odd series of $(2\,n\,+\,1)$-*foils*. Their underlying graph skeletons are the corresponding odd cycles, and fully double-edged. For the smallest nontrivial knot $\mathfrak{T}$refoil, C_3 = Triangle = K_3. K_4 also succeeds in supporting a knot, the figure-of-8: descriptively-named. Upon inspection, K_5 fails to support a knot. No further K_p can support a knot, by violating 4-sub-regularity.

Observing the increasing convenience of naming [graphs] via their sparser representative, instead $\overline{P_6}$ and $\overline{\text{Pan}(5)}$ support knots on 6 crossings. The first of these gives us a name for the third prime 6-knot, which has hitherto remained unnamed. The second is also called the *Miller Institute knot*, having been adopted there as a sigil. The third 6-knot has the good fortune of having a commonly-used truer name, $\mathfrak{S}$tevedore, meaning a dock worker, since these traditionally used this knot. On this occasion, it is the underlying 6-graph that would appear not to have a name, so we call it *Stevedore graph*.

Most larger knots, and most larger graphs, lack names, so this cross-naming strategy will have fairly limited shelf life. Knots are moreover *far* rarer than graphs. Being a knot shadow's underlying skeleton is thus a notable feature to name graphs after. Even if the knot has no more of a name than p_q: the qth knot on p crossings. $\mathfrak{S}$quare and $\mathfrak{G}$ranny exemplify (not just chirally) distinct knots sharing both underlying graph skeleton and which edges are multiple. They are both built on the Boat = $\overline{\text{Domino}}$ 6-graph skeleton.

For links, the Hopf link is just the 1-link: the first nontrivial member of the *linear n-link* family. The Solomon link is alias C_4 link, and the Whitehead link is alias Wheel link

48.18 Knot colouring

Structure 1 Colourability for knots usually refers to edge-colourability under the same rules as for general graphs.

Exercise 32 What is χ_1 for each knot with $\leq$ 6 crossings?

48.19 Pointers on knots

Pointer 5 Book 1 quantifies extent of linking by use of *Gauss' linking number*. And outlines a few of the simpler knot polynomial invariants (see also [250, 254, 255]). Book 4 introduces *knot groups*; see also [247]. Some further elementary topics in Knot Theory – *Seifert surfaces* and *torus knots* – are briefly covered in Volume 6 on Differential Geometry These developments indicate that basic Knot Theory follows the usual path of growth parallelling Linear Algebra, Group Theory and Differential Geometry.

Pointer 6 Applications of knots include [277] to DNA and protein modelling in Biology, and to some comparable purely Chemical modelling. Some properties of electric circuits, and more occasionally of flows (literal or abstract), depend on knotting and linking. Knots also enter some generalizations of Feynman diagrams, and in Statistical Mechanical modelling. At a deeper level, knots play a substantial role in the Topological and Differential-Geometric study of 3- and 4-manifolds: in many ways the hardest two dimensions in which to study manifolds.

Part VII

Introduction to Order Theory

Chapter 49

Posets. I. Introduction and simplest examples

49.1 Recap

Definition 1 A *poset*

$$\mathfrak{p} \;=\; \langle\, \mathfrak{X},\;\; \preceq \;\rangle$$

[49, 224, 216] is a set $\mathfrak{X}$ equipped with a partial order $\preceq$. Recollect that this means that $\preceq$ has the following properties.

i) *Reflexivity*:

$$\forall\, a \;\in\; \mathfrak{X}\;,\;\; a \;\preceq\; a\;.$$

ii) *Transitivity*:

$$\forall\, a, b\, c \;\in\; \mathfrak{X}\;,\;\; a \;\preceq\; b\;,\;\; b \;\preceq\; c \;\;\Rightarrow\;\; a \;\preceq\; c\;.$$

iii) *Antisymmetry*:

$$\forall\, a, b \;\in\; \mathfrak{X}\;,\;\; a \;\preceq\; b\;,\;\; b \;\preceq\; a \;\;\Rightarrow\;\; a \;=\; c\;.$$

Also recollect that the term 'poset' is shorthand for *partially-ordered set*.

Remark 1 The simplest posets are chains.

Definition 2 Also recollect that a *chain* is a poset for which $\preceq$ is additionally total. In this case, $\preceq$ is termed a *total ordering*. Finally recollect that this means that

$$\text{one or both of } \;x \;\preceq\; y\; \text{ and }\; y \;\preceq\; x\; \text{ must hold } \quad \forall\, x, y, \;\in\; \mathfrak{X}\;.$$

Structure 1 *Incomparable elements* are possible in posets: pairs of elements which are unrelated by $\preceq$. In contrast, in chains, pairs of distinct elements must always be comparable.

49.2 Chain and poset morphisms

Definition 1 A *poset morphism* [224] is a map between posets

$$\phi \;:\; \mathfrak{p} \;\longrightarrow\; \mathfrak{p}'\;. \tag{49.1}$$

Definition 2 A poset morphism ϕ is *order-preserving* alias *isotone* or *monotone* if

$$\forall\;\; u, v \;\in\; \mathfrak{X}\;,\;\; u \;\preceq\; v \;\;\Rightarrow\;\; \phi(u) \;\preceq\; \phi(v)\;.$$

It is *order-reflecting* if

$$\forall\ u, v\ \in\ \mathfrak{X}\ ,\quad \phi(u)\ \preceq\ \phi(v)\quad \Rightarrow\quad u\ \preceq\ v\ .$$

Definition 3 More specifically, a poset morphism that is both order-preserving and order-reflecting is said to be *a poset embedding*. In some texts, this is alias *order-embedding*, though this is a less precise name since not all orders are posets.

A *poset isomorphism* ψ is a bijection between posets

$$\psi\ :\ \mathfrak{p}\ \longrightarrow\ \mathfrak{p}' \tag{49.2}$$

that is also a poset embedding.

Exercise 1$^-$ Show that poset embeddings must be injective.

Definition 4 A poset morphism is an *order-reversing* alias *antitone* or just *anti-isomorphism* [224] if it is a bijection between posets

$$\alpha\ :\ \mathfrak{p}\ \longrightarrow\ \mathfrak{p}' \tag{49.3}$$

that preserves every aspect of the ordering except that it is reversed. I.e. such that

$$\forall\ x, y\ \in\ \mathfrak{X}\ ,\quad x\ \preceq\ y\quad \Rightarrow\quad \alpha(y)\ \preceq\ \alpha(x)\ .$$

Remark 1 Much as chains are simple subcases of poset, chain (anti-)(iso-)morphisms are simple subcases of poset (anti-)(iso-)morphism.

Exercise 2$^-$ Show that isotone maps compose. What happens with anti-isotone maps?

49.3 Small examples of chains

♠ ◇ ♣

Example 0

The *0-chain* alias *unchain* is just the empty set.

Example 1 The *1-chain* is just a loose point.

♠ ◇ ♣

Definition 1 A *terminal element* of a chain or poset is an element with one or both of no successor or no predecessor. It is an *initial element* if it has no predecessor, and a *final element* if it has no successor[1]

Definition 2 A chain is called *bounded* if it possesses both initial and final elements. It is *semi-bounded* if it possesses just one of the two. It is *unbounded* if it possesses neither.

Remark 1 Among chains, the 1-chain is the only case for which there is a terminal element that is both initial and terminal. The following is the minimal case of chain that is non-trivial in this way.

Example 2 The *2-chain* is the pair of points with the relation $\preceq$ holding between them. This can be modelled by P_2's ori-graph version, with the arrow pointing in the direction that $\preceq$ holds in.

[1] Be warned: some authors use 'terminal' for what we here call 'final'.

Remark 2 Chains are always connected.

Remark 3 All chains can be represented by path-graph skeletons made into ori-graphs with all arrows aligned. This is conventionally represented vertically with all arrows pointing downward. This renders chains straightforward, by providing a simple classification for them.

Example 3 Among these, the 3-chain is the minimal for exhibiting a nontrivial middle, i.e. for possessing any non-terminal elements. This example possesses precisely 1 middle element.

Exercise 3 Use minimal examples of chain morphisms to illustrate the following.

a) A chain map that is concurrently isotone and anti-isotone.

b) A chain map that is isotone and not anti-isotone and vice versa.

c) A bijection that is isotone but not anti-isotone and vice versa, as well as a bijection that is neither.

49.4 Chains and posets modulo order reversal

Remark 1 One can view order reversal as a simple kind of duality. It is a C_2 symmetry, whose action we denote by $\uparrow \longleftrightarrow \downarrow$ subscripts (a notation already used for arrow-graphs).

Definition 1 A ⟨chain⟩ is a chain modulo order-reversing morphisms. A ⟨poset⟩ is a poset modulo order-reversing morphisms.

Definition 2 The *dual* of a poset or chain is the same structure with $\preceq$ reversed. This has the same underlying graph, represented with arrows reversed.

49.5 Infinite chains

Remark 1 Chains can be any finite P_n, a semi-infinite path, or an infinite path. While $\mathbb{Z}$ and $\mathbb{N}$ are isomorphic by the map that establishes $\mathbb{Z}$ to be countable (Proposition 2 of Chapter 17), $\mathbb{N}$ has a terminal element while $\mathbb{Z}$ does not. And this is enough to distinguish between the corresponding posets.

Remark 2 We can distinguish also between $\langle\, \mathbb{Z},\ \geq\ \rangle$ and $\langle\, \mathbb{R},\ \geq\ \rangle$: countably- versus uncountably-infinite chains.

Remark 3 In one sense, upper and lower semi-infinite chains are distinct. This reflects that when there is precisely one terminal element, there is some distinction between whether the aligned arrows point into it or away from it. But if one allows for the order-reversing morphism, these are mapped into each other. Thus they are identical as ⟨chains⟩ This applies for instance to $\mathbb{N}$ versus $-\mathbb{N}$ and to $\mathbb{R}_0$ versus $-\mathbb{R}_0$.

Exercise 4 a) What about $\mathbb{R}_+$ versus $-\mathbb{R}_+ \ = \ \mathbb{R}_-$?

b) Which of the finite real intervals $[\,0,\,1\,]$, $(\,0,\,1\,)$, $(\,0,\,1\,]$ and $[\,0,\,1\,)$ provide further cases of chain? Of ⟨chain⟩?

Remark 4 The arena of chains is trivially characterized: $\mathbb{N}$ with 2 or 3 infinite representatives (depending on what?)



49.6 Smallest examples of posets

Remark 1 As compared to chains, small posets require rather more work to distinguish between, and to count up for a given N.

Remark 2 While chains always have to be connected, the availability of incomparability means that posets can have multiple components.

Example 0 The simplest example of poset that is not a chain is the *incomparable pair of points* (Fig 49.1.a). I.e. the 2-point point-cloud pt^2 alias the discrete graph D_2, which has no edges and thus supports no arrows.

Example 1 The *point and 2-chain poset* (Fig 49.1.b) is minimal as regards jointly realizing both comparable and incomparable points.

Example 2 = Naming Remark 1 The *minimally nontrivial posets*, in the sense of not just being chains or disjoint unions thereof, are the P_3-v of Fig 49.1.c). I.e. the bent 3-paths, of which there are two. One is the downwards-bending path, whose down-flowing arrows form the sink ori-graph. The other is the upwards-bending path, whose down-flowing arrows form the source ori-graph. These constitute a first *heteroreversing pair of posets*, the notion of reversal now consisting of both arrow reversal and up-down reversal. In contrast, all chains and disjoint unions thereof are *homoreversing* singlets.

Remark 3 Below, we shall be encountering a number of further examples of bent paths, since these constitute quite a high proportion of the smaller posets.

Example 3 = Naming Remark 2 The smallest posets which are not some bending of a path occur for $N = 4$, as afforded by the underlying graph skeleton C_4. There are in fact two such posets: the 'natural' 121 square (Fig 49.1.d) and the 22 crossed square (Fig 49.1.e). Both are homoreversing singlets. These are also minimal as regards posets exhibiting a cycle.

Example 4 = Naming Remark 3 The bent paths supported by $N = 4$ are the chain, the single-bend pair of hockey sticks (Fig 49.1.f) and the two-bend N (Fig 49.1.g).

49.7 Cover diagrams (alias Hasse diagrams). I. Definitions

Definition 1 w *covers* u whenever the following hold.

a) $u \preceq w$.

b) There is no vertex v playing an intermediary role

$$u \preceq v \preceq w \ .$$

Remark 1 One idea is then to represent posets as ori-graphs in which covering relations in this sense are arcs from covered to covering vertex, and no other arcs are present.

Definition 2 Let O be an ori-graph. Then a *transitive reduction* [215] of O is another ori-graph O_{tr} with the following properties.

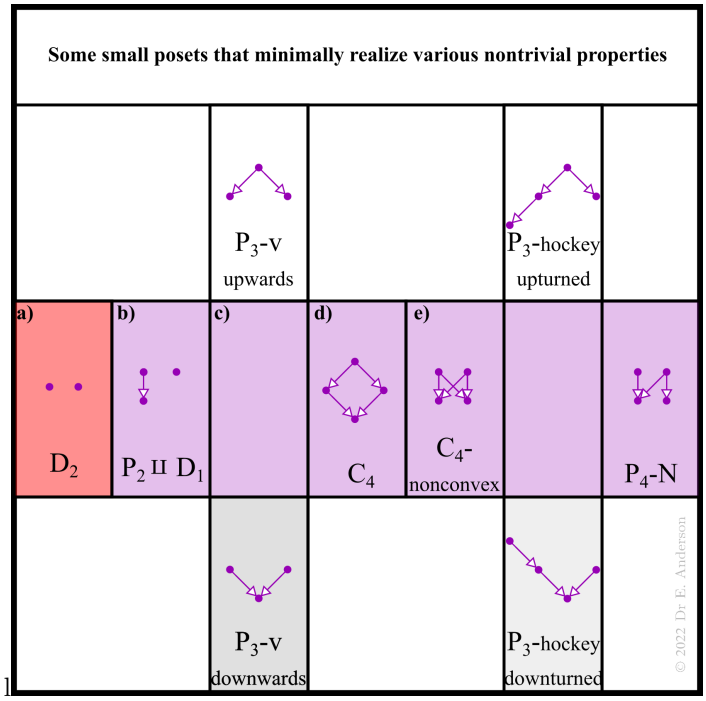


Figure 49.1:

i)

$$\mathfrak{V}(\mathrm{O}_{\mathrm{tr}}) \;=\; \mathfrak{V}(\mathrm{O}) \;.$$

ii) O_{tr} and O have the same *reachability relation*. I.e. given two distinct vertices, there is an oriented path between them in O_{tr} iff there is one in O.

iii) O_{tr} is size-minimizing.

Definition 3 Given a poset, its *cover diagram* is a presentation of its underlying ori-graph's transitive reduction that is built in the following way.

I) Each element of our poset's underlying set is represented as a vertex in $\mathbb{R}^2$.

II) Two vertices u, w are joined by an edge flowing down from u to w whenever w covers u in the sense of Definition 1.

Remark 2 It turns out that small finite posets are conveniently represented by cover diagrams. See Figs 49.1 and 49.4 for examples of cover diagrams.

Exercise 5 Give examples of elements covering other elements in a finite chain and in $\mathbb{N}$. Can you give examples of this in $\mathbb{R}$?

49.8 II. Discussion

Naming Remark 4 Cover diagrams is a truer name used by e.g. West [76] for what are probably still more widely-known under the name of Hasse diagrams [224]. Davey and Priestley [224] call them just *diagrams*. Though books which are not just about posets and specializations thereof like lattices are likely to contain other types of diagram, e.g. the current Volume also contains Young diagrams. One resolution to this is to call cover diagrams *poset diagrams* and Young diagrams

partition diagrams. [Though there is still also the Young versus Ferrers aesthetic distinction in type of representation in the latter case.]

Notation Remark 1 Representing posets by cover diagrams flowing upwards, rather, is also more common. In building up larger posets from smaller posets, however, it makes more sense to flow downwards. This is so that the new information appears after the already-established information in the human 'reading down the page' ordering. This explains the current Volume and Series' downwards-flowing convention.

Remark 1 Finite posets are fully-determined by their cover relations and thus cover diagrams. The current Series only makes any detailed use of posets which are finite, so cover diagrams encode everything that we need. A more advanced point is that cover relations additionally completely determine posets that are *locally-finite*. This is meant in the sense that every interval of the poset is finite. [Which holds for say $\mathbb{N}$ but not for $\mathbb{R}$, given that we have mentioned these as being among the types of infinite chain.]

Exercise 6 Prove this in the finite case.

♠ ◇ ♣

Remark 2 It is unfortunately far from straightforward to construct cover diagrams which manifest further properties.

Example 1 Cover diagrams that exhibit complementation symmetry.

Example 2 Cover diagrams that exhibit further automorphisms.

Example 3 Cover diagrams which are planar graphs, are termed *downwards planar* [222, 194]. This is a more stringent condition than planarity of the underlying graph skeleton, out of the downwards structure also needing to be preserved.

Naming Remark 5 Following Notation Remark 1, it should not be surprising that most other authors term this *upward(s) planarity*.

49.9 The forbidden triangle diagnostic

Proposition 1 Triangles are forbidden in posets.

Proof This follows in a couple of lines from the transitivity and reflexivity properties of posets (**Exercise 7**). □

Remark 1 This explains why Triangle does not support a poset, which in turn justifies Example 3's second minimality.

Remark 2 Mantel's Theorem (Sec 43.4) thus places a cut-off on which graph skeletons do not support posets; this cutoff is not necessarily sharp.

Exercise 8 a) Find an example of order-preserving poset morphism that is not order-reflecting and vice versa.

b) By splitting the partial ordering structure into the underlying graph order, and adjacency relations, arrowing and downwards-bending structure, it is clear that isotone poset morphisms can be more stringent than ori-graph morphisms. Similar applies to anti-isotone poset morphisms versus ori-graph arrow-reversal morphisms. Find an ori-graph isomorphism that is not an isotone poset

isomorphism, and an ori-graph arrow-reversing isomorphism that is not an anti-isotone poset isomorphism.

[Nonetheless, in the current Volume we can get away with denoting both ori-graph and poset reversals by $\uparrow \longleftrightarrow \downarrow$.]

Remark 3 This can be rephrased as posets necessarily being of girth $g \geq 4$. This points to posets being a potential target for Ex 48.8's crossing number inequality, which turns out to be quite helpful for our examples in Part VIII.

49.10 Towards systematically tabulating the smallest posets. I. Arena-level diagrammatic notation.

Remark 1 We use reversibility as an organizing principle for tabulating the diversity of small posets. Fig 49.1's background colours encode this as orange for homoreversing singlets and yellow-and-grey pairs for heteroreversing posets. We furthermore choose to concentrate upper-bending copies into the yellow copy. Though one would need more specific rules than this to handle posets containing both upward and downward bends.

Remark 2 Since we are using the downwards-flowing convention, it is unnecessary to draw arrows. To keep posets immediately visually distinct from their underlying graph skeletons, posets are presented in purple to graphs' in blue.

Remark 3 We also use a levelled presentation convention. This is then of some aid in counting edges in larger posets. For there is a well-defined sense in which we can count edges emanating down level by level.

49.11 II. A first few relational variables for posets

Structure 1 We can for instance pass from the notion of an element's level within a given poset to the *relative level* between comparable elements of that poset. This is a path-length notion for our poset's underlying graph skeleton. More precisely, it is a *minimal* path-length notion, since for instance the convex-pentagon C_5 contains both a 3 path and a 4-path from its top to its bottom.

Structure 2 In looking to systematically tabulate posets, however, we are more interested in comparing between whole posets rather than between elements within a given poset. In this regard, we can pass from the notion of depth $depth(\mathfrak{P})$ for a poset $\mathfrak{P}$ to the *relative depth* between posets,

$$Rdepth \ := \ depth(\mathfrak{P}) \ - \ depth\left(\mathfrak{P}'\right) \ .$$

This quantifier is at the ori-graph level of structure, where we already considered a version of it.

Structure 3 A second chain cardinal of note for posets is

$$c(\mathfrak{P}) \ := \ (\text{minimum number of chains that a poset can be partitioned into}) \ . \tag{49.4}$$

Naming Remark 6 We choose this lower-case c notation to signify minimum chain number. For reasons that will subsequently become apparent, we now also elect to re-denote depth for posets by

$$C(\mathfrak{P}) \ , \tag{49.5}$$

using the upper-case C notation to signify maximum chain.

Remark 1 We thus also re-denote *Rdepth* by *RC*. And introduce

$$Rc \ := \ c(\mathfrak{P}) \ - \ c\left(\mathfrak{P}'\right) \tag{49.6}$$

so as to have another comparer between posets $\mathfrak{P}$ and $\mathfrak{P}'$.

Finally, one independent relative variable

$$d \ := \ C \ - \ c \tag{49.7}$$

and one independent ratio variable

$$\mathcal{C} \ := \ \frac{c}{C} \tag{49.8}$$

are supported by our two chain cardinals for posets.

Exercise 9 What are C and c for each poset on ≤ 4 vertices?

♠ ◇ ♣

Structure 4 Example 4 begins to illustrate that the *bending number* B that counts the number of bends exhibited by a poset is a useful variable.

The *excess bending number* EB is even more useful, due to applying to non-paths as well. We define this relative variable to be

$$EB(\mathrm{G}, \preceq) \ := \ B(\mathrm{G}, \preceq) \ - \ B_{\min}(\mathrm{G}) \ ,$$

The minimization here is over all the posets supported by the underlying graph skeleton G.

For instance, C_4-nonconvex has

$$B \ = \ 4$$

but

$$EB \ = \ 4 \ - \ 2 \ = \ 2 \ .$$

[For paths, the chain gives $B_{\min} \ = \ 0$, so EB just returns B itself.]

Structure 5 Let us also introduce the *upwards* and *downwards* bending numbers $B_{\pm}$, which count the number of upwards and of downwards bends respectively. Aside from supporting their own excess-bending counterparts, these enable the following further relative variable.

The '*bending index*'

$$\vartriangle B \ := \ B_+ \ - \ B_- \ . \tag{49.9}$$

This is for instance nontrivially 0 for P_4-N.

Clearly also

$$B \ = \ B_+ \ + \ B_- \ .$$

Remark 2 Up and down bends must moreover alternate, with e.g. P_4-N being $+-$ and P_5-M being $+-+$. This clarifies that $\vartriangle B$ is merely C_2-valued, thus not adding much information to B.

Remark 3 The previous two entries cannot discern between P_3-v and P_4-hockey, which differ metrically: as regards how long the subpaths between bends are. This example can be remedied by using N alongside bending variables. More generally, there is some use in characterizing paths with B bends by an ordered string of $B \ + \ 1$ subpath-lengths.

49.12 The actual tabulation of smallest examples of posets

Structure 1 We now have enough information to classify the posets on 0 to 4 vertices. Checking our list of graphs for suitable skeletons, for $N = 3$, all bar Triangle pass. For $N = 4$, all connected graphs pass bar Paw, Diamond and K_4. Fig 49.2 ensues.

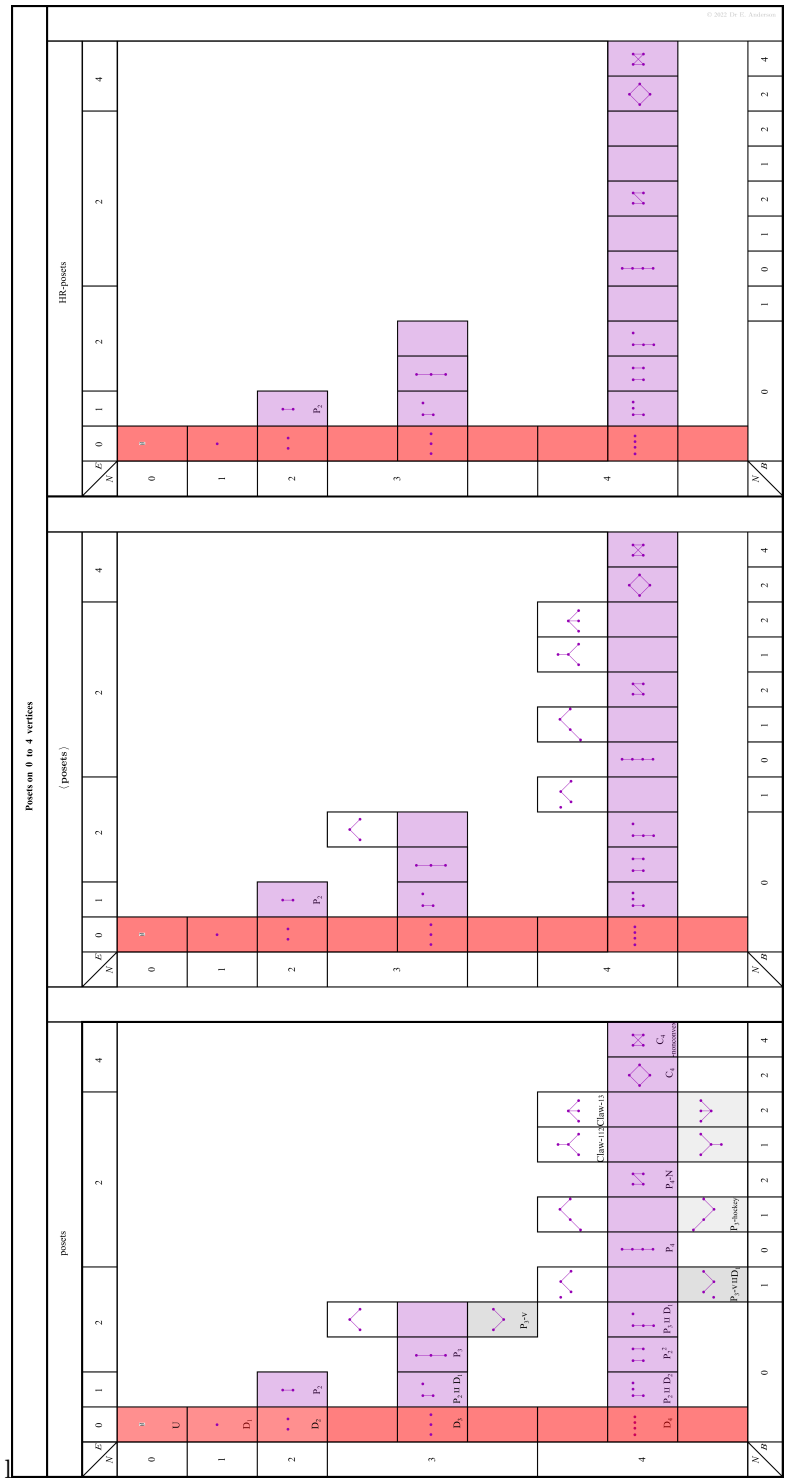


Figure 49.2:

Structure 2 = Naming Remark 7 As regards the connected posets on 5 vertices, the possible underlying connected graph skeletons are P_5, Chair, S_4, C_5, Banner and $K_{3,2}$. [Working this out was set as part of Exercise 43.3. Also observe that it has become shorter to list those that pass

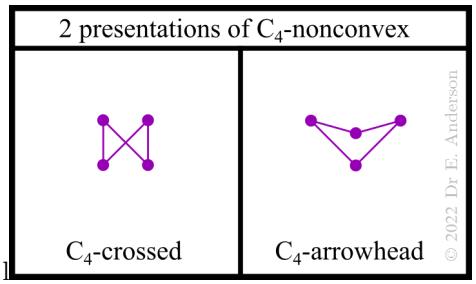


Figure 49.3:

rather than those that do not.] Fig 49.4 ensues.

♠ ◇ ♣

Exercise 10 How many fully distinctly labelled posets are there on each of 0 to 4 vertices? How many of these are homodual?

Exercise 11 Find the automorphism groups of each of the unlabelled posets on 0 to 4 vertices.

Exercise 12 How many non-connected posets are there on 5 vertices? How many posets are there on 6 vertices?

Exercise 13 a) Show that the C_4-crossed presentation is poset-isomorphic to the C_4arrowhead presentation (Fig 49.3). This accounts for why we henceforth call this poset 'nonconvex' rather than either of the above presentations' names, on account of those being merely presentation-specific. It also provides us with a downwards-planar presentation of C_4-nonconvex.

b) Does $K_{3,2}$ admit a downwards-planar presentation? Does $K_{p,2}$? Does $K_{p,3}$ for $p \geq 3$?

End-Remark 1 (Semi-)infinite posets are also possible; the current Series concentrates on small finite posets, however.

End-Remark 2 We have here built up the following nest of specializations.

$$\text{Arrowgraphs} \supset \text{Ori-graphs} \supset \text{Triangle-free Ori-graphs} \supset \text{Posets} . \tag{49.10}$$

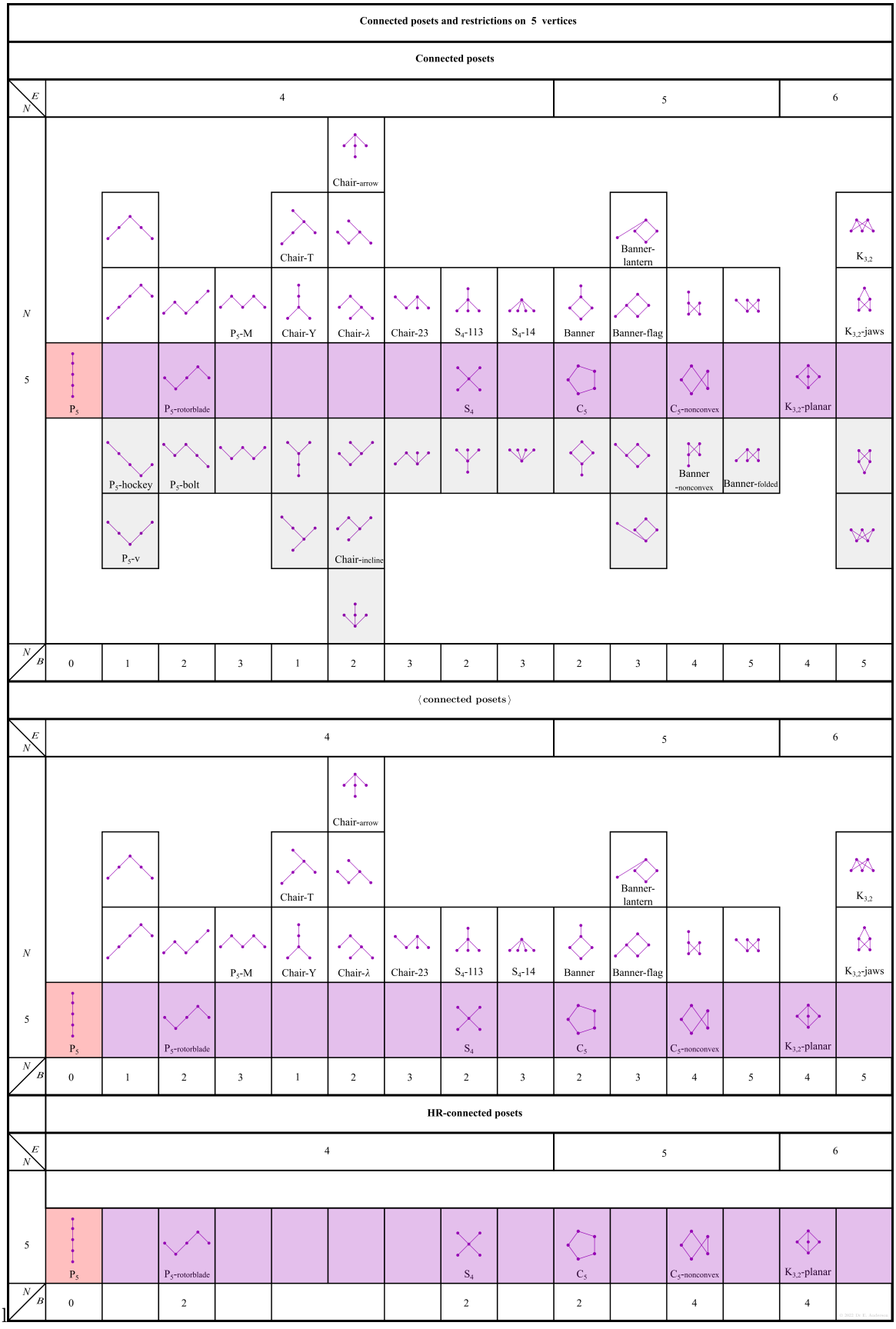

Connected posets and restrictions on 5 vertices
Connected posets
E
N
4
5
6
N
5
P5
P5-rotorblade
P5-M
Chair-T
Chair-arrow
Chair-Y
Chair-λ
Chair-23
S4-113
S4-14
Banner
Banner-flag
Banner-lantern
K3,2
K3,2-jaws
P5-hockey
P5-bolt
P5-v
Chair-incline
S4
C5
C5-nonconvex
Banner-nonconvex
Banner-folded
K3,2-planar
N
B
0 1 2 3 1 2 3 2 3 2 3 4 5 4 5
⟨connected posets⟩
HR-connected posets
0 2 2 2 4 4

Figure 49.4:

Chapter 50

Posets. II. Further structure

A. Topological structure

50.1 Placing a topology on posets

Remark 1 The topology on a graph (Sec 31.7) will not by itself serve for posets, due to now also needing to obey the ordering structure.

Structure 1 One approach to placing an order-respecting topology on posets is to take our topology's constituent open sets to be the *lower sets* $\mathfrak{L}$ alias *isotones*. I.e. sets $\mathfrak{L}$ such that if $u \in \mathfrak{p}$ then any $w \in \mathfrak{p}$ with $u \prec w$ is also in $\mathfrak{p}$.

Remark 2 This approach suffices for the case of finite posets that the current Series concentrates on.

Naming Remark 1 People drawing cover diagrams the other way up would call these *upper sets*.

Naming Remark 2 The ensuing topology is widely referred to as an *Alexandrov topology* [208]. Truer names for this are *open-set-intersectional topology* and (already widely used) *(finitely-generated topology)*. The first of these refers to the property that Alexandrov studied; the spaces thus arising were subsequently shown to be equivalent to posets [212]. The second of these refers to a later reformulation (though for us it is trivial that our *finite* posets have to be *finitely*-generated).

♠ ◇ ♣

Exercise 1 Show that down-sets are indeed intersectional.

[It suffices for you to do this for finite posets and thus for finite intersections.]

Worked Example 1 What are the down sets of the Banner-spear-downturned poset depicted in Fig 50.1.a)]? What is the ensuing topology?

Answer 1 Flowing down any poset, we can read off an isomorphic poset of down-sets. For our Banner-spear-downturned, this gives Fig 50.1.b).

We need to check unions; the down-sets are really a *base* for our topology. Which consists of them, their unions, and whichever of the whole set and the empty set if these were not to be already on our list; this gives one new union. Alongside Exercise 1), this guarantees that the type of family of subsets that we are considering is indeed a topology.

For Banner-spear-downturned, checking unions gives one new subset (column 7). The whole set is already on our list of subsets, but the empty set needs adding (column 8). We thus arrive at the family of subsets in columns 2 to 8, which constitute our desired topology.

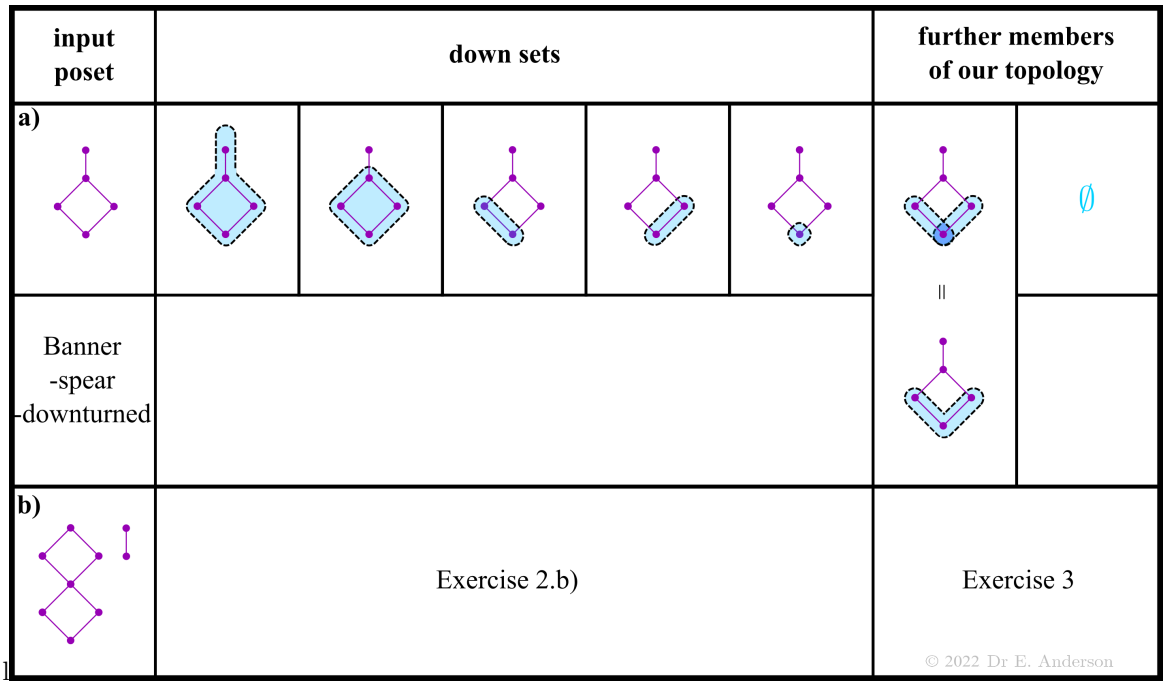


Figure 50.1:

Exercise 2 What does our notion of topology give on

a) P_3-v?

b) The poset in Fig 50.1.b)?

Exercise 3 Show that for P_3-v, the underlying ori-graph's in- and out-neighbourhoods can additionally serve as poset-level neighbourhoods. But that this is not the case for Banner-spear-downturned. Finally, for the poset in Fig 50.1.b), show that unions of down sets are needed so as to have in- and out-neighbourhoods about at least one point.

50.2 Connectivity for posets

Remark 1 We already explained that posets do not in any sense need to be connected, in Sec 49.6.

Remark 2 Most further considerations of notions of connectivity for posets follow from those for the underlying ori-graphs (Sec 35.8). The following result is thus really about *weak* connectivity.

Remark 3 Posets that are at least semi-bounded are connected. For the only starting point for the order is the top element, so there is a path from this to any other element. Thus ignoring ordering, there is a path between any two elements since there is a path from each of these to the top and then applying concatenation. This result does not in general extend to *strong* connectivity.

B. New posets from old

50.3 Subposets

Remark 1 This can be done in two particularly meaningful ways as follows.

Definition 1 A *subposet*

$$\mathfrak{S} \ = \ \langle \subset, \ \preceq_{\mathrm{W}} \ \rangle$$

of a poset $\mathfrak{P} \ = \ \langle \mathfrak{X}, \ \preceq \ \rangle$ is *weak* if the following hold.

i) $\mathfrak{S} \ \subseteq \ \mathfrak{P}$ as a set, i.e. $\subset \ \subseteq \ \mathfrak{X}$.

ii) $\mathfrak{S}$ carries a partial ordering $\preceq_{\mathrm{W}}$ with the following property

$$\forall \, u, v \ \in \ \mathfrak{S} \ , \ \ u \ \preceq_{\mathrm{W}} \ v \ \in \ \mathfrak{S} \ \ \Rightarrow \ \ u \ \preceq \ v \ \in \ \mathfrak{P} \ .$$

Definition 2 A *subposet* $\mathfrak{S} \ = \ \langle \mathfrak{A}, \ \preceq_{\mathrm{I}} \ \rangle$ of a poset $\mathfrak{P}$ is *induced* if i) and the following hold.

iii)

$$\forall \, u, v \ \in \ \mathfrak{S} \ , \ \ u \ \preceq_{\mathrm{I}} \ v \ \in \ \mathfrak{S} \ \Leftrightarrow \ u \ \preceq \ v \ \in \ \mathfrak{P} \ .$$

$\preceq_{\mathrm{I}}$ is then said to be the *induced order*, i.e. the order induced on $\mathfrak{S}$ by the underlying order $\preceq$ on $\mathfrak{P}$.

Remark 2 We warn the Reader that some texts refer to either of these notions unqualifiedly as subposets. We follow [49] in using 'subposet' to mean specifically the induced case.

Exercise 4 a) Find a minimal counterexample to weak subposets also being induced subposets.

b) How do refinements fit in as a subcase within the above analysis?

50.4 Example 1: Antichains

Definition 1 *Antichains* are subposets of mutually-incomparable elements. I.e. following [224] the elements $\{ \, u_i \, \}_{i \in \mathfrak{J}}$ of a poset constitute an antichain only if

$$u_j \ = \ u_k \ \ \forall \ \ j, k \ \in \ \mathfrak{J} \ .$$

Exercise 5$^-$ How can one equip any set to be an antichain?

Structure 1 Any chain can intersect any antichain at most once.

For suppose that there were two intersection points, u and v. Then by the definition of intersection, on the one hand, u and v belong to our chain. And chains are totally ordered. So u must be above v or below v. But on the other hand, since u and v must also belong to our antichain, they must be incomparable. Contradiction!

This observation provides a premise by which chains and antichains are a dual pair of concepts.

Naming Remark 3 This duality explains the name 'antichain'.

Remark 1 A given poset can be partitioned into a bunch of chains. It can also be dually partitioned into a bunch of antichains.

Exercise 6$^-$ Such partitions are not necessarily unique. a) Illustrate this nonuniqueness with the natural C_4 and C_5 posets.

b) How many ways can these be partitioned into i) chains and ii) antichains?

c) How many ways can the planar-graph realizations of $K_{3,2}$ and $K_{4,2}$ as posets be partitioned thus?

d) What is the smallest number of chains that the above four examples of posets can be partitioned into? What is the longest chain-length supported by the poset? What are the antichain counterparts of these two counts?

Remark 2 Antichains are clearly a subcase of independent sets for graphs. Namely the case in which the graphs carry the extra structure of a poset (i.e. are ori-graphs and with up-down arrow significance). They thus constitute a notion of separation.

50.5 Antichain cardinals for posets

Remark 1 Each antichain inherits a notion of cardinality from its underlying set, namely the count of how many point-elements the antichain consists of.

Structure 1 For a poset, we can set about listing all the antichains therein and finding the maximum cardinality among these. This gives what is conventionally termed the *width* of the poset,

$$width(\mathfrak{P}) \ . \tag{50.1}$$

Exercise 7$^-$ What is the longest possible antichain in any 5-poset? ... that is connected? ... and bounded?

♠ ◇ ♣

Remark 2 Width is the chain-antichain dual concept to height.

Remark 3 Width is furthermore the poset counterpart of Graph-Theoretic edge independence number, α_1.

Naming Remark 4 On the one hand, the names 'height' and 'width' fit both our up-down and the more commonly used down-up presentation of posets (Fig 50.2). On the other hand, we have subsequent reasons to prefer the notation

$$A(\mathfrak{P})$$

for the width of the poset. Where the capital A encodes that this is a maximum antichain notion.

Structure 2 A second poset antichain cardinal of note is

$$a(\mathfrak{P}) \ := \ (\text{ minimum number of chains that a poset can be partitioned into }) \ . \tag{50.2}$$

Naming Remark 5 We choose this lower-case a notation to signify minimum chain number; this is the chain-antichain dual of the poset chain cardinal c.

Exercise 8 a) What are A and a for each poset on $\leq$ 4 vertices?

b) Compare your answer to a) with that to Exercise 49.9 for the corresponding values of C and c.

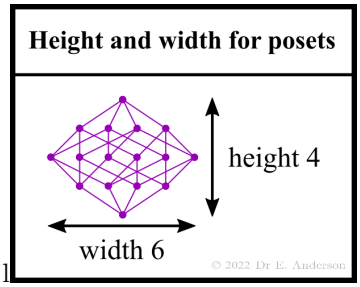


Figure 50.2:

Remark 4 When we introduced independence numbers α_i for graphs, we accompanied them with covering numbers β_i. We could also define such for posets.

What we find below, however, is that it is the quartet A, a, C, c that carry the new theorems that restrict specifically poset-level structure. So each independence number has been introduced alongside *what it partners with in new theorems at the corresponding level of mathematical structure*. Which partners *do not* match between the graph level of structure and the poset level of structure.

I.e. our conceptual analysis in the current section and in Chapter 37 has in each case been *technically* restricted to 'the beef' (i.e. what turns out to be mathematically interesting) rather than being an unrestrictedly-broad conceptual overview. The feature that each pairing *does* retain is that it is a *minimax pairing*.

50.6 Further relational variables for posets

Structure 1 Envisaging width as a type of independence number suggests introducing *width per unit order*,

$$\frac{A}{N} \tag{50.3}$$

for use in comparing posets. This now comes with a dual notion of *height per unit order*,

$$\frac{C}{N} \ . \tag{50.4}$$

Structure 2 Let us also now partner relative height between posets RC with *relative width* between posets,

$$RA \ := \ A(\mathfrak{P}) \ - \ A\left(\mathfrak{P}'\right) \ .$$

We furthermore introduce

$$Ra \ := \ a(\mathfrak{P}) \ - \ a\left(\mathfrak{P}'\right) \ .$$

Structure 3 Our two antichain cardinals furthermore support one independent relative variable

$$b \ = \ A \ - \ a$$

and one independent ratio variable

$$\mathcal{A} \ = \ \frac{a}{A} \ ,$$

each of which are single-poset notions.

Structure 4 We can furthermore contemplate mixed chain-antichain relative, ratio and relational quantities. In fact, all of the below analysis makes use of all 4 of the following ratios that complete

$\mathcal{A}$, and eq. (49.8)'s $\mathcal{C}$ set. Though at most 3 of the ratios in this set can be independent, as per Appendix A.

The *poset global aspect ratio* is

$$\mathcal{G} \ := \ \frac{C}{A} \ .$$

The *poset local aspect ratio* is

$$\mathcal{L} \ = \ \frac{c}{a} \ .$$

This is local in the sense of counting minimal numbers of disjoint pieces involved. The *maximally-mixed ratios* are

$$\mathcal{M} \ := \ \frac{c}{A} \tag{50.5}$$

and

$$\mathcal{D} \ := \ \frac{a}{C} \ . \tag{50.6}$$

These are maximally-mixed by involving one min and one max each, as well as one chain cardinal and one antichain cardinal. $\mathcal{M}$ is meant to stand for 'maximally-mixed' and $\mathcal{D}$ for 'dual-maximally-mixed'.

Structure 5 For later use, let us also introduce the relative difference variables

$$\Delta \ := \ C \ - \ A$$

and

$$\delta \ := \ c \ - \ a \ .$$

Structure 6 And two independent relational variables: the global

$$\mathcal{R} \ := \ \frac{\delta}{\Delta} \ = \ \frac{c \ - \ a}{C \ - \ A}$$

and the local

$$\mathcal{S} \ := \ \frac{d}{b} \ = \ \frac{C \ - \ c}{A \ - \ a} \ .$$

50.7 Dilworth's and Mirsky's Theorems

Theorem 1 (Dilworth 1950) For any poset $\mathfrak{p}$,

$$|\,\text{maximal antichain}(\mathfrak{p})\,| \ = \ (\ \text{minimal number of disjoint chains in } \ \mathfrak{p}\ ) \ . \tag{50.7}$$

Theorem 2 (Mirsky 1971) For any poset $\mathfrak{p}$,

$$|\,\text{maximal chain}(\mathfrak{p})\,| \ = \ (\ \text{minimal number of disjoint antichains in } \ \mathfrak{p}\ ) \ . \tag{50.8}$$

Remark 1 In Dilworth's Theorem, the left-hand-side is the width A of the poset $\mathfrak{p}$, while the right-hand-side is eq. 49.4's c. In Mirsky's Theorem, the left-hand-side is the height C of the poset $\mathfrak{p}$, while the right-hand-side is eq. 50.2's a. These two theorems can thus be recast as

$$A \ = \ c$$

and

$$C \ = \ a \ .$$

This symbolic form is brief. It emphasizes the chain-antichain duality between the two theorems. And it does not sacrifice displaying minimax information provided that one is aware that the upper and and lower cases retain this information.

Naming Remark 6 We have also by now undertaken enough conceptual exploration to rename these Theorems as the *Dual (Anti)chain Minimax Theorems.*

Proof of Mirsky's Theorem.

Induction hypothesis: $\exists$ a partition of $\mathfrak{P}$ into C pieces.

For $C = 1$, the poset is just a point. Then on the one hand, our point clearly supports a maximal chain consisting of just our point itself: of order 1. While on the other hand, the minimum number of disjoint antichains that the point supports is 1 as well. So we are in agreement.

Suppose that the Theorem holds for $C = K$. Let $\mathfrak{P}$ be any poset whose maximal chain contains $K + 1$ points. Then the set

$$\mathfrak{M} := \{ \text{maximal elements of } \mathfrak{P} \}$$

is an antichain of $\mathfrak{P}$.

Consider also the subposet $\mathfrak{P} - \mathfrak{M}$. This must be bereft of $(K + 1)$-point chains. So the maximal chain therein has to contain $\leq K$ points (Pigeonhole Principle!)

But suppose that a maximal chain in $\mathfrak{P} - \mathfrak{M}$ contains $< K$ points. And let $\mathfrak{D}$ be a maximal chain in $\mathfrak{P}$. Then $\mathfrak{M} \cap \mathfrak{D}$ contains ≥ 2 points. Contradiction!

Thus any maximum chain in $\mathfrak{P} - \mathfrak{M}$ must contain precisely K points. But then the induction hypothesis applies, so $\mathfrak{P} - \mathfrak{M}$ can be partitioned into K antichains. To extend this partition to our target $\mathfrak{P}$ itself, all that we need to do is add the antichain $\mathfrak{M}$ itself to the above partition of $\mathfrak{P} - \mathfrak{M}$. We thus have a partition into $K + 1$ antichains, thus establishing the Theorem by induction. $\square$.

Proof of Dilworth's Theorem [213].

We induct on the size of our set, $|\mathfrak{S}|$.

When this is zero, the Theorem clearly holds.

Next suppose that our Theorem holds for some $|\mathfrak{S}| \geq 1$, with $\mathfrak{C}$ being a maximal antichain therein. The following two cases then exhaust.

Case A) No antichain in $\mathfrak{S} - \mathfrak{C}$ contains m elements. Then, by our induction hypothesis,

$$\mathfrak{S} - \mathfrak{C} = \bigcup_{i=1}^{m-1} \mathfrak{C}_i \text{ chains } .$$

Thus, including $\mathfrak{C}$ itself, $\mathfrak{S}$ is a union of m chains, so we are done.

Case B)

$$\mathfrak{S} - \mathfrak{C} \supseteq \mathfrak{A} = \{a_1, ...a_m\} :$$

an antichain of length m. Let

$$\mathfrak{S}^- := \{ x \in \mathfrak{S} \,|\, x \preceq a_j \text{ for some } j \in [1, ..., m] \} .$$

And similarly, let,

$$\mathfrak{S}^+ := \{ x \in \mathfrak{S} \,|\, x \succeq a_k \text{ for some } k \in [1, ..., m] \} .$$

Also denote the maximal element of $\mathfrak{C}$ by c^+.

Suppose that $c^+ \in \mathfrak{S}^-$. Then $c^+ \prec a_j$, contradicting the maximality of $\mathfrak{C}$! Thus $c^+ \notin \mathfrak{S}^-$. Our induction hypothesis then gives that

$$\mathfrak{S}^- = \bigcup_{i=1}^{m} \mathfrak{D}_i^- \,. \tag{50.9}$$

The $\mathfrak{D}_i^-$ here being chains such that $a_i \in \mathfrak{D}_i^-$.

Suppose that $x \in \mathfrak{D}_i^-$. This means that $x \preceq a_l$ for some $l \in [1, \ldots, m]$. So if $a_i \prec x$, then $a_i \prec a_j$: contradiction!

Thus a_i is $\mathfrak{S}_i^-$'s maximal element.

Evoking $+ \leftrightarrow -$ exchange symmetry,

$$\mathfrak{S}^+ = \bigcup_{i=1}^{m} \mathfrak{D}_i^+ \,. \tag{50.10}$$

Here, $\mathfrak{D}_i^+$ is a chain whose minimal element is a_i. Next use that

$$\mathfrak{S} = \mathfrak{S}^- \cup \mathfrak{S}^+ = \left(\bigcup_{i=1}^{m} \mathfrak{D}_i^- \right) \cup \left(\bigcup_{i=1}^{m} \mathfrak{D}_i^+ \right) .$$

The second equality here follows by plugging in (50.9) and (50.10). Rearrange the order of this union of terms to obtain

$$\mathfrak{S} = \bigcup_{i=1}^{m} (\mathfrak{S}_i^- \cup \mathfrak{S}_i^+) \,.$$

Observing that this is a union of m chains, then we are done. □

Exercise 9$^+$ What happens if you try to 'dualize' the above proof of Mirsky's Theorem to establish Dilworth's Theorem (and vice versa)?

Exercise 10$^+$ Show that Dilworth's Theorem $\Rightarrow$ Hall's Pairs-Matching Theorem.

50.8 Ratio, relative and relational reformulations of the Dual Theorems

Ratio version Using Definitions (50.6) and (50.5), our dual Theorems to be recast in ratio form as[1]

$$\mathcal{D} = 1 \quad \text{(Dilworth)}$$

and

$$\mathcal{M} = 1 \quad \text{(Mirsky)} \,.$$

So two of our ratio quantities are in fact frozen to have fixed values of 1 by our dual Theorems. This leaves us with just one independent ratio. Indeed,

$$\frac{a}{C} = \mathcal{D} = 1 = \mathcal{M} = \frac{c}{A} \,, \tag{50.11}$$

[1] This gives a further reason to denote $\mathcal{M}$ and its dual $\mathcal{D}$ in that order. I.e. that these stand furthermore for *Mirksy ratio* and *Dilworth ratio* respectively.

whose first-to-last equality can be rewritten as

$$\frac{c}{a} \ = \ \frac{C}{A} \ .$$

I.e. that

$$\mathcal{G} \ \equiv \ \mathcal{L}$$

for posets. So by virtue of our Dual Theorems, our a priori distinct notions of global and local aspect ratios for posets are forced to coincide.

Relative version

$$\delta \ = \ -\Delta \ ,$$

$$d \ = \ -b \ .$$

This is by

$$\delta \ = \ c \ - \ a \ = \ A \ - \ C \ = \ -\Delta$$

and

$$d \ = \ C \ - \ c \ = \ a \ - \ A \ = \ -b \ .$$

So by (50.8), global difference equals local difference for posets, up to sign. While by (50.8), the antichain difference equals the chain difference up to sign.
There are thus no mismatches in extensivity between our four chain-or-antichain quantities.

Relational version

$$\mathcal{R} \ = \ -1 \ = \ \mathcal{S} \ .$$

This is by

$$\mathcal{R} \ = \ \frac{\delta}{\Delta} \ = \ -1 \ = \ \frac{d}{b} \ = \ \mathcal{S} \ .$$

Remark 1 So our 4 chain-antichain minimax cardinals provide only 1 independent ratio (or relative) quantity for posets We celebrate by henceforth dropping the local and global prefixes to talk unqualifiedly about *poset aspect ratio*, and *poset aspect difference*. While relational aspect *variables* are not further talked about at all for posets, these having been rigidly fixed to constant value by our Dual Theorems.

Remark 2 This rigidity motivates considering further cardinals so as to be able to discern between (and attempt to classify) posets, toward which Sec 49.11 provides a small start.

50.9 Example 2. Poset middles

Definition 1 The *middle* of a bounded poset consists of all of its elements other than its top and its bottom, alongside all arrows between these other elements.

Remark 1 The middle of a poset is in one sense nontrivial if it is not a chain.

Example 1 The square poset is additionally minimal for being bounded while having a nontrivial poset middle. The square poset's middle is however an antichain.

Exercise 11 A more advanced duality-invariant criterion of nontriviality is for the middle to be neither a chain nor an antichain. What is the smallest bounded poset whose middle is each of the following?

a) Neither a chain nor an antichain.

b) Not a disjoint union of chains and of antichains.

c) Neither a (possibly bent) path, nor a union thereof.



50.10 Sums of posets

Remark 1 The viability of disjoint unions of posets rests upon posets not necessarily needing to be connected. I.e. it is in this way that disjoint unions of posets manage to be posets themselves. One thus just returns to the multi-component posets.

Exercise 12 The *vertical sum* [224] of a downwards-bounded poset and an upwards-bounded poset is the result of linking the first to the second via a bottom-to-top down-arrow.

The *linear sum* [224] of two posets is the result of joining all the minimal elements of the first with all the maximal elements of the second. This is a democratically-natural joining condition.

a) Are these operations commutative?

b) When can vertical sums of three or more posets be defined? In this setting, is the vertical sum associative? Is the linear sum associative?

c) Which posets on 5 vertices cannot be expressed as vertical sums?

d) What is the smallest depth $\geq$ 4 connected poset that cannot be expressed as either of these sums of 2 connected nontrivial (exclude chain pieces) posets? Likewise but for 3 such pieces?

50.11 Products of posets

Definition 1 [49, 224] A *Cartesian product* $\square$ can be placed on posets by making their product sets into a poset with the following ordering. $\underline{u} \preceq \underline{w}$ iff

$$(\text{every vector component } u_i) \leq (\text{ the corresponding } w_i) .$$

Remark 1 This can be envisaged as placing a poset order on the Cartesian product $\square$ of their underlying graph skeletons.



Exercise 13 Is this commutative? Associative?

Example 0

$$\mathfrak{p} \square \mathrm{D}_1 = \mathfrak{p} .$$

Example 1 $\mathrm{P}_2 \square \mathrm{P}_n = \downarrow \text{Ladder}(n)$, starting with P_2, $\downarrow \mathrm{C}_4$ and Fig 50.3,a)'s $\downarrow$ Domino poset.

Example 2

$$\mathrm{P}_3 \square \downarrow \mathrm{C}_4 = \text{Boxstack}(2) ,$$

as per Fig 50.3.b).

Example 3 $\downarrow C_4 \ \square \ \downarrow C_4$ is as per Fig 50.3.c).

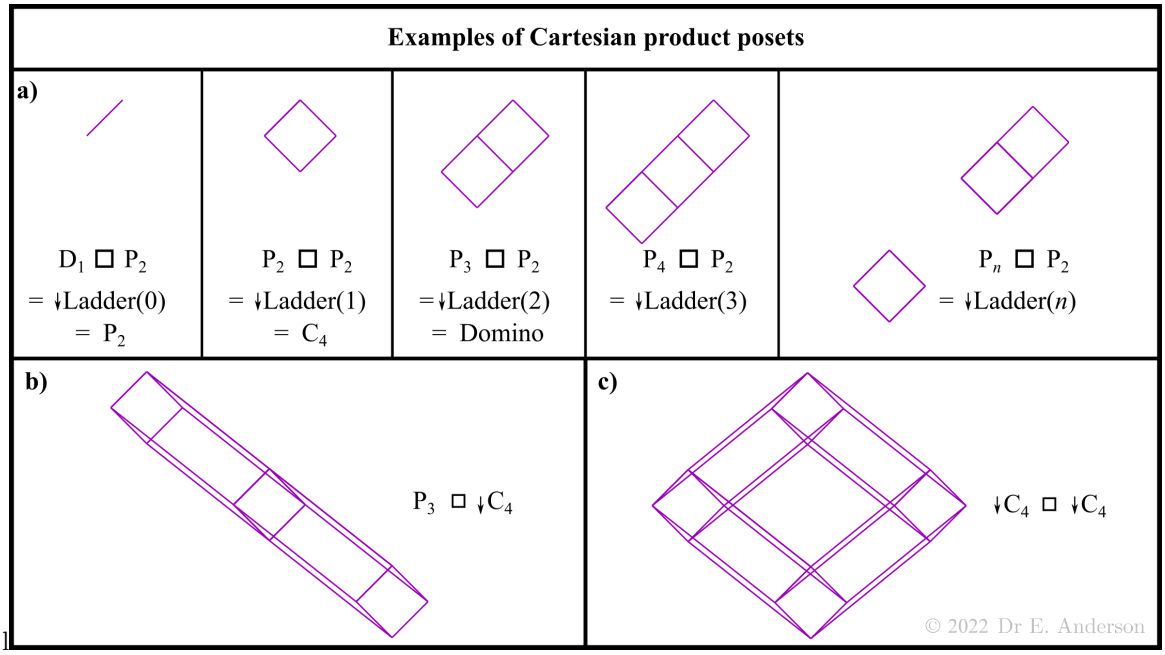


Figure 50.3:

Exercise 14 What are the following posets? P_3-v $\square$ P_2, P_3-v $\square$ P_3, P_3-v $\square$ P_3-v, P_4-N $\square$ P_4-N and C_4-nonconvex $\square$ P^4-nonconvex.

Example 4 See Sec 50.13 for

$$\square_{i=1}^{n} P_2 \ .$$

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Exercise 15 Lexicographic ordering gives another natural way $\circ_{\mathrm{L}}^{m}$ of forming products of posets. What are $P_m \ \circ_{\mathrm{L}} \ P_n$, $P_3 \ \circ_{\mathrm{L}} \ \downarrow C_4$ and C_4-nonconvex $\circ_{\mathrm{L}}$ C_4-nonconvex? with itself?

Question 1 Can a product poset be placed on Sec 34.9's direct product $\times$ of the underlying graph skeletons?

Remark 2 At this point in this Series, we have no systematic comment to make about quotients of posets. See Part VIII for some examples.

C. Examples of families of posets

50.12 Some simple families of posets

Example 1 The *fence family* Fence(n) [224] is depicted in Fig 50.4.a). This is built by maximally bending the P_n graphs. For $n = 2$ and above, the n-odd fences are heteroreversing while the n-even fences are homoreversing. These are equivalently the $\Delta B = 1$ and $\Delta B = 0$ fences respectively. So bending index B is realized precisely as vertex-number parity here. Also $B = N - 2$.

Example 2 The *crown family* Crown [224] is depicted in Fig 50.4.b). This is built by maximally bending the $C_{2\,n}$ graphs. These crowns form a purely homoreversing family. It is unrelated to the graph-level use of the name 'crown' in Sec 34.9. $\Delta B \ = \ 0$ throughout, while bending number itself $B \ = \ N$ and excess bending number $E \ = \ N \ - \ 2$ for $N \ \geq \ 2$.

Example 3 The *(square) bunting poset family* are the $(\,3\,n \ + \ 1\,)$-element posets depicted in Fig 50.4.c). [2] These can be characterized by cycle number C, and more accurately as

$$+_{\mathrm{L}\ i\ =\ 1}^{\ n}\,\mathrm{P}_3\text{-v-upturned}\ .$$

The second row gives each square bunting poset's underlying graph skeleton's homeomorph prime. In this sense, our first new poset is not Graph-Theoretically new to us, by reducing to just Bowtie. We thus present one further poset, so as to obtain a new homeomorph prime.

Example 4 The *candelabra poset family* is illustrated in Fig 50.4.d). These are of order $4\,n \ + \ 2$. They are characterized by cycle number C once again, or more accurately now as

$$+_{\mathrm{V}\ i\ =\ 1}^{\ n}\,\mathrm{C}_4\ .$$

Each's bridge number

$$Bri \ = \ C \ - \ 1\ .$$

The first new poset here is once again not Graph-Theoretically new to us at the level of its homeomorph prime, returning just Bridge. Thus once again we draw out one further member of this series.

Exercise 16 We can re-run Examples 3 and 4 again for other cycles and for nonconvex poset presentations of cycles. This gives e.g. Fig 50.4.e) as an option for polymerizing uncrossed pentagons.

a) Show that this is the only such option that is a vertical sum

b) Is any polymer of nonconvex C_4's by single bonds (Fig 50.4.e) a vertical sum?

c) What do we get if we re-run Example 4 for $K_{3,2}$ in place of C_4? [Consider single bonds only! Your answer should include which of these are vertical or linear sums.]

d) What are the smallest new homeomorph primes produced among a) to c)'s underlying graph skeletons?

50.13 The power set, free or hypercube posets

Structure 1 Sets of subsets of a fixed finite set $\mathfrak{X}$ form posets under the subset ordering $\subseteq$.

A) In the unlabelled case, these posets are just chains as totally-ordered by the subsets' cardinality.

B) In the labelled case, the set of subsets is the power set. For an N-element subset, this takes the form of the hypercube $\downarrow \mathrm{Hypercube}_N$ as decorated with down-flowing arrows.

Example 1 $\downarrow \mathrm{Hypercube}_1$ is just the line segment (1-cube) P_2, the poset decoration of which is just $\vec{\mathrm{P}}_2$.

[2] This is the smallest bunting polymer – a type of cyclic polymer – for posets. For posets do not support triangles, so the graph-level triangular bunting polymer of Sec 50.12 cannot be adorned with a partial ordering.

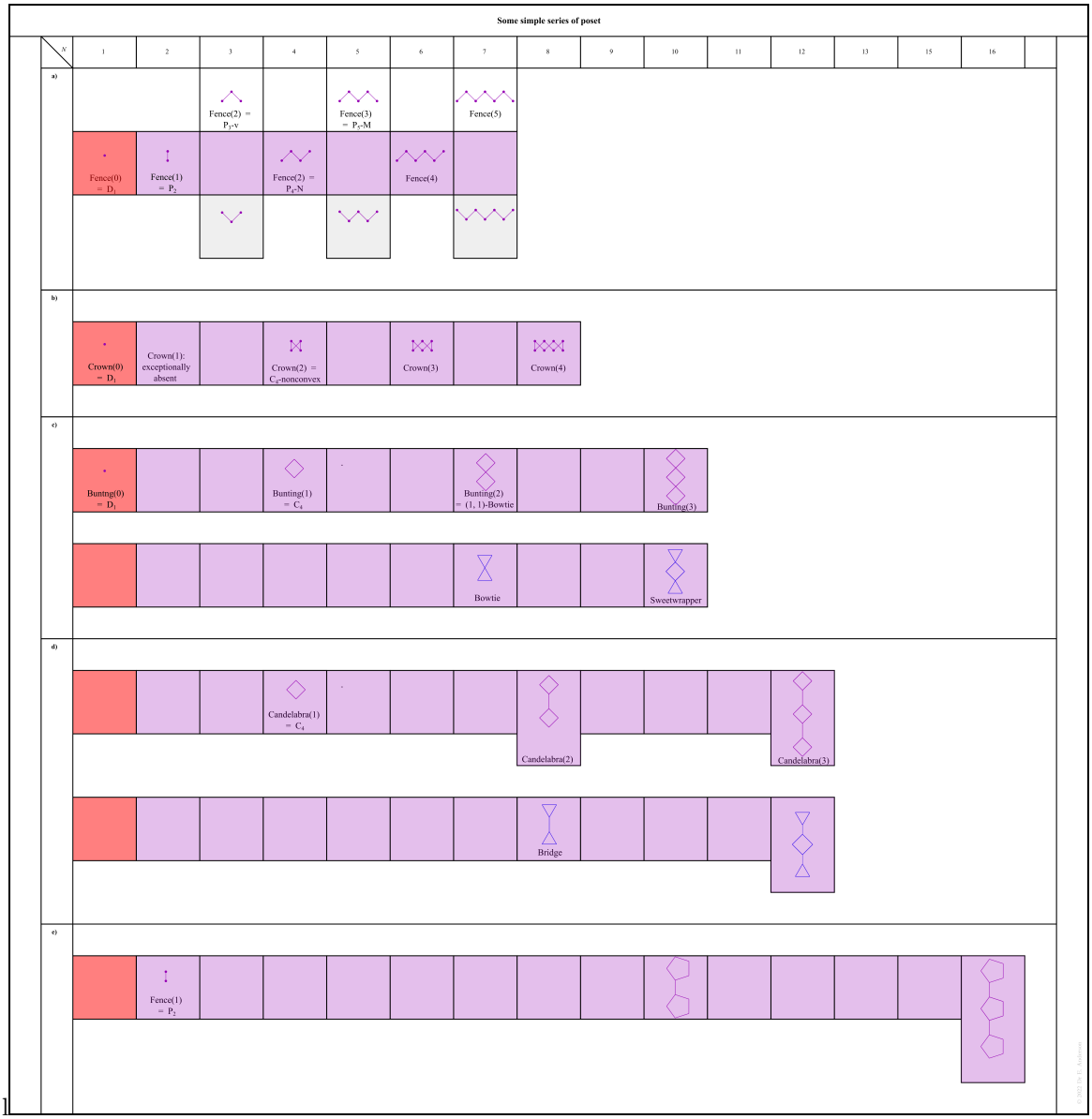


Figure 50.4:

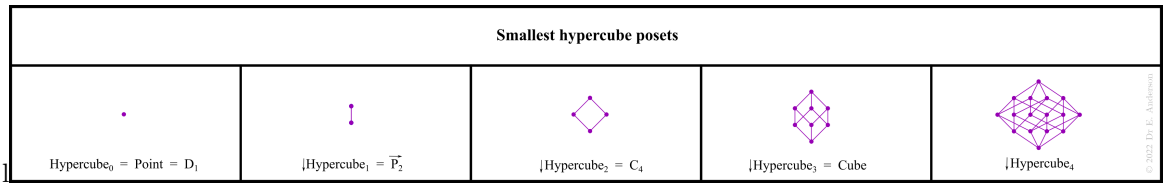


Figure 50.5:

Example 2 ↓ Hypercube_2 is the natural square poset ↓ C_4 (2-cube). For which we choose the natural 121 (rather than crossed 22) poset presentation (Fig 50.5.b), whose underlying ori-graph is source-sink.

Example 3 ↓ Hypercube_3 is the usual (3-)cube, for which we select the 1331 down-flowing poset

presentation (Fig 50.5.c).

Example 4 See additionally Fig 50.5.d) for $\downarrow \text{Hypercube}_4$.

Remark 1 We can see that each's second-highest antichain has one further element than the preceding, corresponding to being able to select one further distinctly-labelled element. The third antichain has $\begin{pmatrix} N \\ 2 \end{pmatrix}$ elements corresponding to the number of ways of choosing two distinctly-labelled elements. Thus

$$\text{P}_N \ = \ \text{Hypercube}_N \ \text{ in down-flowing-arrowed}$$

$$\begin{pmatrix} N \\ 0 \end{pmatrix} \ + \ \begin{pmatrix} N \\ 1 \end{pmatrix} \ + \ \ ... \ \begin{pmatrix} N \\ N \end{pmatrix} \ = \ \text{Pascal's triangle's } N\text{th row} \ \text{ poset presentation .}$$

Naming Remark 7 Having complete freedom in which combinations of elements we pick here is one reason to also call these *free posets*, which we denote by F_N. This also carries some connotations of *Algebraic freeness*, which generally means that a structure is all generators and no relations. More specifically for the current example, we know that $\mathfrak{P}(N)$ is also the commutative product group ${C_2}^N$. This does have the C_2 relation

$$g_i{}^2 \ = \ 1$$

for each generator g_i. But it has *no further* relations, in particular meaning that there are *no mutual* relations between generators.

Remark 2 After this, it becomes increasingly hard to follow drawings by eye, out of the arrowing being quite dense between middling-level layers. One way forward is to store poset information Linear-Algebraically, rather (see Volume 1 [441]).

Remark 3 Other books [49] use the notation B_N

50.14 Posets of divisors

Example 0 For primes, this just returns the 2-chain (Fig 50.6.a)

Example 1 For prime powers p^n, we get the $(n \ + \ 1)$-chain (Fig 50.6.b).

Example 2 For numbers of the form $p\,q$, such as $6 \ = \ 2 \ \times \ 3$, we get the natural square poset $\downarrow \text{C}_4$ of Fig 50.6.c).

Example 3 For numbers of the form $p^n\,q^m$, we get the natural *grid poset*. I.e. the downwards-arrows-adorned fully-uncrossed tilted presentation $\downarrow \text{Grid}(n, m)$ of the grid graph
$\text{Grid}(n, m)$. We can furthermore view this as a product of Example 1's chains (Fig 50.6.d).

Example 4 For numbers of the form $p\,q\,r$, such as $30 \ = \ 2 \ \times \ 3 \ \times \ 5$, we get the natural cube poset $\downarrow$ Cube of Fig 50.6.e).

Example 5 $p^n\,q^m\,r^l$ gives Fig 50.6.f)'s natural 3-d grid poset $\downarrow \text{Grid}(n, m, l)$.

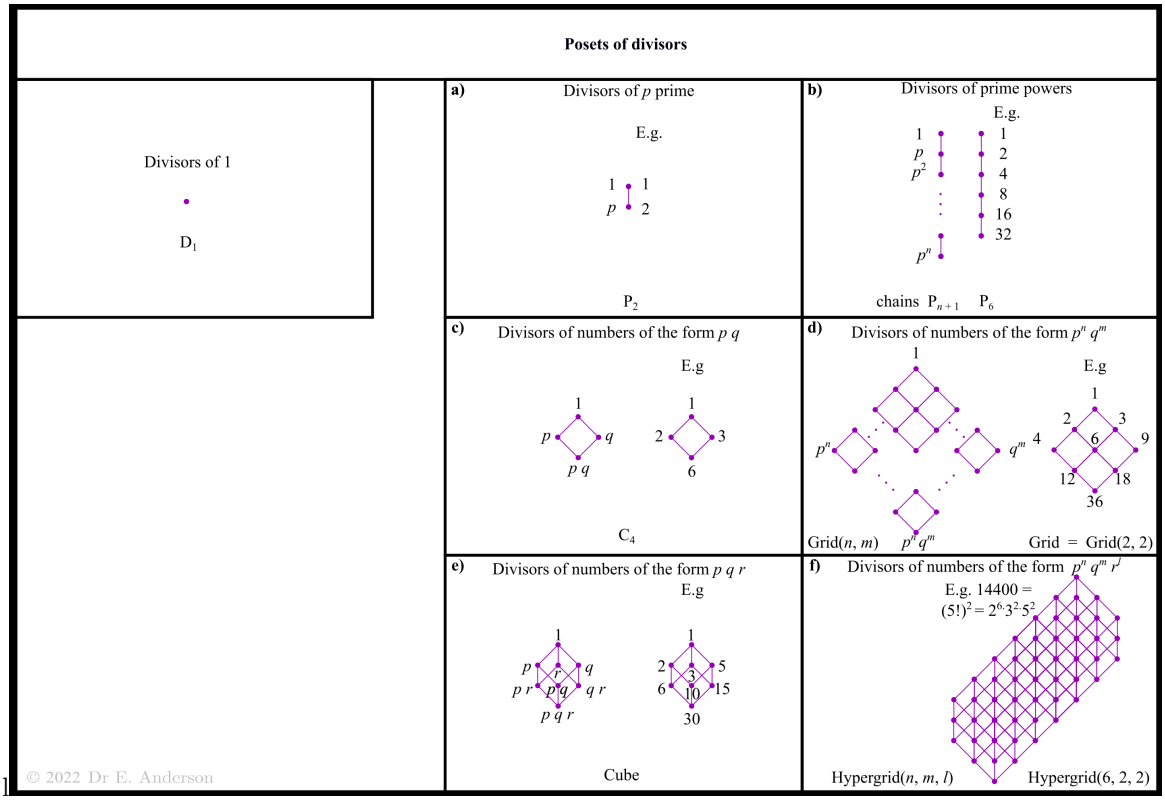


Figure 50.6:

Example 6

$$\prod_{i=1}^{n} p_i$$

gives the natural *hypercube poset*, $\downarrow \text{Hypercube}(n)$.

Structure 1 What of the general case? Here one has

$$\prod_{i=1}^{n} {p_i}^{a_i} \ .$$

This gives the natural *n-d hypergrid poset* $\downarrow \text{Hypergrid}\,(\underline{b})$. This has sidelengths of

$$b_i \ := \ a_i \ - \ 1$$

edges in the ith direction.

50.15 Quadrangulations and Quadriculations

Structure 1 *Quadrangulations* are the simplest remaining possibility once triangulations are banned, which they are for instance in graph skeletons that support posets. They describe a large subclass of posets, which, as we shall see in Chapter 52, is additionally a mathematically-meaningful restriction.

Example 1 = Naming Remark 8 A *trellis* is either a) a periodically-infinite single face-fused squared poset that is symmetric about its midline. Or a growing family of approximands b) that

gradually sweeps out a such. These are determined by a recurring sequence of level lengths. And by which finite approximand they are in case b). See 50.7.a) for small examples.

Remark 1 These are additionally *quadriculations*, meaning 4-regular quadrangulations, which can be described using the gridlines of an independent pair of coordinates.

Example 2 Linelike quadriculations, such as *kinked ladders*. These can be linelike in the sense of what they look like from afar. Or more restrictedly in the sense of being limited in the breadth of its levels by the underlying graph skeleton needing to be outer-planar. See 50.7.b) for small examples of this which are also periodic or approximands to a growing periodic structure.

Example 3 Any Young diagram in 45 degree presentation is also placing a poset interpretation on a quadriculation. Though from a poset point of view, we do not need to adhere to this nonincreasing row lengths restriction (Fig 50.7.c).

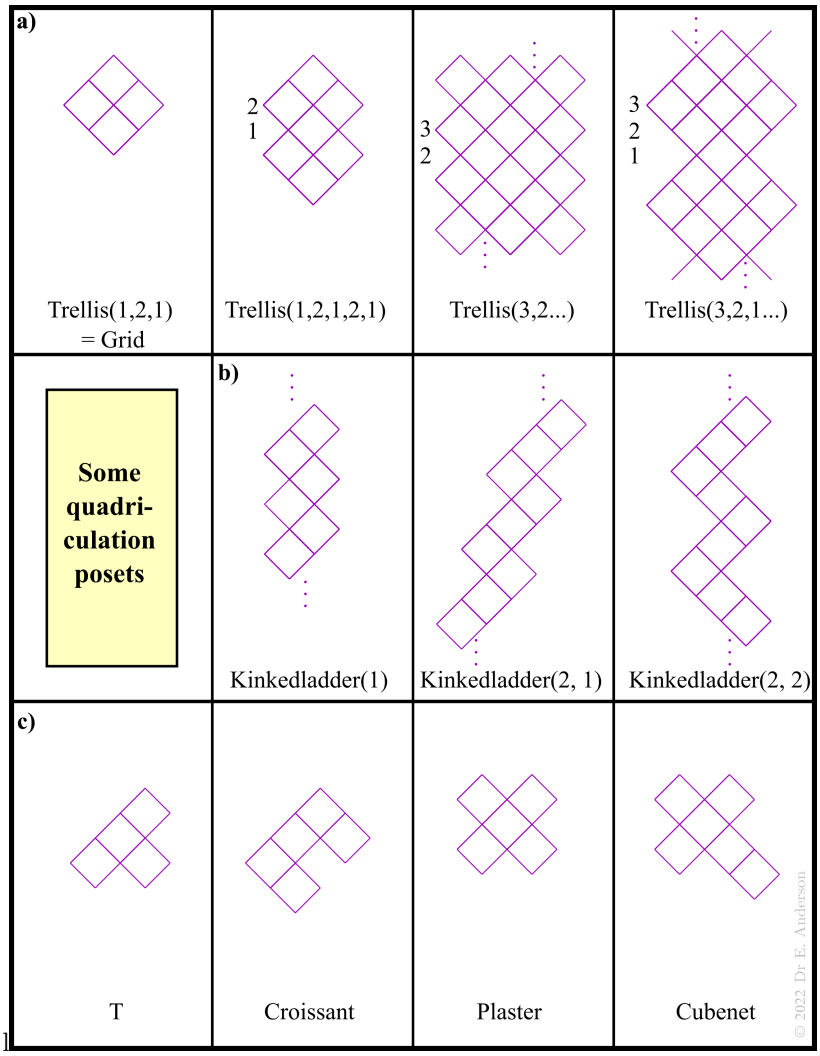


Figure 50.7:

Example 4 See Fiq 50.8.a) for a poset that is a quadrangulation but not a quadriculation.

Pointer 1 In Computer Graphics, using quadrangulation – alias 'quad-modelling' in this context – has quite some future as an alternative to triangulation, as follows. This allows for stable and reasonably continuous representation of curves across a surface. This extends to representing mesh gridlines thus; in fairly flat local regions, quadriculation does well. But elsewhere and globally, more general quadrangulation can be useful Quadrangulation also enables modelling, encodement and direct visualization with moving or deformable surfaces. (Such as animations or video-level visualizations of phenomena [231]).

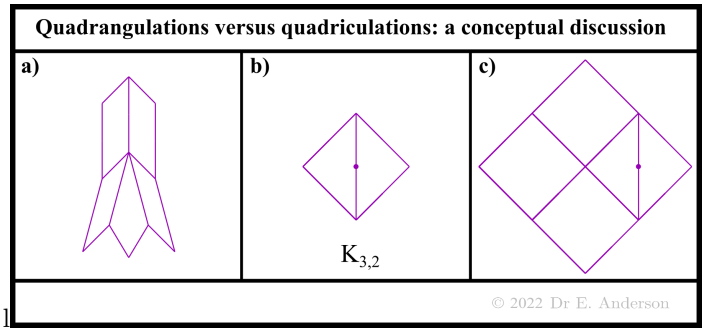


Figure 50.8:

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Structure 2 For some practical purposes, one would exclude quadrilaterals that share two adjacent faces (Fig 50.8.b-c). Doing this is of middling strength between quadriculation and fully-general quadrangulation. Let us call this regime $K_{3,2}$-free quadrangulation.

50.16 Motivation for posets

Motivation 1 These first two Chapters' lists of examples are all of broad, basic and widely applicable. Set Theory, Number Theory and Applied Topology are visibly dancing therein.

Motivation 2 Further large examples lists have been postponed until we have introduced further subcases. In particular, each of the following nested subcases of posets carry further examples lists in the next two Chapters.

$$\text{Posets} \supset \text{Lattices} \supset \text{Modular Lattices} \supset \text{Distributive Lattices} . \tag{50.12}$$

Motivation 3 We shall furthermore show in Part VIII that many basic mathematical objects' arenas are posets or the above three nested specializations thereof. This often proceeds via the following Exercise.

Exercise 17 Let $\mathfrak{A}$ be an operation between mathematical objects consisting of forming one object from the other by adding some fixed kind of piece.

a) Check directly that $\mathfrak{A}$ cannot produce triangles.

b) From first principles (i.e. checking each axiom), show that $\mathfrak{A}$ induces a poset ordering on the mathematical objects.

[Examples of what we shall be adding are a subset, a family member, an element or vertex, or a graph edge.]

Motivation 4-6) Three further applications of note that have not been mentioned yet are as follows.

In Computer Science, *denotational semantics* [220] can be usefully formulated in terms of posets. *Face relations* [219] – used in layer-by-layer building up of complexes layer by layer for Topolgical modelling – also have a poset structure. As does *stratification* [274, 310, 311]: another type of layering in advanced Differential Geometry, in particular of quotient spaces (see [332] for a pre-stratificational introduction) and dynamical systems applications thereof [313]. The first string of references here are all introductory – explaining what stratification is; for a poset formulation thereof, see e.g. [230].

End-Remark 1 If you would like to know more about posets, consult in particular [49, 224, 44, 216].

Chapter 51

Lattices

51.1 Definitions

Definition 1 A *lattice* $\mathfrak{L}$ [224, 49, 207, 206, 210, 226, 211] is a poset for which each pair of elements a, b possesses both of the following.

i) A *join* (least upper bound), denoted

$$a \ \wedge \ b \ .$$

ii) A *meet* (greatest lower bound), denoted

$$a \ \vee \ b \ .$$

Definition 2 If join alone is present, one has an *upper semi-lattice*. If meet alone is present, one has a *lower semi-lattice*.

Naming Remark 1 The above uses of 'upper' are upper in order but lower in the diagram in our downwards-flowing convention.

Remark 1 A lattice is thus a lower semi-lattice that is also an upper semi-lattice.

Remark 2 We caution the Reader that these Order-Theoretic lattices can be defined in multiple equivalent ways. See in particular Davey and Priestley's *Introduction to Lattices and Order* [224] for more about ways of defining an Order-Theoretic lattice.

51.2 Some already-familiar examples of join and meet operations

Three such pairs are as follows.

Example 1 In the basic study of numbers,

$$\text{lowest common multiple} \ \ \text{lcm} \ = \ \wedge$$

and

$$\text{highest common factor} \ \ \text{hcf} \ = \ \vee \ .$$

Example 2 In basic Logic's, Electronics' and Computer Science's truth tables,

$$\text{OR} \ = \ \wedge$$

and

$$\text{AND} = \vee \ .$$

Example 3 In considering subsets of sets,

$$\cup \ = \ \wedge$$

and

$$\cap \ = \ \vee \ .$$

Notational Remark 1 Indeed, the symbols for join, $\wedge$, and meet, $\vee$, are chosen to quite closely resemble those for union, $\bigcup$ and intersection, $\bigcap$ in particular bending in the same direction as them. Compare also how posets' $\preceq$ looks quite a lot like the set-theoretic $\subseteq$ as well as like the usual inequality $\leq$ on totally-ordered sets such as $\mathbb{N}$. In both cases, the slightly modified shape indicates 'generalized concept': helpfully chosen to have approximately the same shape as some of its most widely-known subcases.

51.3 Dual lattices

Definition 1 The *dual* of a given lattice is another lattice in which the roles of $\vee$ and $\wedge$ are interchanged. The corresponding covering diagrams are upside-down versions of each other. If the arrows are furthermore reversed, the *antitone dual* is realized.

Duality Principle for Lattices $\vee$ and $\wedge$ are dual operations.

$$\text{Upper} \ \leftrightarrow \ \text{lower} \ ,$$

$$\text{last} \ \leftrightarrow \ \text{first} \ ,$$

$$\text{top} \ \leftrightarrow \ \text{bottom} \ .$$

Exercise 1 What is the meaning of this duality for each of our three examples of join-and-meet pairs above?

Definition 2 A lattice is *homodual* if it is equal to its own dual. Lattices which are not homocomplementary form *heterocomplementary* pairs.

51.4 Lattices as algebras

Useful Lemma 1 (called *Connecting Lemma* in [224]). For l, m any two elements of a lattice, the following are equivalent

i)

$$l \ \preceq \ m \ .$$

ii)

$$l \ \vee \ m \ = \ l \ .$$

iii)

$$l \ \wedge \ m \ = \ m \ .$$

Proof Out of l and m, i) gives that l is the upper element and m the lower element. ii) and iii) thereby follow.

If ii) holds, then l is an upper bound for l and m. So l lies above m, which is what i)'s $l \preceq m$ statement means. While if iii) holds, then m is a lower bound for l and m. So m lies below m, which is i)'s $l \preceq m$ statement read backwards. □

Proposition 1 $\wedge$ and $\vee$ form an algebra. More specifically,

a) each of these operations is commutative, associative, and idempotent (defined in Sec 6.2).

b) They also obey the *absorption conditions*

$$a \vee (a \wedge b) = a \text{ and } a \wedge (a \vee b) = a \,. \tag{51.1}$$

Proof By lattice duality, we need only establish half of these conditions.

a) Commutativity is trivial since $\wedge$ is symmetric.

Associativity: Suppose that $u \in$ the upper set of a, b and c, $\sup(a, b, c)$. This can be rephrased split up into the two conditions $u \in \sup(a, b)$ while also $c \preceq u$. But the first of these means that $a \wedge b \preceq u$. So we can furthermore rephrase our condition as $u \in \sup(a \wedge b, c)$. These rephrasings are furthermore reversible. Thus

$$\sup(a, b, c) = \sup(a \wedge b, c) \,.$$

So

$$(a \wedge b) \wedge c = \sup(a, b, c) \,.$$

The right-hand-side here being manifestly brackets-independent,[1]

$$a \wedge (b \wedge c) = \sup(a, b, c)$$

must also hold. So equating the previous two expressions,

$$(a \wedge b) \wedge c = \sup(a, b, c) = a \wedge (b \wedge c) \,.$$

Whence omitting our sneaky manifestly-brackets-independent intermediary term, the associativity law for joins is established.

Idempotency: set $a = b$ in iii) of Useful Lemma 1.

b) Absorption: The following two cases exhaust.

Case A) If $b \preceq a$, the following's first equality holds,

$$a \wedge (a \vee b) = a \wedge a = a$$

the second following from idempotency.

Case B) If instead $a \preceq b$, the following's first equality holds.

$$a \wedge (a \vee b) = a \wedge b = a \,,$$

The second equality here follows from iii) of the Useful Lemma applying in Case B). □

Exercise 2 Prove the other half of these results from first principles, then obtaining the above half of the results by duality.

[1] E.g. [224] proceeds by establishing this manifest formulation.

Exercise 3 Show that $u \preceq v \quad \Rightarrow$

$$u \vee w \preceq v \vee w \,,$$

$$u \wedge w \preceq v \wedge w$$

for any u, v, w collectively belonging to any lattice $\mathfrak{L}$. I.e. that we can 'meet multiply' and 'join multiply' any of the poset's ordering-relation generalizations of inequalities.

51.5 Unit and zero elements of a lattice

Definition 1 An element 1 of $\mathfrak{L}$ is a *unit element* if the following holds.

$$\forall \; l \in \mathfrak{L} \,, \;\; l \preceq 1 \,. \tag{51.2}$$

An element 0 of $\mathfrak{L}$ is a *null* alias *zero element* if the following

$$\forall \; l \in \mathfrak{L} \,, \;\; 0 \preceq l \,. \tag{51.3}$$

Exercise 4$^-$ Show that these need not exist for all lattices. Also show that when they do exist, then each of 0 and 1 are unique within a given lattice.

Definition 2 A lattice that possesses both of these is termed a *bounded lattice*.

Remark 1 Finite lattices must be bounded (showing this is part of Exercise 10 below).

51.6 Lattice morphisms

Definition 1 A *lattice morphism* is a map between lattices,

$$\phi \;:\; \mathfrak{L} \longrightarrow \mathfrak{L}' \,. \tag{51.4}$$

Definition 2 A *lattice isomorphism* is an order-, join- and meet-preserving bijection between lattices,

$$\psi \;:\; \mathfrak{L} \longrightarrow \mathfrak{L}' \,. \tag{51.5}$$

Definition 3 An *antitone lattice isomorphism* is an order-reversing, join- and meet-swapping bijection between lattices,

$$\alpha \;:\; \mathfrak{L} \longrightarrow \mathfrak{L}' \,. \tag{51.6}$$

Exercise 5$^-$ Find one poset morphism between lattices of each of the following kinds.

a) It is additionally a lattice morphism.

b) It fails to preserve $\vee$ while respecting $\wedge$.

c) Vice versa.

d) It fails to preserve both $\vee$ and $\wedge$.



51.7 Narrowing down from posets to lattices

Corollary 1 For finite lattices, the underlying posets must have a single terminal element at either end.

Corollary 2 A lattice's underlying graph skeleton must be connected.

Corollary 3 We are thus restricted to within the connected, triangle-free graphs.

Remark 1 Proving these is also left until Exercise 10.

Remark 2 We can quickly read off Figs 49.2 and 49.4 which graph skeletons are not forbidden from supporting connected posets. We then consider the posets supported by each of these graph skeletons, removing those that do not have a unique top and a unique bottom.

Remark 3 This does not exhaust removals of posets that fail to meet the lattice axioms. For instance, we have the following restriction.

Exercise 6 a) Show that the C_4-nonconvex poset (Fig 51.1) is not a lattice.

b) Show furthermore that posets containing this as a subposet are not lattices.

[This is a result which we use repeatedly in Part VIII.]

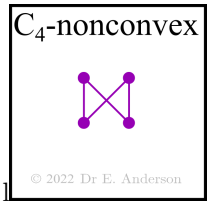


Figure 51.1:

51.8 Relational variable for lattices

Remark 1 Since lattices have to be (weakly-)connected, the minimal edge number for lattices is the *tree value*,

$$E_{\text{min}} \;=\; E_{\text{tree}} \;=\; N \;-\; 1 \;. \tag{51.7}$$

This bound on E is furthermore always saturated. This is since all finite chains are not only trees but also lattices when decorated with a down-path of arrows.

Structure 1 A convenient relational variable for discussing lattices then turns out to be the 'super-treeness'

$$\overline{T} \;:=\; E \;-\; E_{\text{tree}} \;=\; E \;-\; N \;+\; 1 \;=\; 1 \;-\; \chi_1 \;. \tag{51.8}$$

51.9 Smallest examples of lattices

Structure 1 = Naming Remark 2 In terms of N and $\overline{T}$, we now present the lattices on $N \;=\; 0$ to 6 points in Fig 51.2.

Exercise 7$^+$ a) Check that there are no more lattices on 0 to 6 vertices.

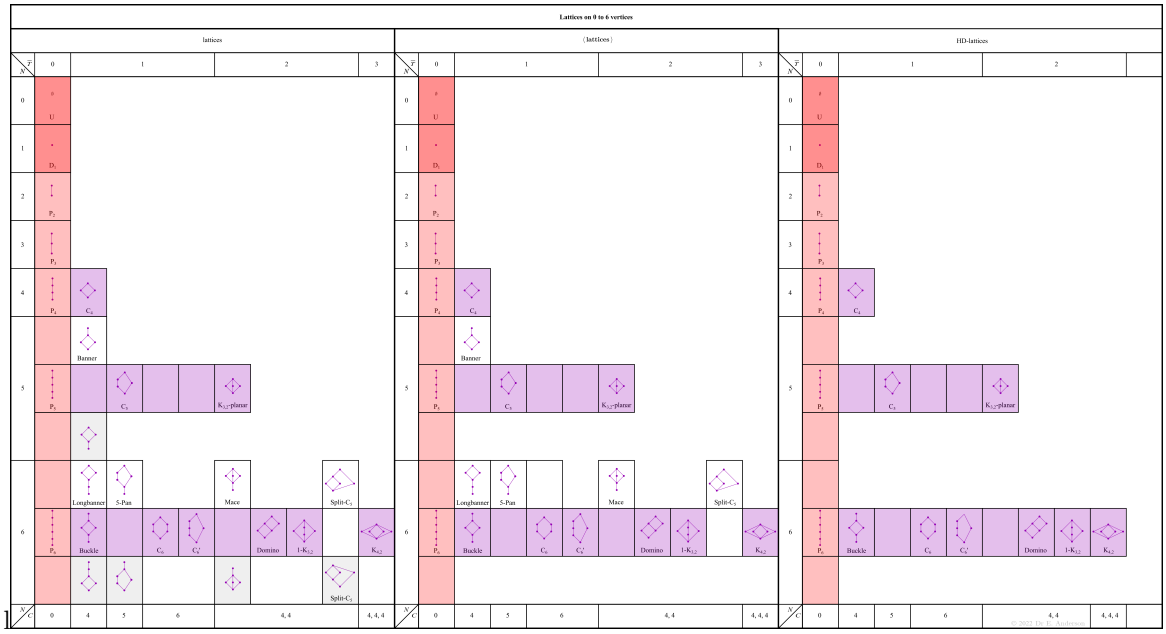


Figure 51.2:

b) Via considering which 5-graphs (Fig 30.3) and then which 5-posets (Fig 49.4) support them, find all the semi-lattices on 0 to 5 vertices

51.10 Further examples of lattices

Exercise 8 For N finite, show that the hypercube posets $\downarrow \text{Hypercube}_N$ alias free posets F_N, and thus the power set posets $\mathfrak{P}(N)$ that realize these, are a fortiori lattices. What are the meet and join operations here? What are the top and bottom elements?

Exercise 9 Show that the m-dimensional grid posets, and thus the divisor posets that realize these, are also a fortiori lattices. What are the meet and join operations here? What are the top and bottom elements?

Pointer 1 In Chapter 58, we show furthermore that the subgroups of a group form a lattice. As do the normal subgroups of a group. And the quotient groups, in the sense of the quotients of the group itself by all of its normal subgroups. Finally, given a space that a group acts upon, that space's quotient by each subgroup of the group also forms a lattice.

Exercise 10 a) Why must the subsets of a finite set form a *bounded* lattice? Why must the divisors of a finite set?

b) Prove that any finite lattice must be bounded.

c) Deduce Corollaries 1 to 3 of Sec 51.7.

d) Give also two examples of unbounded lattice and two examples of semi-lattices that are semi-bounded (i.e. bounded at just one end).

Exercise 11 Let $\mathfrak{L}$ be a lattice in which every subset possesses a sup and an inf. Then $\mathfrak{L}$ is termed *complete*.

a) Show that every finite lattice is complete.

b) Show that every power set $\mathfrak{P}(\mathfrak{X})$ is complete, even if its underlying set $\mathfrak{X}$ is not.

c) Give an example of a lattice which is not complete.

Exercise 12 A lattice $\mathfrak{L}$ is *complemented* if for each element $a \in \mathfrak{L}$, there is another element $b \in \mathfrak{L}$ such that the following hold.

$$a \vee b = 0$$

and

$$a \wedge b = 1 .$$

a) Are power set lattices complemented?

b) Are lattices of divisors complemented?

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Exercise 13 A *sublattice* $\mathfrak{K}$ of a lattice $\mathfrak{L}$ is a nonempty subset of $\mathfrak{L}$'s underlying set such that the following holds.

$$k_1, k_2 \in \mathfrak{K} \Rightarrow k_1 \wedge k_2 , \quad k_1 \vee k_2 \in \mathfrak{K} .$$

a) Show that any sublattice is itself a lattice.

b) Find a minimal counterexample to all subsets of a lattice being sublattices.

c) Also find a minimal counterexample to subsets of a lattice that are not sublattices not being lattices.

Exercise 14 a) Which of $\amalg$, and the previous Chapter's $+_V$ and $+_L$ operations on posets preserve lattice structure?

b) Which preserve semi-lattice structure?

c) Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be lattices. Define the *product lattice* [224] on the underlying product set $\mathfrak{L}_1 \times \mathfrak{L}_2$ by introducing the following operations. The *product join* is to be

$$(l_{11}, l_{21}) \wedge l_{21}, l_{22}) = (l_{11} \wedge l_{21}, l_{21} \wedge l_{22}) .$$

And the *product meet* is to be

$$(l_{11}, l_{21}) \wedge l_{21}, l_{22}) = (l_{11} \wedge l_{21}, l_{21} \wedge l_{22}) .$$

Show that product lattices are indeed lattices.

Chapter 52

Introduction to Lattice Theory

We consider here various further useful adjectives for lattices.

A. Modular Lattices

52.1 Introduction to modular lattices

Definition 1 A lattice $\mathfrak{L}$ is *modular* if it satisfies the following condition.

$$\forall\, a, b, c \in \mathfrak{L}\,, \quad \text{if } c \preceq a \text{ then } i = f\,. \tag{52.1}$$

s and e are here two further elements obtained as triple products of our incipient elements a, b, and c. 'Triple products' is meant here in the sense of involving a meet $\vee$ and a join $\wedge$, as follows. The *initial bracketing*

$$i := (a \vee b) \wedge c\,.$$

The *final bracketing*

$$f := a \vee (b \wedge c)\,.$$

Remark 1 Modularity is thus a 5-element condition: on a, b, c, i and f.

Example 0 All lattices on ≤ 4 elements are therefore modular.

Example 1 The pentagonal lattice C_5 is not modular. For, representing it as in Fig 52.1, firstly

$$c \preceq a\,. \tag{52.2}$$

Secondly, however,

$$i := (a \vee b) \wedge c = 0 \wedge c = c \preceq a = a \wedge 1 = a \vee (b \wedge c) =: f\,.$$

We have here used the definitions of: i in the first step, join in the second, and meet in the third. Eq. (52.2) in the central step. And then the definitions of: join in the antepenultimate step, meet in the penultimate step and f in the final step.

52.2 Diagnostic for modular lattices

Remark 1 Below, we show furthermore that, firstly, this C_5 lattice is the unique minimal counterexample to lattices being modular. Secondly, that this is the unique forbidden sublattice present in any other non-modular lattice. I.e. that we have the following Theorem and Corollary.

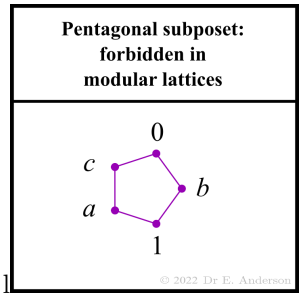


Figure 52.1:

Theorem 1 A lattice is modular iff it does not contain the C_5 lattice as a sublattice.

Remark 2 The above is how this Theorem is usually presented. C_5 only admits two posets, however, and the other – C_5-nonconvex – fails to even be a lattice. So this Theorem can be rephrased at the level of a forbidden subgraph of underlying graph skeleton, as follows.

Theorem 1′ A lattice is modular iff its graph skeleton contains no C_5 subgraphs.

Corollary 1 The only cycles supported by modular lattices are 4-cycles.

Proof of Corollary 1. Cycles of size > 5 would contain 5-cycle sublattices, which are forbidden. Also triangles are prohibited in posets by Proposition 1 of Chapter 51. This leaves just 4-cycles standing. □

52.3 The smallest modular lattices

Structure 1 Using the above Proposition to cut down on our table of smallest lattices, we are left with the following table of modular lattices on 0 to 6 elements.

Remark 1 For $N = 6$ itself, comparing with Fig 51.2, we have lost the following.

a) The 5-Pans: the smallest heteroreversing pair of nonmodular lattices.

b) The two lattices supported by C_6.

c) 1-$K_{3,2}$: the first homeomorph of $K_{3,2}$.

d) split-C_5.

Exercise 1 The Reader should identify at least one pentagon in each of these. And also be satisfied that there are no pentagons anywhere in Fig 52.2 itself.

52.4 Other simple examples of modular lattices

Exercise 2 Show that the hypercube lattices alias free lattices, and thus the power set lattices that realize these are furthermore modular.

Exercise 3 Show that the m-dimensional grid lattices, and thus the divisor lattices that realize these are furthermore modular.

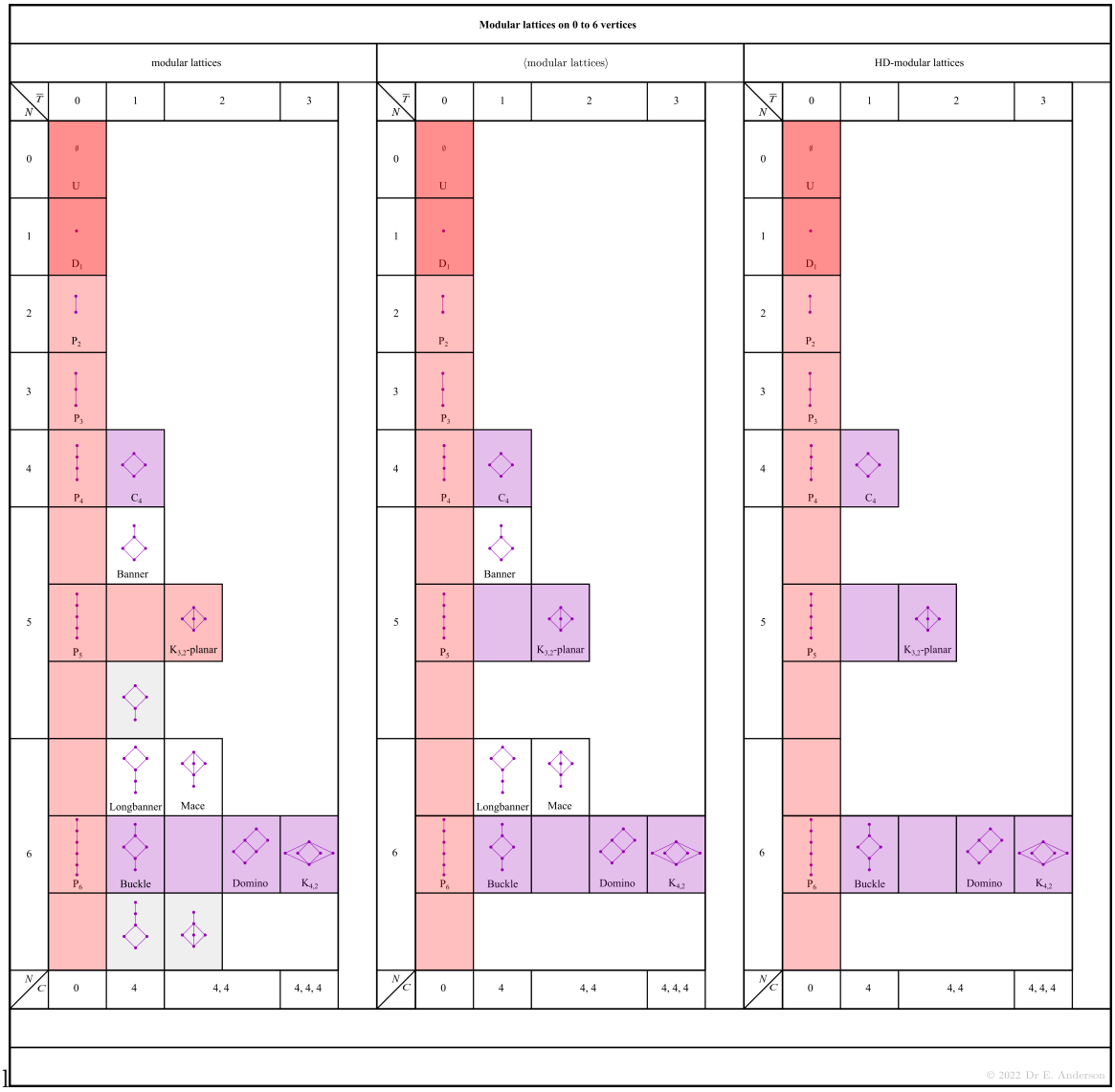


Figure 52.2:

Exercise 4 Find at least three further separate infinite families of modular lattices.

Pointer 2 See 58 for further examples of modular lattices, of certain subgroups of groups.

Naming Remark 1 = Pointer 3 The above use of 'modular' exhibits some alignment with that of the modules of Abstract Algebra [288]. Namely, that the submodules of a module form a modular lattice under the $+$ and $\cap$ operations.

52.5 Proof of the C_5 diagnostic for modularity

A) We have shown that C_5 is not itself modular.

B) From $C_5 \leq L$ (C_5 nonmodular) $\Rightarrow$ $\mathfrak{L}$ (not modular) follows as the contrapositive.

C) Finally assume that $\mathfrak{L}$ is nonmodular. Then it must contain a quintet of elements a, b, c, i and f such that $c \prec a$ and $i \prec f$, with i and f defined as before.

Claim: Suppose that

$$b \vee i = b \vee f$$

and

$$b \wedge i = b \wedge f .$$

Then our quintet forms a C_5. So (nonmodular $\mathfrak{L}$) $\Rightarrow$ (C_5-containing $\mathfrak{L}$). Thus applying the contrapositive, (C_5-free $\mathfrak{L}$) $\Rightarrow$ (modular $\mathfrak{L}$).

It remains to prove our claim.

1)
$i \wedge b = ((a \vee b) \wedge c) \wedge b = (a \vee (b \wedge c)) \wedge b = c \wedge (a \vee b) \wedge b) = c \wedge b .$

Here the first step uses the definition of f, the second commutativity, the third associativity and the fourth absorption.

2)
$f \vee b = (a \vee (b \wedge c) \vee b = a \vee ((b \wedge c) \vee b) = a \vee (b \vee (b \wedge c) = a \vee b .$

Now the first step uses the definition of i, the second associativity, the third commutativity and the fourth absorption.

3)
$a \vee b = (a \vee b) \vee b \preceq ((a \vee b) \vee c) \vee b = i \vee b \preceq f \vee b = a \vee b .$

Our ordering step is powered by Exercise 51.3. Our first equality is just a trivial insertion paralleling that of any set being the intersection with itself, or more generally of an idempotent picking up an extra power. We do this so as to insert a meet with c between the two meets with b, so as to peel off our second equality by the definition of i. Our last equality is just 1).

4)
$b \wedge c = i \wedge c \preceq f \wedge c = (a \vee (b \wedge c)) \wedge c \preceq c \wedge b \wedge c = b \wedge c .$

Our inequalities here are powered likewise; we go through 4)'s equalities backwards so as to maximally align this and 3)'s working (why does this work?) The last equality is another trivial insertion, now of a power of c and followed by use of commutativity to send it to the other end of the string. Doing this now permits us to use the definition of f in the penultimate equality. Finally, the antepenultimate equality is just 2). □

B. Distributive Lattices

52.6 Introduction to distributive lattices

Definition 1 A lattice is *distributive* if it satisfies the following condition.

$$\forall\, a, b, c \in L , \quad \text{if } c \preceq a \text{ then } f = d . \tag{52.3}$$

f is here the same triple product of our incipient elements a, b, c as before. The *distributive bracketing* is a new quadruple product thereof,

$$d := (a \vee b) \wedge (a \vee c) .$$

Remark 1 Distributivity is thus also a 5-element condition: on a, b, c, i and d.

Remark 2 Recollect that, given a pair of operations O_1 and O_2, there are two distributivity questions: whether O_1 is distributive over O_2 and vice versa. In the case of lattices' meet and join operations, however, the other of these is just the dual of the above:

$$d^* := (a \wedge b) \vee (a \wedge c) .$$

Duality then saves us some work in studying distributivity for lattices.

Example 0 All lattices on ≤ 4 elements are therefore distributive.

Counterexample 0 All nonmodular lattices are not distributive either. This is since

$$i = (a \wedge b) \vee c \preceq (a \vee c) \wedge (b \vee c) = d . \tag{52.4}$$

Thus

$$f = d \succeq i ,$$

where the first step is the distributivity condition and the second step is (52.4). Which reads overall

$$f \succeq i .$$

This then includes as its saturated case the modularity condition

$$f = i .$$

It remains to establish (52.4). This is straightforward. $\forall\, p, q \in \mathfrak{L}$,

$$p \vee q \preceq p , q \tag{52.5}$$

from the definition of $\vee$. And

$$p , q \preceq p \wedge q \tag{52.6}$$

from the definition of $\wedge$.

So firstly,

$$a \vee c , b \vee c \preceq c \tag{52.7}$$

as special cases of (52.5). Secondly,

$$a \vee c \preceq a \preceq a \wedge b , \tag{52.8}$$

by concatenating (52.5) and (52.6). Then finally

$$(a \wedge b) \vee c \succeq (a \wedge b) \vee (b \vee c)(a \vee c) \vee (b \vee c) .$$

The first step here uses (52.7) while the second uses (52.8).

Example 1 Examples in which $\vee$ and $\wedge$ are represented by $\bigcap$ and $\bigcup$ of set elements are distributive.

Exercise 5 Prove this.

52.7 Graph skeleton diagnostic

Theorem 2′ A modular lattice is a fortiori distributive iff it contains no $K_{3,2}$ subgraphs. .

Remark 1 While there are 5 ways of realizing $K_{3,2}$ as a poset (Fig 49.4), the two heteroreversing pairs are not even lattices due to containing C_4-nonconvex. Thus, on the one hand, all that we need to show is that the remaining poset $K_{3,2}$-planar cannot occur. On the other hand, this means that the result can be viewed as being about $K_{3,2}$ being a forbidden subgraph in the underlying graph skeleton. [The result is usually stated, rather, as $K_{3,2}$-planar being a forbidden sublattice.]

Remark 2 This violates the distributive condition by the following argument alongside hereditarity.

For, representing it as in Fig 52.3, the following holds.

$$f := a \vee (b \wedge c) = a \vee 1 = a \neq 0 =$$

$$(a \vee b) \wedge (a \vee b) =: d .$$

Here, the second step reads off the join from the Figure, and the third likewise for its meet. The fourth can also be read off the Figure, while the fifth is an obvious property of the bottom element. Finally, the sixth's two meets can then be read off the figure as well.

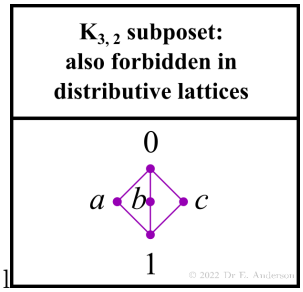


Figure 52.3:

Remark 3 C_5-and-$K_{3,2}$-free lattices are distributive.

Corollary 1 Distributive lattices can only contain 4-cycles, which moreover cannot share 2 contiguous edges.

Remark 4 Much like the proof of the Modular Theorem – phrased in triple products – needs quadruple products, the proof of the Distributive Theroem – phrased in up to quadruple products – necessitates up to hexuple products. So we postpone its algebraic manipulations until the Readers are a bit more experienced with Algebra. We thus include this proof instead as a Worked Example and supporting Exercises in our Volume 1 on Linear Algebra, after various more familiar triple- to hexuple-product identities for vectors have been played out.

52.8 The smallest distributive lattices

Structure 1 We now apply the above restriction to our table of $N = 0$ to 6 modular lattices to obtain Fig 52.4.

Remark 1 For $N = 6$ itself, comparing with Fig 52.2, we have now also lost the following.

a) The Maces: the smallest heteroreversing pair of modular and yet non-distributive lattices.

b) $K_{4,2}$.

Exercise 6 The Reader should identify at least one $K_{3,2}$ in each of these. And also be satisfied that there is no $K_{3,2}$ lurking anywhere in Fig 52.4 itself.

52.9 Other simple examples of distributive lattices

Exercise 7$^-$ Show that the hypercube lattices alias free lattices, and thus the power set lattices that realize these are additionally distributive.

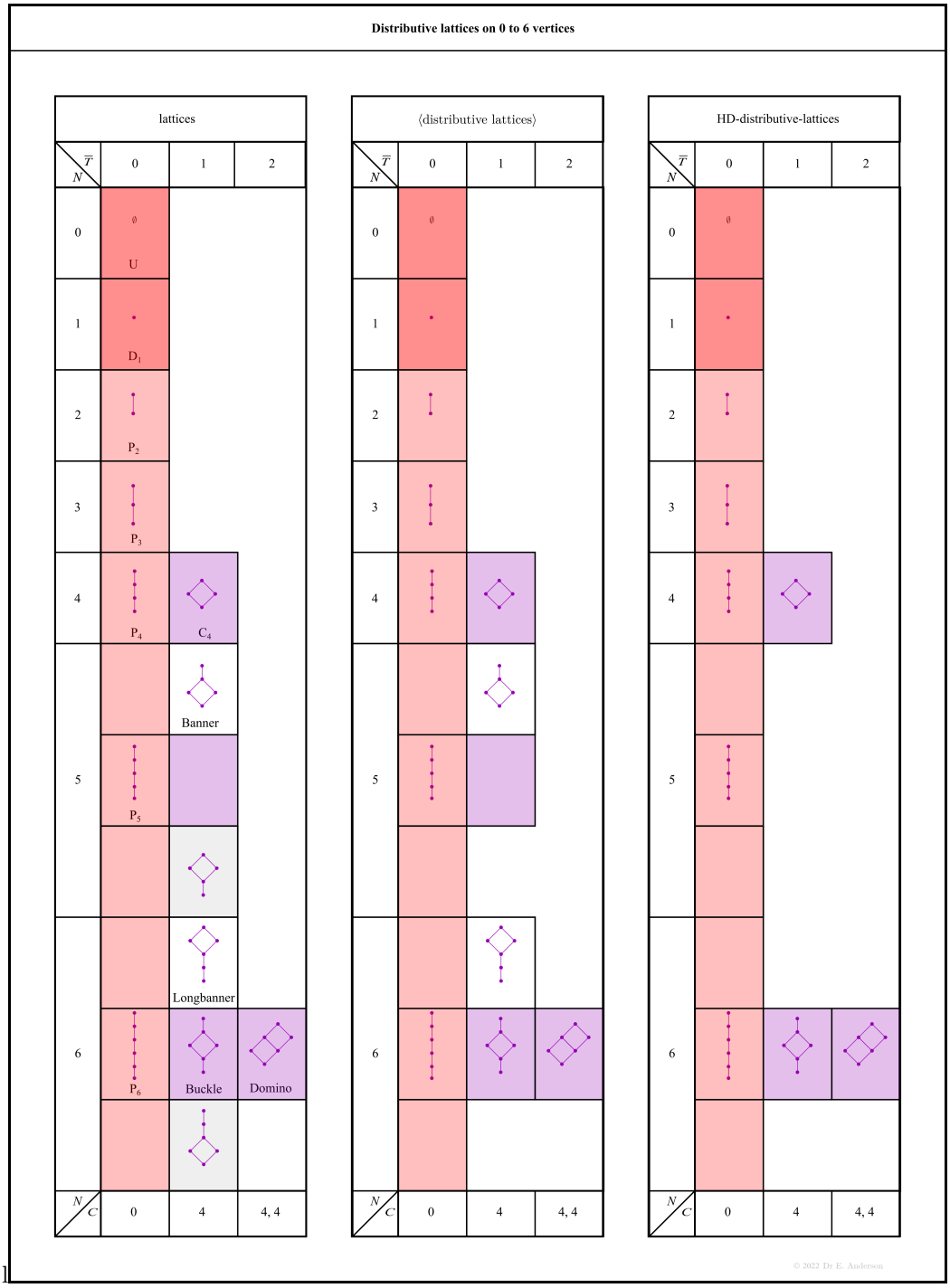


Figure 52.4:

Exercise 8^{-} Show that the m-dimensional grid lattices, and thus the divisor lattices that realize these are also distributive. This includes trellis posets being a fortioiri examples of distributive lattices.

Exercise 9 Extend Exercise's 4's answer to at least three separate infinite families of modular lattices, three of distributive and yet not modular lattices and three of non-distributive lattices.

Pointer 4 Lattices of normal subgroups are modular but not necessarily distributive (Exercise 58.4).

52.10 Heat map of types of lattice within the connected posets

Let us end Part VII by presenting this in Fig 52.5.

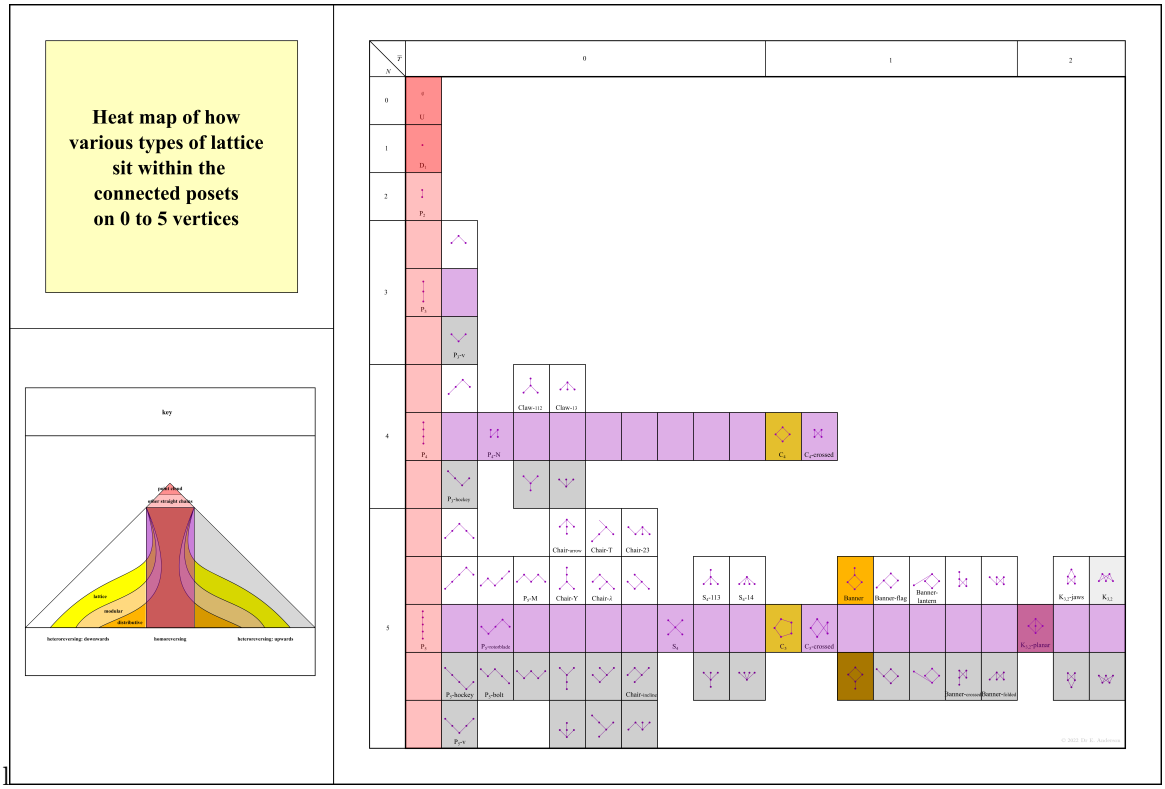


Figure 52.5:

52.11 On the word 'lattice', and other meanings for it

Naming Remark 2 'Lattice' has come to mean an increasingly sophisticated construct made out of pieces of wood. It has an Indo-European stem 'slat'. Meaning a log or beam: long and yet also comparably thick to the tree trunk that it came from.

Which became the Germanic 'latto' for a plank. I.e. still a long piece of wood that has however now been sawn down to being considerably thinner. Which in turn became the Frankish 'latta' for a lath. This being a smaller, thinner piece of wood, a series of many of which would be used as a base layer in plastering walls.

Some designs for which start to look like some of the mathematical objects studied in these last two Chapters, for instance our trellis example. Indeed becoming known by the similarly-structured French word 'lattis', without deviating from 'latta' in meaning. Compare also our trellis with the lattice pastry shape familiar from many pies; these are in particular modular examples of lattices.

Naming Remark 3 The word 'lattice' is furthermore used *somewhat* differently from the technical definition that we gave above, in some texts on Order Theory. Finally, such Order Theory lattices are not to be confused with crystalline lattices, lattices of physical atoms and so on, which is an entirely distinct *Geometrical* concept.



52.12 Further Exercises

Exercise 10 a) In a lattice, an *atom* is an element that covers 0. What are these for the power set? For the lattice of divisors? For a finite chain? For the semi-infinite chain $\mathbb{N}$? For the infinite chain of integers, $\mathbb{Z}$?

b) *Co-atoms* are the lattice-dual notion to atoms. Define co-atoms explicitly. What are these for each of the above examples?

c) A lattice is *atomic* if every element therein it has some atom above it. Is Young's lattice of partitions 57.1 atomic? Coatomic? What about the lattice of topologies (Fig 56.5)?

d) A lattice is *atomistic* if every element therein is a join of atoms. Show that power sets are atomistic, but that some lattices of divisors are while others are not.

e) Compare this notion of atom with the notions in Remark 1 of Sec 44.3.

Exercise 11 Prove that a lattice that is both atomic and complemented is also co-atomic.

Exercise 12 A lattice that is bounded, distributive and complemented is said to be *Stanley* or to constitute a *Boolean Algebra* [229, 201].

a) Show that the classical logic gates form a Boolean Algebra.

b) Show that the power set on a finite set constitutes a Boolean Algebra.

c) Which of the following are Boolean? The lattice of divisors of 6, of 36 and of 60, and Young's lattice of partitions.

Exercise 13$^+$ In which ways do Boolean algebras manifest duality? Explore first with what Boolean algebras arise from the previous Exercise.

52.13 Pointers and End-Remarks

Pointer 5 One can quite readily imagine truth tables of differing form, much like group times tables can differ from (modular) arithmetic times tables. Indeed, one can set up a truth table for any Boolean Algebra. Modifications of Boolean algebras include *Heyting algebras* [228, 205]: the minimal amount of structure to support an implication operation and thus at least an 'intuitionistic' logic.

Pointer 6 Such lines of thinking leads to substantial generalizations of logic, and of some subjects whose everyday cases lean on truth tables, (more Computer Science than Electronics, and yet also even Philosophy!) Other lines also including *multi-valued logics* and *context-dependent* notions of truth. Amazingly, some advanced theories of logic manage to be unintended unifications with other

branches of mathematics, most notably forming *Topos Theory*. The leading mathematician of the 20th century, Alexander Grothendieck, introduced this name, deriving from the same $\tau o \pi o \varsigma$ – Greek for 'place' or 'location' that Topology's name derives from. This is however too complicated for either of our Series to entertain. The world's main introductory textbook on this subject that is widely approachable by advanced undergraduates in STEM is [225].

A simpler application of lattices is to *domains* [224]. These are a type of lattice that model information systems. More specifically, these are a tracking system for which information tokens do and do not carry mutually-consistent information. This is largely a Computer-Science specific application. For instance, the previously-mentioned posets of Denotational Semantics arise as maps between domains.

End-Remark 1 Order Theory originated in the works of Boole [229], Schröder [202] and Dedekind [203] in the middle and late 19th Century. In Birkoff's and von Neumann's capable hands, it became a coherent subject in the late 1930s [207, 206, 210], with Birkhoff's book on it titled *Lattice Theory*.

End-Remark 2 See in particular [224] for more about the fascinating subject of Lattice Theory!

End-Remark 3 We have thus formed the following nest of structural specializations over Parts V to VII.

$$\text{Arrowgraphs} \supset \text{Ori-graphs} \supset \text{Triangle-free Ori-graphs} \supset \text{Posets}$$

$$\supset \text{Connected posets} \supset \text{Lattices} \supset \text{Modular Lattices} \supset \text{Distributive Lattices} . \tag{52.9}$$

This turns out to be quite a useful sequence of notions to look out for in studying the arenas formed by basic mathematical objects of Set Theory, Finite-point-set Topology, Combinatorics, Finite Group Theory, Graph Theory and Order Theory. Establishing this with copious examples is the raison d'être for Part VIII below.

Part VIII

Arenas

Chapter 53

Subset arenas

In this last Part, we take types of object introduced in previous Parts, and then proceed to consider *arenas*: the mathematical spaces formed by each of these types of objects. In this Part, we typically only consider unlabelled objects, though we make exception for this in this first Chapter, due to its containing a very well-known case.

53.1 Unlabelled-subset arenas

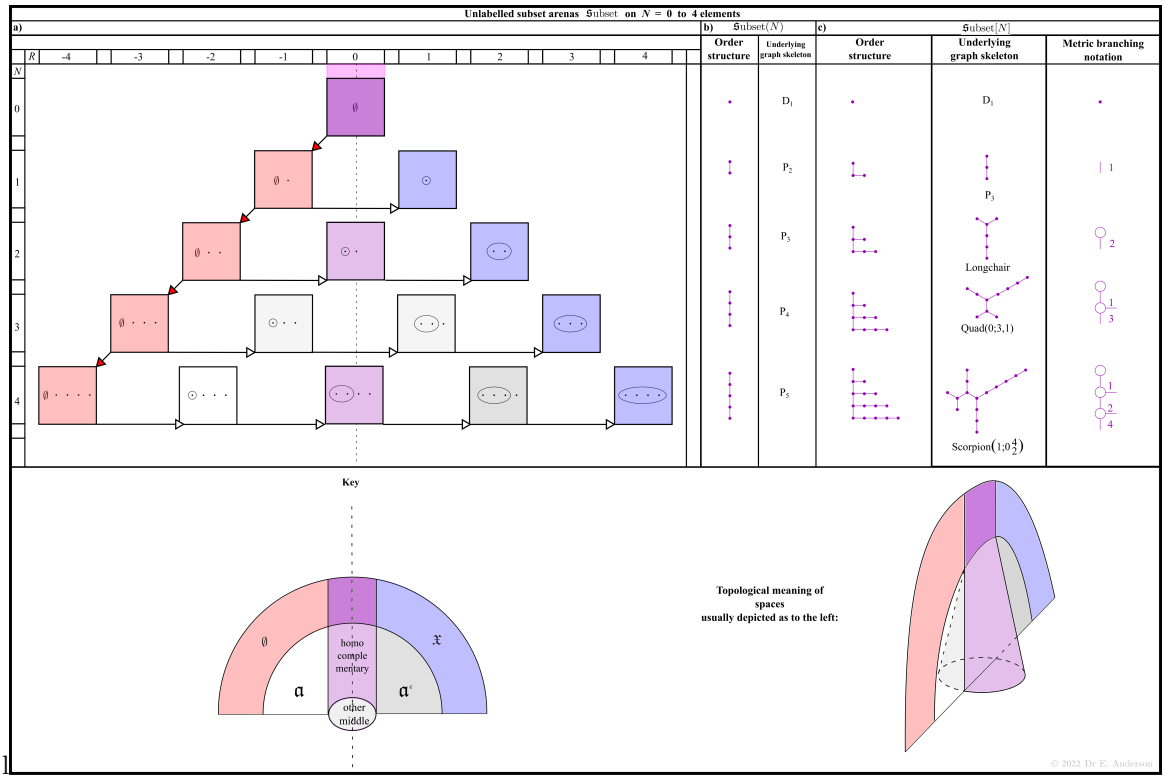


Figure 53.1:

Arena 1-( ) Let us denote the *unlabelled subsets arena* corresponding to the N-object set by

$$\mathfrak{Subset}(N) \ .$$

Ordering by the cardinality of the subset, M, this immediately takes the form of an $(N+1)$-chain for each N. This is picked out by the white arrows in Fig 53.1.a), and then in isolation in Fig 53.1.b). Subfigure a) displays these using the centering variable R of eq (8.5).

Remark 1 Picking out what the top and bottom elements of all arena order structures in this Part are shall be very illuminating.[1] For our first arena,

$$\mathfrak{S}\text{ubset}(N)_1 \; = \; \emptyset$$

and

$$\mathfrak{S}\text{ubset}(N)_0 \; = \; \{ 1, ..., N \} \; :$$

the corresponding universe set. In accord with our conceptual diagram key at the bottom of our Figure, we colour these red and blue respectively. The midline that we centre about, of homocomplementary subsets, is picked out in lilac. The heavier purple apex signifies that the universe set *is* the empty set, *and* is homocomplementary: a triple intersection of special cases.

Arena 1- _ The *arena of labelled subsets of finite sets* is then

$$\mathfrak{S}\text{ubset} \; = \; \amalg_{N \in \mathbb{N}_0} \mathfrak{S}\text{ubset}(N) \; .$$

Arena 1-[ ] The *arena of labelled subsets up to fixed order* N,

$$\mathfrak{S}\text{ubset}[N] \; = \; \amalg_{n \, = \, 0}^{N} \mathfrak{S}\text{ubset}(n) \; . \tag{53.1}$$

Remark 1 $\mathfrak{S}\text{ubset}[N]$ can be viewed as the Nth cumulative approximand to $\mathfrak{S}\text{ubset}$.

Remark 2 Studying this as a connected whole requires a further ordering operation inter-relating subsets of universe subsets of different cardinalities. A first possibility for this involves expanding the universe-set by one element. A minimalist way of implementing this is to link adjacent chains' tops: with an arrow from each empty subset of the N-set to that of the corresponding $(N+1)$-set. This is denoted by the red arrows in Fig 53.1.a).

Worked Example 1 What order structures and underlying graph skeletons does this give the $\mathfrak{S}\text{ubset}[N]$ for $N \leq 4$?

Answer 1 The red arrows by themselves form the chain of empty subsets of the N-sets. This is clearly isomorphic to (and indeed is being used as a stand-in for) the chain of N-sets as ordered by their cardinality N

For $\mathfrak{S}\text{ubset}[N]$ itself, one is to concurrently order using red and white arrows. On the top side,

$$\mathfrak{S}\text{ubset}[N]_1 \; = \; (\text{ the } \emptyset \text{ subset of } \subset \; : \; \text{the empty universe-set }) \; = \; \mathfrak{S}\text{ubset}_1 \; .$$

While on the bottom side, $\mathfrak{S}\text{ubset}[1]$ onward have no unique bottom, and $\mathfrak{S}\text{ubset}$ itself has no bottom at all. In such orderings involving ≥ 2 input types of ordering as indicated by $C \geq 2$ colours of arrow, various interpretations are possible.

1) The least structured interpretation is just that one has a *C-fold multi-order*.

2) More structured interpretations include viewing this instead as some single product order.

In Interpretation 1), we need no longer always adhere to the downwards-flowing convention. For instance, in the current Example, the empty subset stand-in for the chain of N-sets 'spine' could

[1] If any, since at most semi-lattices can in general be expected in cumulative arenas.

flow downwards. While the subset-size M-chain 'teeth' could then flow rightwards. This gives the *linear comb* chain polymer in the upright presentation of Fig 53.1.c) for $\mathfrak{S}\text{ubset}[N]$. .

In the current Example, both input orders give chains. Thus we can straightforwardly use the lexicographic order (N, M) as a product order in Interpretation 2), This being a single-order interpretation, we are therefore to use our standard downward-flowing convention. So we now get the same linear comb chain polymer but in, say, the diagonally-downward-flowing presentation of Fig 53.1.c). [This choice has maintained the perpendicularity of the 'spine' relative to the teeth.]

53.2 Complementation-quotiented counterparts

Arena 2 Quotienting out subset complementation, we arrive at

$$[\mathfrak{S}\text{ubset}](N) \text{ and } [\mathfrak{S}\text{ubset}][N] \, .$$

Worked Example 2 What order structures and underlying graph skeletons does the above place on the $[\mathfrak{S}\text{ubset}](N)$ and $[\mathfrak{S}\text{ubset}](N)$ for $N \leq 4$?

Answer 2 These are picked out by the white arrows with the same meaning as before, in Fig 53.2.a), and then in isolation in Fig 53.2.b). Subfigure a) displays these using the unsigned centering variable U of eq. (8.6). These are

$$\frac{N+2}{2}\text{-chains for } N \text{ even}$$

and

$$\frac{N+1}{2}\text{-chains for } N \text{ odd} \, .$$

On the top side,

$$[\mathfrak{S}\text{ubset}](N)_1 = [\emptyset] \, ,$$

i.e. the equivalence class consisting of the empty set and the universe set, whose sparse presentation is as indicated. On the bottom side, we must have something that is not inherited from $\mathfrak{S}\text{ubset}(N)_1$ or $\mathfrak{S}\text{ubset}(N)_0$ since $[\mathfrak{S}\text{ubset}](N)_\top$ has already greedily eaten up both of these! In the even case,

$$[\mathfrak{S}\text{ubset}](2\,p)_0 = \text{ the homocomplementary subset} \, .$$

In the odd case, we have the slightly more convoluted

$$[\mathfrak{S}\text{ubset}](2\,p+1)_0 = [\text{ the heterocomplementary pair that is closest to realizing homocomplementarity }] \, .$$

These two cases can be jointly summarized by

$$[\text{ the subsets that most closely realize homocomplementarity }] \, .$$

Next furthermore bring in a 'spine' of black arrows with the same meaning as the previous section's red arrows. Then we get another comb graph polymer. Namely, the *step-width-2 comb* of Fig 53.2.c) in the 2-order interpretation, or equivalently of Fig 53.2.d) in the lexicographic single-order interpretation.

53.3 Complementation-symmetric ordering $\mathfrak{subset}_\mathrm{S}[N]$

Arena 1-S We shall also find it useful to extend the red arrow ordering to be complementation-respecting, by adding the blue arrows which correspond to adding an element to each subset (Fig 53.3.a). We denote this arena by

$$\mathfrak{S}\text{ubset}_\mathrm{S}[N]$$

where the S subscript stands for *(complementation-)symmetrized*.

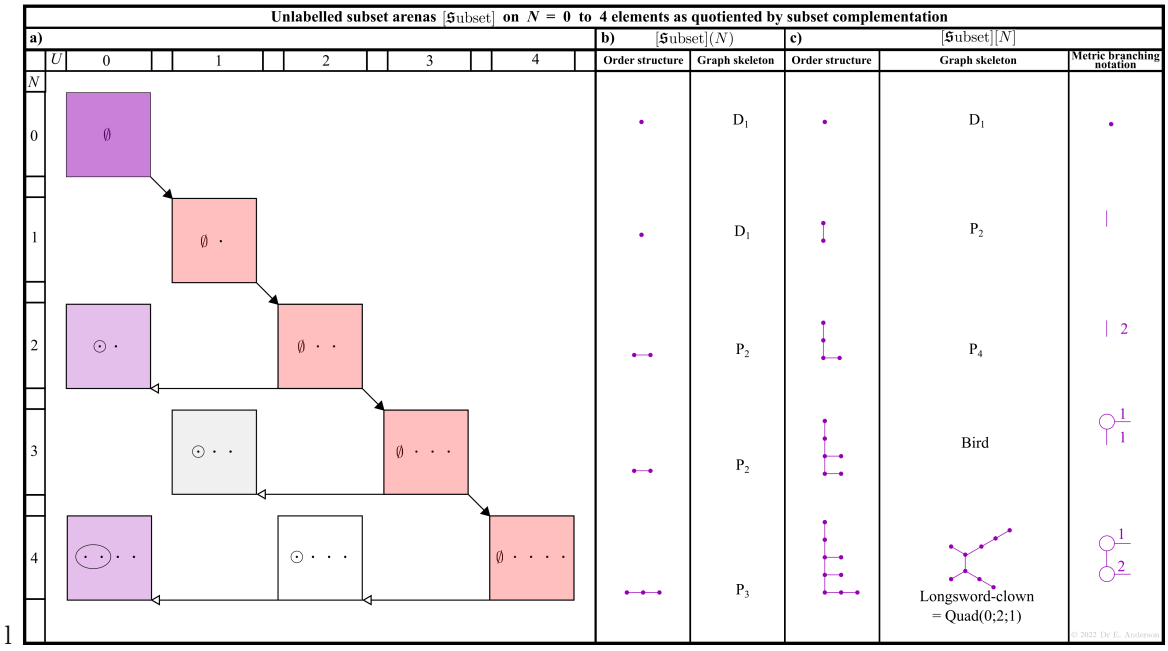


Figure 53.2:

Remark 1 In the current Volume, we do not subsequently bother with separately defining the (N), _ and $[N]$ versions of each arena. We at most study $\mathfrak{G}$raph$[N]$ rather than $\mathfrak{G}$raph itself.

Remark 2 We are now in a position to explain why Sec 53.2 used black arrows. Our convention is that ordering of the equal-N objects forming $\mathfrak{A}$rena(N) is indicated with white arrows. While our default for further arrows linking distinct-N objects is black.

Recall however that C_2 symmetries are common among the current Volume's Arenas. This means that our black-arrow default is often overridden by red arrows whose complements are blue arrows. In cases without C_2 symmetry, however, our black-arrow default shines through. This is to include all of the C_2-quotient arenas themselves. For in these, the corresponding unreduced arena's red and blue arrows have been *identified*. And are thus better-denoted by the complementation-coloration-unbiased black arrows.

Worked Example 3 Describe the order structure of the $\mathfrak{S}$ubset$_S[N]$, and their underlying graph skeletons.

Answer 3 The $\mathfrak{S}$ubset$_S[N]$ are now attached via their empty-set end by a red-arrowed ray and about their universe-set end by a blue-arrowed ray (Fig 53.3.b). These two rays emanate from a common point, since the empty subset *is* the whole universe-set when the whole universe is the empty set!

Firstly, observe in Fig 53.3.b) that this no longer gives a comb. For $\mathfrak{S}$ubset$[N]$'s 'spine' has been replaced by two such emanating from a common point: These arenas are thus, rather, growing sequences of corner-nested triangles.

Secondly, observe that while the 2-order interpretation continues to be available, a lexicographically-ordered presentation is not. Availability of a lexicographic interpretation is furthermore non-generic since our input orders are generically more complex than chains. Therefore these Series' default is to present 2-orders in the downwards-and-then-rightwards convention.

Thirdly, observe that each $\mathfrak{Subset}_S[N]$ for $N \geq 1$ contains a triangle. Thus these arenas cannot even be posets. This is the first alternative for a price to pay for demanding a complementation-symmetric ordering for the arena of subsets. Let us explain the origin of this problem. Preliminarily, add vertex and add subset are compatible structure-addition operations, thus guaranteeing poset structure (and distributivity as well). In contrast, in the summit, the white arrow can be interpreted as switching a red addition for a blue addition, for which a triangle is possible. Thus in pushing for symmetrization, we have gone too far and dinted our guarantor of a poset ordering.

The alternative price one could pay is excision of one of the triangle's vertices. There are two reasons for this to be the apex. Firstly, this is the only choice that preserves our motivational intent to find a complementation-symmetric ordering. Secondly, this is the most neatly formulable sacrifice. I.e. we now have to start our arena not at $N = 0$ but at 1. The discarded vertex's description being the empty set subset of the empty set itself. This excision knocks out not only the apex but also the two edges emanating from it. These are collectively highlighted in orange in Fig 53.3.c).

If one pays the second price, – the remaining purple parts in Subfigure c), which we denote by

$$\mathfrak{Subset}_{S*}[N] \ .$$

is a poset. A vestige of asymmetry is also retained in the sense of whether the new top layer's right or left vertex should be taken as initial element. A pentagonal suborder is also clearly visible in the new summit, ending distributivity.

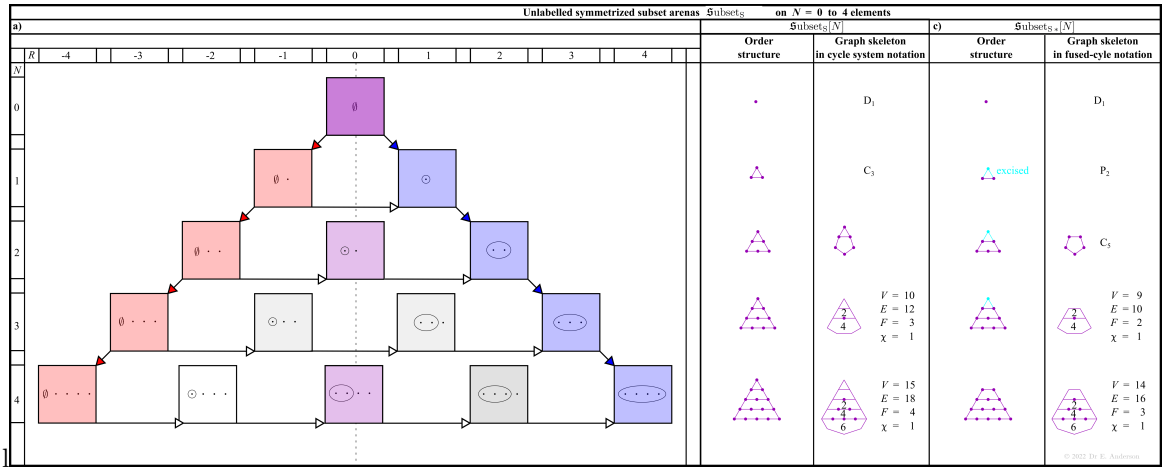


Figure 53.3:

Exercise 1$^-$ Depict the general N's

i) $\mathfrak{Subset}[N]$ using Sec 44.9's metrized branching point notation of the last column of Fig 53.1.

ii) $\mathfrak{Subset}_S[N]$ and $\mathfrak{Subset}_{S*}[N]$ using Sec 46.4's excess-cycle-length labelled cycle system notation of the last column of Fig 53.3.

53.4 Labelled-subset arenas, including power sets

Arena 3 Let us denote the *labelled subset arenas* by

$$\mathfrak{LSubset} \ .$$

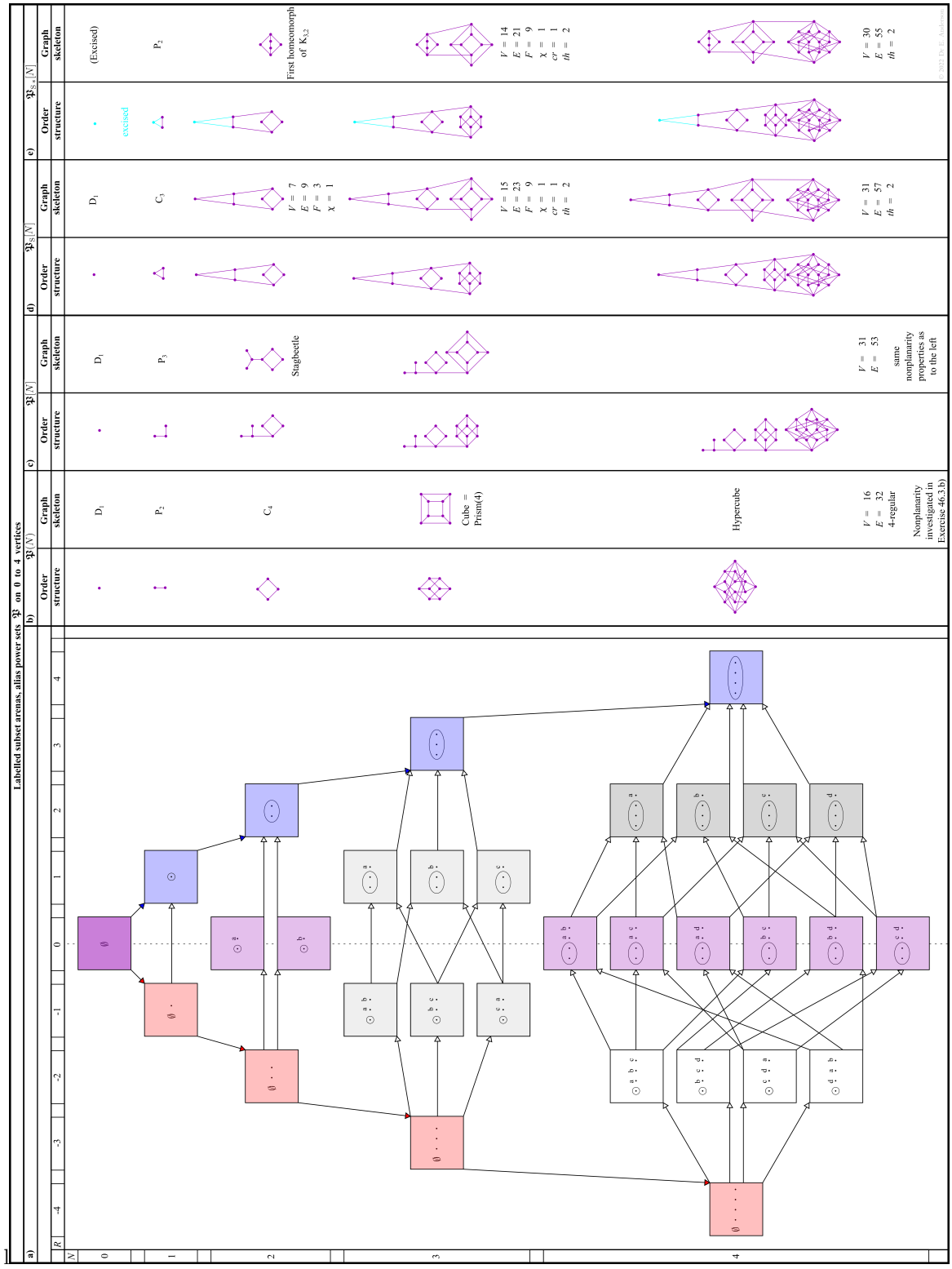


Figure 53.4:

One of these is well-known:

$$\mathfrak{LS}\text{ubset}(N) \;=\; \mathfrak{P}(N) \;:$$

the *power set* of the set of N distinguishably-labelled elements. This concurrently carries the following structures.

Structure 1 and Worked Example 4 Using the subset-expansion order that generalizes the above white arrows, $\mathfrak{P}(N)$ is (Fig 53.4.b) the free lattice F_N alias hypercube lattice H_N that we encountered in Sec 51.10.

Power sets are bounded, with

$$\mathfrak{P}(N)_0 \ = \ \emptyset$$

and

$$\mathfrak{P}(N)_1 \ = \ \mathfrak{X}(N) \ :$$

the N-element universe set.

Power sets are lattices by Exercise 51.8. Their meets and joints are furthermore unstructured intersection and union based, so these lattices are a fortiori distributive. For our labelled subsets, subset complementation furthermore ensures that the definition of complemented lattice is satisfied (answering Ex 51.12.a). Finally, the previous four sentences combine to establish that power sets $\mathfrak{P}(N)$ are Boolean algebras (thus answering Ex 52.11.b).

Structure 2 $\mathfrak{P}(N)$ is also the commutative group ${C_2}^N$ as per Sec 19.8).

Exercise 2$^-$ What happens to the above arguments for boundedness, distributivity and complementation for the unlabelled $\mathfrak{S}\mathrm{ubset}(N)$?

Exercise 3 The antichains supported by the power set are often called *Sperner sets* or *clutters* [218, 204]. It is probably clearer to just refer to these as *power set antichains* or *hypercube lattice antichains*. The name *independent sets* is often used for them, which is Graph-Theoretically, and more generally Topologically, correct, if less specific. How many of these are supported by the power sets on each of 0 to 4 objects?

♠ ◇ ♣

Structure 3 Combined with the previous blue-arrowed element addition operation, $\mathfrak{LS}\mathrm{ubset}[N]$ realizes the *N-combs of hypercubes* of Fig 53.4.c).

Structure 4 Additionally combining with the red-arrowed operation, the hypercubes are attached to a pair of rays emanating from a common origin. The resulting $\mathfrak{LS}\mathrm{ubset}_\mathrm{S}[N]$ once again contains a triangle at its summit. This either prevents it from being a poset. Or requires excision of the apex and edges emanating therefrom, leaving us with the purple truncations in Fig 53.4.d). This is a first indication that our previous example's triangular summit problem will be somewhat of a recurrent feature in Part VIII.

53.5 Complementation-quotiented counterparts

Arena 2 We next quotient out set complementation, giving the further ordering structures in Fig 53.5.a).

Worked Example 5 What order structures and underlying graph skeletons are induced on the $[\mathfrak{LS}\mathrm{ubset}](N)$ and $[\mathfrak{LS}\mathrm{ubset}][N]$ for $N \ \leq \ 4$?

Answer 5 The top element of $[\mathfrak{LS}\mathrm{ubset}](N)$ is the equivalance class of $\mathfrak{P}(N)$'s top and bottom elements. I.e. the empty set and the universe set, or, more succinctly, the indiscrete topology $\mathcal{T}_\mathrm{i}(N)$. Bottom elements are not unique for $N \ \geq \ 2$. The $[\mathfrak{LS}\mathrm{ubset}](N)$ are thus at most semi-lattices.

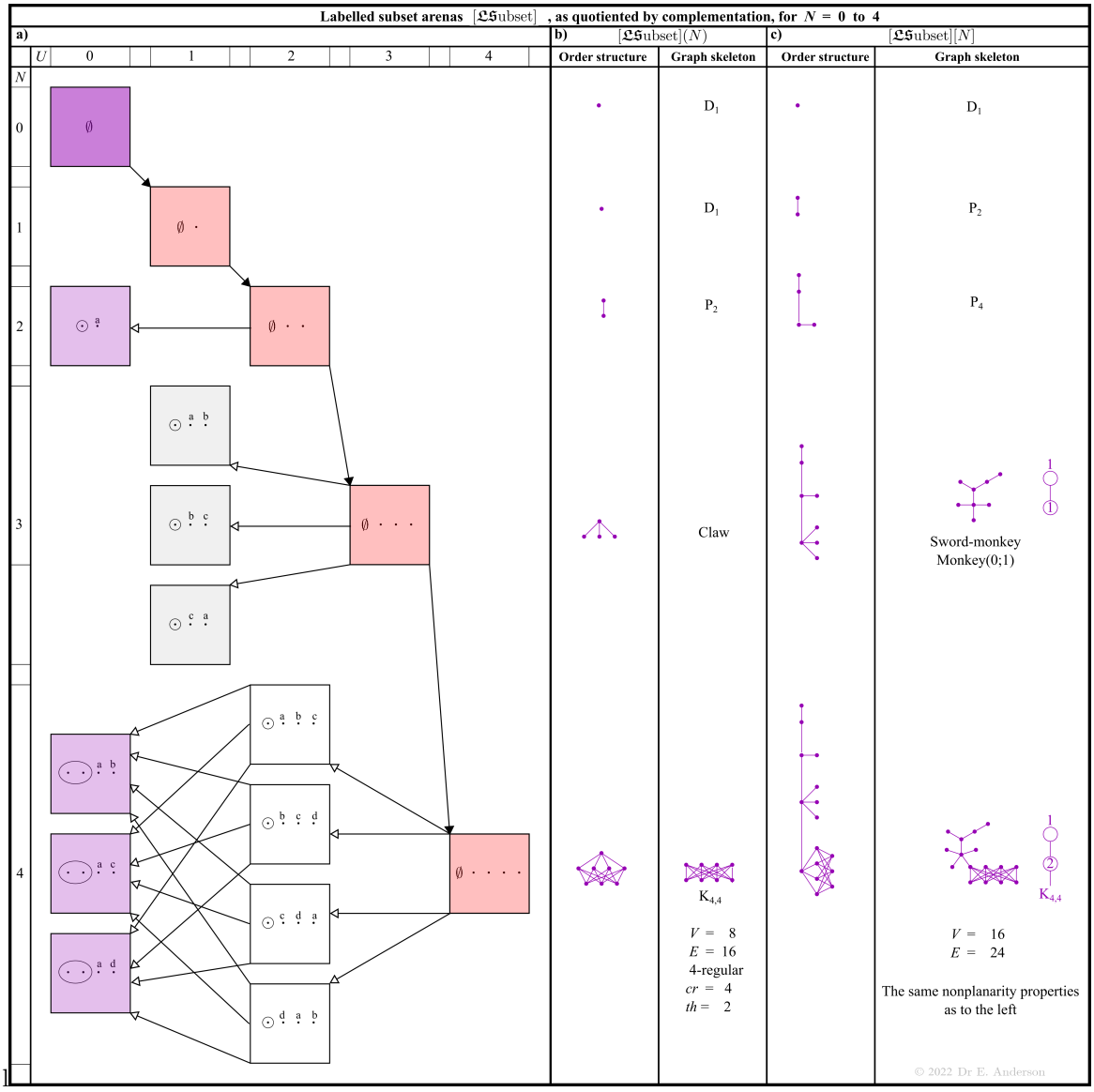


Figure 53.5:

The poset property is inherited. The complementation quotient has furthermore used up the power set's complementedness. Complementation for $[\mathfrak{LS}\text{ubset}](N)$ cannot even be posed for $N \geq 2$ by these not having unique bottom elements. For our smallest N, the $[\mathfrak{LS}\text{ubset}](N)$ yield the point, point, chain, semi-lattice and poset forms of Fig 53.5.b). The last of these is a poset that is not a semi-lattice due to its containing C_4-nonconvex subposets such as the one with layer by layer vertex set

$$\{\, a\,b\,c,\, b\,c\,d;\, a\,b,\, a\,c \,\}\ .$$

Nor is it distributive, as is clear from any of the various $K_{3,2}$ that it contains, such as

$$\{\, \mathrm{pt}^4;\, a\,b\,c,\, b\,c\,d,\, c\,d\,a;\, a\,b,\, \}\ .$$

Let this serve as an early warning that latticeness and distributivity are not necessarily preserved under quotienting of posets.

The cumulative counterparts

$$[\mathfrak{LS}\text{ubset}][N]$$

then form the combs in Fig 53.5.c).

53.6 End-Remarks

Aside 1 If we needed an arena of subsets of a set with partly-distinctly labelled elements, then we would label it by the corresponding subgroup G of the permutation group S_n:

$$\mathfrak{S}\text{ubset}(n, G) \ .$$

From this point of view, $\mathfrak{L}$ is shorthand for id and $_$ for S_n.

End-Remark 1 Let us next detail our comment that semi-lattices are usually the most that can be expected to arise in cumulative-approximand arenas. On some occasions, these manage furthermore to be lattices. They can also however fail to be semi-lattices, for instance by containing a C_4-nonconvex. More occasionally, they can even fail to be posets due to containing triangles. Non-cumulative arenas are also on some occasions no more than posets due to failing to be connected. The extra arrow type(s) for cumulative-approximand arenas have some capacity to regain connectivity.

While we know that a simple and reasonably meaningful excision removes triangular summits, some of these other infractions can occur both infinitely often and fairly irregularly. So we almost always have to live with such limitations on the niceness of the orders that naturally arise on discrete mathematical objects' arenas.

End-Remark 2 Let us finally comment that subset arenas, with their chains, hypercubes and simple compositions thereof, are quite generally too structurally-regular to be a source of much Mathematics. Deviations from perfect structural regularity are more interesting than perfect structural regularity itself. As are highly-irregular structures that have some underlying mathematical significance. Moreover even just passing from subset arenas to subset family arenas already produces interesting structural irregularity.

End-Remark 3 By being not only Boolean algebras but also somewhat surprisingly commutative groups, and by the meaningfulness and applicability of their clutters, power sets count among the simpler citizens of Kallista.

Chapter 54

Recurrence orders and selection arenas

54.1 Recurrence orders

Structure 1 Let $\mathfrak{S}$ be a set of mathematical objects in possession of a recurrence relation from which all the objects in $\mathfrak{S}$ may be computed (given some initial-data objects to start things off). Then a *recurrence ordering* for $\mathfrak{S}$ is an ordering of its objects in which each object's predecessors are those objects directly used to compute it in accord with the recurrence relation.

54.2 Combination arenas

.

Worked Example 1 Place the order induced by Pascal's identity (13.6) viewed as a recurrence relation on the arena of combinations. What specific order structures and underlying graph skeletons does this produce for $n = 0$ to 4? And for the general n? And for the version quotiented by midline reflection symmetry?

Remark 1 As Fig 54.1 details, the arena of combinations has a very similar conceptual diagram to that for the previous Chapter's arenas of subsets. Albeit with less significant initial and terminal elements: here these are all 1's whereas there they were empty sets $\emptyset$ and universe sets $\mathfrak{X}(N)$. So we do not bother to shade these 1's in red and blue. The purple apex 1 is all of a first entry, a last entry and a unique maximal entry. The difference between the two arenas is that the subset one admits further hetercomplementary pairs of middle objects.

Answer 1 For the combinations, the now horizontal rather than vertical pair of predecessors of Pascal's recurrence relation (13.6) produces a widening structure. This is thus our first purely black arrow ordering structure. The $n = 0$ to 4 ordering structures and underlying graph skeletons are displayed in Fig 54.2.a).

Fig 54.2 has a high amount of structural regularity. In the unreduced semi-infinite case, the triangle's interior vertices are quartic while its two sides' vertices are cubic. Solely the apicial initial-datum root-vertex is of degree 2. Thus the underlying graph skeleton here is almost-everywhere quartic. In the finite approximands, there are precisely 2 leaves (degree 1): the bottom corners, while the rest of the triangle's third 'base' side is of degree 2.

This highly regularity permits the general-n case's explicit form to be written down, as indicated. This is manifestly a poset and a fortiori a distributive lattice.

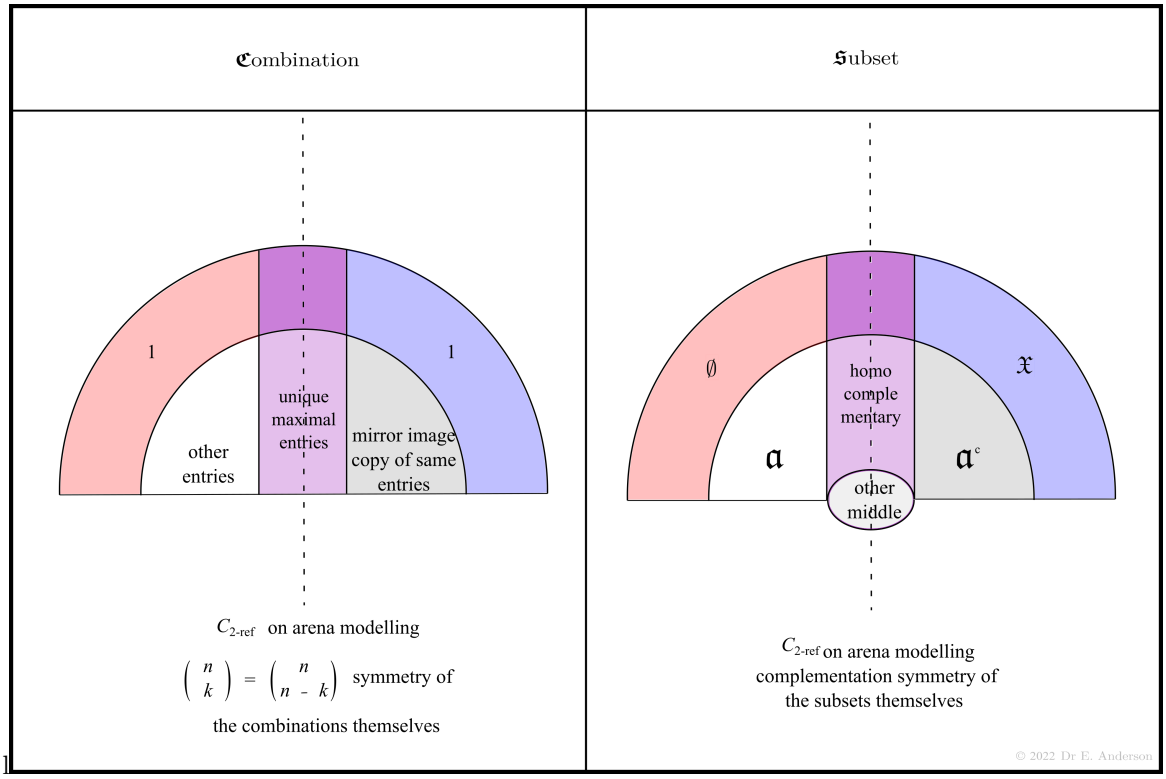


Figure 54.1:

Quotienting by the midline reflection symmetry, we obtain the order of Fig 54.2.b) on the relational arena of combinations. Because the universality of Pascal's recurrence relation would cease to be manifest going into the midline of unique maxima, we counter with the following notation. When both a combination and its reflection contribute to a midline element, we place a double edge between their identification and the midline element. See Fig 54.2.b).

Exercise 1 We leave it as an Exercise for the Reader to detail the extent to which Fig 54.2.b)'s relational case is regular.

Exercise 2 a) Determine how V, E and F go with n for the underlying graph skeletons in Fig 54.2. Prove the general-n case's formulae for these for both Subfigures a) and b).

b) What are the maximal outer-planar cases of this for each of a), b) with n odd and c) with n even?

54.3 Other selection arenas

.

Worked Example 2 What are the corresponding order structures for the Stirling arenas?

Answer 2 The 'Stirling lattices' are isomorphic to the 'Pascal lattice' (Possible modulo labelling the arrows with the multiplicity that they contribute via the slightly less trivial Stirling recurrence relations). No information-preserving quotienting is possible here, so there is no accompanying relational version.

Examples 5-7 The Bell, 'para-Bell' and Catalan recurrence structures – all of which make exhaustive cumulative use of prior entries – are just K_n at the level of graph skeletons. Thus for $n \geq 3$

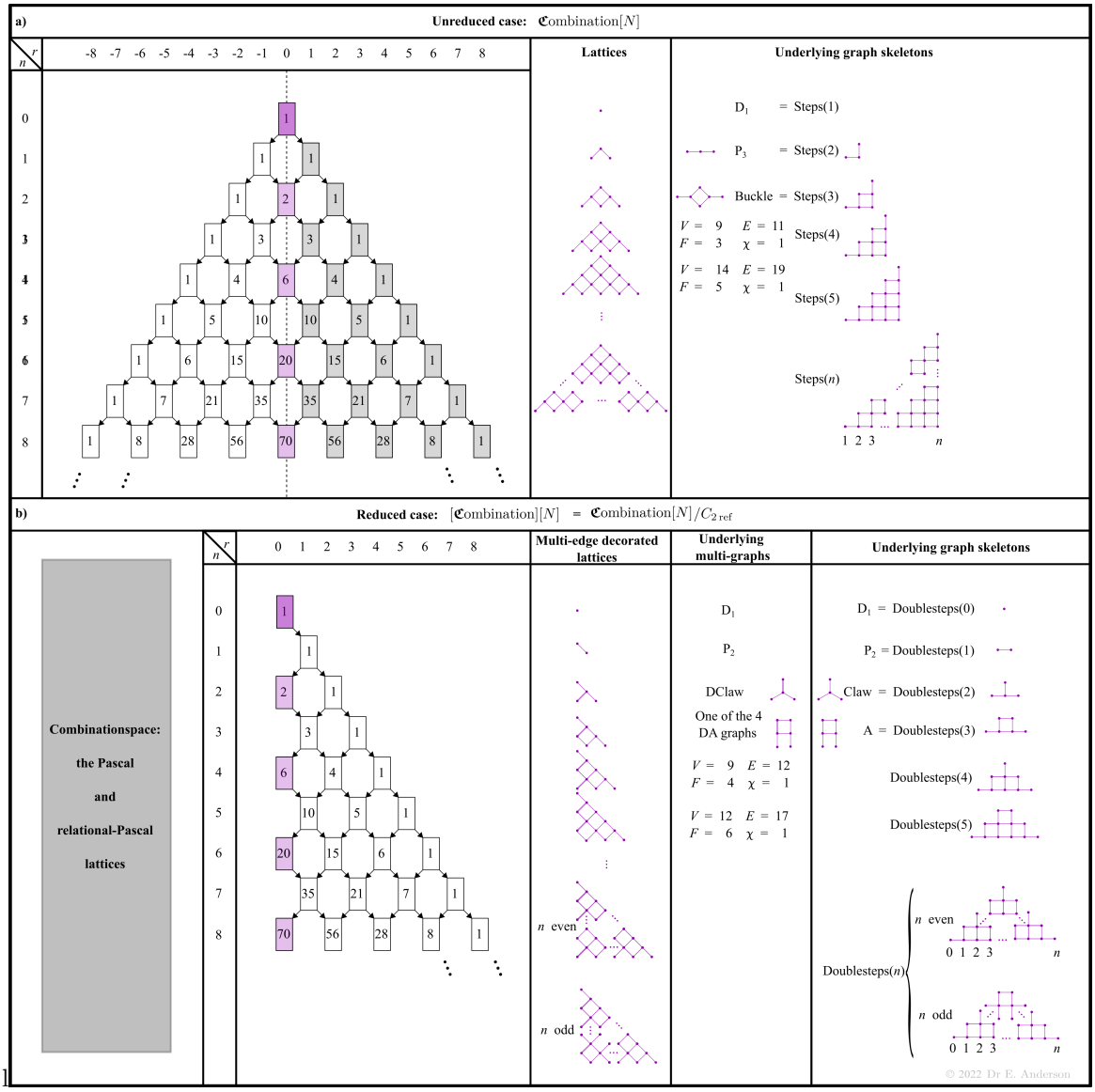


Figure 54.2:

these contain triangles, and thus cannot even be posets. Indeed, as n goes to infinity, these provide further examples of (54.1).

54.4 Other recurrence orders

.

Example 1 The first-order linear recurrence relation (14.1) for some fixed initial datum just gives the semi-infinite chain as its recurrence order.

Worked Example 3 What about the EUSOLDE alias Fibonacci problem (14.2)?

Answer 3 The recurrence ordering of the solution for this is as indicated in Fig 54.3. This ordering fails to be a poset, since it contains triangles. In fact, its cycles are entirely triangles: the n-triangulations of a strip Strut_n ensue, bar a single leaf rooting the structure at the initial datum.

This has a high level of regularity: in the semi-infinite case all vertices are cubic bar the zeroth (leaf) and first (degree 2). While additionally the last vertex is of degree 2 in the nth approximand case.

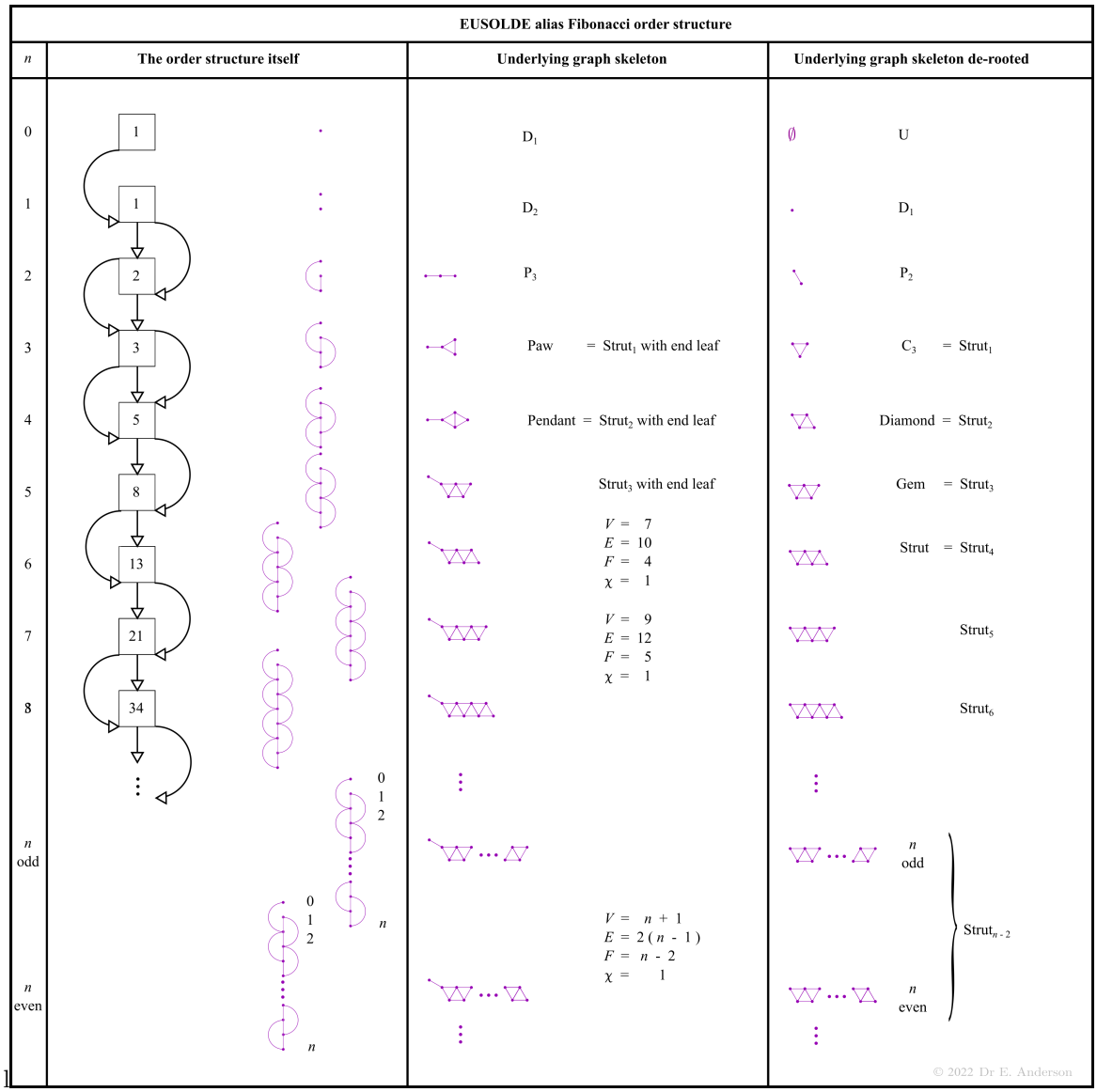


Figure 54.3:

54.5 Discussion

Remark 1 EUSOLDE's order structure does not only fail to be a poset but also serves as a first example of

an arena whose natural ordering structure contains arbitarily many not meaningfully excisable triangles! (54.1)

Remark 2 While Fig 54.2 has a semi-infinite wedge lamina (2-d) large-scale structure, Fig 54.3 exhibits instead a semi-infinite line large-scale structure.

Remark 3 While subset arenas and combination arenas have the same conceptual diagram, they do not coincide in their order structure, or even in whether their principal arrowing is white or black. This illustrates some of the value in passing from such conceptual diagrams – which can already imply some topological adjacencies and preclude some impossible overlaps – to actual order structures for our arenas. Which is a summary description of what we do throughout this final Part VIII.

End-Remark 1 At least among the current Volume's examples of recurrence relations' corresponding recurrence structures, the Pascal case is Kallista. Insofaras it alone is nontrivially reflection symmetric and a distributive lattice while the other cases fail to even be posets. The many routes involved are here well-known: basic Combinatorics, Algebra and Probability all hit upon the combinations.

End-Remark 2 All of this and the previous Chapter's order structures are however too highly regular to be of much interest as examples of order structures or as regards their underlying graph skeletons. For large enough N, this limitation will not apply in any subsequent Chapter.

Chapter 55

Family arenas

55.1 The unreduced case

Definition 1 Let us denote arenas of families of subsets by

$$\mathfrak{F}\text{amily} \ .$$

Two significant ordering operations for families are as follows.

Structure 2 Let us denote 'add element' once again by blue arrows; this now corresponds to forming a red nest (rather than a red cone). For some purposes, 'add element' requires a complementary operation (red-arrowed): 'add element and all subsets containing that element'. The symmetric presentation also brings in the corresponding blue nest.

Structure 1 Internally to each N, we have the natural operation 'add subset'. This is what our white arrows now signify.

Worked Example 1 Study $\mathfrak{F}\text{amily}(N)$ and $\mathfrak{F}\text{amily}[N]$ for $N \ = \ 0$ to 2.

Answer 1 The $\mathfrak{F}\text{amily}(N)$ are bounded. With

$$\mathfrak{F}\text{amily}(N)_0 \ = \ \emptyset \ : \ \text{ the empty family}$$

And

$$\mathfrak{F}\text{amily}(N)_1 \ = \ (\text{ the completed family }) \ \text{ alias } (\text{ the discrete topology }) \ = \ \mathcal{T}_{\mathrm{d}}(N) \ .$$

For $\mathfrak{F}\text{amily}(N)$, Exercise 50.17 guarantees that our 'add element' operation gives a poset. Complementation is guaranteed, since each family bi-partitions the completed family.

Fig 55.1.b) presents the $\mathfrak{F}\text{amily}(N)$ for $N \ = \ 0$ to 2. Top and bottom do not even coincide for $N \ = \ 0$, by the empty collection $\emptyset \ \neq \ \{\emptyset\}$. So $\mathfrak{F}\text{amily}(0)$ is already a 2-chain.

$\mathfrak{F}\text{amily}(1)$ is then minimal for having a nontrivial poset middle (forming the natural C_4). While $\mathfrak{F}\text{amily}(2)$ is the first Graph-Theoretically nontrivial case. Though this straightforwardly still manages to be planar. Its cycle system is furthermore a quadrangulation in the more stringent sense, consisting of 9 internal and 1 external 4-cycles, as per Fig 55.1.b). Thus

$$Y \ = \ 10 \ \text{ and } \ Z \ = \ \frac{10 \ \times \ 2}{2} \ = \ 10 \ .$$

Fig 55.1.b) then presents the cumulative

$$\mathfrak{F}\text{amily}[N] \;=\; \text{Comb}(\mathfrak{F}\text{amily}(N)) \;.$$

The $N \;=\; 2$ case straightforwardly manages to retain planarity. Across a bridge edge, this picks up 1 further disjoint 4-cycle carrying a P_3 sidechain.

$\mathfrak{F}\text{amily}_{\mathrm{S}}[2]$ (Fig 55.1.d) manages to be nonplanar, albeit just with crossing number 2. Finally observe that the $\mathfrak{F}\text{amily}_{\mathrm{S}}[N]$ now a priori have flat rather than triangular summits. So the excision versus no poset saga for subset arenas has no counterpart here. The $\mathfrak{F}\text{amily}_{\mathrm{S}}[N]$ retain the choice between not having unique tops and having to be interpreted asymmetrically anyway.

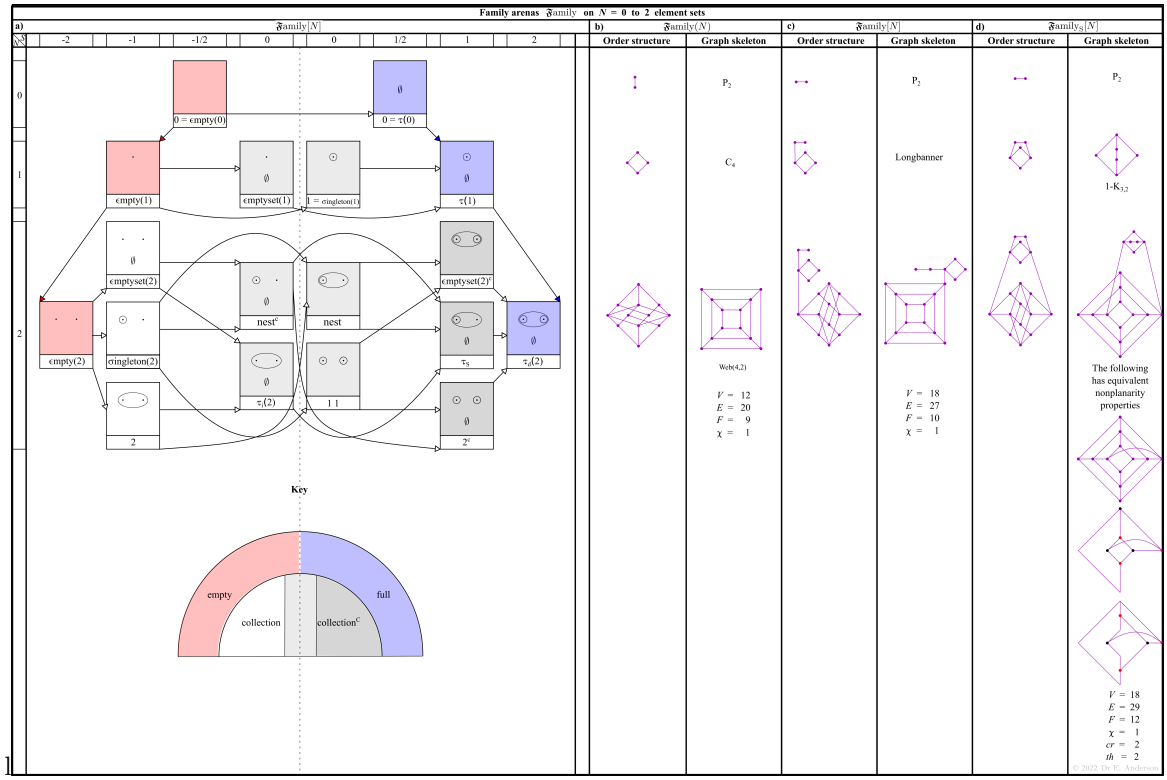


Figure 55.1:

55.2 The family-complementation quotient

Arena 6 and Worked Example 2 What order structures and underlying graph skeletons are induced on $\mathfrak{F}\text{amily}(N)$ and $[N]$ for $N \;=\; 0$ to 2?

Answer 2 Now

$$[\mathfrak{F}\text{amily}](N)_1 \;=\; [\,\epsilon\text{mpty}(N)\,] \;:$$

the equivalence class consisting of $\epsilon\text{mpty}(N)$ and $\phi\text{ull}(N) \;=\; \mathcal{T}_{\mathrm{d}}(N)$. Also bottom elements are not unique for $N \;\geq\; 2$. So one has at most semi-lattices. $N \;=\; 2$ furthermore suffices to manifest a C_4-nonconvex suborder

$$\{\,\emptyset(2),\, 2;\, \text{nest},\, 1\,1\,\} \;.$$

By which we have at most a poset here, and indeed a poset by hereditarity.

For $N = 0$ to 2, we furthermore observe that all the underlying graph skeletons are highly standard (Fig 55.2.b).

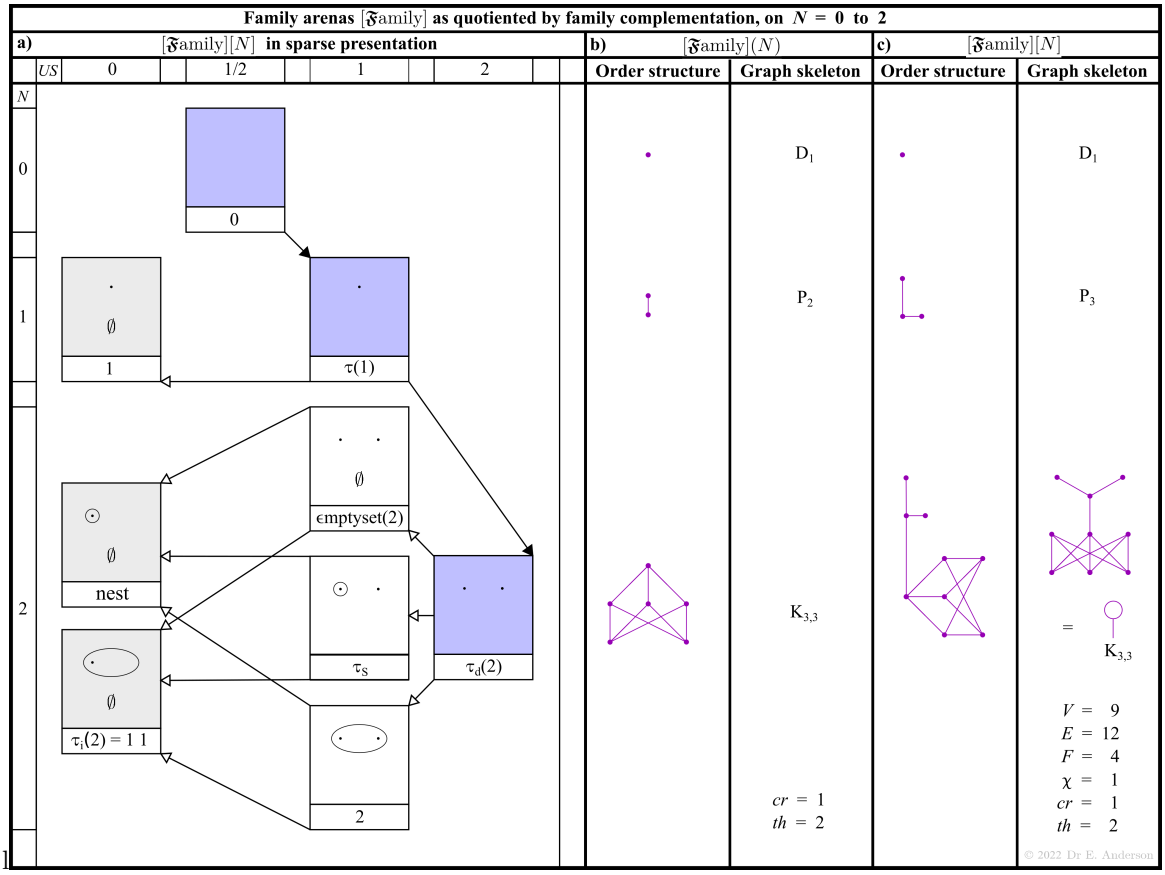


Figure 55.2:

The [𝔉amily][N] are combs of the preceding. Unlike for 𝔉amily[N], there is now a top element

$$[\mathfrak{F}\text{amily}][N]_1 = [\,0\,] :$$

I.e. the equivalence class consisting of $0 = \epsilon\text{mpty}(0)$ and $\{\,\emptyset\,\}$. The underlying graph skeletons remain standard (Fig 55.1.c).

Exercise 3 Work out the corresponding unreduced and reduced arenas for subset families on 0 to 2 labelled-element sets.

55.3 The fourfold and eightfold quotients

Structure 1 The fourfold quotient is

$$\widetilde{\mathfrak{F}\text{amily}}_4 = \frac{\mathfrak{F}\text{amily}}{C_2 \times C_2}$$

with the group acting as forgetting whether the family contains whichever of $\mathfrak{X}$ or $\emptyset$. The eightfold quotient is

$$\widetilde{\mathfrak{F}\text{amily}} = \frac{\mathfrak{F}\text{amily}}{C_2 \times C_2 \times C_2}$$

with the further C_2 acting by local complementation.

Remark 1 The discussion in can be interpreted in terms of a nontrivial stabilizer subgroup for $N = 0$ to 2. For 0 and 1, the resulting arenas are just points, while for 2 they are a 2-chain and a 3-chain. For 3, they are the posets in Figs 55.3 and 55.4, with underlying graph skeletons and planarity analysis provided. These are not lattices due to containing C_4-nonconvex subposets, the one common to both Figures being

$$\{\, \varsigma\text{ingleton}, \delta\text{oublet}; 2\,1, \mathcal{T}_{\mathrm{nest}} \,\} \ .$$

The grey shading in these figures is slightly darker than its counterparts in previous figures due to its denoting local rather than primary complementation.

Remark 2 Let us coin the name CompassRose for $\mathfrak{F}\mathrm{amily}_4(3)$'s underlying 20-graph skeleton: the minimal nontrivial $\mathfrak{F}\mathrm{amily}_4$ graph.

Remark 3 $\widetilde{\mathfrak{F}\mathrm{amily}}_4$ is the most useful quotient since the distinguished subarenas of essential covers and topologies are not complementation-symmetric. Furthermore, we observe the simple product structure

$$\mathfrak{F}\mathrm{amily} \ = \ \widetilde{\mathfrak{F}\mathrm{amily}}_4 \,\square\, \mathrm{C}_4$$

both as a graph and as a poset using $\downarrow \mathrm{C}_4$. And all the topologies reside in one copy of the quotient, with all the essential covers residing in another. In particular, as graphs,

$$\mathfrak{F}\mathrm{amily}(2) \ = \ \mathrm{P}_3 \,\square\, \mathrm{C}_4 \ = \ \mathrm{Boxstack}(2)$$

and

$$\mathfrak{F}\mathrm{amily}(3) \ = \ \mathrm{CompassRose} \,\square\, \mathrm{C}_4 \ .$$

Though comparing with Fig 50.3, as posets

$$\mathfrak{F}\mathrm{amily}(2) \ \neq \ \mathrm{P}_3 \,\square\, \downarrow \mathrm{C}_4 \ .$$

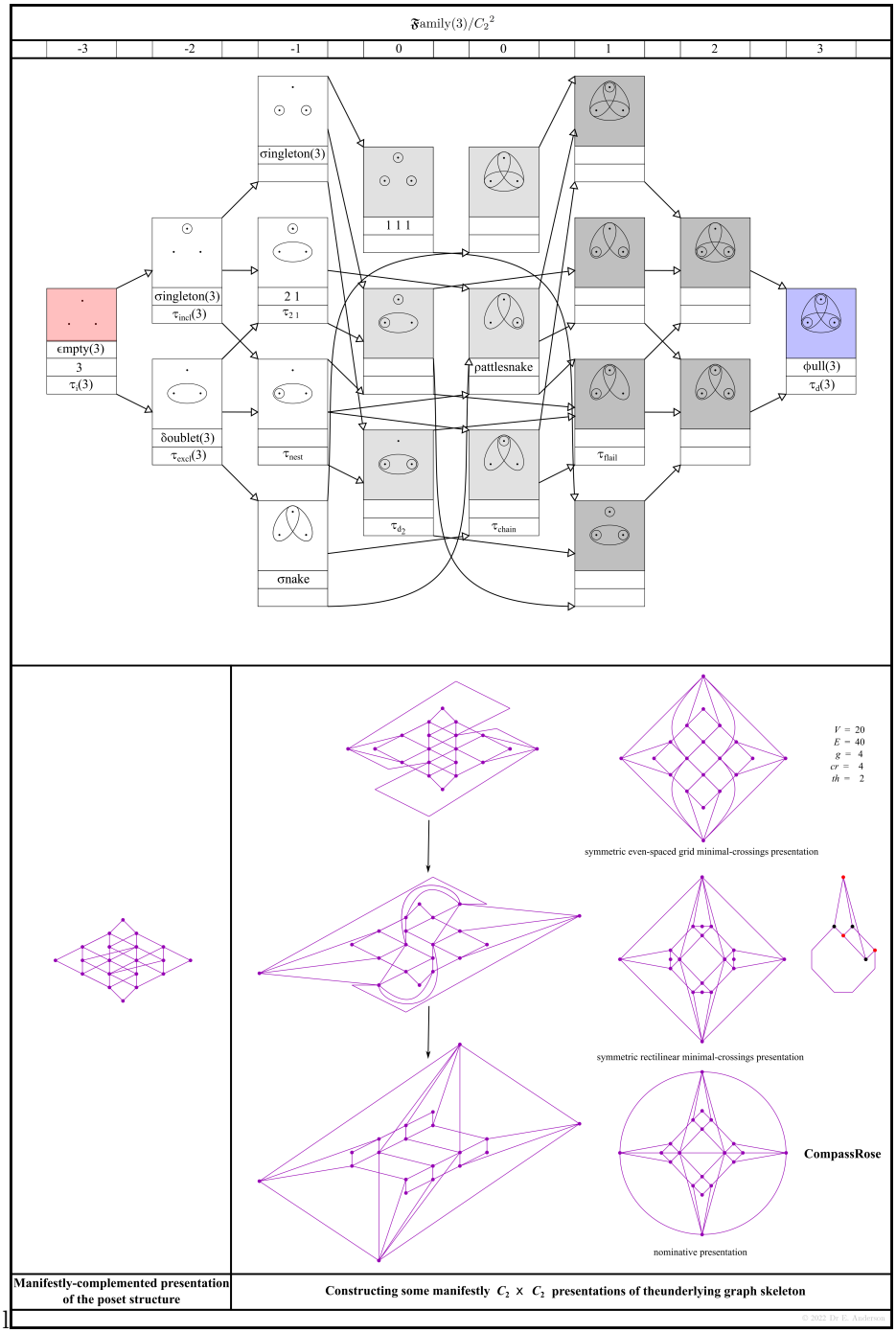


Figure 55.3:

Remark 4 𝔉amily(N) is all of a poset, structurally-irregular while conceptually well-motivated, and rich in many even more exhalted subarenas. Since I am not aware of this structure arising by many routes, however, I do not presently consider it to reside in Kallista. Observing, rather, that it is a part of Platonia that contains quite an accumulation of Kallista subarenas, of which the next Chapter details a first two.

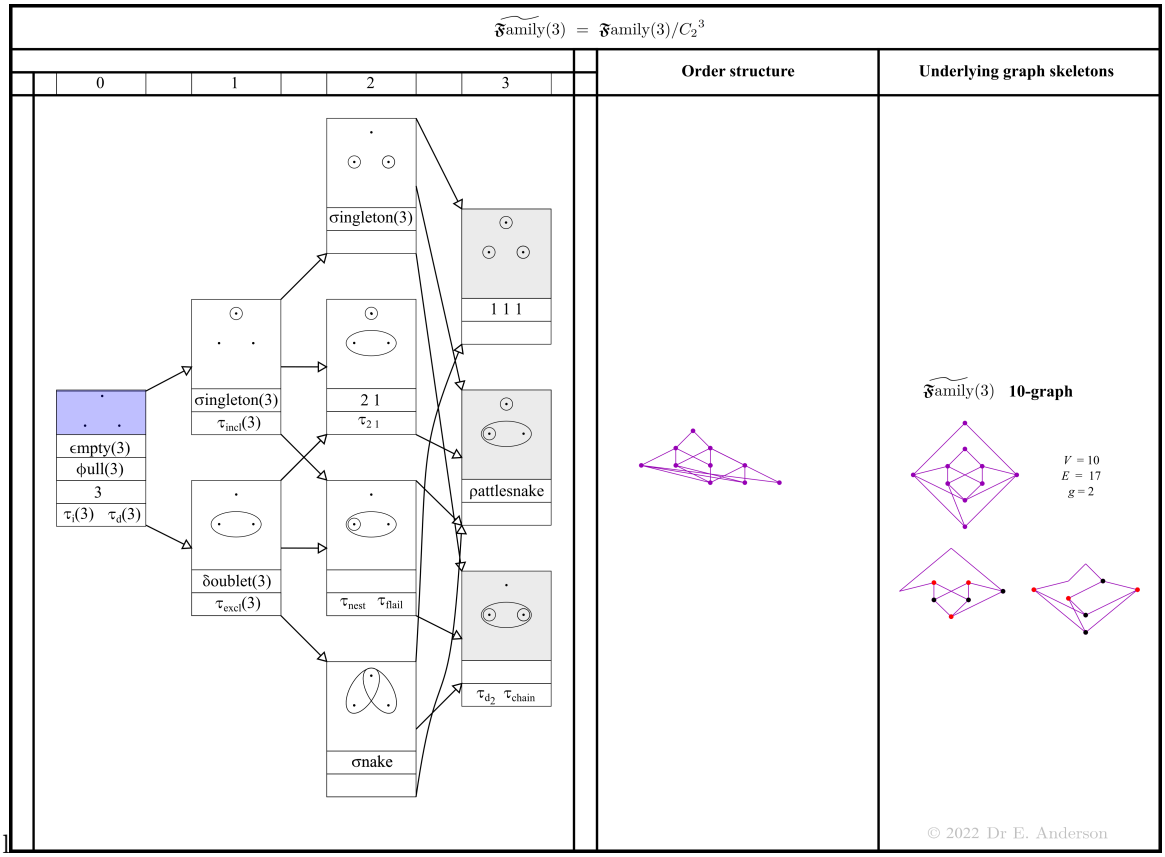

$\widetilde{\mathfrak{F}\text{amily}}(3) = \mathfrak{F}\text{amily}(3)/C_2{}^3$
0
1
2
3
Order structure
Underlying graph skeletons
σingleton(3)
1 1 1
σingleton(3)
τincl(3)
2 1
τ2 1
ϵmpty(3)
ϕull(3)
3
τt(3) τd(3)
ρattlesnake
δoublet(3)
τexcl(3)
τnest τflail
τd2 τchain
σnake
$\widetilde{\mathfrak{F}\text{amily}}(3)$ 10-graph
V = 10
E = 17
g = 2
© 2022 Dr E. Anderson

Figure 55.4:

Chapter 56

Cover and Topology arenas

56.1 Plain cover arenas

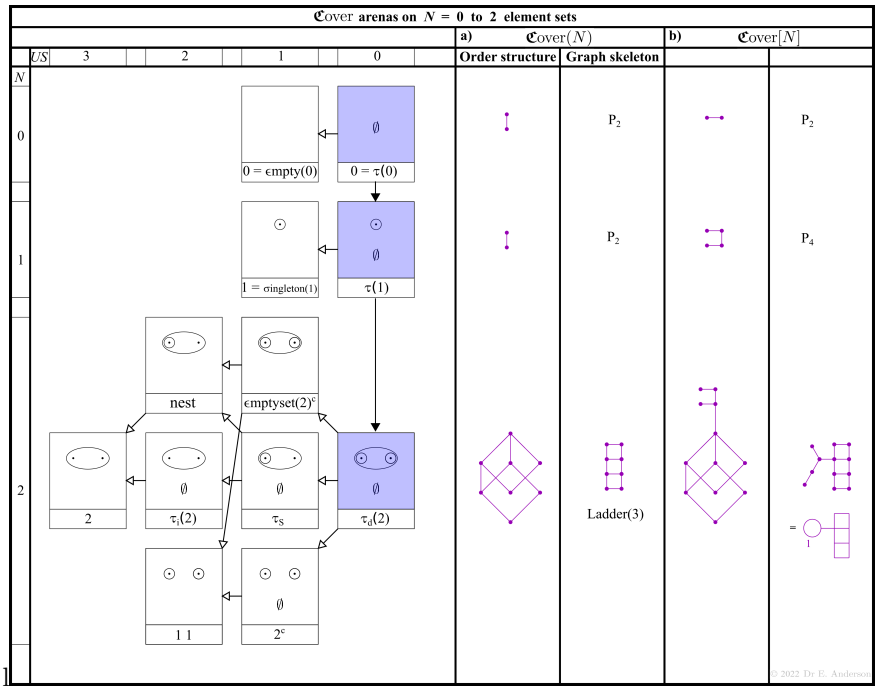


Figure 56.1:

Structure 1 The *arena of covers* is

$$\mathfrak{Cover} \ .$$

Worked Example 1 Induce $\mathfrak{Family}(N)$'s natural ordering structure on $\mathfrak{Cover}(N)$ for $N = 0$ to 2.

Answer 1 Let us now use the 'remove subset' reversed version of families' poset-forming white arrows. For with these we have strong connectivity of $\mathfrak{Cover}$.

$$\mathfrak{Cover}(N)_1 \ = \ \mathcal{T}_{\mathrm{d}}(N)$$

is inherited from the larger arena of families. But the empty-family bottom elements therein are

clearly not covers. Now instead, the bottom elements are

$$\mathfrak{Cover}(N)_0 \;=\; N \;=\; \{\,\mathfrak{X}\,\}:$$

the coarsest partition alias the family consisting of the universe set alone. So boundedness is attained.

Observe in Fig 56.1.b) that $N = 2$ is minimal for complementation No C_4-nonconvex, $K_{3,2}$ or C_5 subposets are in evidence. There is however another reason why $\mathfrak{Cover}(2)$ is not a lattice: there are element pairs such as $\mathcal{T}_i(2)$ and 1 1 which fail to have a meet.

The graph skeletons thus emerging are very standard for our range of N. This is one of the places where letting N take higher values is more necessary.

Remark 1 The main reason for studying family arenas is to subsequently study therein the subarenas of various rather more mathematically interesting objects that so happen to be subcases of families. How is $\mathfrak{Cover}(2)$ extrinsically realized within $\mathfrak{Family}(2)$? We pick this out in orange in Fig 56.2. And Fig 56.1 for an intrinsic view.

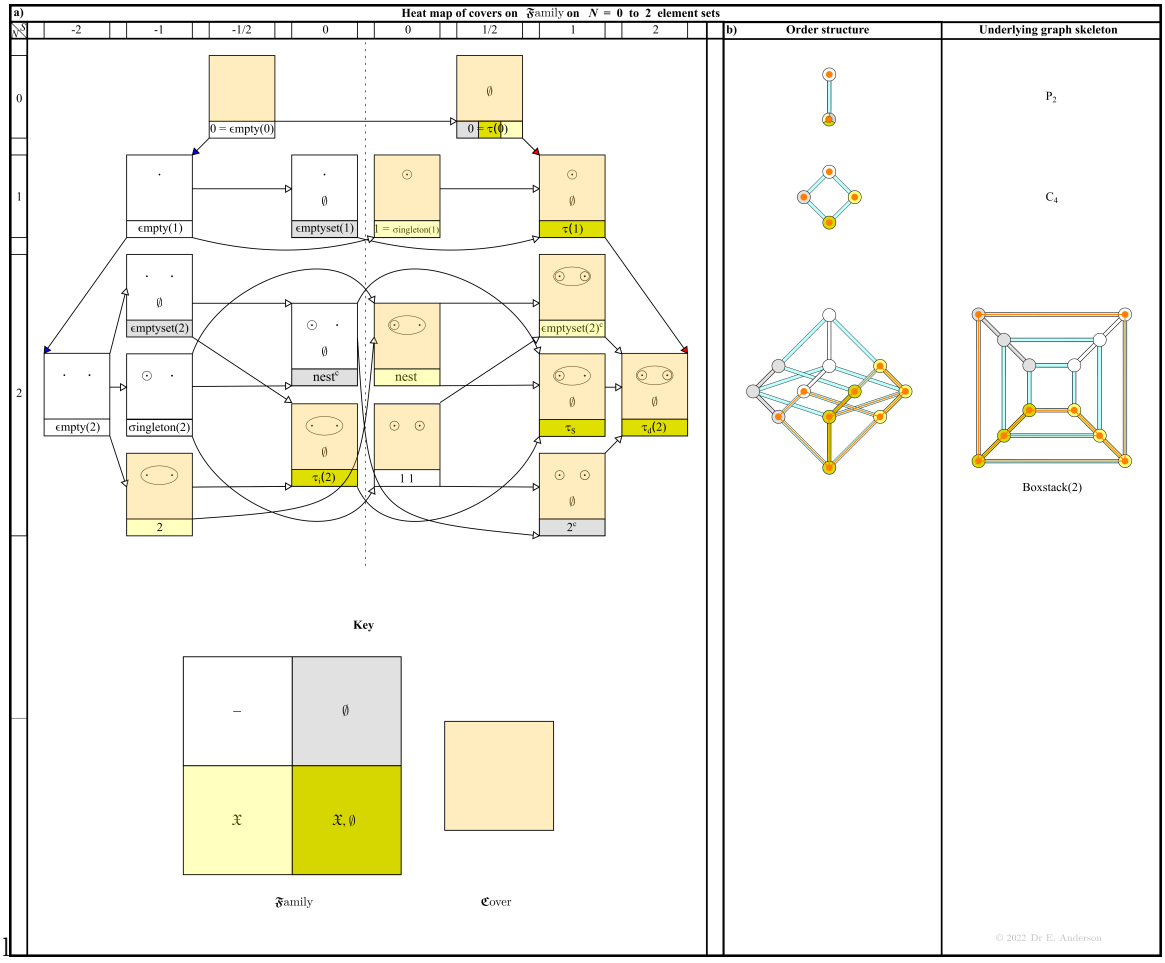


Figure 56.2:

Remark 2 Since covers do not respect complementarity, there is no [$\mathfrak{Cover}$] option to entertain.

Exercise 1 Decribe how families' blue-arrowed operation is highly incompatible with covers, while explaining why their red-arrowed operation is suitable. Thus give the order structure on $\mathfrak{F}\text{amily}[N]$ for $N = 1$ and 2, identifying the underlying graph skeletons in the process. How is $\mathfrak{C}\text{over}[2]$ extrinsically realized within $\mathfrak{F}\text{amily}[2]$?

Remark 3 As compared to the previous Chapter, there is less motivation for concurrently using red and blue arrows. For now our arena has no midline reflection symmetry from $N = 2$ onward.

56.2 Examples of essential cover arenas

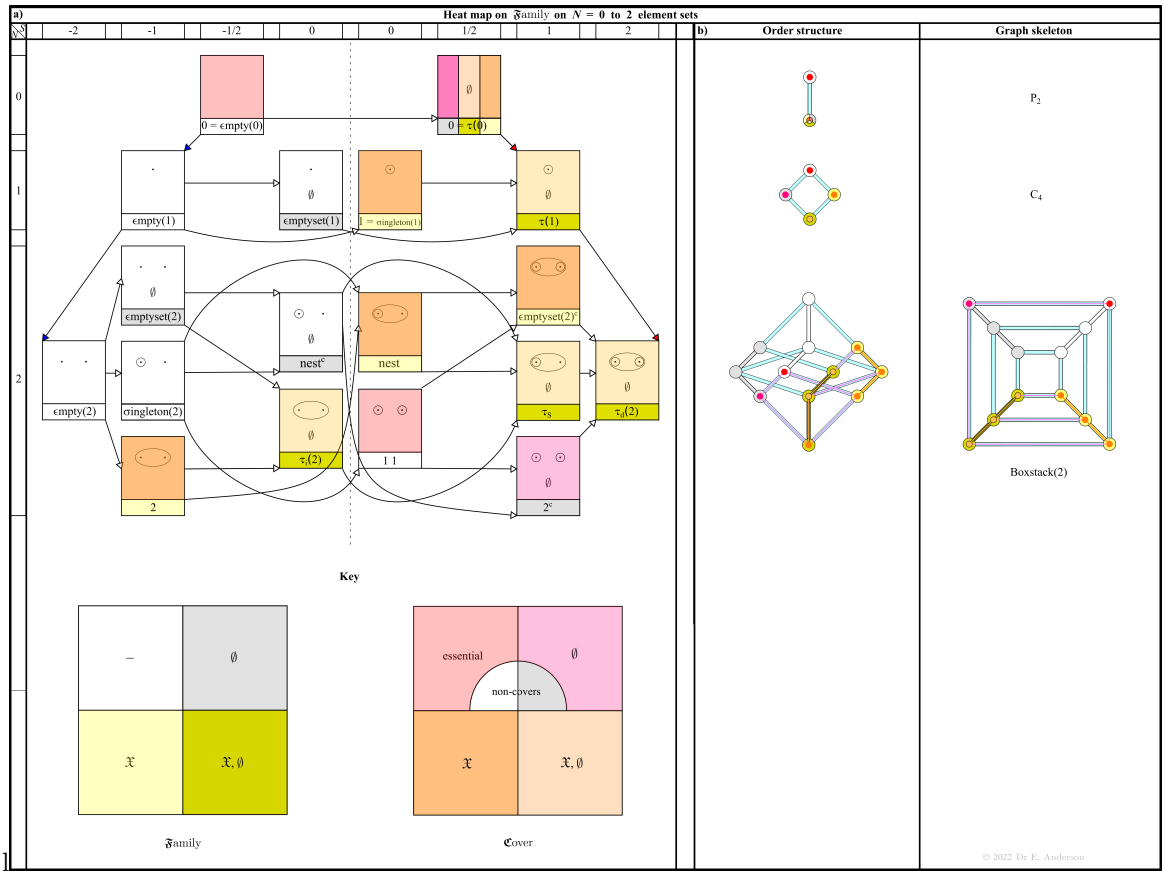


Figure 56.3:

Structure 1 The *arena of essential covers* is

$$\mathfrak{e}\mathfrak{C}\text{over} \leq \mathfrak{C}\text{over} \leq \mathfrak{F}\text{amily} .$$

Remark 1 $\mathfrak{e}\mathfrak{C}$over can furthermore be envisaged as the quotient arena

$$\mathfrak{e}\mathfrak{C}\text{over} = \frac{\mathfrak{C}\text{over}}{C_2 \times C_2} ,$$

acting in the same manner as in the quartering of $\mathfrak{F}$amily.

Worked Example 2 Extend our analysis of cover arenas up to $N = 3$ for essential covers.

Answer 2 $N = 2$ is minimal for there to be any essential covers at all. There is precisely 1: the partition 1 1. Thus

$$\mathfrak{ec}\mathfrak{C}\text{over}(2) = \text{pt} :$$

the trivial order. While

$$\mathfrak{ec}\mathfrak{C}\text{over}(0) = \mathfrak{ec}\mathfrak{C}\text{over}(1) = \emptyset ,$$

viewed as the unorder.

$N = 3$ is the minimal nontrivial case in quite a few ways. Using the same poset-forming white-arrowed order as above induces the order structure in 56.4.a)-b). Since both $\mathfrak{C}\text{over}(N)$'s top and bottom – $\mathcal{T}_{\text{d}}(N)$ and N – contain the universe set, these are not available among the essential covers. Instead,

$$\mathfrak{ec}\mathfrak{C}\text{over}(N)_1 = \mathcal{T}_{\text{d}}(N) - \{ \mathfrak{X}, \emptyset \} :$$

the 'essentially truncated' family version of the discrete topologies $\mathcal{T}_{\text{d}}(N)$. While for $N \geq 3$ there is now no unique bottom. The partitions into ≥ 1 part are among the final configurations. Yet $N = 3$ is also minimal for there to be a final configuration in excess of these: σnake.

One of the various C_4-nonconvex subposets here is

$$\{ 2, \varsigma\text{ingleton}(2); \mathcal{T}_{\text{i}}(2), \text{ nest} \} .$$

$\mathfrak{ec}\mathfrak{C}\text{over}(3)$ is thereby not a lattice. A $\text{K}_{3,2}$ subposet is present:

$$\{ 3 \text{ 2-subsets with 1 singleton}; \overline{1\,1\,1}, \rho\text{attlesnake, chain}; \sigma\text{nake} \} .$$

$\mathfrak{ec}\mathfrak{C}\text{over}(3)$ is thereby non-distributive.

$\mathfrak{ec}\mathfrak{C}\text{over}(3)$'s underlying graph skeleton manages to be planar by the moves in the second column of Fig 56.4.b). Its cycle system is a quadrangulation in the more stringent sense (paralleling that for triangulations), consisting of 10 internal 4-cycles and an external 4-cycle. Thus

$$Y = 11 \text{ and } Z = \frac{11 \times 2}{2} = 11 .$$

It also carries one leaf: the partition 1 1 1.

$\mathfrak{ec}\mathfrak{C}\text{over}(3)$ is planar $\mathfrak{ec}\mathfrak{C}\text{over}[3]$ simply adds a third leaf – the unique essential cover on 2 elements – by which it is immediately planar too.

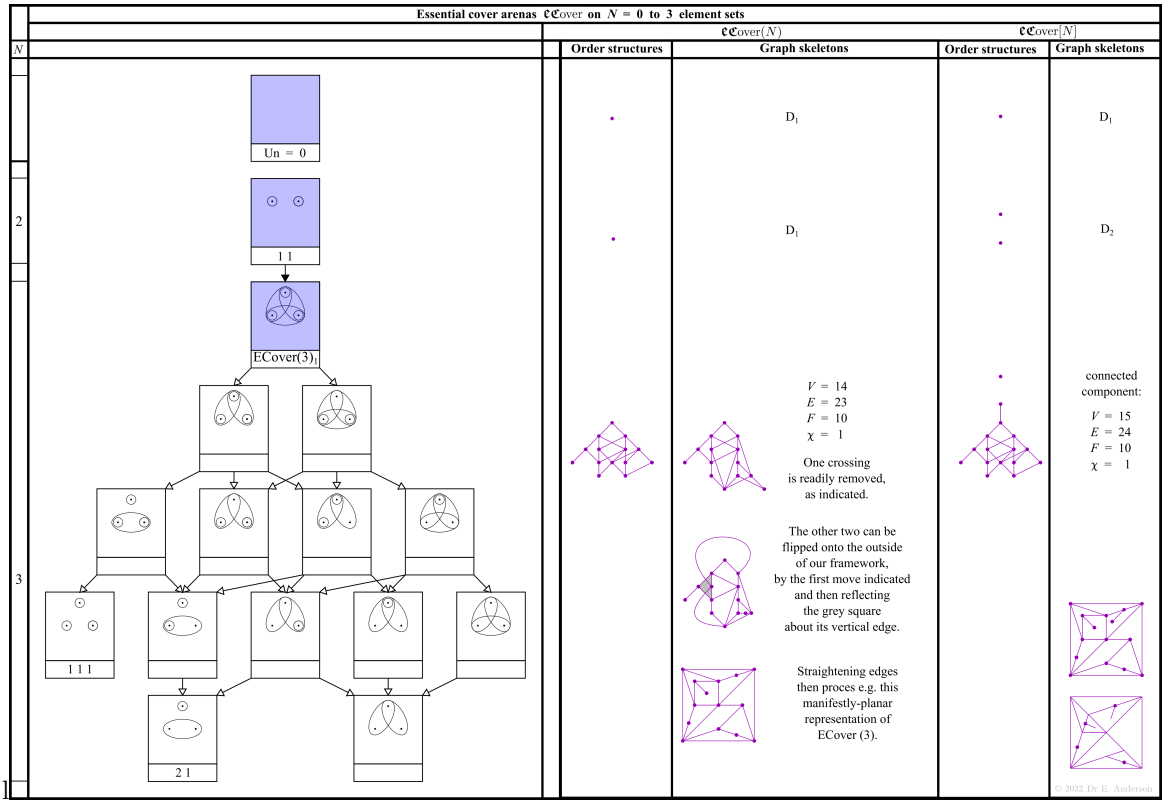


Figure 56.4:

56.3 Topology arenas

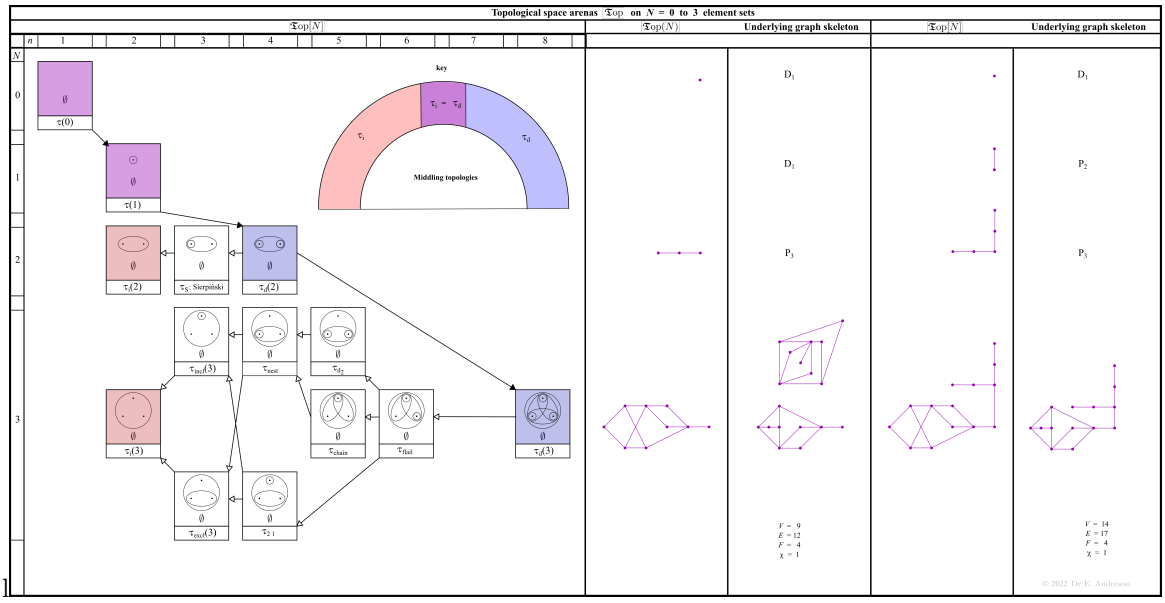


Figure 56.5:

Worked Example 3 Induce a natural ordering structure on $\mathfrak{Top}(N)$ and $\mathfrak{Top}[N]$ for $N = 0$ to 3.

Answer 3 Let us once again use a complementation asymmetry black-arrowed rendition of $\mathfrak{F}$amily's red arrows. In tandem with the topological space variant of the 'remove subset' reversed version of $\mathfrak{F}$amily's white arrows. The black arrows link adjacent discrete topologies, which are the top topologies in our ordering's arrow-orientation. Unlike for covers, each N also has a unique bottom: the corresponding indiscrete topology.

$\mathfrak{T}$op(2) is minimal for top-bottom distinction. As well as for exhibiting a nontrivial chain middle: the Sierpiński topology.

$\mathfrak{T}$op(3) is minimal as regards possessing a nontrivial poset middle. And for not exhibiting complementation symmetry. The $\mathfrak{T}$op(N) are thus bounded but not complemented $\mathfrak{T}$op(3) also contains a C_4-nonconvex subposet, whose vertices are the included-point and excluded-point topologies followed by the nest and 2 1 topologies. Thus $\mathfrak{T}$op(3) is not a lattice.

'Topological space variant' is being used above in the following sense. Some of $\mathfrak{F}$amily's subset removals breach topological space axioms. Suppose that this occurs and yet topological spaces can be reached by further subset removals. Then amalgamate a minimal set of these removals with the original one into a single white arrow.

The minimal nontrivial example for this occurs for $N = 3$ as follows. In $\mathcal{T}_{\text{flail}}$, the 'chain-link' singleton and the 'head' 2-subset can only be removed concurrently. For without the 'chain-link', the two 2-subsets breach the intersections-included axiom. While without the 'head', the two singletons breach the unions-included axiom.

With these arrows, strong connectivity also ensues for $\mathfrak{T}$op(N) within our smallest examples range ($N \leq 3$).

See Fig 56.6 for the topologies as a blue heat map within $\mathfrak{F}$amily[2]. And Fig 56.5 for an intrinsic view up to $N = 3$. $N = 2$ is minimal to have a chain middle. Finally, $N = 3$ is minimal to exhibit a pentagonal sublattices. E.g. flail, chain, nest, excl, 2 1. $\mathfrak{T}$op(N) is thus not distributive.

Exercise 2$^+$ Work out the fully distinguishably labelled counterpart. In which Order-Theoretic ways is this notably different?

Exercise 3$^+$ Find all the topologies on the unlabelled 4-element set. Then place the same ordering structures on this as above.

56.4 Heat map summary so far

Structure 1 See Fig 56.7 for a heat map of the locations of all the covers, essential covers, topologies and partitions within $\mathfrak{F}$amily(3). Where all but one of the squares holding together the bottom four quarter-spaces have been suppressed.

Exercise 4 a) Systematically count out the number of edges in $\mathfrak{F}$amily(3).

b) Deduce a weak bound on the crossing number from calculation of its density.

c) Place a stronger bound by taking into account the girth of the graph as well.

Exercise 5 Provide the orbit-stabilizer analysis for the counts of covers and of essential covers.

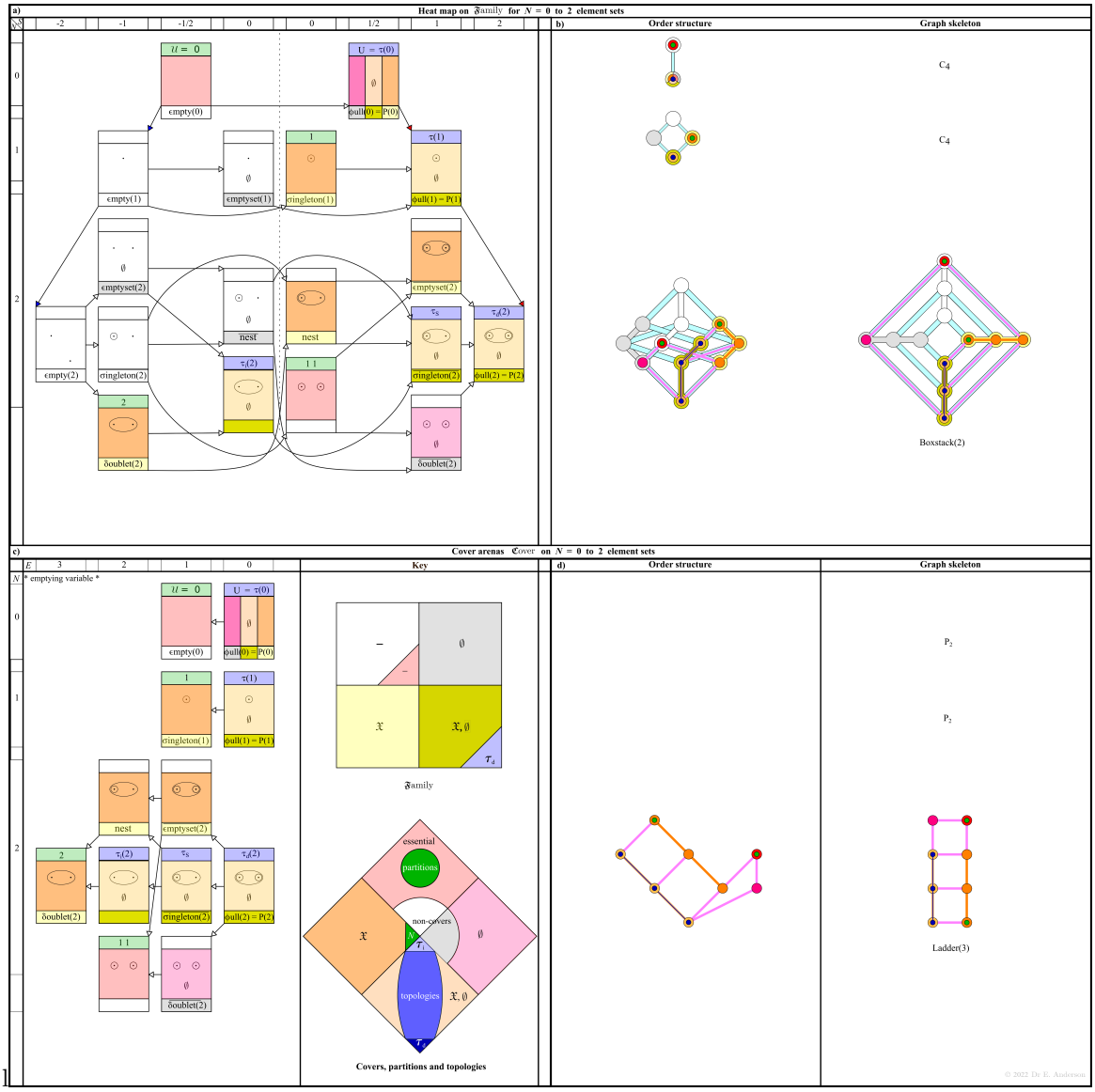


Figure 56.6:

56.5 Discussion and pointers

End-Remark 1 Both **Cover** and **Top** meet the structurally-irregular and yet conceptually-well-motivated criterion for mathematical interestingness. There are many routes to [449] topologies and to covers, and these are furthermore very widely applicable (as Chapter 9 outlined). **Cover** and **Top** are thus complex citizens of Kallista.

End-Remark 2 **Cover**(N) and **Top**(N) sit in an *asymmetric* manner within **Family**(N) as regards complementation. So it is specifically **Family**(N) and not [**Family**](N) that we need for an extrinsic study of these more exhalted subarenas.

♠ ◇ ♣

Exercise 6$^+$ Fig 56.8, obtained via the ZIPHoN Theorem on the planar subgraph of the first presentation in the last column with its curved edges removed, shows that **Family**$_4(3)$ = CompassRose

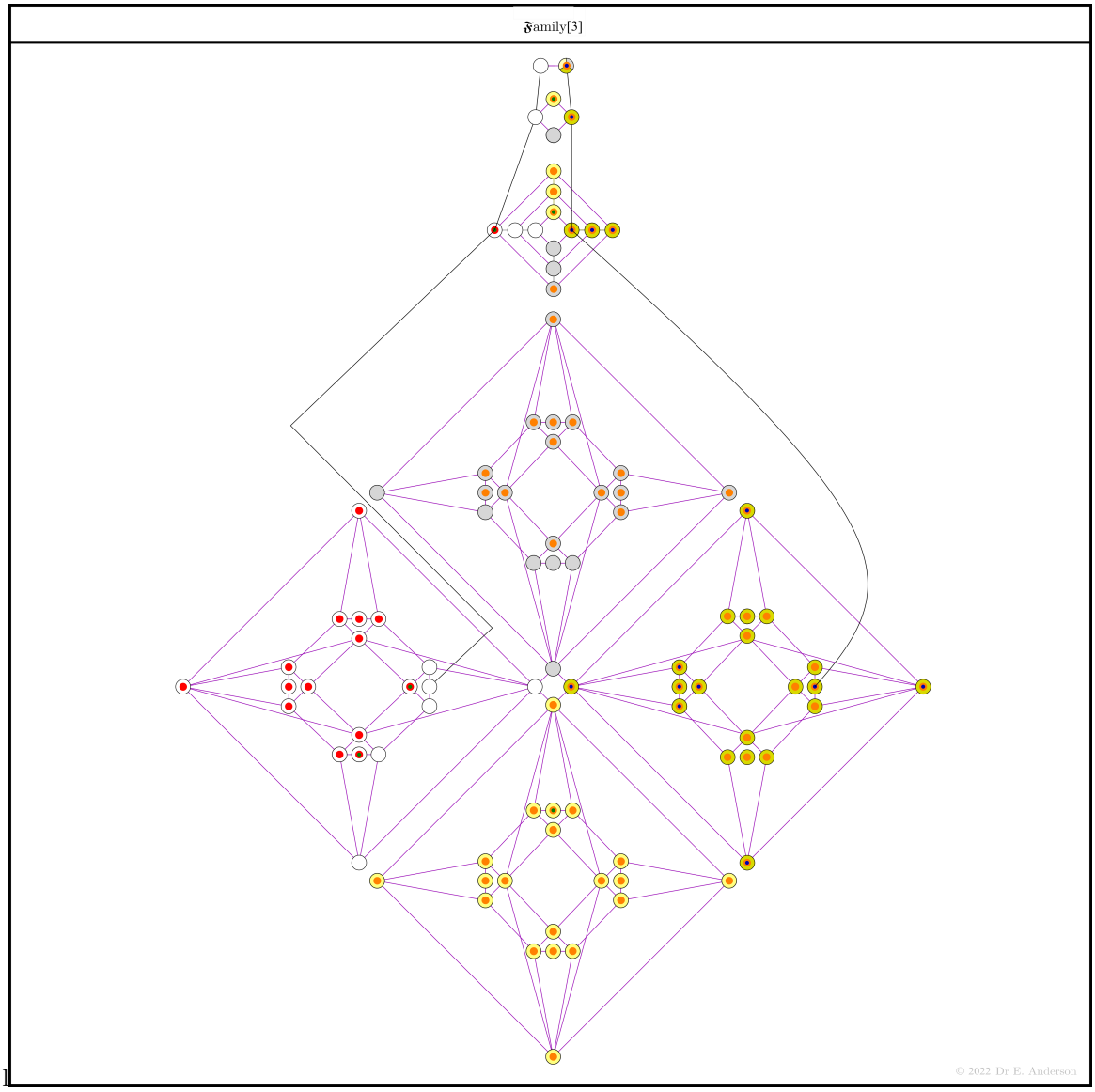


Figure 56.7:

is Hamiltonian, answering Exercise 46.18.a). Show that $\mathfrak{F}$amily(3) = CompassRose □ C_4 is also Hamiltonian.

Remark 1 Assorted arenas from this and the previous Chapter are in Fig 56.8. $\mathfrak{F}$amily$_4$(3) = CompassRose is furthermore Eulerian (**Exercise 7**$^-$).

♠ ◇ ♣

End-Remark 3 $\mathfrak{C}$over form an attractor set within $\mathfrak{F}$amily: once a family covers, adding further subsets to it will never breach the cover condition. Partitions and topologies do not have this permanency.

Pointer 1 The distribution of topologies within $\mathfrak{F}$amily requires further study. Also see e.g. [212] and [452] for more about order structures intrinsic to $\mathfrak{T}$op.

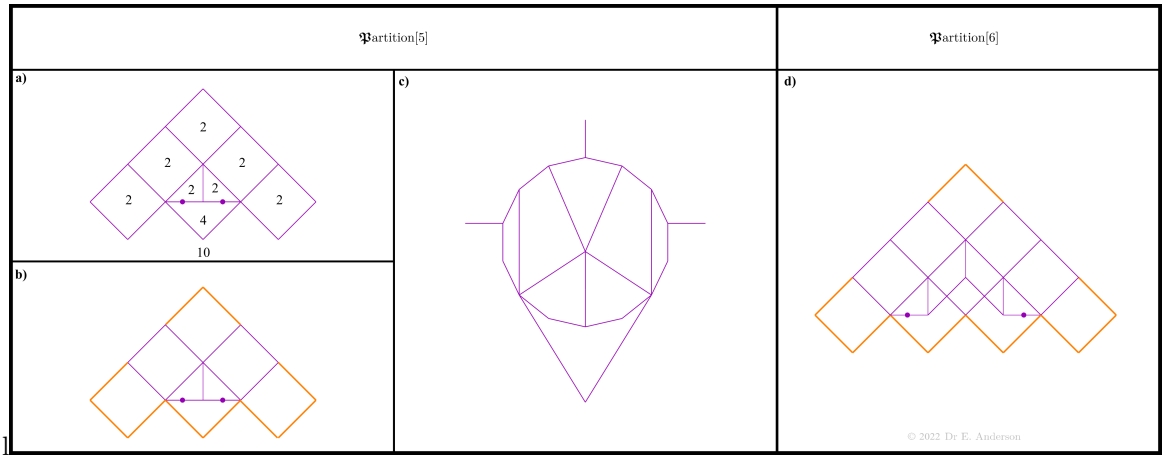


Figure 56.8:

Exercise 8 Since $\mathfrak{Top}$ itself has a vast number of interesting subarenas, it is reasonable to consider further orders natural to $\mathfrak{Top}$.

a) One simple possibility is to use 'add element to the indiscrete topology' in place of the above black arrows. Give the ensuing order structures and underlying graph skeletons for $N \leq 3$.

b) What if both of these types of black arrow are used at once? Show in particular that the ensuing $\mathfrak{Top}[3]$ is nonplanar.

Pointer 2 The family arrows do *not* manage to produce a connected arena of partitions $\mathfrak{Partition}$ (the objects labelled in green in the heat map). The partitions do better under further orders that are intrinsic rather than inherited from $\mathfrak{Family}$; see the next Chapter for a start.

Chapter 57

Partition arenas

57.1 The Young lattice on $\mathfrak{P}$artition

Structure 1 The *arena of partitions* is

$$\mathfrak{P}\text{artition} \leq \mathfrak{C}\text{over} \leq \mathfrak{F}\text{amily} .$$

Furthermore, the *arena of nontrivial partitions* – to the exclusion of the 1-part partitions – is

$$\mathfrak{P}\text{artition}_* \leq \mathfrak{ec}\text{over} .$$

While $\mathfrak{F}$amily has a white-arrowed 'add subset' order, this cannot possibly map between partitions. So within each $\mathfrak{F}$amily(N), $\mathfrak{P}$artition(N) furthermore forms an antichain. $\mathfrak{F}$amily's black-arrowed 'add element' operation combined with adding the singleton around that element does map between partitions with adjacent N. This is however rather limited as an ordering on partitions. Partitions thus require further intrinsic ordering rather than induced ordering.

Structure 2 The most commonly pursued option is to place a Young diagram box-add operation on $\mathfrak{P}$artition. This is another black-arrowed operation: the current Volume's second example of a purely black-arrow ordering of an arena. The resulting structure is the well-known *Young lattice* [217].

Worked Example 1 Show that this is a semi-bounded lattice. What basic order properties does it have? What explicit order structures and underlying graph skeletons arise thus for $N \leq 6$?

Answer 1 Such addition of structure guarantees a poset. The meet and join of two partitions are here provided by the intersection and the union of their Young diagrams. These operations fulfill the definition of lattice (without the lower boundedness criterion), and enable distributivity.

Our order structure has a unique top element, the unpartition:

$$\mathfrak{Y}\text{oung}(N)_1 = \mathfrak{P}\text{artition}(N, \text{box-add})_1 = \mathfrak{U} .$$

It has no bottom elements, while its Nth approximands for each $N \geq 2$ do not have unique bottoms. Thus it is semi-bounded, by which we have a semi-bounded lattice. Semi-boundedness also renders the entire Young semi-bounded lattice acomplemented for $N \geq 2$.

See Fig 57.1 for the finite Young lattice on $\mathfrak{P}$artition's Nth approximand, $\mathfrak{P}$artition[N]. The $N = 0$ and 1 cases that elude the above bans on boundedness and being complemented are clearly just trivial points.

For $N = 4$, we have, as depicted, a cycle system of 3 internal 4-cycles with an external 8-cycle. With 4 leaves. Thus

$$Y = 4 \text{ and } Z = \frac{3 \times 2 + 6}{2} = 6 .$$

For $N = 5$, we have, as depicted, a cycle system of 7 internal 4-cycles and 1 6-cycle with an external 12-cycle. With 3 leaves. Thus

$$Y = 9 \text{ and } Z = \frac{7 \times 2 + 4 + 6}{2} = 12 .$$

The 6-cycle here is not a poset cycle.

Finally, $N = 6$ is minimal for nonplanarity. Answering Ex 48.10.b), this has 2 $K_{3,3}$ subgraphs (midline-reflection) symmetrically placed without sharing any crossings. So by Worked Example 48.2, $cr(\mathfrak{Y}\text{oung}[6]) \geq 2$. But we furthermore exhibit a 2-crossing presentation. Thus $cr(\mathfrak{Y}\text{oung}[6]) = 2$ is ascertained.

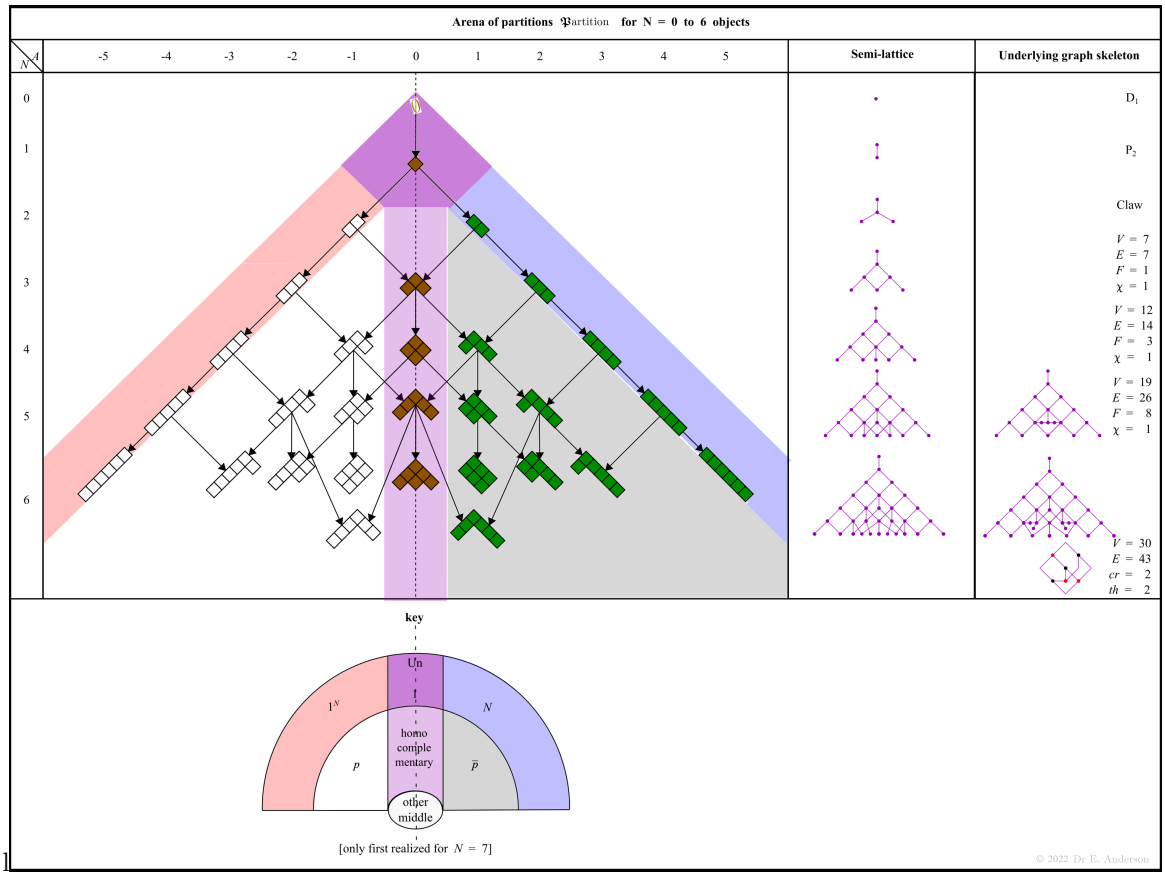


Figure 57.1:

Remark 1 Observe that the box-add ordering operation already manages to respect partitions' own particular C_2 symmetry: that with partition-conjugation action. Thus unlike for sets and families, there is no need to further symmetrize the N-increasing order structure. [This also justifies our having shaded our arrows black rather than blue.]

Remark 2 Also observe that the Young semi-lattice is not structurally-regular and yet conceptually well-motivated, by which it is mathematically interesting. It furthermore arises in the Representation

Theory of symmetric groups (Volume 5), and again, with labels, in that of Lie groups (Volume 7). All in all, it is a citizen of Kallista.

57.2 Its quotient by partition conjugation

Worked Example 2 Study this for $N \leq 6$.

Answer 2 By being partition-conjugation-compatible, the box-add operation continues to serve for [$\mathfrak{P}$artition]. Union and intersection of sets continue to serve as joint and meet. Semi-boundedness is inherited.

See Fig 57.2 for [$\mathfrak{P}$artition][N] with $N \leq 6$. For $N = 5$, we have, as depicted, a cycle system of 3 internal 4-cycles with an external 8-cycle. With 2 leaves and a 2-path sidechain. Thus

$$Y = 4 \text{ and } Z = \frac{3 \times 2 + 6}{2} = 6 .$$

For $N = 6$, we have, as depicted, a cycle system of 5 4-cycles and a 6-cycle internally, with an external 10-cycle.

With 2 leaves and a 2-path sidechain. Thus

$$Y = 7 \text{ and } Z = \frac{5 \times 2 + 4 + 6}{2} = 10 .$$

The 6-cycle here is not a poset cycle.

These models are not large enough to manifest nonplanarity.

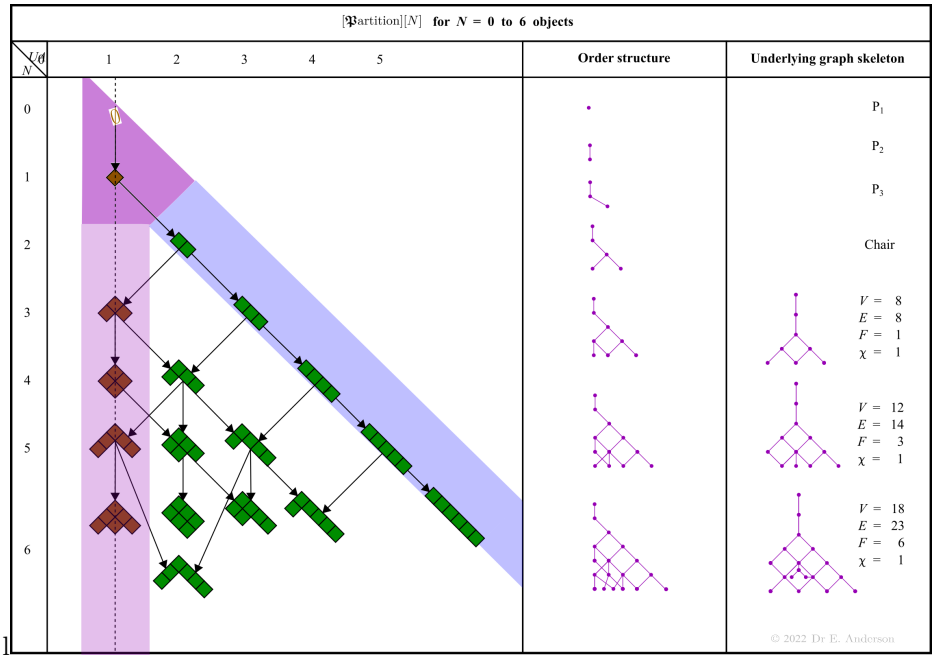


Figure 57.2:

57.3 Other lattices on 𝔓artition

Exercise 1 We can either use 'add box' or 'slide down box', in the following sense.

a) Were we to use both at once, show that we would not even have a poset.

b) Were we to define 'slide down box' without Operation 2 of Sec 15.6's comment, show that not even a poset would ensue.

c) Show that passing to [partitions] scarcely ameliorates this situation.

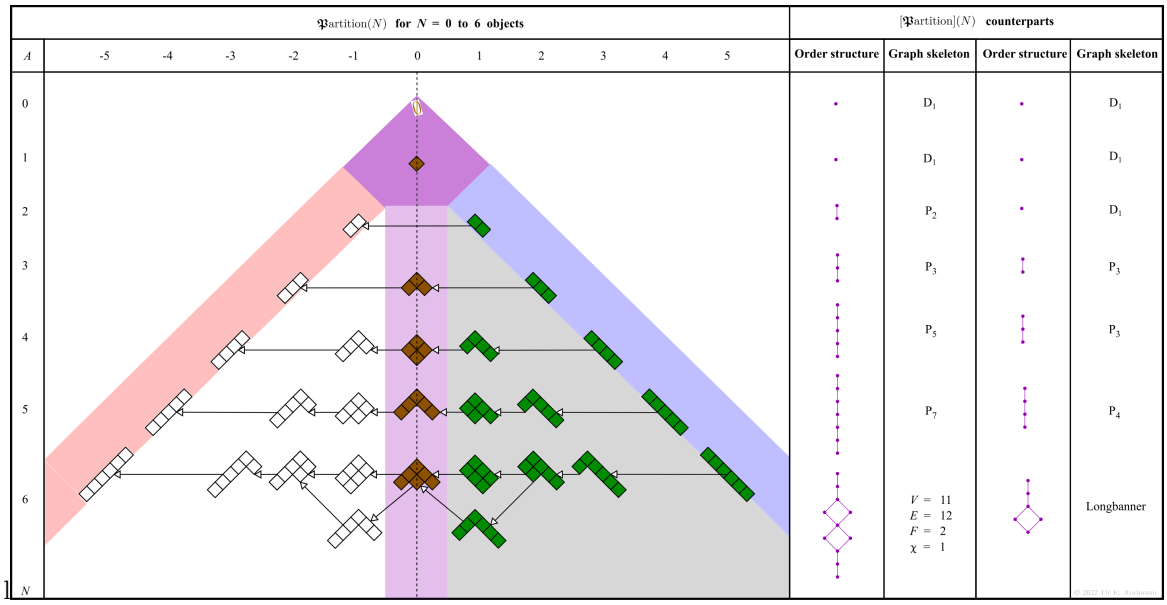


Figure 57.3:

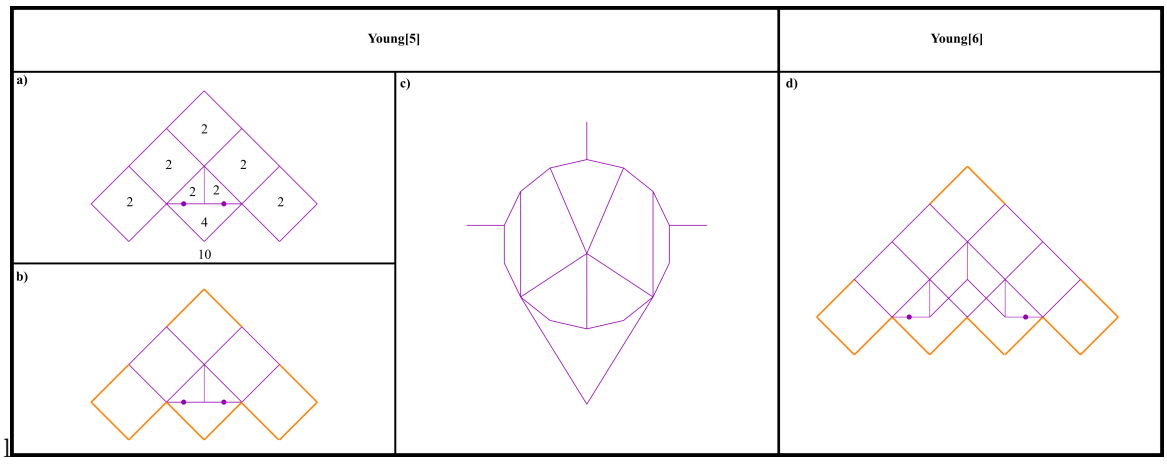


Figure 57.4:

Remark 1 For 𝔜oung[5] by Fig 57.4.a), the cycle system has $Z = 14$, which is inconclusive. Fig 57.4's forced edges however straightforwardly preclude a Hamiltonian cycle. A 14-cycle is supported,

giving the alternative presentation in Fig 57.4.c). Fig 57.4.d)'s forced edges similarly confer non-Hamiltonianness on $\mathfrak{Y}$oung[6].

Exercise 2 Is [$\mathfrak{Y}$oung][6]'s cycle system Hamiltonian?

Remark 2 The above operations thus only afford a choice to either order each $\mathfrak{P}$artition(N) or $\mathfrak{P}$artition[N] as a whole, but not both. [Or at least not both without losing almost all order structure in the process.]

Remark 3 Contrast with the situation for subsets and for families, where we can concurrently use 'downwards' and 'rightwards' operations.

Exercise 3 Extend the current Chapter's analysis to $\mathfrak{Y}$oung[7].

Pointer 1 The above white-arrowed operation is conceptually an exchange of parts. Another option would be to split parts: a fine-graining operation. [50] presents a third option based on a notion of domination, and [43] presents two more.

57.4 More specialized partition arenas

Exercise 4 Find the order structure and underlying graph skeletons for

a)

$$\mathfrak{P}\text{artition}(\, N;\ \leq\ k\ \text{ sized pieces}\,)\ \text{ with } N\ \leq\ 7\ .$$

b)

$$\mathfrak{P}\text{artition}(N;\, m\ \text{ parts exactly })\ \text{ with } N\ \leq\ 7\ .$$

c)

$$\mathfrak{P}\text{artition}(\, N;\ \text{all parts odd}\,)\ \text{ with } N\ \leq\ 9$$

and

$$\mathfrak{P}\text{artition}(\, N;\ \text{all parts even}\,)\ \text{ with } N\ \leq\ 10\ .$$

Remark 1 The nontrivial partitions, in the sense of consisting of more than one piece,

$$\mathfrak{P}\text{artition}(\, N; \geq\ 1\ \text{ piece}\,)\ \leq\ \mathfrak{C}\mathfrak{C}\text{over}(N)\ .$$

Exercise 5$^+$ Find the lattices of homoconjugate partitions into all parts odd for $N\ \leq\ 21$.

Exercise 6 Find the lattices of partitions with no parts equal for $N\ \leq\ 7$.

Exercise 7 a) Represent the $N\ \leq\ 6$ restrictions of your answers to Exercises 4 to 6 as a heat map on our depiction of the Young lattice.

b)$^+$ Extend this to include $N\ =\ 7$ as well.

Exercise 8 Which of these more specialized partitions are realized as single-connected-component pieces of Young's lattice (at least for this range of values of N)? Which are nested within which others, which are disjoint, and which exhibit intersections?

Exercise 9$^+$ Place a coarse-graining order on $\mathfrak{P}$artition[N] with $N\ \leq\ 6$. And on $\mathfrak{C}$over[N] with $N\ \leq\ 2$. And on $\mathfrak{T}$op(N) for each $N\ \leq\ 3$.



Chapter 58

Lattices of subgroups

58.1 Introduction

Exercise 1 a) Show that the set of subgroups $\{ H_i \}_{i \in \mathfrak{I}}$ of a given group G form a poset under the 'is a subgroup or equal to' interpretation of the symbol $\leq$.

b) Show furthermore that the intersection $H_1 \cap H_2$ of two subgroups H_1, H_2 of a group G is itself a subgroup.

c) Show that the union $H_1 \cup H_2$ of two subgroups H_1, H_2 of a group G is not necessarily itself a subgroup.

Thus intersection is available as a meet operation, but union is not available as a join operation. Show however that the structure generated by the union of two subgroups, which we denote by $H_1 + H_2$, is a subgroup of G. Use this as a join operation to establish that the subgroups of a group G constitute a lattice,

$$\mathfrak{L}(\text{subgroups}; G) \ .$$

What are the top and bottom elements?

d) Show furthermore that

$$H \cap K \leq H_1 \ , \quad H_2 \leq H_1 + H_2 \ .$$

Exercise 2 What is the lattice of subgroups for each of the trivial group, C_2, C_3, C_4, C_5, C_6, $C_2 \times C_2$ and $D_3 \cong S_3$?

Exercise 3 What is the lattice of subgroups for each of the following?

a) C_p for p prime.

b) $C_{p\,q}$ for p, q distinct primes.

c)

$$C_{\prod_{i=1}^{k} p_i^{n_i}}$$

for p_i all distinct primes occurring with multiplicities $m_i \in \mathbb{N}$.

Exercise 4$^+$ a) For G a commutative group, show that its lattice of subgroups must be modular, but is not necessarily distributive.

b) For G a non-commutative group, show that its lattice of subgroups is not necessarily modular.

Exercise 5 a) Show that the normal subgroups of a group must form a lattice.

b)$^+$ What is this lattice for D_n, A_4, S_4 and Q_8?

Exercise 6$^+$ Show that the lattice of normal subgroups of a group must furthermore be modular but is not necessarily distributive.

Exercise 7 What is the full lattice of subgroups of A_4, S_4 and Q_8?

58.2 Smallest examples of lattices of subgroups

We provide here figures 58.1- which answer some of the opening section's Exercises, for their subsequent usefulness.

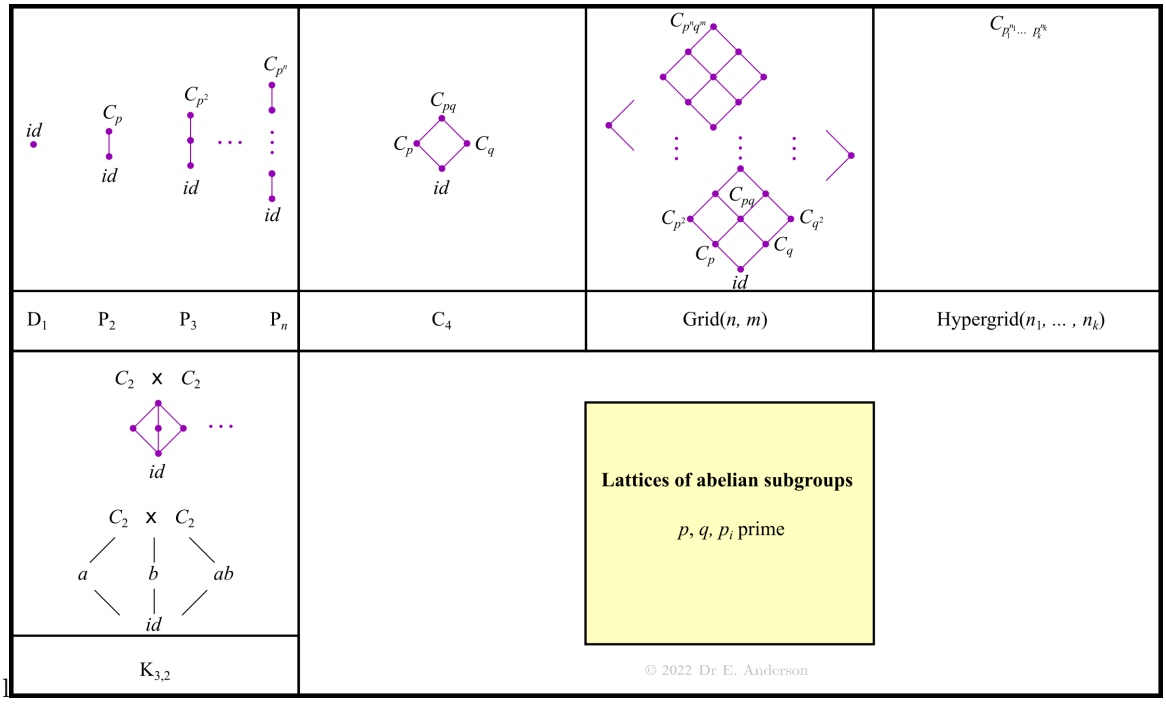


Figure 58.1:

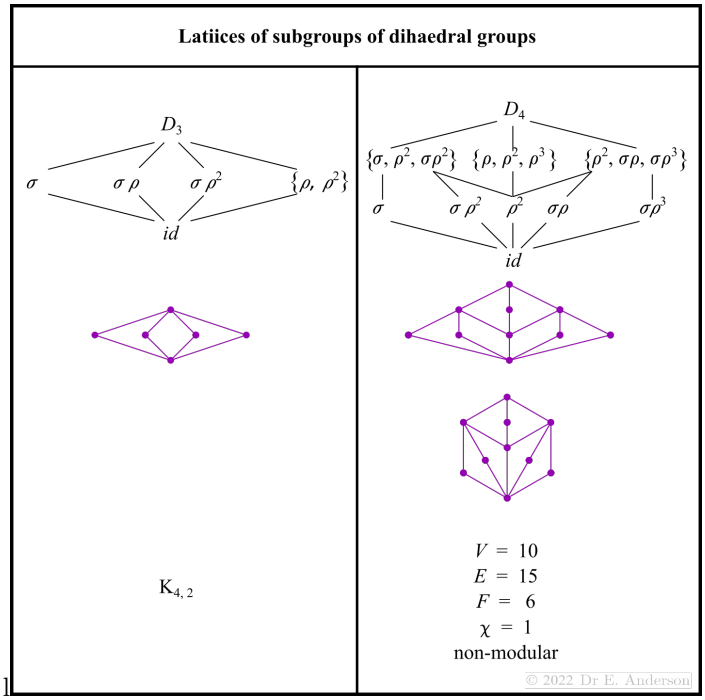


Figure 58.2:

Exercise 8 Give the lattices of normal subgroups, and the full lattices of subgroups, for the following.

a)$^+$ $Sym(\mathrm{C}ube)$.

b)$^+$+ A_5 and S_5.

Pointer 1 See e.g. [221] for somewhat more advanced reading on this topic.

58.3 Lattices of distinct subgroup actions

Given an arena $\mathfrak{Ob}$ of objects Ob that a group $\mathfrak{g}$ acts upon, some subgroups of $\mathfrak{g}$ may not act distinctly. In this case, a lattice of *distinct subgroup actions* on $\mathfrak{Ob}$

$$\mathfrak{L}\left(\vec{G}\ \mathfrak{Ob}\right) \ ,$$

that is smaller than the lattice of subgroups itself, is of relevance. Its top and bottom elements are the whole group's action on $\mathfrak{Ob}$ and the trivial action on $\mathfrak{Ob}$. id acting on each Ob to simply return that same Ob again).

Worked Example 1 What is the lattice of actions of subgroups of $SO(3)$ on a regular dodecagon?

Answer 1 This object is only rotationally-symmetric in the 2-d plane it is contained in. So nothing but $SO(2)$ and its subgroups can survive as distinct actions. SO(2)'s subgroups are discrete rotations, i.e. the cyclic groups C_n (including the identity, corresponding to the trivial action). The lattice realized by the surviving distinct such is that of the divisors of 12. [Which you can work out from Fig 51.6.]

Examples 2 to 5 See Fig 58.5 for symmetry-breaking sublattices of subgroup action lattices.

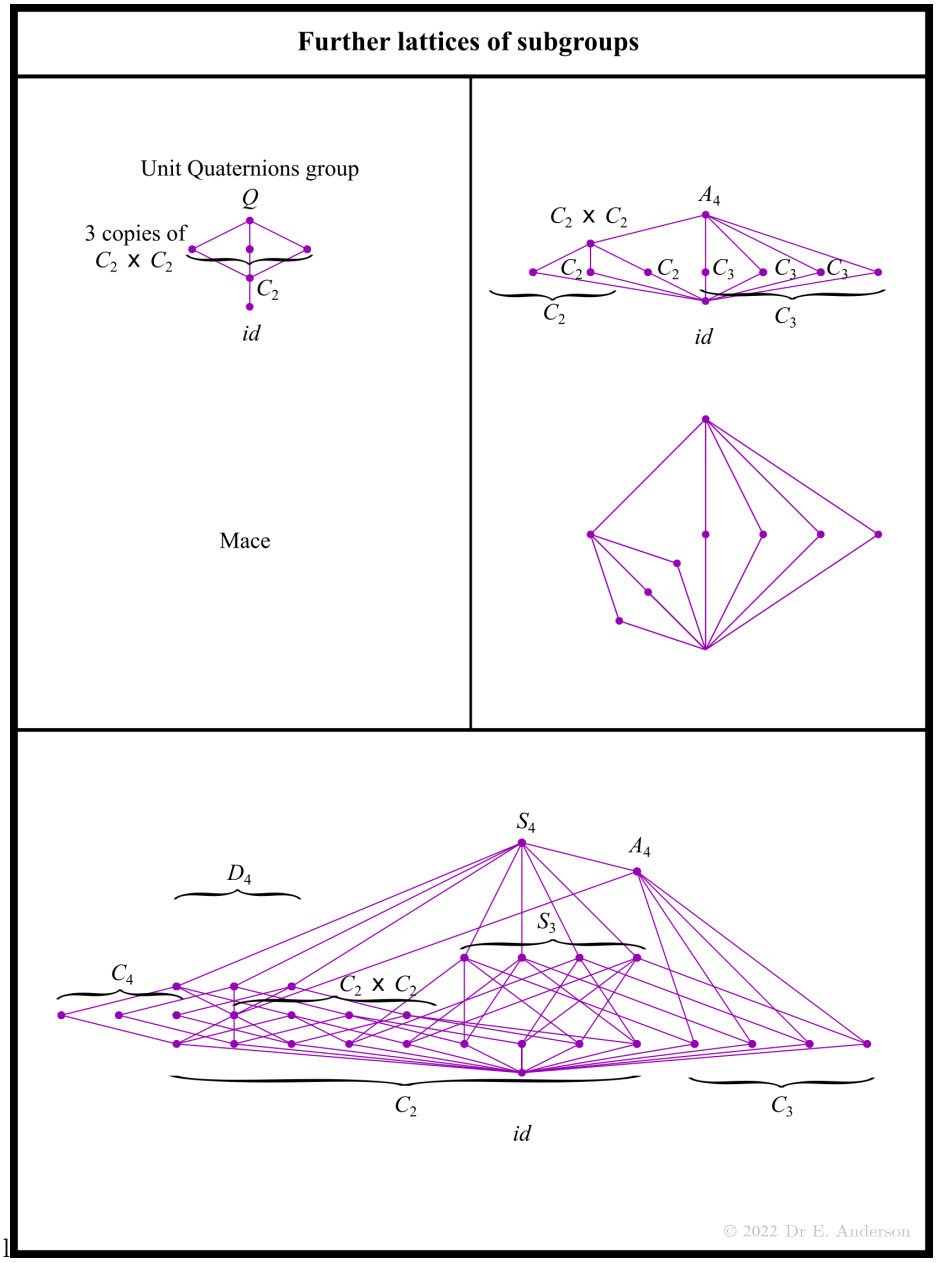


Figure 58.3:

Exercise 9 What is the lattice of actions of $O(3)$ on sulphur hexafluoride SF_6?

58.4 Lattices of quotient groups

These form the dual lattice to the lattice of normal subgroups. See Fig 58.6 for examples.

58.5 Quotient spaces

These now form the corresponding dual lattice to the full lattice of subgroups.

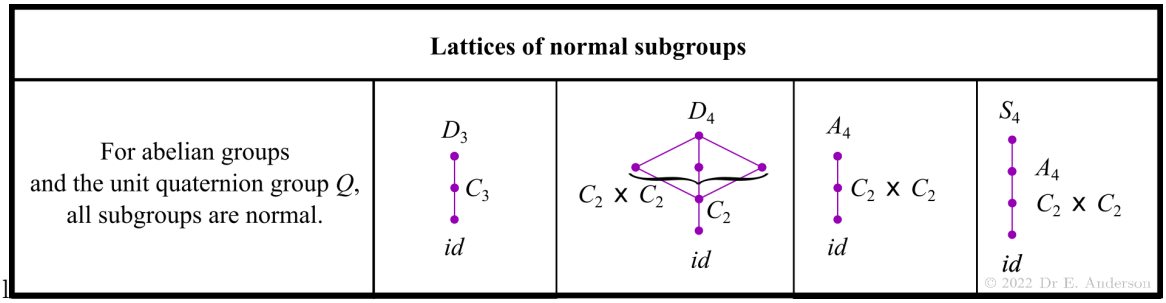


Figure 58.4:

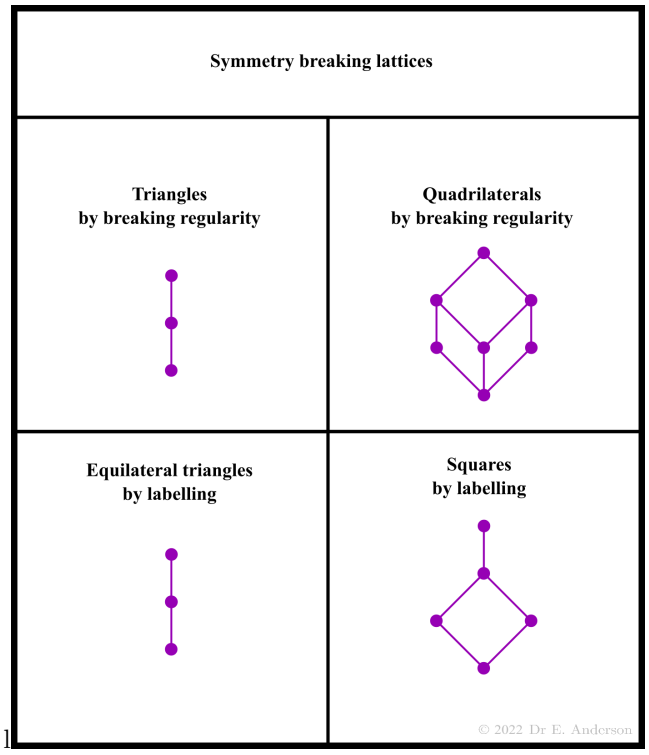


Figure 58.5:

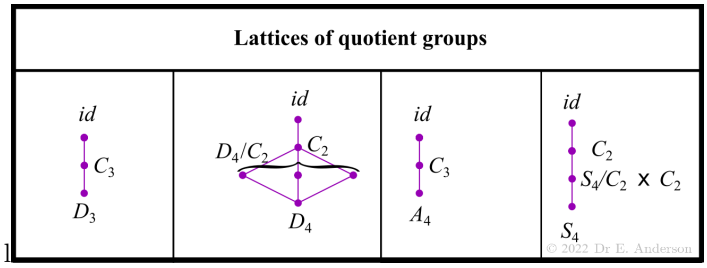


Figure 58.6:

Example 1 Quotienting the 1-d 3-body problem by $Sim(1)$ gives an eightfold of quotients depending on which of the translation, dilation and reflection are included. These form the dual lattice Cube as per Fig 58.7.

This lattice also occurs for $Sim(d)$ with $d \geq 2$ is under the restriction of taking the translations as a single block and the rotations as a single block. Lifting this restriction, the possibilities are more extensive, though still straightforward to work out.

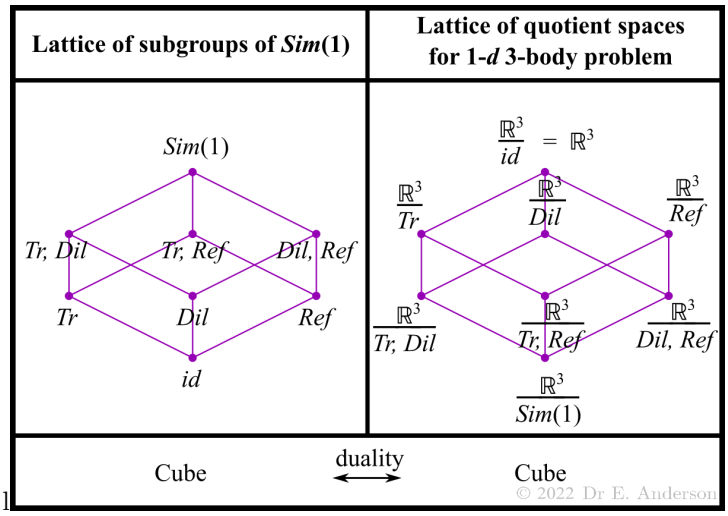


Figure 58.7:

Exercise 10 a) What happens if we quotient the 2-d 3-body problem's naïve configuration space

$$(\mathbb{R}^2)^3 \;=\; \mathbb{R}^6$$

by $Eucl(2)$? Assume that each of rotations, dilations and reflections can be dropped separately.

b) As a bonus, prove that each of these can dropped separately.

[If you do not know how to do b) at this stage, try again after studying Part I of Volume 1 [441].]

Pointer 2 See e.g. [396, 433, 434, 443, 445, 448] for further examples of quotient spaces and lattices thereof.



Chapter 59

Graph arenas

59.1 Introduction

Definition 1 Let us denote the *arena of point clouds* by

$$\mathfrak{P}\text{oint}\mathfrak{C}\text{loud} \ .$$

And the *arena of graphs* by

$$\mathfrak{G}\text{raph} \ .$$

Remark 1 Since counting suffices to characterize point clouds,

$$\mathfrak{P}\text{oint}\mathfrak{C}\text{loud} \ = \ \mathbb{N}_0 \ .$$

Structure 1 Within each fixed N, arenas of graphs carry the natural 'add-edge' ordering operation. For compatibility with complementation, we rephrase 'add edge' as 'switch edge colour from red to blue in the Ramsey presentation [232].

Structure 2 In passing between the arena of graphs of one order N_1 and that of another distinct order N_2, a first natural operation is 'add a disconnected vertex'. This operation is not however compatible with complementation. This motivates viewing 'add disconnected vertex' as forming a red cone. We can then complementation-symmetrize by additionally forming a blue cone.

Worked Example 1 What order structures and underlying graph skeletons do the above orders give on the $N \ = \ 0$ to 4 graphs?

Answer 1 The add-edge operation provides poset structure by Exercise 50.17. As per Fig 59.1.a)-b), up to $N \ = \ 3$, these are furthermore chains.

$$\mathfrak{G}\text{raph}(N)_1 \ = \ \text{D}_N \ = \ \text{pt}^N$$

and

$$\mathfrak{G}\text{raph}(N)_0 \ = \ \text{K}_N \ .$$

So the $\mathfrak{G}\text{raph}(N)$ are bounded posets. Graph complementation furthermore guarantees that the $\mathfrak{G}\text{raph}(N)$ posets are complemented.
Because the rightmost elements have the same edge number, the $\mathfrak{G}\text{raph}(4)$ crossings actually form part of not a C_4-nonconvex but a $\text{K}_{3,2}$ subposet. Namely,

$$\{ \, \overline{\text{Paw}}; \ \text{Claw}, \ \overline{\text{Claw}}, \ \text{P}_4; \ \text{Paw} \, \} \ .$$

The $\mathfrak{G}\text{raph}(N)$ are thereby not in general distributive. The Reader should check how 'the intersections and unions argument giving distributivity' fails here because of edges' nontrivial structural content (**Exercise 1**). Finally, observe that $\mathfrak{G}\text{raph}(4)$ manages to be a modular lattice.

A single vertex move renders $\mathfrak{G}\mathrm{raph}(4)$'s underlying graph skeleton manifestly planar. As depicted, its cycle system consists of 4 internal 4-cycles with an external 8 cycle. There are also 2 leaves. Thus

$$Y = 5 \text{ and } Z = \frac{4 \times 2 + 6}{2} = 7 .$$

For now, we consider 'add vertex' operations as acting on whichever of top or bottom elements only. The ensuing comb and suspension versions of $\mathfrak{G}\mathrm{raph}[N]$ and $\mathfrak{G}\mathrm{raph}_{\mathrm{S}}[N]$ add nothing to the planarity analysis. The $\mathfrak{G}\mathrm{raph}_{\mathrm{S}}[N]$ are moreover not even posets because of their paw subgraph summit's triangle. Excising this is slightly less motivated than for subsets, as we would now need to kick out the point as well as the unpoint.
$\mathfrak{G}\mathrm{raph}_{\mathrm{S}}(4)$'s cycle system, as depicted, has an 11-cycle, a 6-cycle, 4 4-cycles and a triangle. Its external cycle is a 12-cycle. Thus

$$Y = 8 \text{ and } Z = \frac{10 + 9 + 4 \times 2 + 4 + 1}{2} = 16 .$$

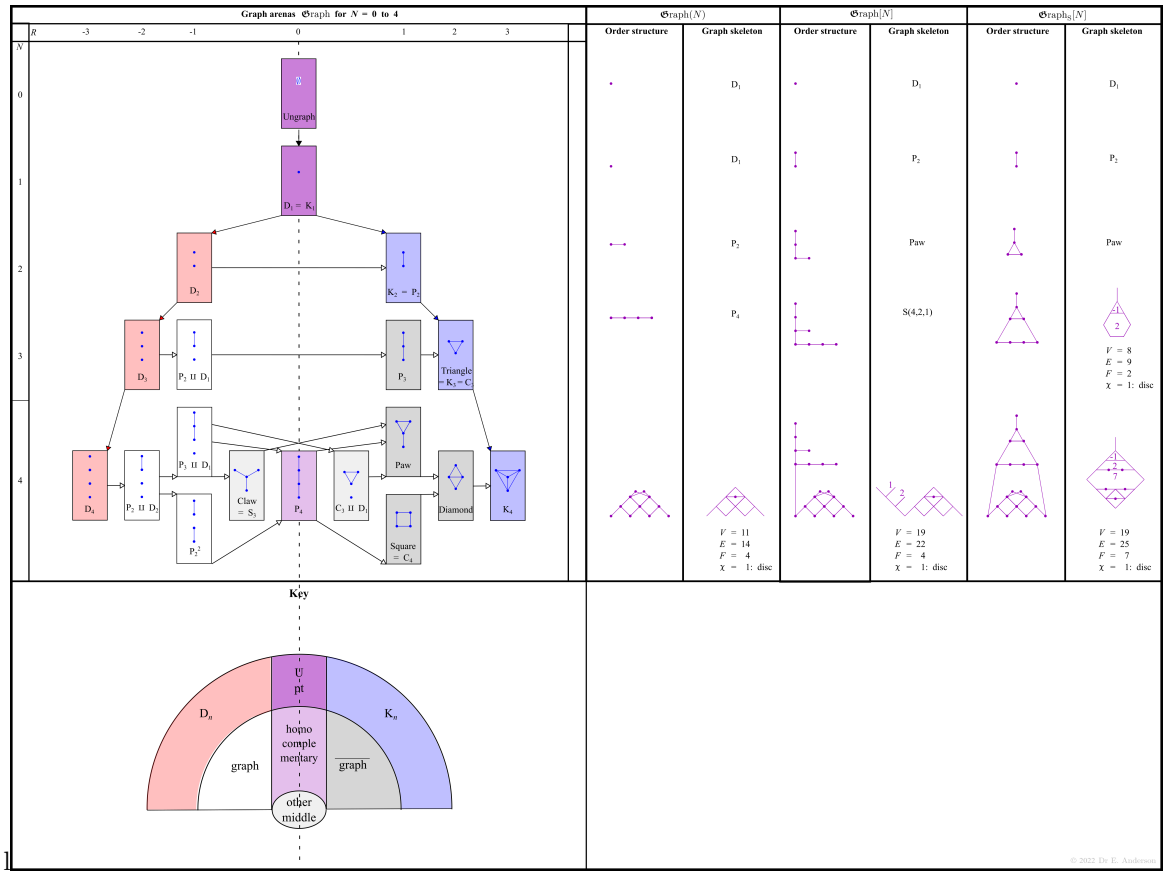


Figure 59.1:

59.2 Democratic orders on 𝔊raph

Structure 3 The idea here is to apply whichever of 'add loose vertex', or its complement to all graphs rather than just to the D_N and K_N. This is a simplex-dimension democratization of how we treat vertices and edges for graphs. Let us denote the resulting democratic cumulative graph arenas by

$$\mathfrak{G}\mathrm{raph}_{\mathrm{D}}[N] \text{ and } \mathfrak{G}\mathrm{raph}_{\mathrm{DS}}[N] .$$

Worked Example 2 What are these on $N = 0$ to 4 vertices?

Answer 2 The first nontrivial $\mathfrak{G}\mathrm{raph}_{\mathrm{D}}[N]$ occurs for $N = 4$. As depicted, internally this has 8 4-cycles and externally a 14-cycle. So, invariantly

$$Y = 9 \text{ and } Z = \frac{8 \times 2 + 12}{2} = 14 .$$

It possesses a $\mathrm{K}_{3,2}$ subgraph and 2 sidechains.

The first nontrivial $\mathfrak{G}\mathrm{raph}_{\mathrm{DS}}[N]$ occurs for $N = 3$. As depicted, internally this has a triangle and 2 4-cycles, and externally a 4-cycle. It also has a single leaf. It thus once again cannot be a poset unless the point configuration is excised. We leave $\mathfrak{G}\mathrm{raph}_{\mathrm{DS}}[4]$'s nonplanarity analysis to Exercise 48.4.

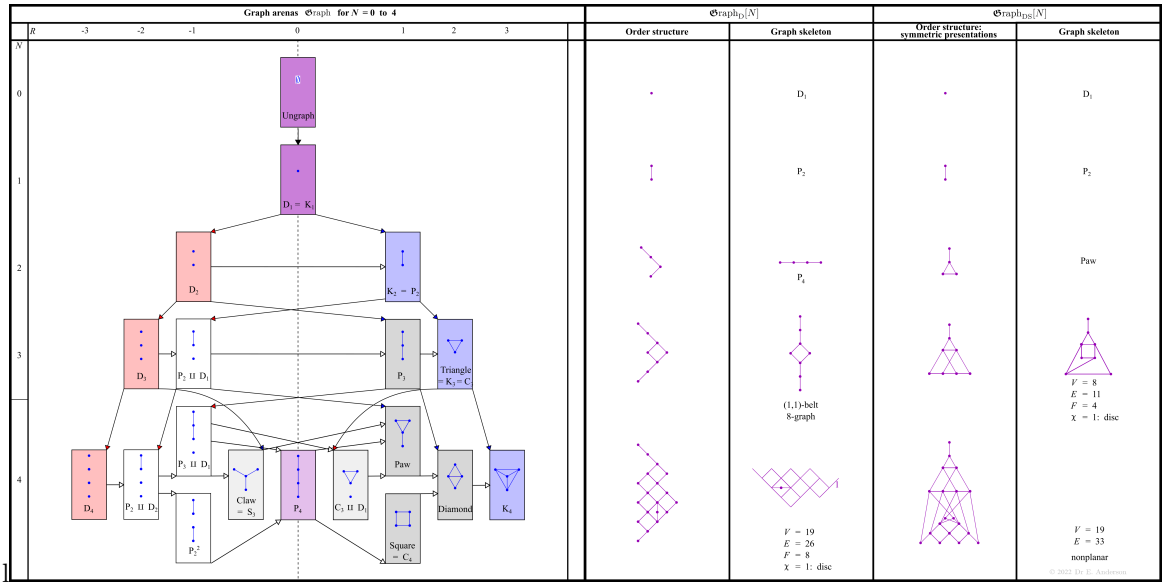


Figure 59.2:

59.3 [𝔊raph](N)

Definition 1 The arena of [graphs] – graphs modulo complementation – is

$$[\mathfrak{G}\mathrm{raph}] . \tag{59.1}$$

Definition 2 The homocomplementary graphs constitute a well-defined subarena of graphs: the *arena of homocomplementary graphs*,

$$\mathfrak{HC} . \tag{59.2}$$

Worked Example 3 What are the [graph] counterparts of Worked Examples 1 and 2's arenas?

Answer 3

$$[\mathfrak{G}\mathrm{raph}](N)_0 = [\mathrm{D}_N, \mathrm{K}_N]$$

while there are no unique bottom elements for $N \geq 4$. These arenas are thus semi-bounded. That these are posets and distributive carries over. Complementation does not.

All our [$\mathfrak{G}$raph](N) give standard posets and standard graphs (Fig 59.3.b). Let us next additionally bring in the blue-cone or democratic orderings. The [$\mathfrak{G}$raph]$_{\rm D}[N]$ remain standard (Fig 59.3.c) [$\mathfrak{G}$raph]$_{\rm DS}[4]$ yields Pincenez – the twin 4-cycle homeomorph of Bowtie – with 2 sidechains (Fig 59.3.d). Finally observe that these $N = 4$ cases are considerably simpler than their unreduced counterparts. They furthermore all manage to be semi-bounded lattices.

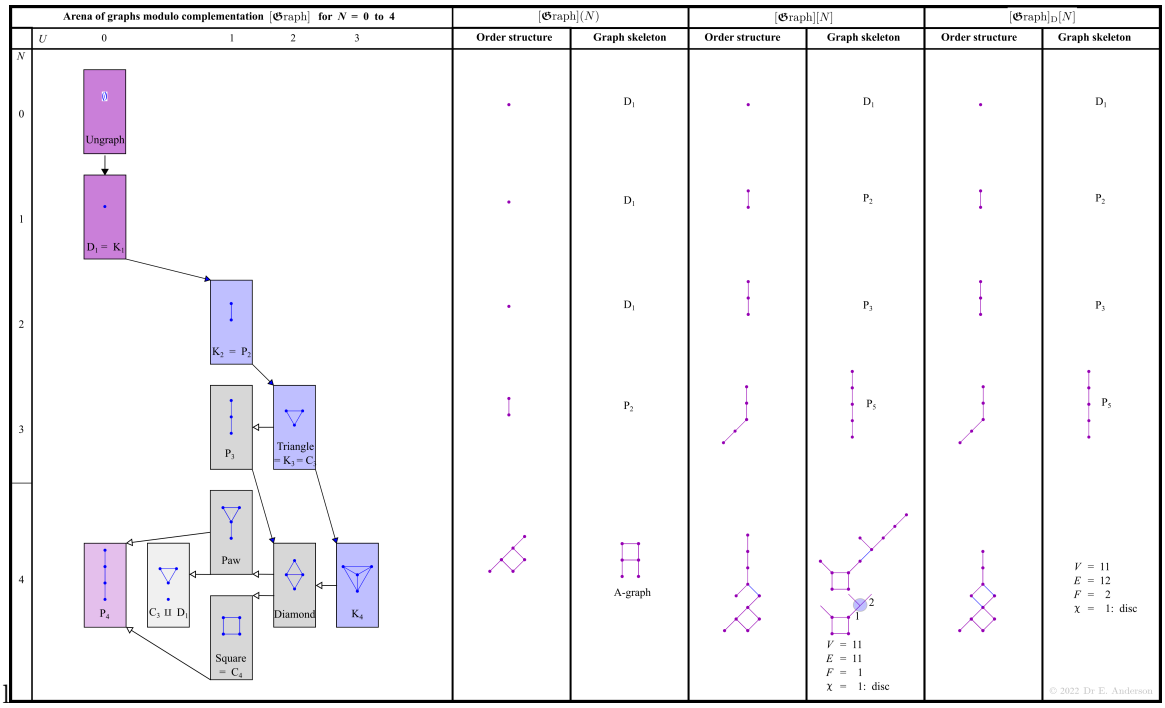


Figure 59.3:

59.4 Order structure on this Book's cover's 'cartoon butterfly arena' of 5-graphs

Worked Example 4 Extend the above structures to $\mathfrak{G}$raph(5) and [$\mathfrak{G}$raph](5). Disprove that the $\mathfrak{G}$raph(N) are in general lattices.

Answer 4 See Fig 59.4.a)-b) for $\mathfrak{G}$raph(5) and Fig 59.5.a)-b) for [$\mathfrak{G}$raph](5). Observe that e.g.

$$\{\,\text{Gem, Spiderhead; Scotshouse, Wheel}\,\}$$

form a C_4-nonconvex subposet. This precludes $\mathfrak{G}$raph(5) from being a lattice. A $K_{3,2}$ is also present.

Exercise 1^{+} Show that the graphs in Fig 59.5.b)-c) are isomorphic.

Exercise 2^{++} Show that the graphs in Fig 59.5.b)-c) are isomorphic. Can you beat the exhibited crossing number?

Exercise 3^{+} Place the preceding sections' orders on [$\mathfrak{G}$raph][5] and $\mathfrak{G}$raph[5]?

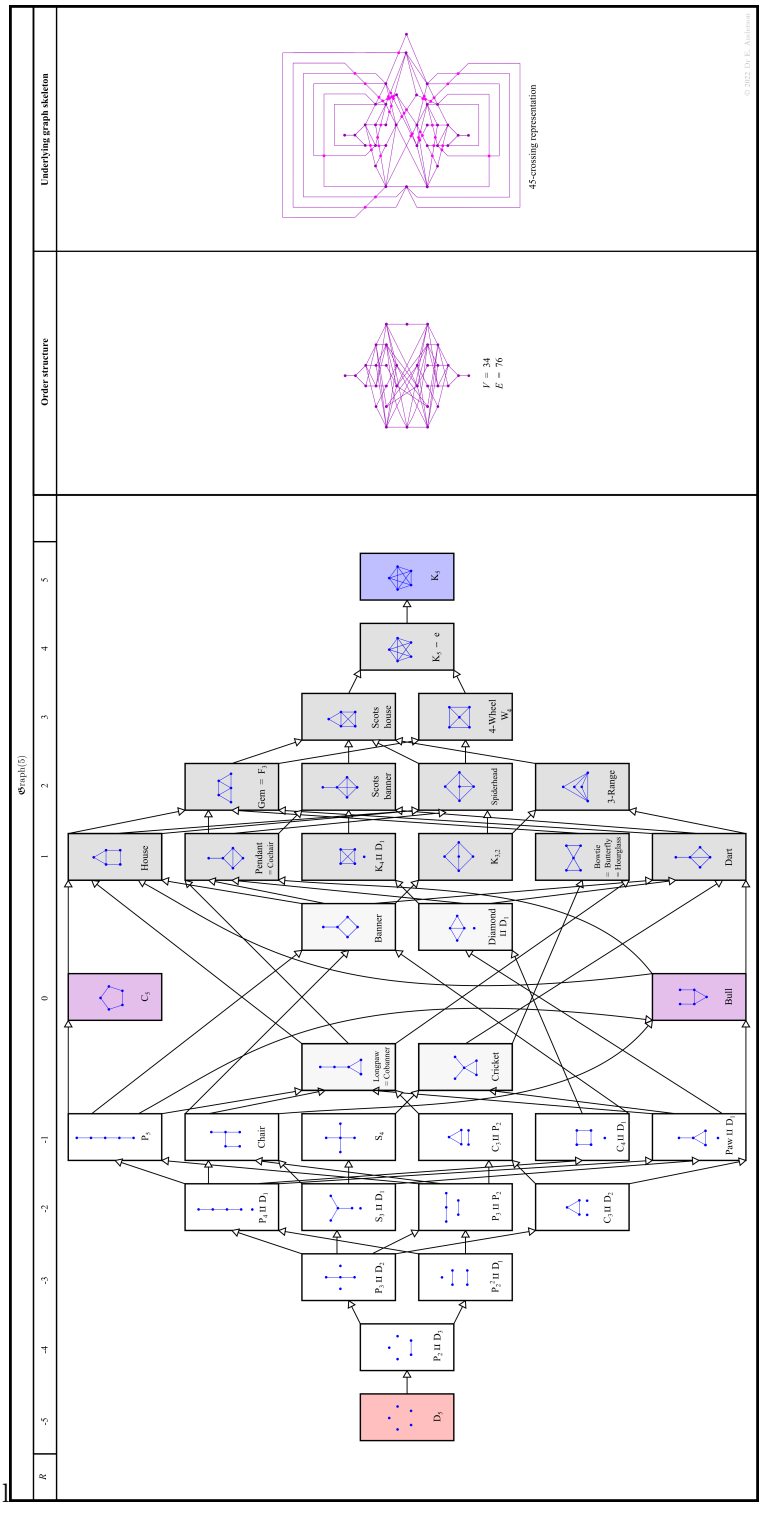

Underlying graph skeleton
45-crossing representation
Order structure
$V = 34$
$E = 76$
$\mathfrak{G}_{\mathrm{Graph}}(5)$
K_5
$K_5 - e$
Scots house
4-Wheel W_4
Gem = F_5
Scots hammer
Spikehead
3-Range
House
Pendant ∪ Cobair
$K_4 \amalg D_1$
$K_{3,2}$
Bowtie ≅ Butterfly ≅ Hourglass
Dart
Banner
Diamond $\amalg D_1$
C_5
Bull
Longbow ∪ Columner
Cricket
P_5
Chair
S_4
$C_4 \amalg P_2$
$C_4 \amalg D_1$
Paw $\amalg D_1$
$P_4 \amalg D_1$
$S_3 \amalg D_2$
$P_3 \amalg P_2$
$C_3 \amalg D_2$
$P_3 \amalg D_2$
$P_2^2 \amalg D_1$
$P_2 \amalg D_3$
D_5
R
-5
-4
-3
-2
-1
0
1
2
3
4
5

Figure 59.4:

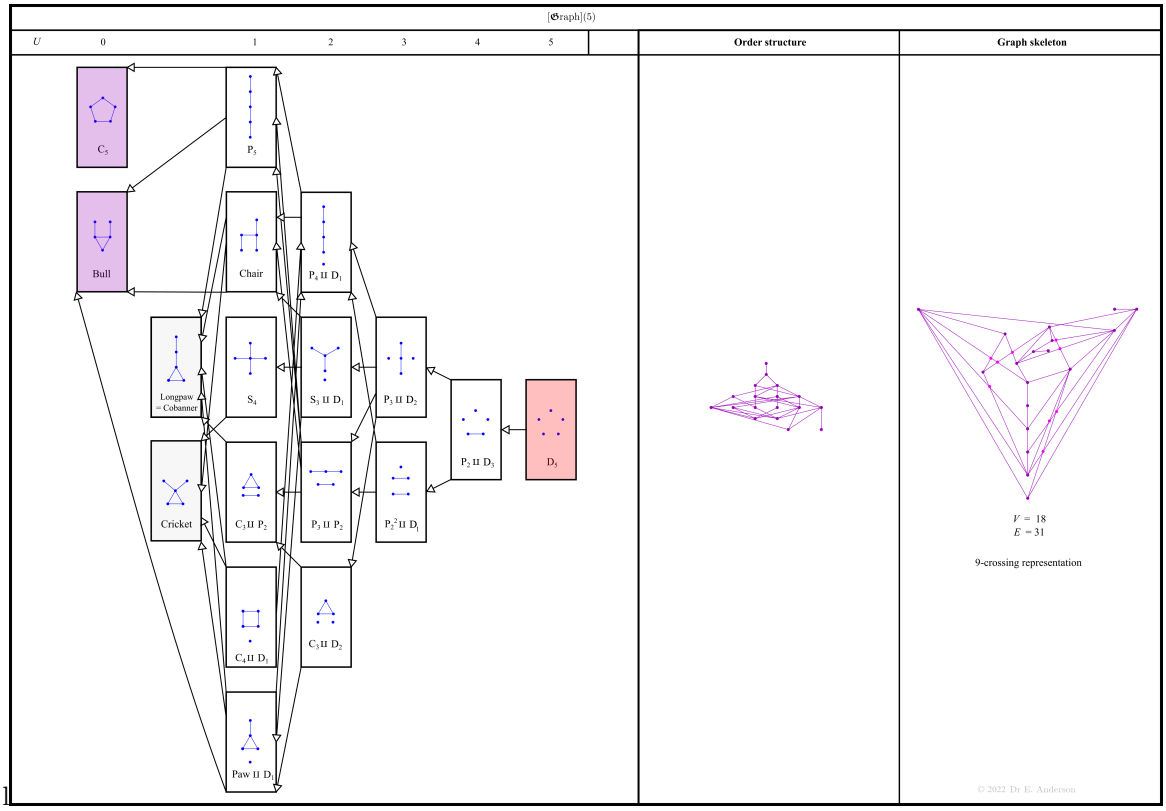

[𝔊raph](5)
U
0
1
2
3
4
5
Order structure
Graph skeleton
C5
P5
Bull
Chair
P4 ⊔ D1
Longpaw = Cobanner
S4
S3 ⊔ D1
P3 ⊔ D2
P2 ⊔ D3
D5
Cricket
C3 ⊔ P2
P3 ⊔ P2
P2²⊔ D1
C4 ⊔ D1
C3 ⊔ D2
Paw ⊔ D1
V = 18
E = 31
9-crossing representation
© 2022 Dr E. Anderson

Figure 59.5:

Chapter 60

Tree arenas

60.1 Some natural order structures on trees

Remark 1 Since all trees with a given vertex number N have the same number of edges by Lemma 1 of Chapter 36, no fixed-N edge-adding operation is supported by $\mathfrak{T}$ree. This can also be viewed as $\mathfrak{T}$ree(N) being an antichain in $\mathfrak{G}$raph(N).

Structure 1 Between trees with vertex count differing by 1, we introduce an 'add single leaf' operation L, whose reverse arrow is a single-leaf defoliation. Let us denote 'add leaf' with L subscripts, and with green arrows in explicit diagrams. 'Add single leaf' can be interpreted as forming the bi-chromatic cone with a single blue edge in the Ramsey presentation.

Structure 2 Defoliation (Sec 38.4), involves removing all of a graph's leaves at once. Let us denote this with F subscripts and brown arrows.

Structure 3 Homeomorphic expansion (Chapter 44), for which we use H subscripts and 'follow the yellow brick road' We largely postpone marking use of this ordering until Chapter 62.

60.2 Path arenas

Worked Example 1 What do each of the above three orders give when applied to the arena of paths, with and without inclusion of the ungraph = untree = unpath U?

Answer 1 Denote the unpath-excluded version with an excision *-subscript. Then in particular

$$\mathfrak{p}\mathrm{ath}_{\mathrm{L}\,*}[N] \;=\; \mathrm{P}_N \;,$$

thus constituting a simple solution to the problem of finding fixed points of *Arenize*. While

$$\mathfrak{p}\mathrm{ath}_{\mathrm{L}\,*}[N] \;=\; \mathrm{P}_{N\,+\,1} \;.$$

$\mathfrak{p}\mathrm{ath}_{\mathrm{F}\,*}[N]$ produces bent-path posets to $\mathfrak{p}\mathrm{ath}_{\mathrm{F}\,*}[N]$'s 3-pronged starlike tree based posets. Finally,

$$\mathfrak{p}\mathrm{ath}_{\mathrm{H}\,*}[N] \;=\; \mathrm{P}_2$$

and

$$\mathfrak{p}\mathrm{ath}_{\mathrm{H}}[N] \;=\; \mathrm{P}_3 \;.$$

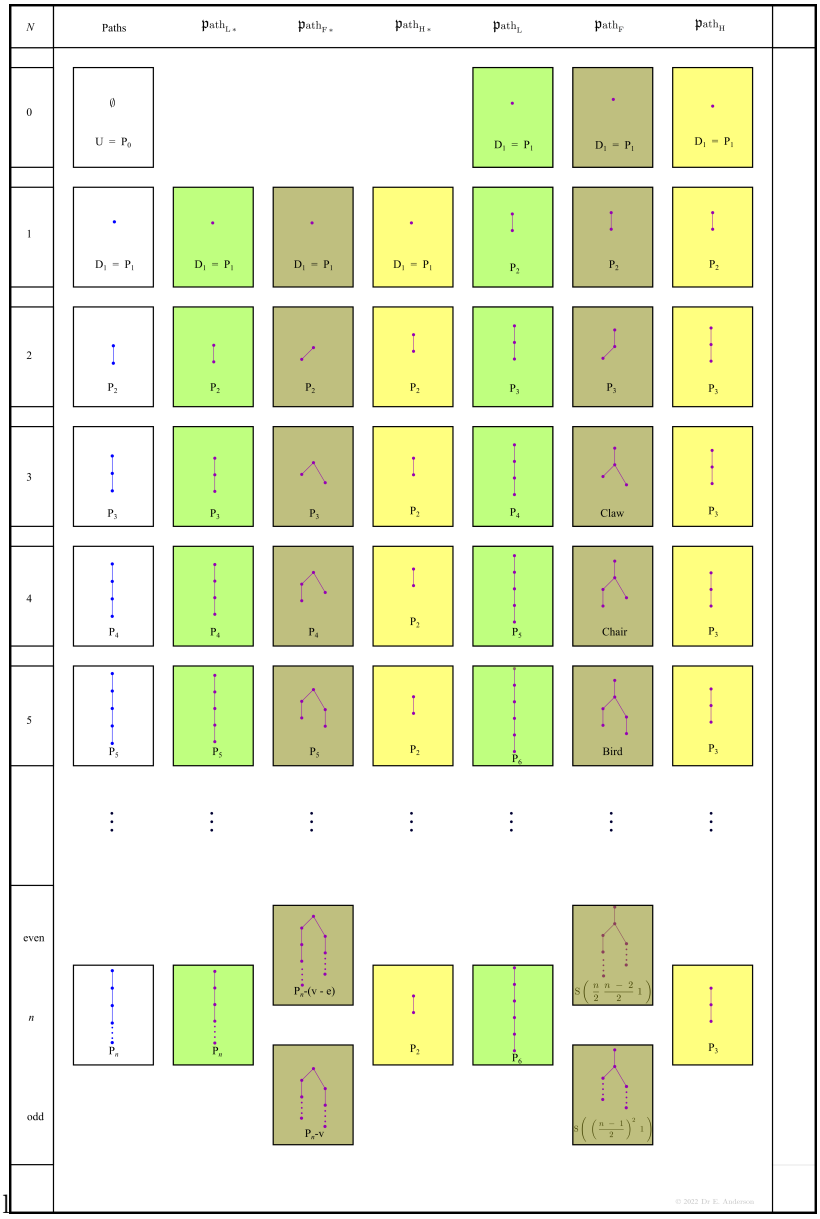


Figure 60.1:

60.3 Single-leaf addition on $\mathfrak{starTree}$

Worked Example 2 What are the starlike tree arenas $\mathfrak{StarTree}_{\mathrm{L}*}[N]$ as order structures and underlying graph skeletons for $N \leq 7$?

Answer 2 The structure-adding operation L is poset-forming. The top element here is

$$\mathrm{S}\mathfrak{Tree}_{\mathrm{L}*}[N]_1 \;=\; \mathrm{pt}\ ,$$

while there is no unique bottom element for $N \geq 4$. [It makes little difference to our analysis whether our posets are crowned by an 'add vertex' black arrow running down from U viewed as the untree.]

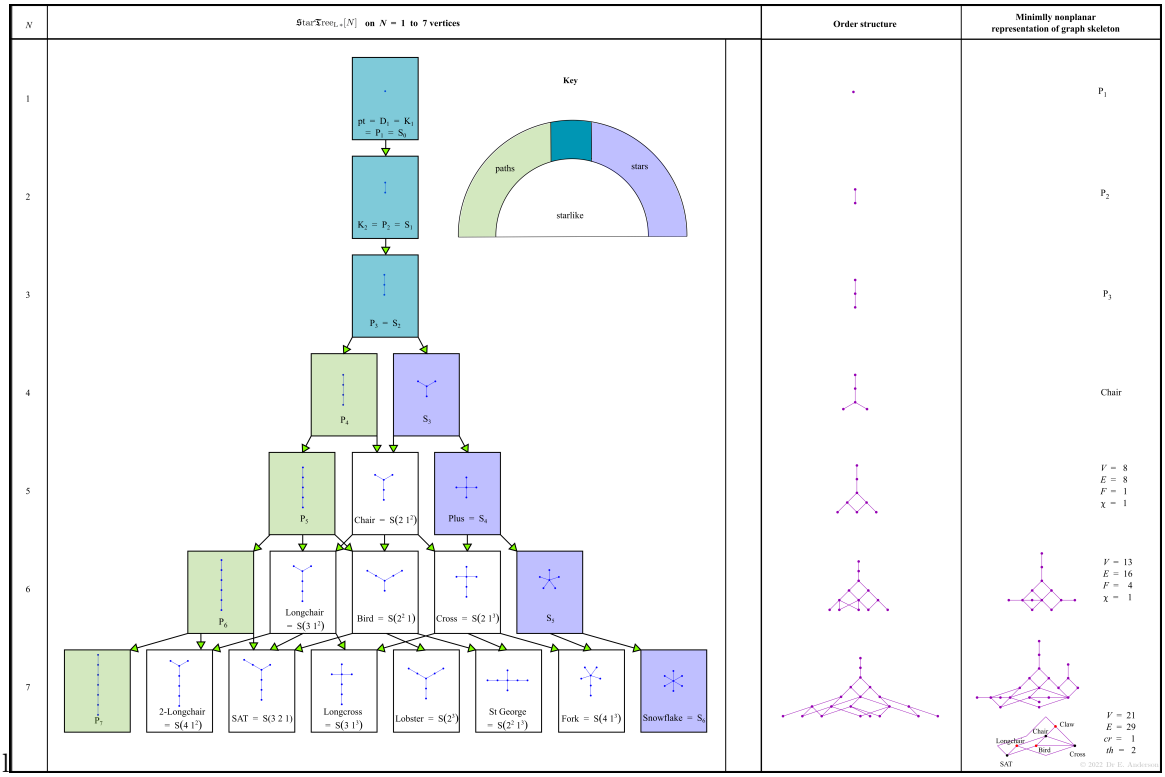


Figure 60.2:

See Fig 60.2 for further details. Observe that $\mathfrak{S}$tar$\mathfrak{T}$ree$_{L*}[6]$ is minimal to exhibit a C_4-nonconvex subposet, with level by level vertex set

$$\{ P_5, \text{Chair}; \text{Longchair}, \text{Bird} \} .$$

This is also minimal to exhibit a $K_{3,2}$ subposet

$$\{ P_5; P_6, \text{Longchair}, \text{Bird}; \text{SAT} \} .$$

The $\mathfrak{S}$tar$\mathfrak{T}$ree$_L$ are thus not in general either lattices or distributive.

Also observe that $N = 6$ is minimal for the arena's underlying graph skeleton to be nonstandard. And $N = 7$ is minimal for nonplanarity by Exercise 45.10).

Remark 1 $\mathfrak{S}$tar$\mathfrak{T}$ree$_{L*}$ and $\mathfrak{S}$tar$\mathfrak{T}$ree$_L$ serve as further examples of a poset morphism and poset quotient, in relation to $\mathfrak{P}$artition. The morphism in question is 1:1 except that all 1-part and 2-part partitions for a given N are identified, since the path graph P_n jointly realizes all of these. This tells us that the only possibility for departures from the latticeness and distributivity of Young's lattice is if paths are involved.

Remark 2 $\mathfrak{S}$tar$\mathfrak{T}$ree$_{L*}$ and $\mathfrak{S}$tar$\mathfrak{T}$ree$_L$ serves furthermore as counterexamples to the hypothesis that *Arenize* always maps symmetric objects to symmetric arenas.

60.4 Single-leaf addition order on $\mathfrak{T}$ree

Worked Example 3 Extend the above analysis to the full tree arenas $\mathfrak{T}$ree$_{L*}[N]$, still for $N \leq 7$.

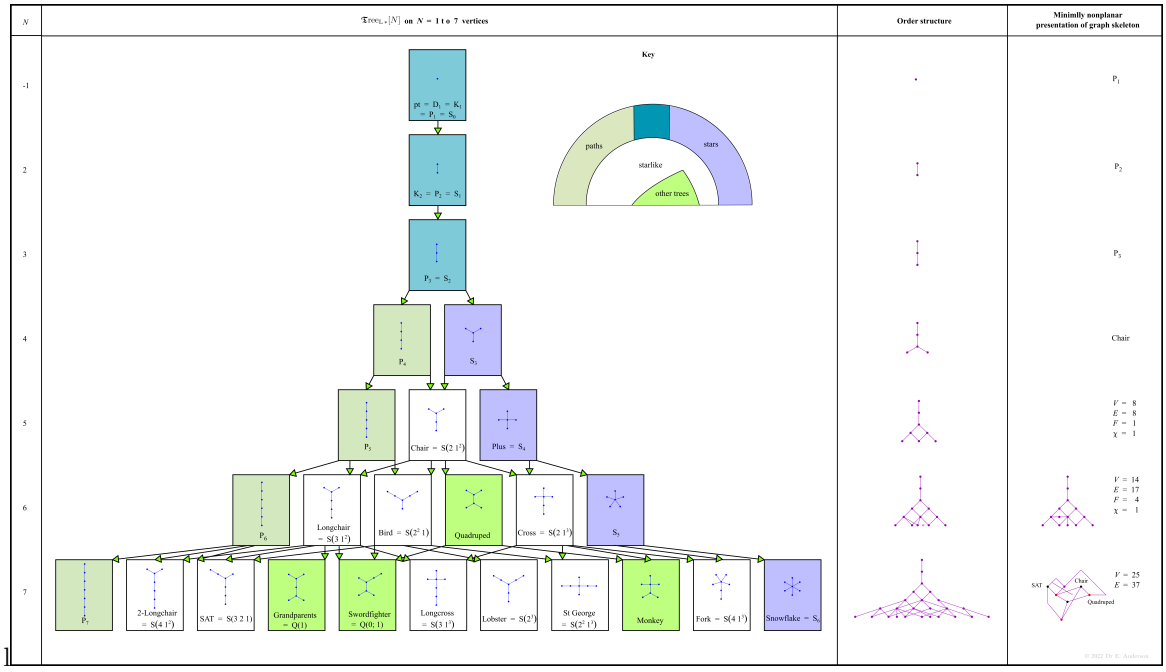


Figure 60.3:

Answer 3 This gives Fig 60.3.a). The $\mathfrak{Tree}_{\mathrm{L}*}[N]$ are thus established to not themselves be trees for $N = 5$ which manifests a first 4-cycle, which then persists for all subsequent N. While a reflection symmetry holds up to $\mathfrak{Tree}_{\mathrm{L}}[5]$, $\mathfrak{Tree}_{\mathrm{L}}[6]$ bucks this trend. One obvious way of noticing this is that the stars S_n for $n \geq 2$, and thus in particular S_4 only ever permit two single-leaf additions: central and peripheral. In contrast, P_5 is the first path to permit at least 3: central, peripheral and middle.

The C_4-nonconvex and $\mathrm{K}_{3,2}$ subposets of $\mathfrak{Star}\mathfrak{Tree}_{\mathrm{L}*}[6]$ carry over to its $\mathfrak{Tree}_{\mathrm{L}*}[6]$ superposet, which is thereby neither a lattice nor distributive either. $\mathfrak{Tree}[7]$ also suffices to exhibit nonplanarity (as per the bottom-right corner of Fig 60.3.b).

Exercise 1 Further analyse $\mathfrak{Tree}_{\mathrm{L}*}[7]$'s nonplanarity.

60.5 Caterpillars, lobsters, hedges and the defoliation order

Remark 1 $\mathfrak{Tree}_{\mathrm{L}}$ can be partitioned into the arenas of paths, of caterpillars, of lobsters, and of $(\, n \geq 3 \,)$-hedges for $n \geq 2$. This partition is by the number of defoliations it takes to attain a path. I.e.

$$\mathfrak{Tree} = \coprod_{F \in \mathbb{N}_0} \mathfrak{Hedge}(n) = \mathfrak{path} \amalg \mathfrak{Caterpillar} \amalg \mathfrak{Lobster} \amalg \coprod_{n \leq 3} \mathfrak{Hedge}(n) \; . \quad (60.1)$$

Caterpillars suffice up to $N = 6$, and lobsters up to $N = 9$.

Exercise 2 a) Rephrase both of these limiting numbers in terms of partitions.

b) Working in $\mathfrak{Tree}_{\mathrm{L}}$, show that the nontrivial caterpillar part is connected for $N \leq 7$. But that the nontrivial lobster part is not connected for $N = 9$.

Exercise 3^{+} a) Explain why studying lobsters, and hedges more generally, is still useful in the F order but not in the H order.

b) Provide the $N \leq 7$ explicit order structures and underlying graph skeletons for $\mathfrak{Tree}_{\mathrm{F}}[N]$. Pick out the nontrivial caterpillar part herein as well.



60.6 Rooted-tree arenas

Remark 1 Try to answer the next five Worked Examples for yourself. Each is to be interpreted as using a suitable modification of the single-leaf ordering L.

Worked Example 4 What explicit order structures and underlying graph skeletons arise from placing the L order on the arena of rooted trees for $N \leq 5$? Placing leaves is here to be restricted such that the root is preserved.

Answer 4 See Fig 36.3 for these on 1 to 5 vertices. And Fig 60.4 for the root-preserving L $*$-order on $\mathfrak{Rooted}\mathfrak{Tree}[N]$ with $N \leq 5$ vertices. $N = 5$ is not large enough to manifest any nonplanarity.

Exercise 4^{+} What are the explicit order structures and underlying graph skeletons for the labelled rooted trees on $N \leq 4$?

60.7 N-ary-tree arenas

Worked Example 5 What is the leaf-addition ordering of all binary trees for $N \leq 11$? [Now as well as preserving the root, we need to add the right amount of leaves, and furthermore licitly placed, so as to remain within the arena of binary trees. Let us denote this order by L(2) $*$.]

Answer 5 See Fig 60.5 for the binary trees on 1 to 13 vertices (answering Exercise 36.5). We furthermore represent these under a single defoliation, as posets formed by bending some smaller trees.

These form Fig Binary-Trees-1-11-Order's $\mathfrak{Bin}\mathfrak{Tree}$ poset of posets.

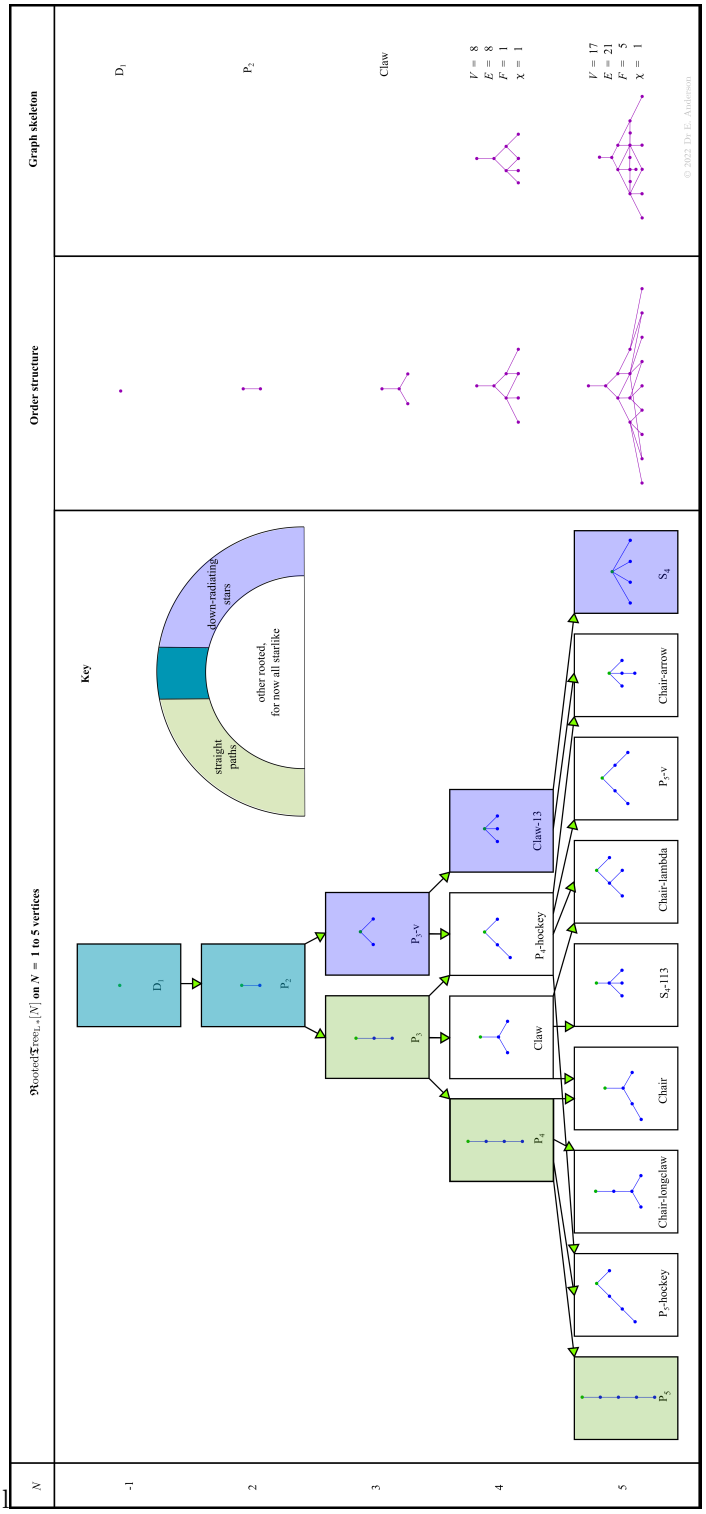


Figure 60.4:

Up to $\mathfrak{B}\text{in}\mathfrak{T}\text{ree}[11]$, a distributive lattice is observed. Planarity is here a trivial matter of slipping two edges, directly involved in crossings presented, into the outer face. Rectilinearizing the resulting graph was set as part of Ex 48.12.

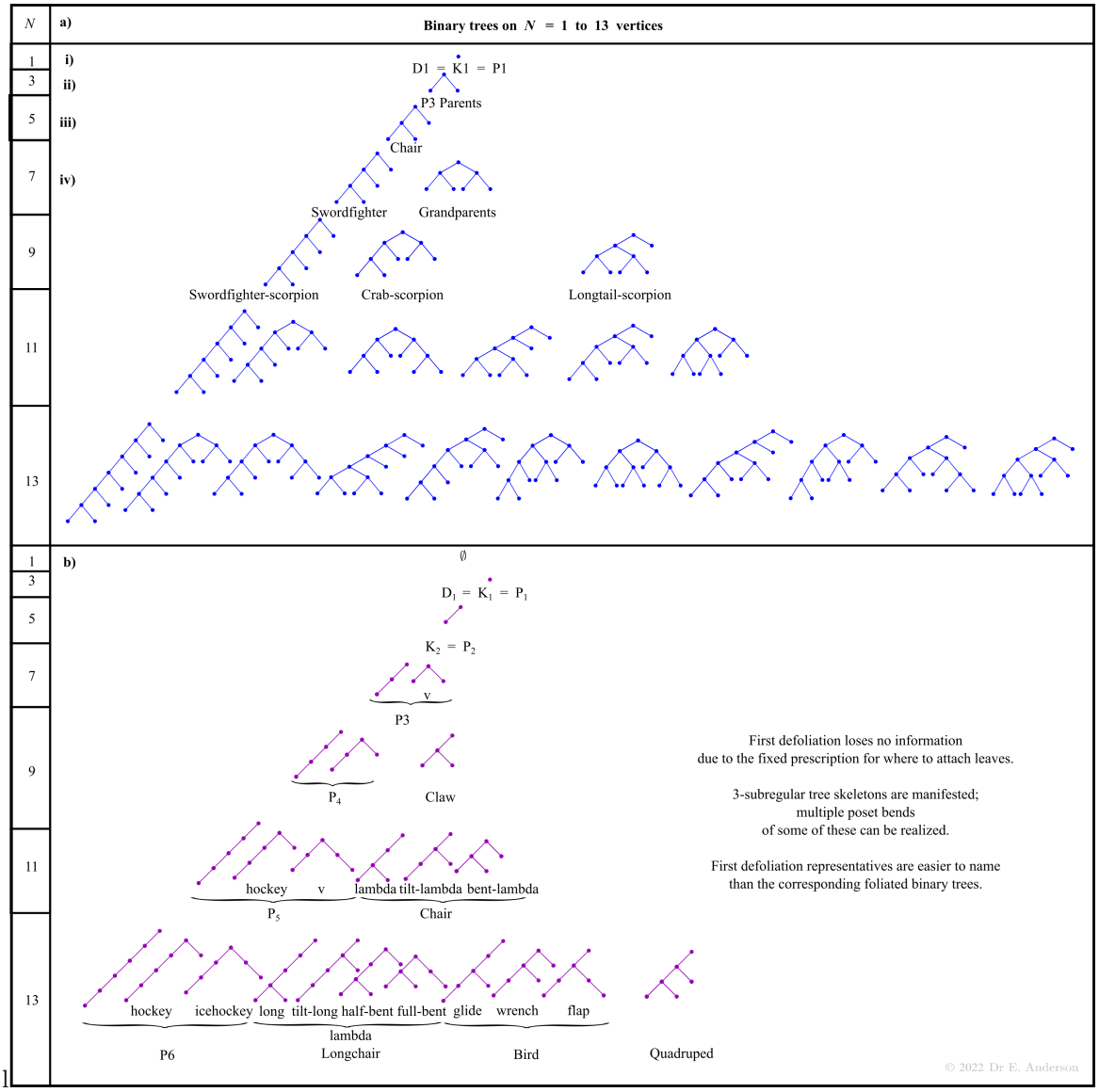


Figure 60.5:

Worked Example 6 What is the leaf-addition ordering of all ternary trees for $N \leq 16$? And of all quad trees for $N \leq 21$? What is special about this value of N for quad trees?

Answer 6 See Fig 60.7 for the ternary trees on ≤ 16 vertices with the appropriate $\mathrm{L}(3) *$ ordering. 16 is sufficient to realize 3 ternary trees (shaded in orange) whose defoliation posets are not realized by binary trees.

See Fig 60.8 for the quad trees on ≤ 21 vertices with the appropriate $\mathrm{L}(4) *$ ordering. 21 minimally realizes quad trees (shaded in red) whose defoliation posets are not realized by ternary trees.

We present $\mathfrak{T}\mathrm{ern}\mathfrak{T}\mathrm{ree}$ lattices of posets under the vertex addition operation in Fig Ternary-Trees-1-11-Order, and $\mathfrak{Q}\mathrm{uad}\mathfrak{T}\mathrm{ree}$ counterparts in Fig Ternary-Trees-1-11-Order. In each case, the previous Worked Example's observation of distributive lattices and trivially-realized planarity perseveres.

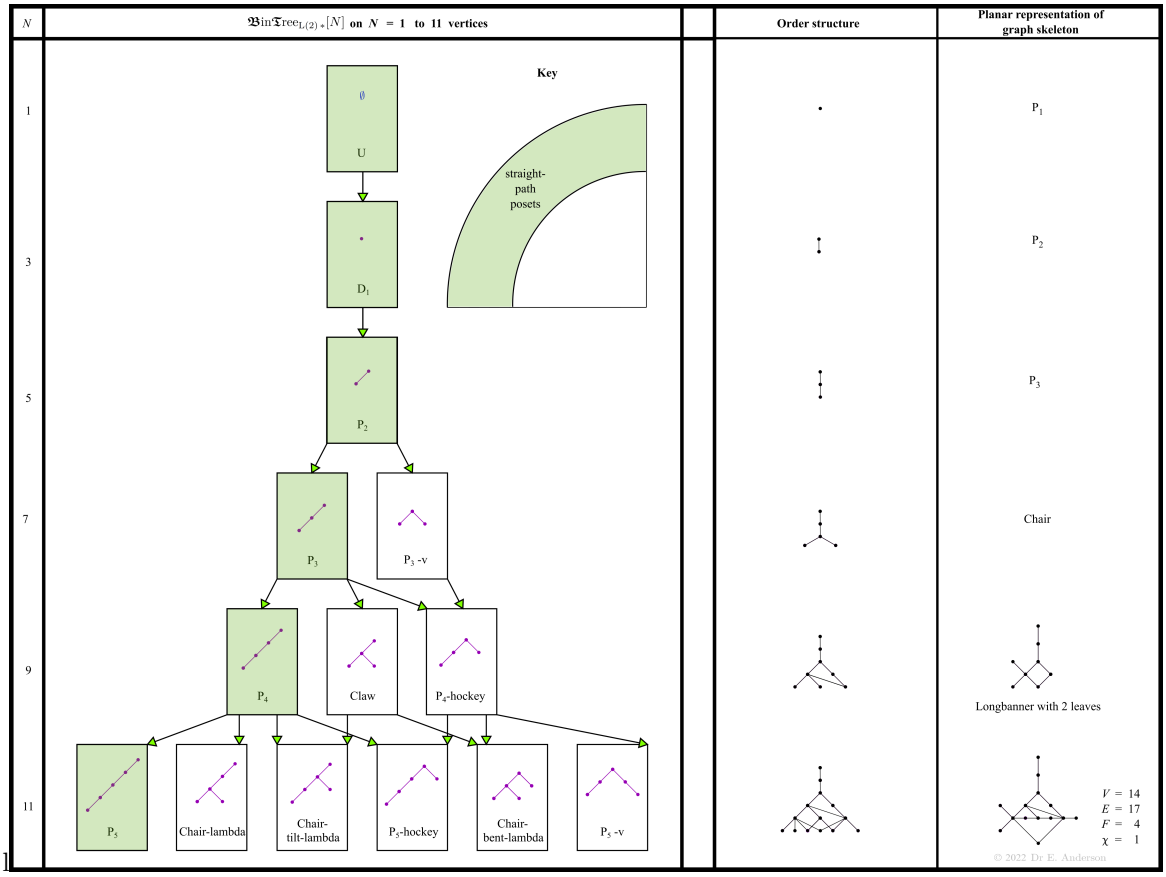


Figure 60.6:

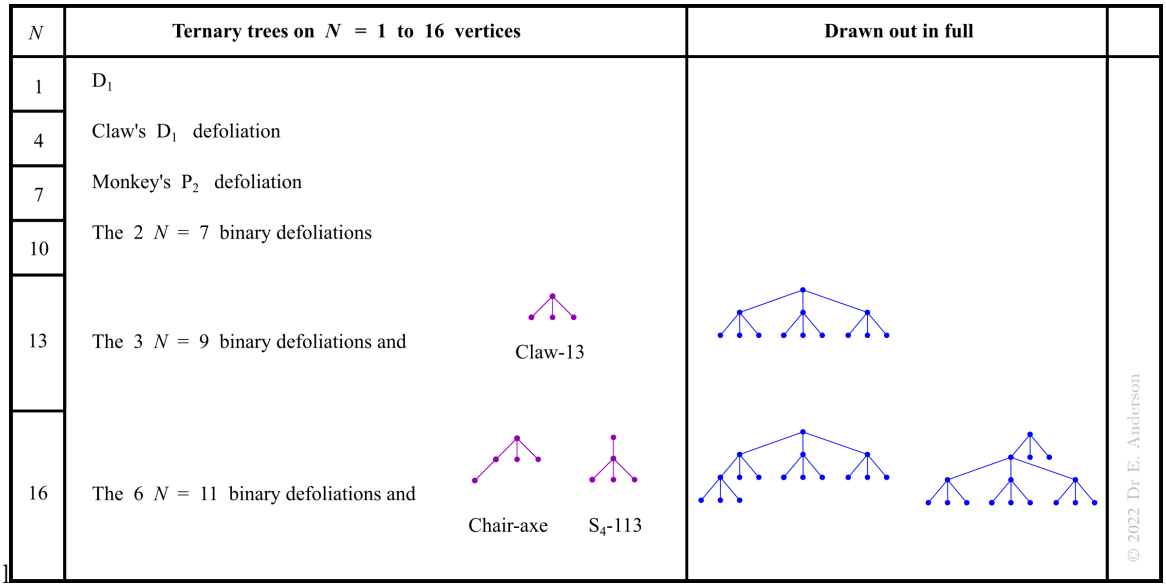


Figure 60.7:

Remark 1 Following Sec 36.15's discussion, Worked Examples 4 to 6 can be taken to be toy models of arenas of computer databases.

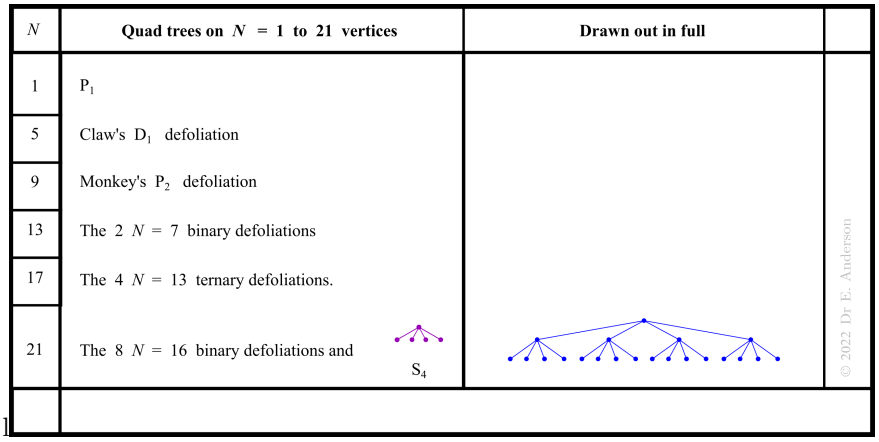


Figure 60.8:

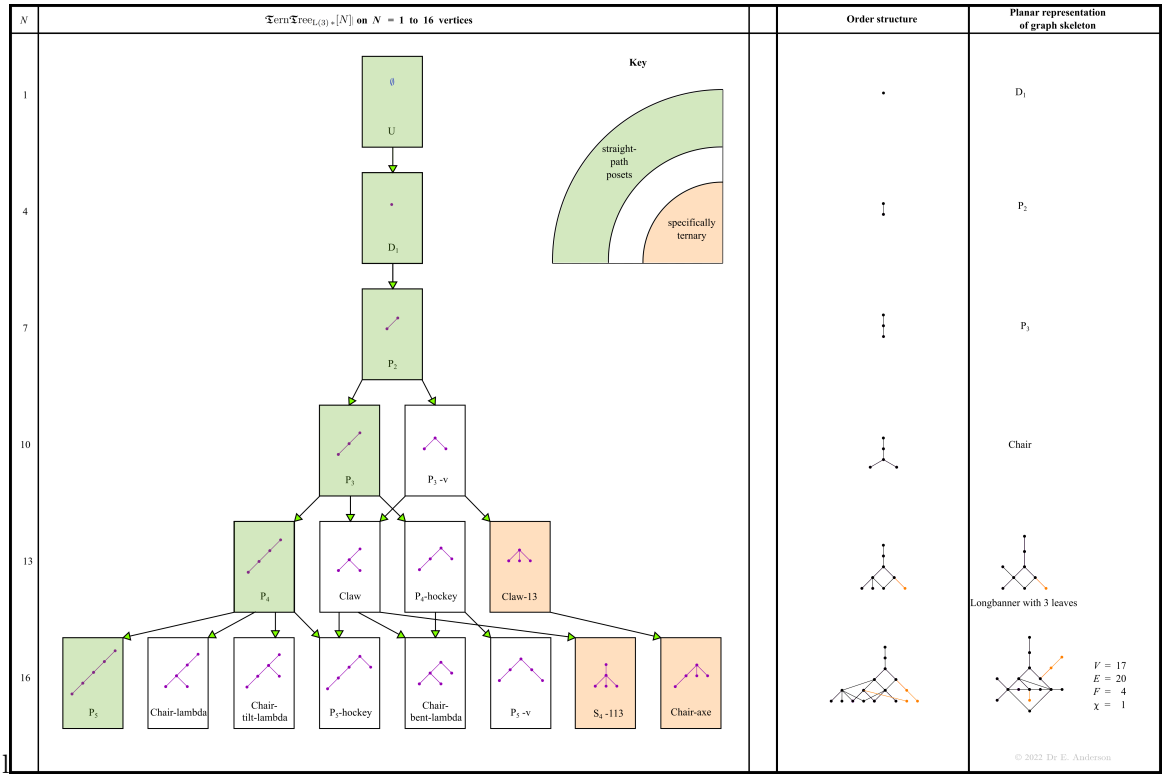


Figure 60.9:

60.8 RL-distinguished at-most-binary tree arenas

Worked Example 7 What are the explicit order structures and underlying graph skeletons for the $\mathfrak{Tree}^{\mathrm{RL}}{}_{\leq 2}[N]$ on $\leq$ 4 vertices?

Answer 7 See Fig 36.5.a) for the corresponding objects. A suitable order here is the root-preserving L($\leq$ 2)$*$. I.e. adding either a single leaf, or a pair of leaves to a common terminal node. At-most binary trees only realize the 112 version of Claw, so we can furthermore drop this detail in naming our configurations.

b) then centres our display, with eq (36.12)'s chirality index I now in the role of horizontal-axis

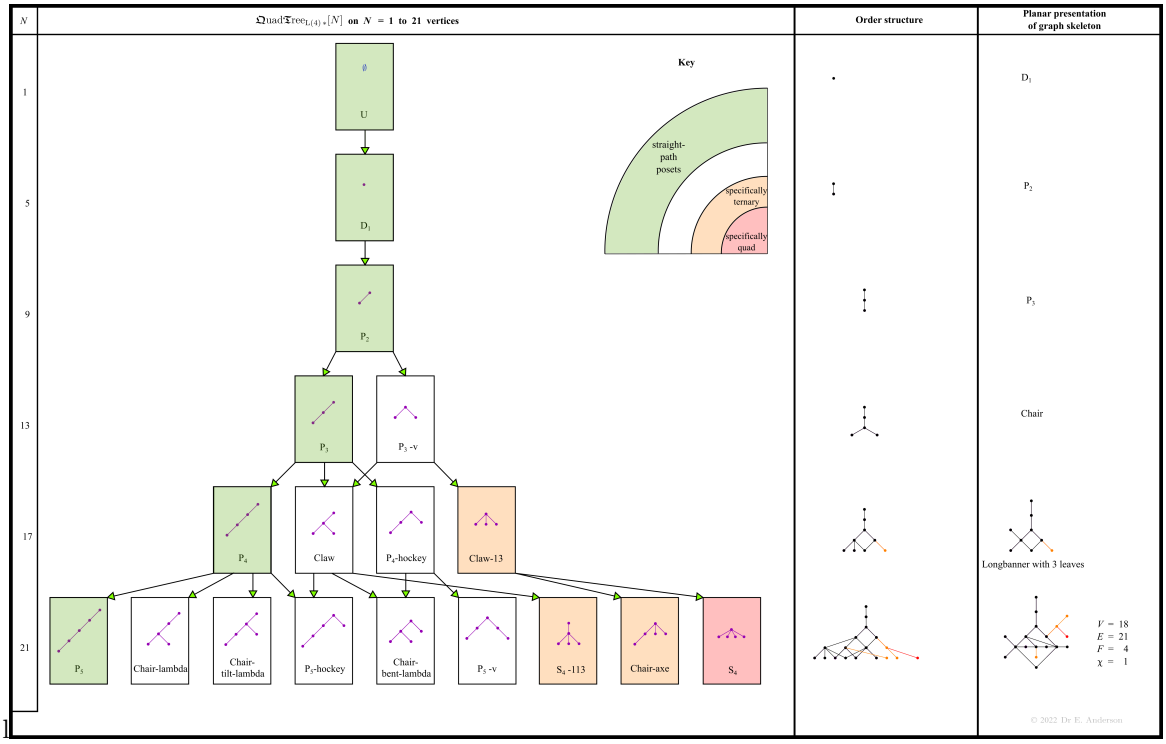


Figure 60.10:

variable. c) is the result of R and L remaining distinguished and yet individually meaningless. This is parametrized by the unsigned chirality index UI of eq. (36.13). This can be zero without RL exchange symmetry being realized, which we pick out in palest grey.

See Fig 60.11 for the root-preserving single-leaf-addition lattices for $\mathfrak{Tree}^{\mathrm{RL}}{}_{\leq 2}[N]$. Working up to $N = 4$ does not suffice to manifest nonplanarity.

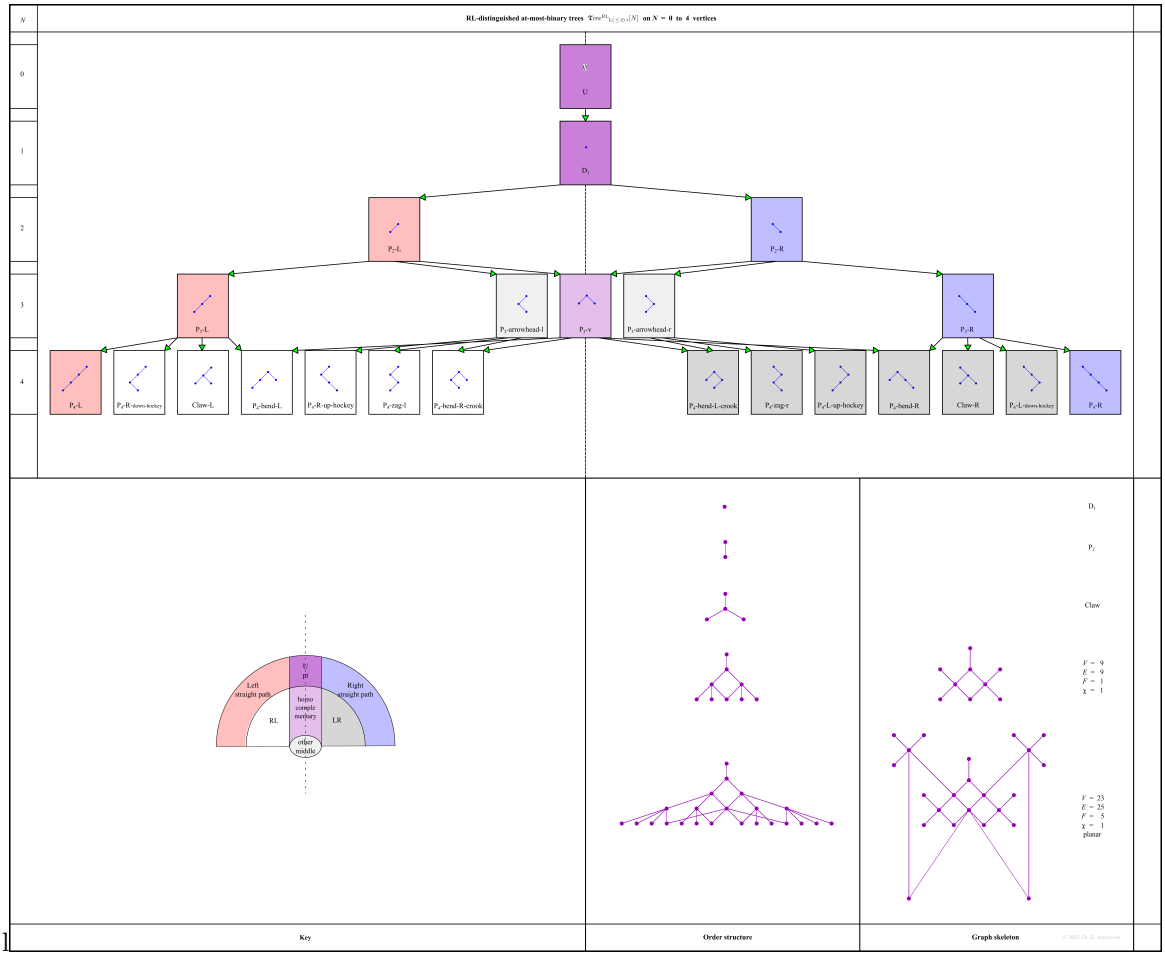


Figure 60.11:

Worked Example 8 What arenas result from the above modelling if the R and L designations are considered to be meaningless?

Answer 8 Fig 60.12 gives the corresponding quotient. Our largest graph skeleton here is just Domino with 5 leaves and a P_3 sidechain.

Remark 1 In attempting to place a Hamiltonian cycle on $\mathfrak{Tree}^{RL}_{\geq 2}[6]$'s cycle system, the 8-cycle in Fig 60.13.a) is forced. So this is not Hamiltonian (answering Ex 46.8.d). Alternative presentations arise by centering about one of the various 8-cycles available. These are all essentially equivalent presentations, of which one is given in Fig 60.13.a).

60.9 Further analysis of graph and tree arenas so far

Remark 1 $\mathfrak{Graph}_D(4)$ has $Z = 14$, which is not ZIPHoN-prohibited, and indeed supports the Hamiltonian cycle depicted in Fig 60.13.b). This then supports the given 16-cycle presentation.

Remark 2 For $\mathfrak{Graph}_S[4]$, $Z = 16$, which is not ZIPHoN-prohibited. However, the cycle indicated in Fig 60.13.c) is forced, so our cycle system cannot be Hamiltonian.

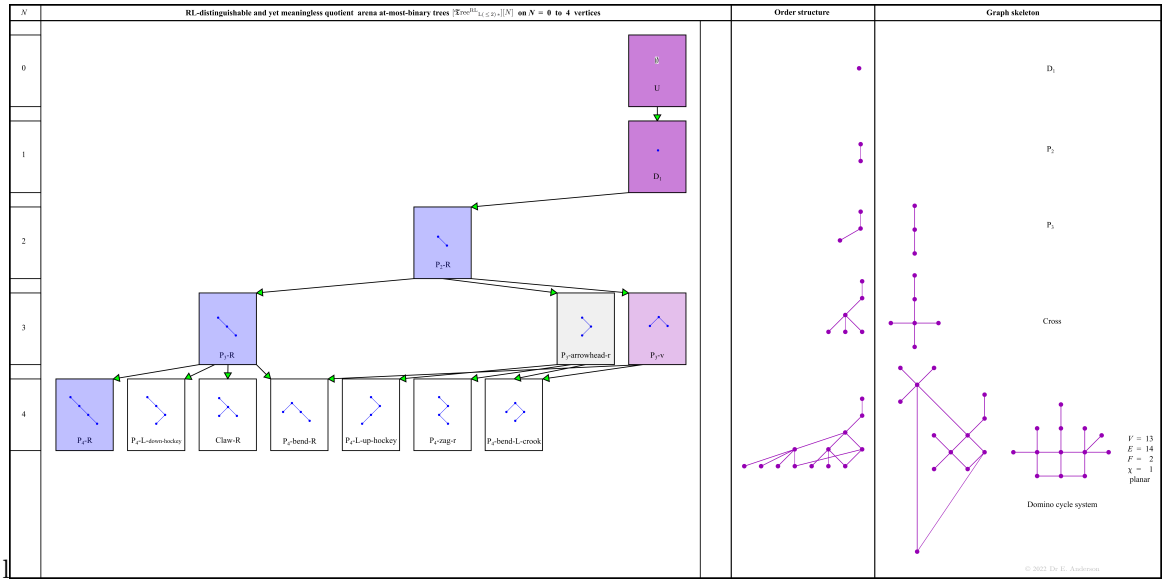


Figure 60.12:

Remark 3 $\mathfrak{T}$ree[6] is not Hamiltonian since the cycle indicated in Fig 60.13.d) is forced.

Remark 4 For $\mathfrak{R}$oot$\mathfrak{e}$d$\mathfrak{T}$ree[5],

$$Z \;=\; \frac{4 \;\times\; 2 \;+\; 4 \;+\; 6}{2} \;=\; 9\;.$$

This odd outcome ZiPHoN-prohibits a Hamiltonian cycle in such a purely even-cycle graph. We can however invert in an 8-cycle or in a 10-cycle. While the 8-cycle is smaller, it has the advantage of maintaining manifest planarity rectilinearly (Fig 60.13.e).

Remark 5 The current Book's largest n-ary tree arena's cycle system is not Hamiltonian by the forced cycle in Fig 60.13.f). An 8-cycle presentation for this is also provided.

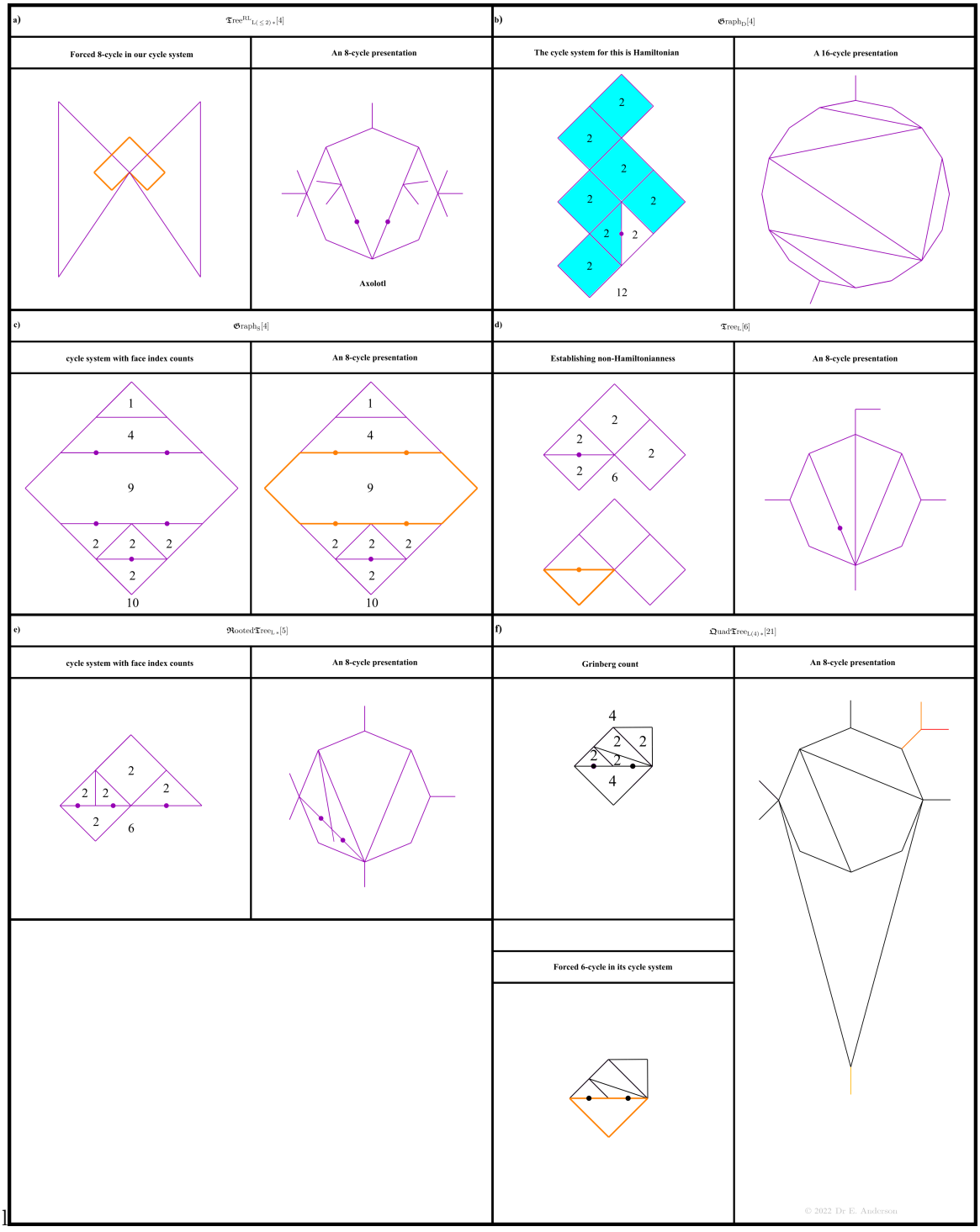

a)
Forced 8-cycle in our cycle system
An 8-cycle presentation
Axolotl
b)
The cycle system for this is Hamiltonian
A 16-cycle presentation
12
c)
cycle system with face index counts
An 8-cycle presentation
10
10
d)
Establishing non-Hamiltonianness
An 8-cycle presentation
e)
cycle system with face index counts
An 8-cycle presentation
f)
Grinberg count
An 8-cycle presentation
Forced 6-cycle in its cycle system
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Figure 60.13:

Chapter 61

More specialized graph arenas

61.1 Connected and component-number graph arenas

Definition 1 The *connected graphs arena* is

$$\mathfrak{Con}\mathfrak{Graph} \ .$$

Worked Example 1 What are the explicit order structures and underlying graph skeletons for $\mathfrak{Con}\mathfrak{Graph}(N)$ with the add-edge ordering for $N \leq 5$?

Answer 1 These posets have

$$\mathfrak{Con}\mathfrak{Graph}(N)_0 \ = \ \mathrm{K}_N \ .$$

But no unique tops for $N \geq 4$, starting rather from $\mathfrak{Tree}[N]$'s bottom antichain, $\mathfrak{Tree}(N)$. The super-treeness relative variable of eq. (51.8) is thus also a suitable relative variable here.

We depict these for $N = 0$ to 5 in Fig 61.1.a)-b). In particular, we observe that $\mathfrak{Con}\mathfrak{Graph}(5)$ contains a C_4-nonconvex subposet:

$$\{ \text{Gem, Scotsbanner, Scotshouse, Wheel} \} \ .$$

The $\mathfrak{Con}\mathfrak{Graph}(N)$ are thus not in general lattices. Exercise 48.13 is about the nonplanarity properties of $\mathfrak{Con}\mathfrak{Graph}(5)$.

Remark 1 Since graph complementation does not in general respect connectedness, this is just a graph as opposed to [graph] property. We thus cannot quotient by the C_2 of graph complementation.

Definition 2 The *k-component graph arena* is

$$\mathfrak{Cpt}_k\mathfrak{Graph} \ .$$

Remark 2 These clearly include

$$\mathfrak{Con}\mathfrak{Graph} \ = \ \mathfrak{Cpt}_1\mathfrak{Graph} \ .$$

The N-argument such form a partition of $\mathfrak{Graph}$,

$$\mathfrak{Graph} \ = \ \amalg_{k \, = \, 1}^{N} \mathfrak{Cpt}_k\mathfrak{Graph} \ .$$

Exercise 1 This is the first of many partitions of $\mathfrak{Graph}$ that the current Chapter considers, all of which can be represented as heat maps of $\mathfrak{Graph}$. Provide a heat map for this over $\mathfrak{Graph}(5)$ as viewed as a poset.

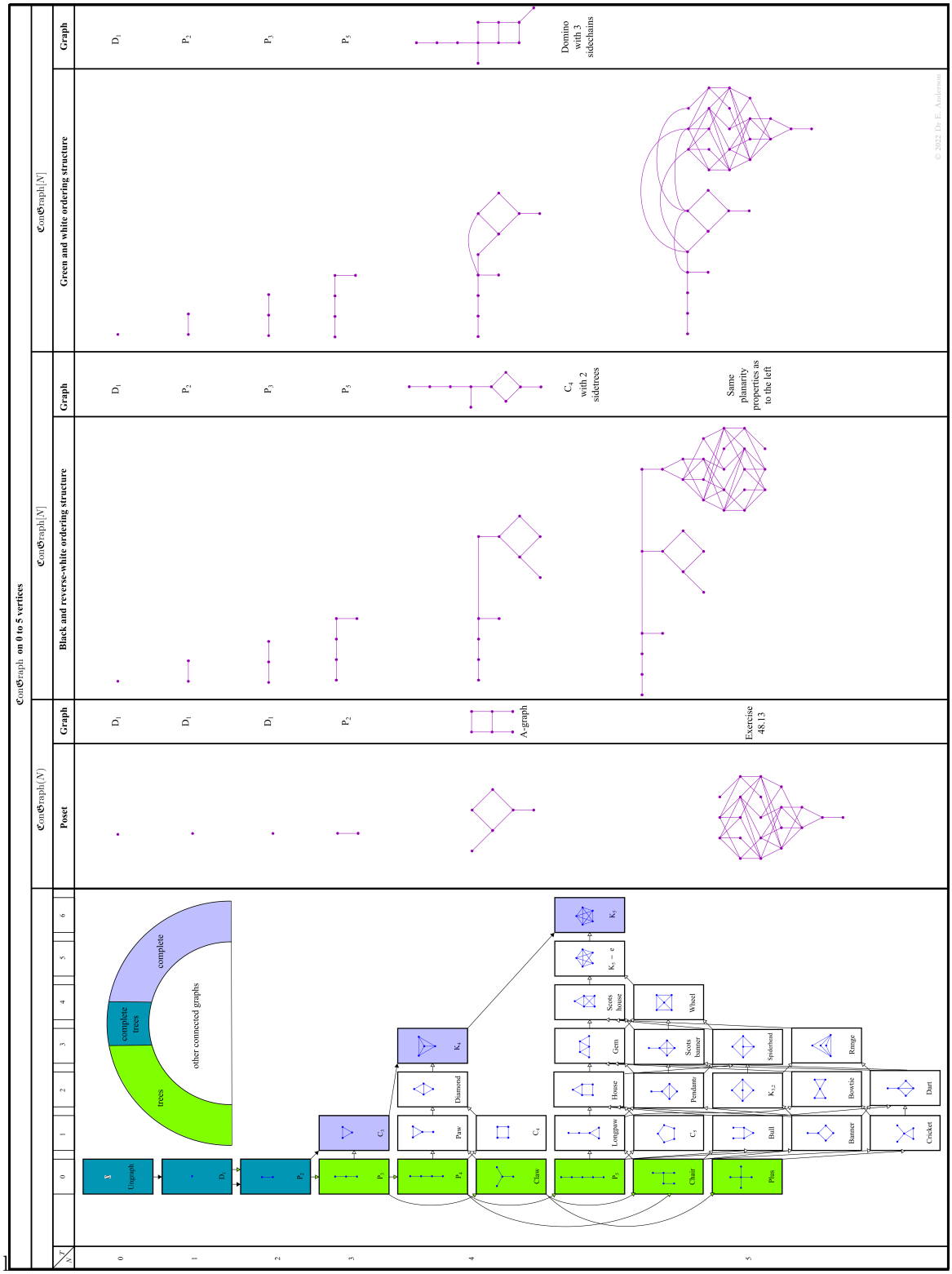


Figure 61.1:

Remark 3 $\mathfrak{Con}\mathfrak{Planar}$ is the arena that we work in for almost all of the next Chapter and a half.

61.2 Cycle number graph arenas

Example 1 The *forest arena* is

$$\mathfrak{F}\mathrm{orest}\ .$$

Example 2 The *cycle-number-#c graph arena* (in Sec 38.8's sense) is

$$\mathfrak{C}\mathrm{ycle}_{\#c}\mathfrak{G}\mathrm{raph}\ .$$

These clearly include

$$\mathfrak{F}\mathrm{orest}\ =\ \mathfrak{C}\mathrm{ycle}_0\mathfrak{G}\mathrm{raph}\ .$$

These furthermore form a partition of $\mathfrak{G}$raph,

$$\mathfrak{G}\mathrm{raph}\ =\ \amalg^N_{\#c\ \leq\ 0}\mathfrak{C}\mathrm{ycle}_k\mathfrak{G}\mathrm{raph}\ .$$

Example 3 The *cycle-number-#c connected graph arena* is

$$\mathfrak{C}\mathrm{ycle}_{\#c}\mathfrak{C}\mathrm{on}\mathfrak{G}\mathrm{raph}\ .$$

These clearly include

$$\mathfrak{T}\mathrm{ree}\ =\ \mathfrak{C}\mathrm{ycle}_0\mathfrak{C}\mathrm{on}\mathfrak{G}\mathrm{raph}\ .$$

These form a partition of $\mathfrak{C}$on$\mathfrak{G}$raph,

$$\mathfrak{C}\mathrm{on}\mathfrak{G}\mathrm{raph}\ =\ \amalg^N_{\#c\ \leq\ 0}\mathfrak{C}\mathrm{ycle}_k\mathfrak{G}\mathrm{raph}\ .$$

Exercise 2 Form the corresponding order structure heat maps for $\mathfrak{G}$raph[4] in the general case and $\mathfrak{C}$on$\mathfrak{G}$raph(5) in the connected case.

Remark 1 Whenever we have two distinct multi-piece partitions of the same space, they can be jointly applied as a partition that refines them both. In this way, we can for instance extend the above partition of $\mathfrak{C}$on$\mathfrak{G}$raph to one of $\mathfrak{G}$raph according to values taken by $(k, \#c)$. Within this refinement, in particular

$$\mathfrak{T}\mathrm{ree}\ =\ \mathfrak{C}\mathrm{on}\mathfrak{G}\mathrm{raph}\ \cap\ \mathfrak{F}\mathrm{orest}\ .$$

Remark 2 $\#c$ is also a graph rather than [graph] property.

61.3 Regular graph arenas

None of the above respect complementarity: these are more specialized kinds of graph, rather than of [graph]. However, the following have a bit more alignment with complementarity.

Example 4 Let us denote the arena of k-regular graphs by

$$\mathfrak{R}\mathrm{eg}_k\mathfrak{G}\mathrm{raph}\ .$$

The arena of $\underline{k}$-multi-regular graphs by

$$\mathfrak{R}\mathrm{eg}_{\underline{k}}\mathfrak{G}\mathrm{raph}\ .$$

The arena of k-sub-regular graphs by

$$\mathfrak{R}\mathrm{eg}_{\leq k}\mathfrak{G}\mathrm{raph}\ .$$

$$\mathfrak{R}\mathrm{eg}_k\mathfrak{G}\mathrm{raph}\ *_{\mathrm{c}}\ \mathfrak{R}\mathrm{eg}_{N\ -\ k\ -\ 1}\mathfrak{G}\mathrm{raph}\ .$$

where $*_{\mathrm{c}}$ denotes 'dual under complementarity. Also

$$\mathfrak{R}\mathrm{eg}_{\underline{k}}\mathfrak{G}\mathrm{raph}\ *_{\mathrm{c}}\ \mathfrak{R}\mathrm{eg}_{(N\ -\ 1)\underline{1}\ -\ \underline{k}}\mathfrak{G}\mathrm{raph}\ .$$

And

$$\mathfrak{Reg}_{\leq k}\mathfrak{Graph} \ *_{\mathrm{c}} \ \mathfrak{Reg}_{>k}\mathfrak{Graph} \ :$$

the *k-super-regular* graphs manifesting no other degrees than those ranging from $k+1$ to $N-1$.

Remark 1 The $\mathfrak{Reg}_{\underline{k}}\mathfrak{Graph}$ are general enough to partition both $\mathfrak{Graph}$ and [$\mathfrak{Graph}$].

Exercise 3 [Long] Partition $\mathfrak{Graph}[4]$, $\mathfrak{Graph}(5)$, [$\mathfrak{Graph}$][4] and [*Graph*][5] into heat maps of multiregularity classes. What order structure do each of the 5 k-regular subcases supported form?

Exercise 4 Partitioning by branching point strength is a subcase of partitioning by multiregularity. Simplify your graph arena heat map so that only graphs with branching points of excess strength 0 and 1 are picked out. How is this reflected in the corresponding [graph] arena heat map?

Exercise 5 [Long] Give the heat map for the $\mathfrak{ConGraph}[5]$ as partitioned by each of the following. a) leaf number, b) twig number, c) leaf n-artiy, d) twig-n-arity, and e) for all four of these at once.

♠ ◇ ♣

Pointer 1 One eventual reason for such heat map distributions will be investigating the extent to which graph properties are persistent in growing network dynamics...

Pointer 2 Some graph subarenas support further ordering operations are not necessarily defined on the whole graph arena.

Exercise 6 Automorphisms are another [graph]-specific structure. Partition the [graph] arenas on $N \leq 5$ vertices by the graphs' automorphism groups.

61.4 (Non-)planar connected graph arenas

Worked Example 3 What is the *planar connected graph arena*

$$\mathfrak{PlanarConGraph}$$

on 0 to 5 vertices, as viewed extrinsically within $\mathfrak{ConGraph}$? Within this, where do the maximally planar, bi-partite maximally-planar, outer-planar and maximally-outer-planar subarenas lie?

Answer 3 Clearly up to $N = 4$,

$$\mathfrak{ConPlanar} = \mathfrak{Con} \ .$$

While for $N = 5$, the only difference with $\mathfrak{Con}$ is that we must excise the terminal leaf K_5. See Fig 61.2, and 61.3 for a heat map of the stated subcases.

Exercise 7 Work out the order structures and underlying graph skeletons for the connected bi-partite graph arenas

$$\mathfrak{ConBipartite}$$

for $N \leq 5$.

Exercise 8 Now working intrinsically, what are the order structures and underlying graph skeletons for the $N \leq 5$ arenas of each of the following? Maximally planar, maximally planar-bi-partite, outer-planar and maximally outer-planar graphs?

Examples 5-6 Connected planar graphs can furthermore be seen as the first part of the partition of $\mathfrak{ConGraph}$ by either crossing number class,

$$\mathfrak{Graph} = \amalg_{cr \leq 0} \mathfrak{Crossing}_{cr} \ .$$

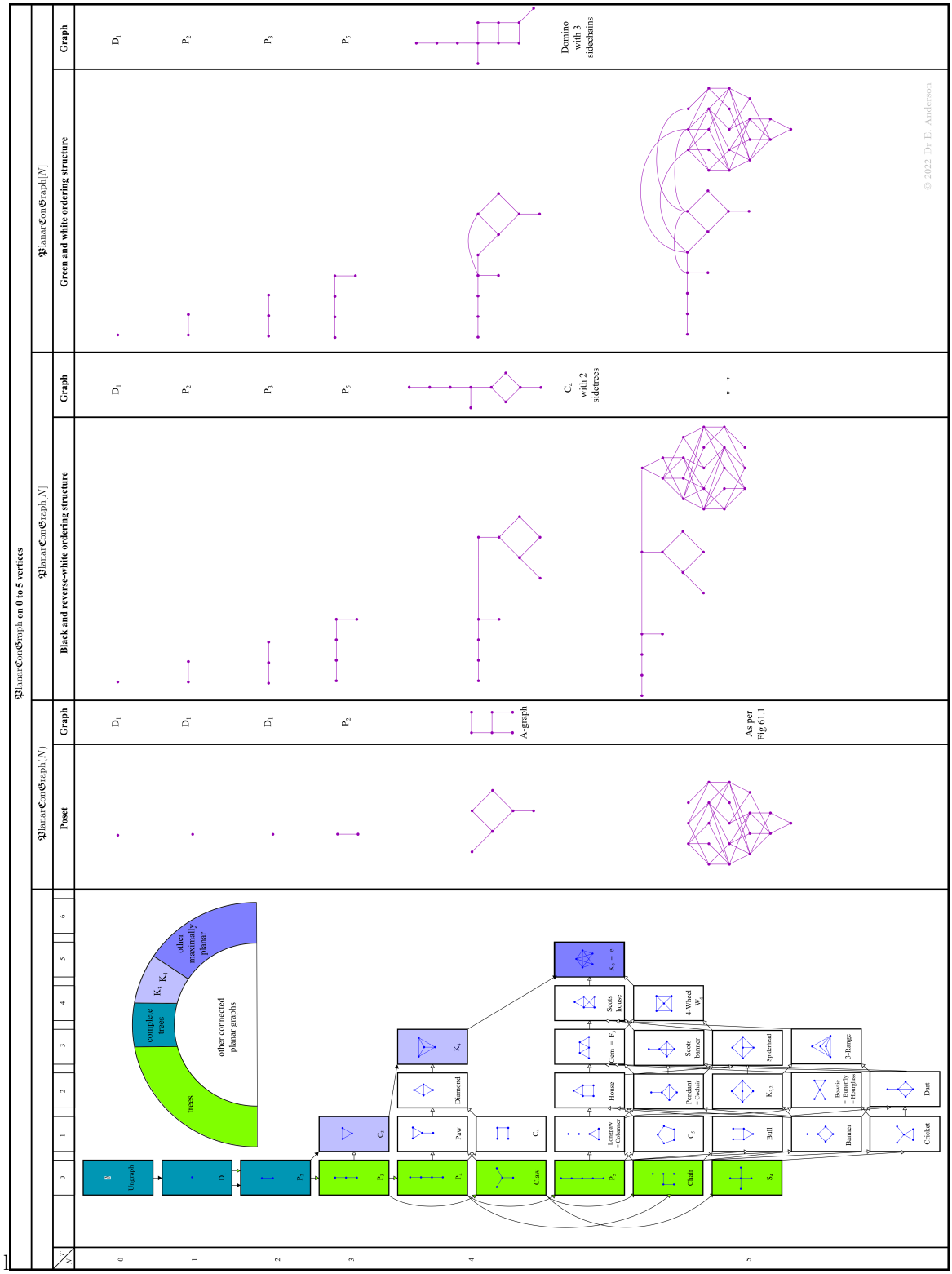


Figure 61.2:

Or by thickness classes

$$\mathfrak{G}\text{raph} \;=\; \amalg_{th \,\leq\, 1}\, \mathfrak{T}\text{hickness}_{th} \;.$$

These are however only worth starting to consider for $N \geq 6$, which lies outside the scope of the current Volume.

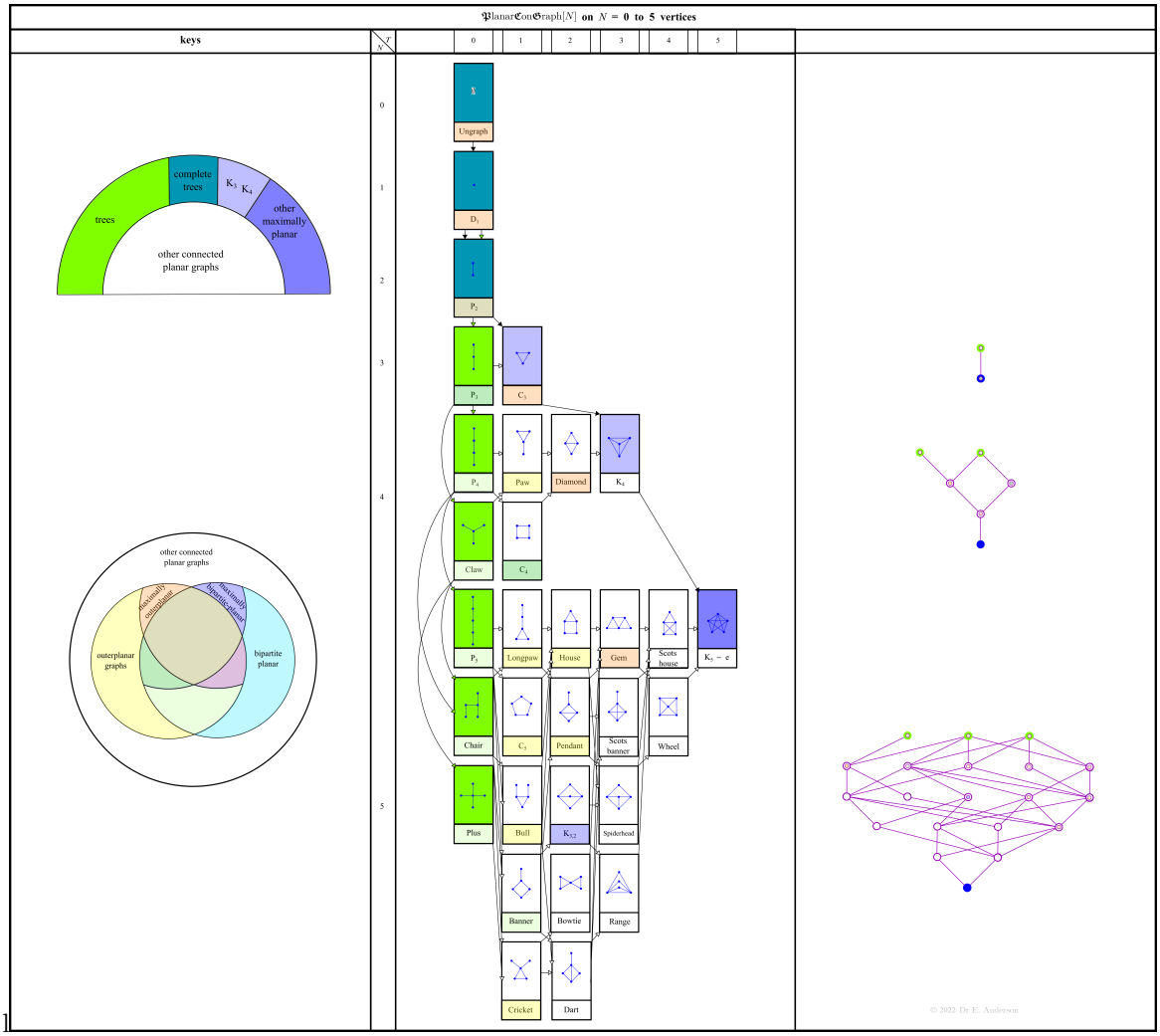


Figure 61.3:

61.5 Coloured graph arenas

Example 7 The *arena of* χ-colourable graphs is

$$\mathfrak{Colour}(N,\, \chi)\ .$$

Definition 1 The *arena of* χ_1*-edge-colourable graphs* is

$$\mathfrak{Colour}(N,\, \chi_1)\ .$$

Remark 1

$$\mathfrak{Graph}(N) \;=\; \sum_{\chi\,=\,0}^{\Delta} \mathfrak{Colour}(N,\, \chi)\ . \tag{61.1}$$

While much more restrictedly by Vizing's Theorem,

$$\mathfrak{Graph}(N) \;=\; \sum_{\chi_1 \,=\, \Delta}^{\Delta \,+\, 1} \mathfrak{Colour}(N,\, \chi_1) \;=\; \sum_{V \,=\, 0}^{1} \mathfrak{Colour}(N,\, \chi_1)\;, \tag{61.2}$$

where V is eq. (47.6) Vizing relative variable.

Exercise 9 Bi-partition $\mathfrak{Graph}[5]$ by Vizing class. Are the pieces connected?

Pointer 3 See Sec 63.3 for simple example of graph colouring arena used in setting up arro-graph arenas.

61.6 Exercises

Remark 1 We can partition any kind of arena of mathematical objects by the values taken by some cardinal function on these objects, or by a set thereof.

Remark 2 Topological cardinal functions for graphs include $\alpha_i, \beta_i, \gamma_i, \omega, \iota, \kappa_i$ and χ_i.

Exercises 10-16 Give heat maps for the partitions by each of these separately for the connected graphs on $N \;\leq\; 5$ vertices. Also give the order structure for each class, including identifying its underlying graph skeleton.

Remark 3 Metric-level cardinal functions for graphs include *diam*, *rad* and girth, g.

Exercises 17-19 Give heat maps for the partitions by each of these separately for the connected graphs on $N \;\leq\; 5$ vertices. Also give the order structure for each class, including identifying its underlying graph skeleton.

♠ ◇ ♣

Exercise 20 On $N \;\leq\; 5$, what ordering structure and underlying graph skeletons on the non-cumulative arenas of a) Eulerian graphs and b) Hamiltonian graphs?

Exercise 21^{-} Show that within $\mathfrak{Graph}[5]$ in our standard ordering, the arena of self-complementary graphs is not connected.

Chapter 62

Graph-homeomorph arenas

62.1 $\mathfrak{HI}$- and $\mathfrak{Top}$-$\mathfrak{Graph}$

Definition 1

$$\mathfrak{Top}\text{-}\mathfrak{Graph} \ := \ \frac{\mathfrak{Graph}}{Homeomorph} \ :$$

the arena of graphs as quotiented by homeomorphing.

Definition 2

$$\mathfrak{HI}\text{-}\mathfrak{Graph}$$

is the arena of homeomorph irreducibles.

Remark 1 One simple difference between $\mathfrak{HI}$- and $\mathfrak{Top}$- is that the first retains meaningful truncations to size N.

Remark 2 We postpone detailed topological and geometrical distinctions between these two arenas to Series 1. We largely only detail $\mathfrak{HI}$-$\mathfrak{Graph}$ arenas in the current Book.

62.2 Incompatible quotients

Remark 1 The operation of taking homeomorphs is not compatible with complementation. The minimal example of this is

$$\mathrm{P}_2 \ \overset{\text{homeo}}{\sim} \ \mathrm{P}_3 \overset{\text{homeo}}{\not\sim} \mathrm{D}_2 \ = \ \overline{\mathrm{P}_2} \ .$$

Quotienting by complementation and by homeomorphs are incompatible. So we can form one or the other but not both at once.

Pointer 1 We shall encounter further incompatible quotients in Flat Geometry (Volume 3), Lie Theory (Volume 7), Natural Law (Volume 9) and in our more detailed Book on discrete objects' arenas (Volume 3 of Series 1).

62.3 Homeomorph-irreducible trees

Worked Example 1 What is $\mathfrak{HI}$-$\mathfrak{Tree}[N]$ for $N \ \leq \ 7$?

Answer 1 We present this in Fig 62.1. 5 chains ensue, then Longchair, and then C_4 with a leaf and a sidechain.

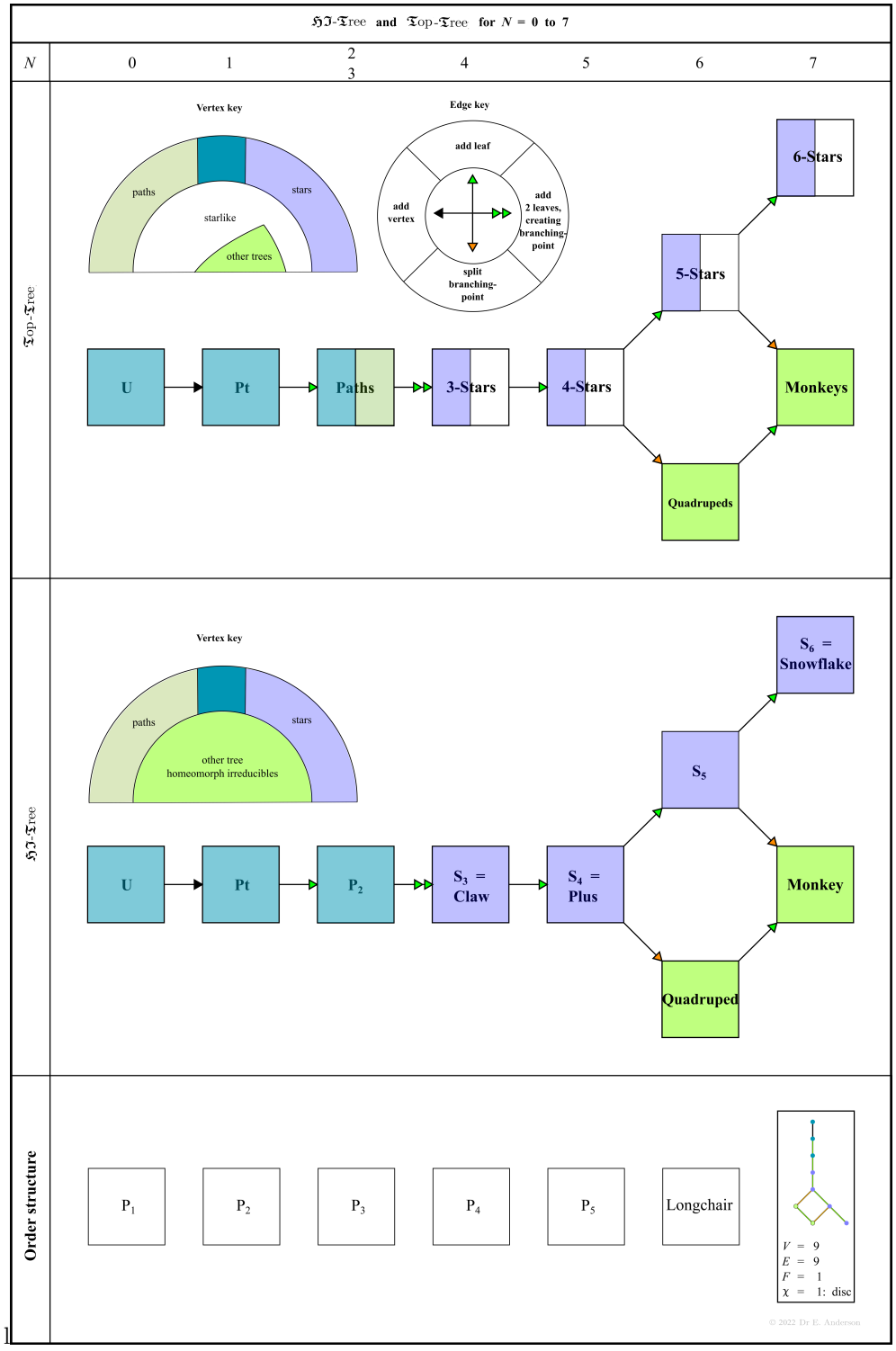


Figure 62.1:

62.4 Topology–metric split arena of trees

Worked Example 2 Give the topology-metric split of $\mathfrak{Tree}$ on 0 to 7 vertices.

Answer 2 See Fig 62.2. These are trees for $N \leq 6$, with only the last of these being new to us. $N = 7$ then picks up 2 isolated 4-cycles. All of these are clearly planar. For $N \leq 5$, the topological-metric split gives a comb. But for $N = 6$, the topological 'spine' becomes a tree, while for $N = 7$, this picks up its first cycle. The other cycle is in the 3-star 'tooth'.

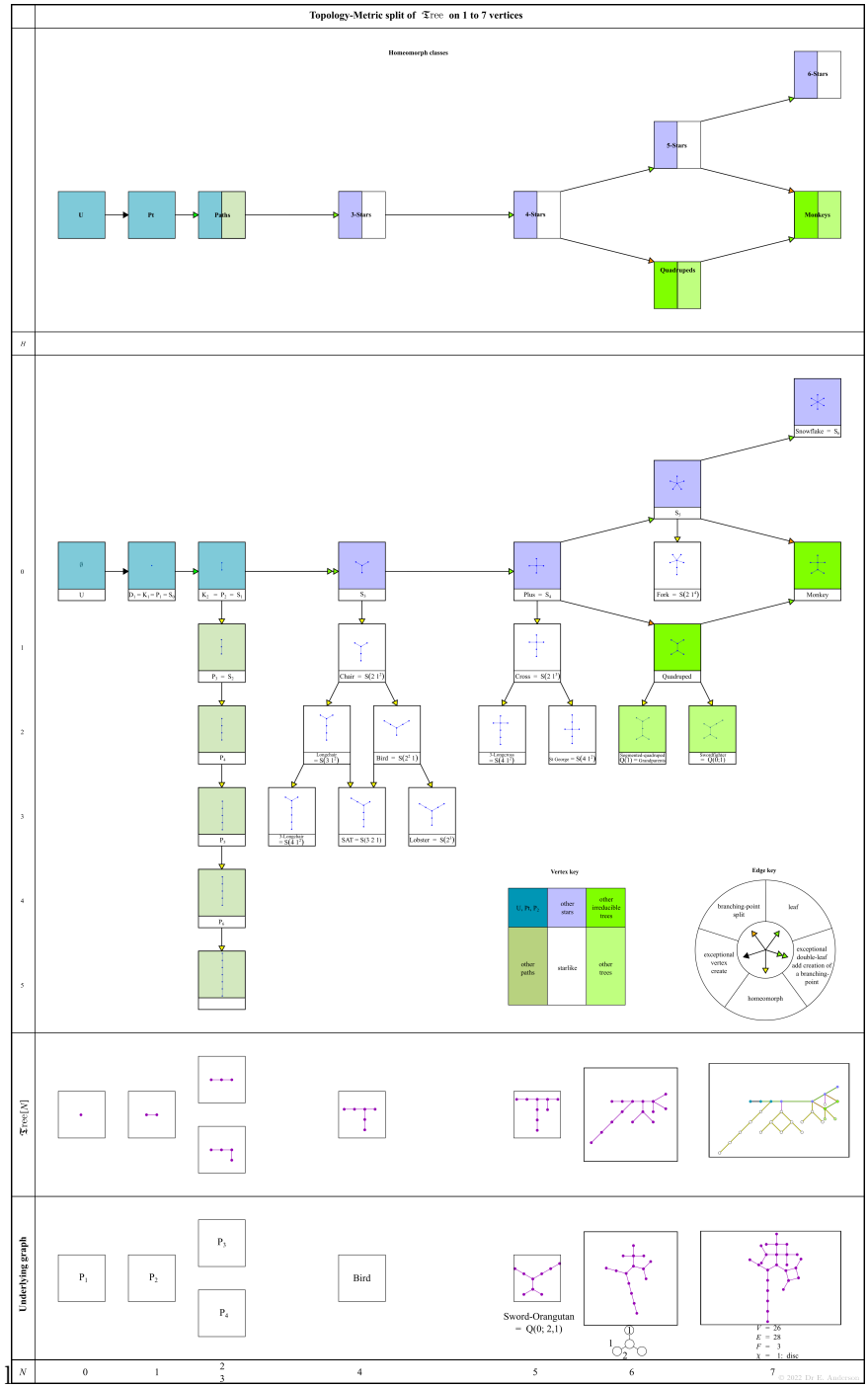


Figure 62.2:

Exercise 1 Extend this analysis to $N = 8$.

62.5 $\mathfrak{HICon}\mathfrak{Graph}[N]$ for $N \leq 5$

Exercise 2 Show that using just 'add leaf' produces a rather disconnected structure on $\mathfrak{HI}$-$\mathfrak{G}$raph(5) and $\mathfrak{HI}$-$\mathfrak{G}$raph[5] (use the graph configurations in Fig 62).

Remark 1 Perhaps such sparseness is unsurprising; after all, the only context in which we have used 'add leaf' as the sole order has been tree arenas. Let us thus contemplate pairing it with another ordering operation, a suggestion for which is as follows.

Structure 1 'Form cycle' consists of adding an extra edge between vertices so as to increase the number of elementary cycles by 1. There is a subtle distinction to make here. This operation turns out to distinguish between which homeomorph irreducibles are related by it, and which homeomorph equivalence classes are. For instance, P_2 and Triangle are not related, but [P_2] and [Triangle] are since $\mathrm{P}_3 \in [\mathrm{P}_2]$ is related to Triangle. We pick the necessarily narrower homeomorph-irreducible case, since this produces a simpler structure in a more controllable way.

Worked Example 3 What explicit order structures and underlying graph skeletons does this place on $\mathfrak{HIC}$on$\mathfrak{G}$raph up to $N = 5$?

Answer 3 $N = 4$ is minimal for possession of more than one homeomorph irreducible. Here 'add cycle' produces a 4-path.

$$\mathfrak{HI}\text{-}\mathfrak{G}\mathrm{raph}(N)_1 = \mathrm{K}_N$$

while there is not in general a unique bottom, due to all of $\mathfrak{HI}$-$\mathfrak{T}$ree(N) being terminal. Each N up to 5 has a unique HIT. With a blip for the trivial $N = 3$ case, which has none. $N = 5$ illustrates another problem in that Bull provides a second final element. We surmise that $\mathfrak{HI}$-$\mathfrak{G}$raph(N) is not in general more than semi-bounded.

By its being based solely on compatible addition of structure, it is a poset. In our small examples, all the elementary cycles are 4-cycles, and no $\mathrm{K}_{3,2}$ subposets are present, conforming with distributivity. Observe furthermore in Fig 62.3 that C_4-nonconvex subposets are present. For instance,

$$\{\text{ Pendant, Dart; Scotsbanner, Gem}\} .$$

Thereby, $\mathfrak{HI}$-$\mathfrak{G}$raph(5) cannot be a semi-lattice.

As a graph, $\mathfrak{HI}$-$\mathfrak{G}$raph(5) is straightforwardly planar. For observe that each of the two crossings in the poset presentation given have one edge that can be concurrently moved into that given presentation's outer face. A corresponding rectilinear presentation is given in Fig 62.3.b)'s second column. As depicted, $\mathfrak{HI}$-$\mathfrak{G}$raph(5)'s cycle system has 7 internal 4-cycles while its external cycle is of length 6. Thus

$$Y = 8 \text{ and } Z = \frac{7 \times 2 + 4}{2} = 9 .$$

The first nonstandard $\mathfrak{HI}$-$\mathfrak{G}$raph[N] is for $N = 5$. Its double poset presentation given (the first column of Fig 62.3.c) has more crossings than $\mathfrak{HI}$-$\mathfrak{G}$raph(N). These render it nonplanar, with $cr = 2 = th$. Rather than presenting our proof of this, we invite the Reader to produce your own (**Exercise 3**).

62.6 Topological-metric split of $\mathfrak{Graph}[4]$

Exercise 4$^+$ What order structure and underlying graph skeleton does the above analysis induce on $\mathfrak{HI}$-$\mathfrak{P}$lanar$\mathfrak{C}$on$\mathfrak{G}$raph[5]? and on $\mathfrak{HI}$-$\mathfrak{P}$lanar$\mathfrak{C}$on$\mathfrak{G}$raph[5]? Also provide minimal counterexamples to bi-partite graphs, maximally-planar graphs and maximally-outer-planar graphs being topologially-invariant notions.

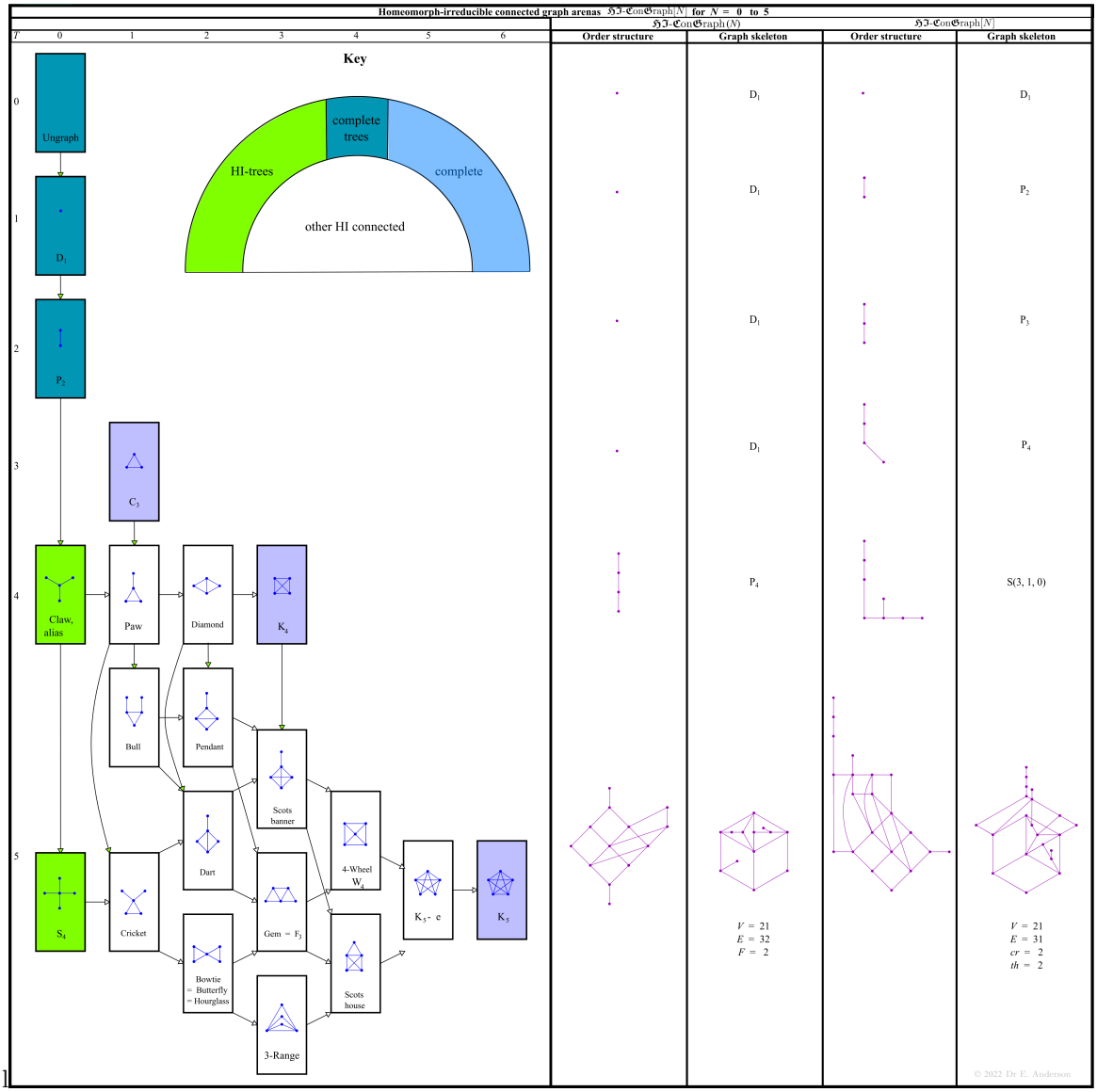


Figure 62.3:

Exercise 5 What is the topological-and-metric split of 𝔓lanar𝔈on𝔊raph[5]?



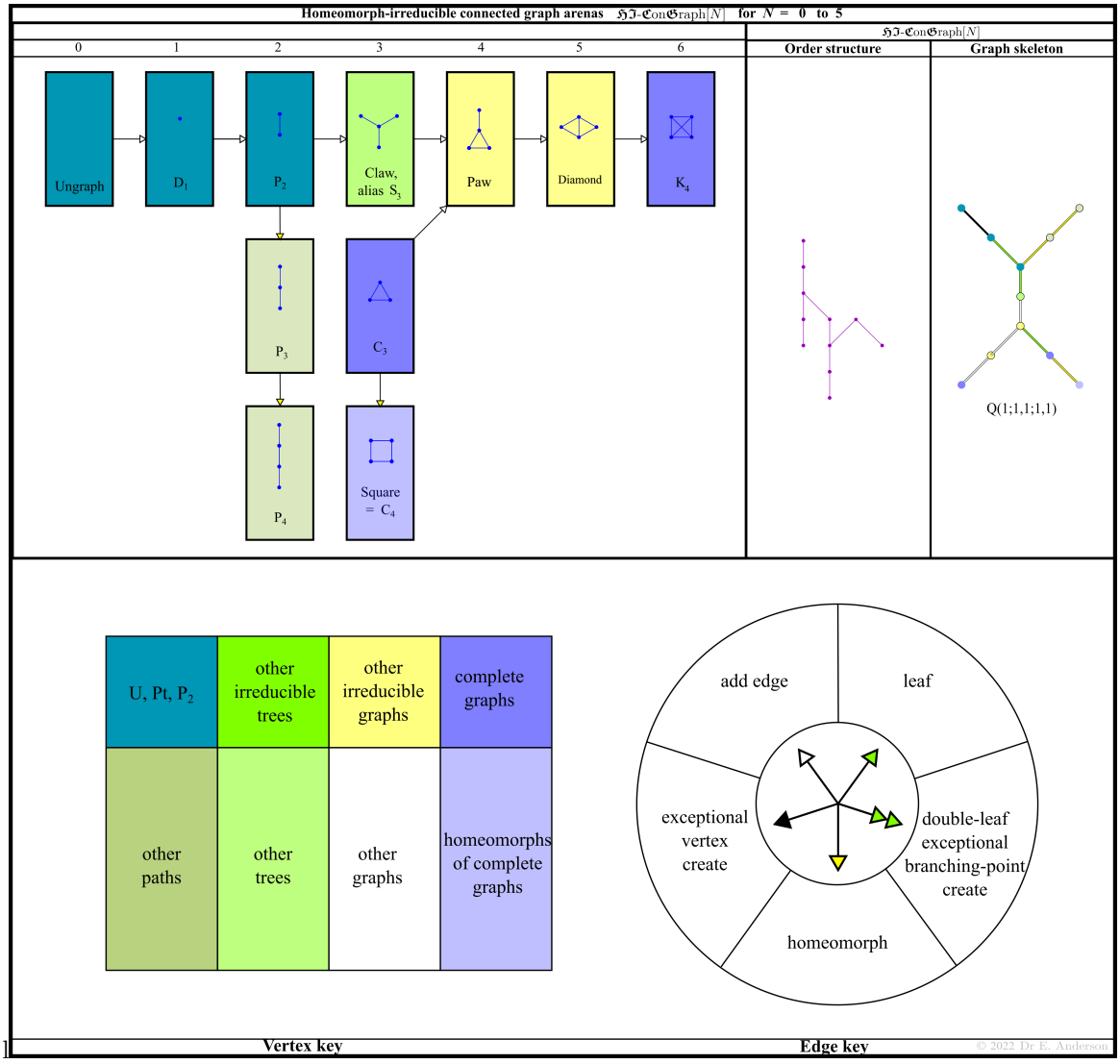


Figure 62.4:

62.7 (Loop-)(multi-)graph arenas

Definition 1 The arena of multi-graphs is

$$\mathfrak{Multi} .$$

Definition 2 The arena of loop-graphs is

$$\mathfrak{Loop} .$$

Definition 3 The arena of loop-multi-graphs is

$$\mathfrak{LoopMulti} .$$

Remark 1 These obey the square of inclusions of Fig 62.5.

Structure 1 We can furthermore define $\mathfrak{Loop}_l\mathfrak{Graph}$, $\mathfrak{Multi}_m\mathfrak{Graph}$ and $\mathfrak{Loop}_l\mathfrak{Multi}_m\mathfrak{Graph}$ of l-loop-graphs, m-multi-graphs, and (l, m)-loop-multi-graphs respectively.

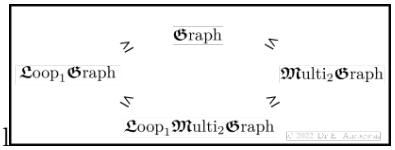


Figure 62.5:

62.8 $\mathfrak{M}$ulti$_2\mathfrak{G}$raph

Worked Example 4 What order structures and underlying graph skeletons result from using 'add edge' on $\mathfrak{M}$ulti$_2\mathfrak{G}$raph(N) for $N = 0$ to 3? What if we combine this with 'add vertex' alone, and then additionally with its symmetrization? Furthermore interpret this symmetrization.

Answer 4 'Add edge' is a poset-forming operation. The $\mathfrak{M}$ulti$_2\mathfrak{G}$raph(N) are bounded. With

$$\mathfrak{M}\text{ulti}_2\mathfrak{G}\text{raph}(N)_1 = \text{D}_N$$

and

$$\mathfrak{M}\text{ulti}_2\mathfrak{G}\text{raph}(N)_0 = \text{TK}_N :$$

the totally-double-edged complete alias 'doubly-complete' 2-multi-graphs. Graph complementation furthermore guarantees a complemented poset.

For $N = 0$ and 1, no double edges are supported, so the corresponding arenas cannot be any different from their simple-graph counterparts. For $N = 2$, a 3-chain in edge number ensues. For $N = 3$, we get a nontrivial poset middle. Consisting of a cycle system of 3 internal 4-cycles between its 2 terminal leaves. The external cycle here is of length 6. This one still manages to be a distributive lattice. We know however that these properties fail for $\mathfrak{G}$raph(N) for higher N; then since $\mathfrak{G}$raph(N) $\leq$ $\mathfrak{M}$ulti$_2\mathfrak{G}$raph(N), these fail here as well.

The first nontrivial $\mathfrak{M}$ulti$_2\mathfrak{G}$raph[N] is for $N = 3$; fused-cycle and sidetree notation covers it (Fig 62.6.c).
The symmetrized operation now mirrors the 'add red double-cone' interpretation of the 'add-vertex' arrows with 'add blue double-cone' arrows. The ensuing $\mathfrak{M}$ulti$_2\mathfrak{G}$raph$_{\text{S}}$[N] are notable for escaping from the $\mathfrak{G}$raph$_{\text{S}}$[N] summit triangle. The first nontrivial such arena is $\mathfrak{M}$ulti$_2\mathfrak{G}$raph$_{\text{S}}$[3]. Whose cycle system, as depicted, consists of 5 internal cycles: one of length 10, and the others all 4's. The external cycle is also of length 10. Alongside an apical leaf (Fig 62.6.d). Thus

$$Y = 6 \text{ and } Z = \frac{2 \times 8 + 4 \times 2}{2} = 12 .$$

Remark 1 For $\mathfrak{HI}$-$\mathfrak{G}$raph(5), $Z = 9$ but all the cycles are even, so Hamiltonianness is ZIPHoN-prohibited. Alternatively, the forced cycle marked in Fig 62.7.a) can be used.

Remark 2 $\mathfrak{M}$ulti$_2\mathfrak{G}$raph[3] also has a forced cycle (Fig 62.7.b), but a symmetry-manifesting 10-cycle presentation is supported.

62.9 Exercises

Exercise 6 What order structures and underlying graph skeletons result from quotienting the above $\mathfrak{M}$ulti$_2\mathfrak{G}$raph by complementation?

Exercise 7$^+$ Using Fig 39.1's configurations, what about for $\mathfrak{L}$oop$_1\mathfrak{G}$raph and [$\mathfrak{L}$oop$_1\mathfrak{G}$raph] for $N \leq 3$? Show in particular that the $\mathfrak{L}$oop$_1\mathfrak{G}$raph$_{\text{S}}$ pick up a different summit triangle, and that cr([$\mathfrak{L}$oop$_1\mathfrak{G}$raph](3) = 2.

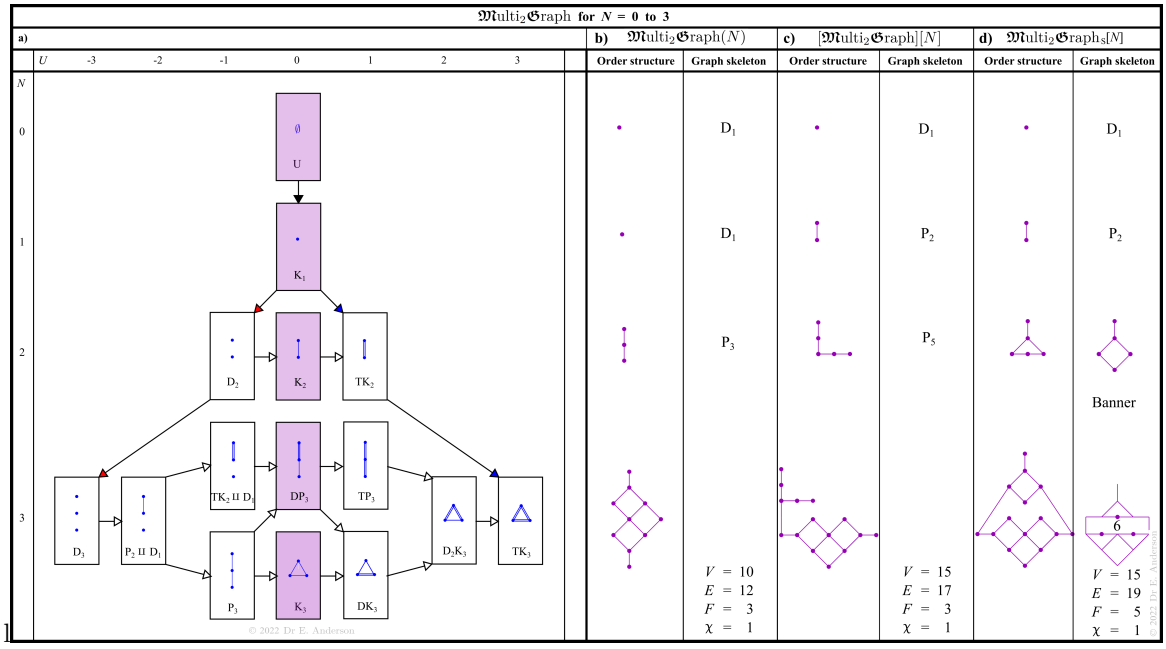


Figure 62.6:

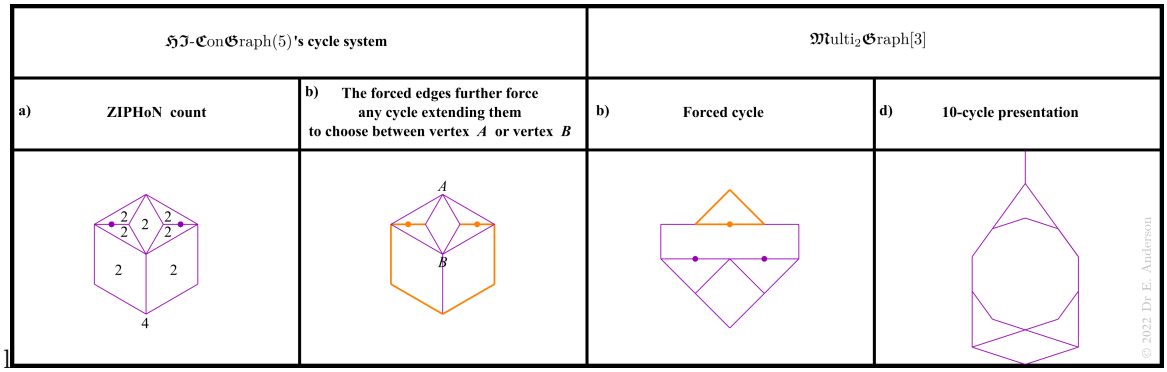


Figure 62.7:

Exercise 8 Study $\mathfrak{Loop}_1\mathfrak{Multi}_2\mathfrak{Graph}$ and $[\mathfrak{Loop}_1\mathfrak{Multi}_2\mathfrak{Graph}]$ using Fig 39.5's configurations up to $N = 2$

[The great swallowtail of 56 configurations for $N = 3$ is too large for the current Volume.]

Exercise 9 Study $\mathfrak{HI}$-$\mathfrak{Multi}_2\mathfrak{Graph}$ up to $N = 3$.

Exercise 10 Study $\mathfrak{HI}$-$\mathfrak{Loop}_1\mathfrak{Graph}$ up to $N = 3$.

Exercise 11$^+$+ Study $\mathfrak{HI}$-$\mathfrak{Loop}_1\mathfrak{Multi}_2\mathfrak{Graph}$ up to $N = 3$, nonplanarity properties included.

Chapter 63

Arrow-graph arenas

63.1 Introduction

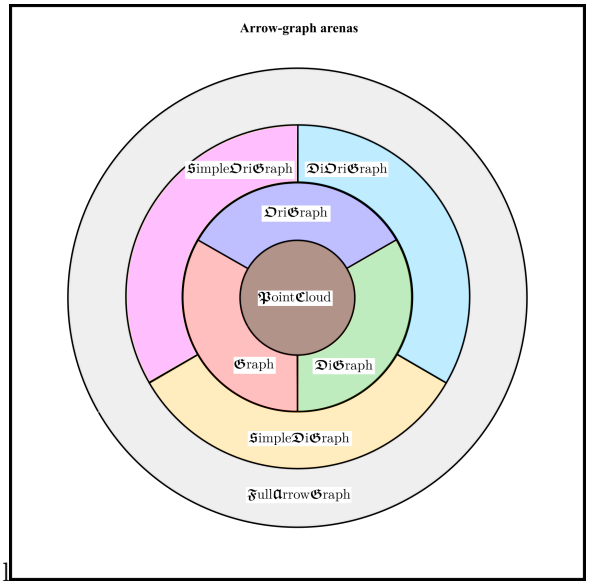


Figure 63.1:

Definition 1 The arena of ori-graphs is

$$\mathfrak{OriGraph} .$$

The arena of di-graphs is

$$\mathfrak{DiGraph} .$$

The arena of simple-ori-graphs is

$$\mathfrak{SimpleOriGraph} .$$

The arena of simple-di-graphs is

$$\mathfrak{SimpleDiGraph} .$$

The arena of di-ori-graphs is

$$\mathfrak{DiOriGraph} .$$

The full arena of arrow-graphs with all three edge types – simple, oriented and directed – is

$$\mathfrak{SimpleDiOriGraph} \text{ alias } \mathfrak{FullArrowGraph} .$$

Remark 1 Together with $\mathfrak{Graph}$ and $\mathfrak{PointCloud}$, these form an octet obeying the inclusions in Fig 63.1.

63.2 Some simple realizations and inclusions

Remark 1 For $N = 0$ and 1, since there are no edges,

$$\mathfrak{P}\text{oint}\mathfrak{C}\text{loud} \cong \mathfrak{G}\text{raph} \cong \mathfrak{D}\text{i}\mathfrak{G}\text{raph} \cong \mathfrak{O}\text{ri}\mathfrak{G}\text{raph} \cong$$

$$\mathfrak{S}\text{imple}\mathfrak{D}\text{i}\mathfrak{G}\text{raph} \cong \mathfrak{S}\text{imple}\mathfrak{O}\text{ri}\mathfrak{G}\text{raph} \cong \mathfrak{D}\text{i}\mathfrak{O}\text{ri}\mathfrak{G}\text{raph} \cong \mathfrak{F}\text{ull}\mathfrak{A}\text{rrow}\mathfrak{G}\text{raph} .$$

Remark 2 For $N = 2$, only one edge can be realized at once, and the oriented case contributes nothing different.

So firstly

$$\mathfrak{P}\text{oint}\mathfrak{C}\text{loud} \subset \mathfrak{G}\text{raph} \cong \mathfrak{D}\text{i}\mathfrak{G}\text{raph} \cong \mathfrak{O}\text{ri}\mathfrak{G}\text{raph} \cong 2\text{-chain} .$$

Secondly, no independent simple-di, simple-ori, or di-ori graphs are possible here, since two edges is minimal for this. Thus

$$\mathfrak{S}\text{imple}\mathfrak{D}\text{i}\mathfrak{G}\text{raph} = \mathfrak{G}\text{raph} \cup \mathfrak{D}\text{i}\mathfrak{G}\text{raph} ,$$

$$\mathfrak{S}\text{imple}\mathfrak{O}\text{ri}\mathfrak{G}\text{raph} = \mathfrak{G}\text{raph} \cup \mathfrak{O}\text{ri}\mathfrak{G}\text{raph} ,$$

$$\mathfrak{D}\text{i}\mathfrak{O}\text{ri}\mathfrak{G}\text{raph} = \mathfrak{D}\text{i}\mathfrak{G}\text{raph} \cup \mathfrak{D}\text{i}\mathfrak{G}\text{raph} ,$$

with all three giving the minimal nontrivial poset P_3-v.

Thirdly, no independent simple-di-ori graphs – i.e. generic arrow-graphs – are possible, since two edges is minimal for this. Thus

$$\mathfrak{F}\text{ull}\mathfrak{A}\text{rrow}\mathfrak{G}\text{raph} = \mathfrak{S}\text{imple}\mathfrak{D}\text{i}\mathfrak{G}\text{raph} \cup \mathfrak{S}\text{imple}\mathfrak{O}\text{ri}\mathfrak{G}\text{raph} \cup \mathfrak{D}\text{i}\mathfrak{O}\text{ri}\mathfrak{G}\text{raph}$$

$$= (\mathfrak{G}\text{raph} \cup \mathfrak{D}\text{i}\mathfrak{G}\text{raph}) \cup (\mathfrak{G}\text{raph} \cup \mathfrak{O}\text{ri}\mathfrak{G}\text{raph}) \cup (\mathfrak{D}\text{i}\mathfrak{G}\text{raph} \cup \mathfrak{D}\text{i}\mathfrak{G}\text{raph})$$

$$= \mathfrak{G}\text{raph} \cup \mathfrak{D}\text{i}\mathfrak{G}\text{raph} \cup \mathfrak{O}\text{ri}\mathfrak{G}\text{raph} .$$

This is realized by the Claw-13 poset.

Remark 3 By (33.5),

$$\mathfrak{D}\text{i}\mathfrak{G}\text{raph} \cong \mathfrak{G}\text{raph}$$

in full generality. We thus do not need to present $\mathfrak{D}\text{i}\mathfrak{G}\text{raph}$, this just being $\mathfrak{G}\text{raph}$ with all the edges carrying the two-way arrow and the names carrying two-way arrow overlines to indicate their pure di-graph edges.

Similarly,

$$\mathfrak{S}\text{imple}\mathfrak{O}\text{ri}\mathfrak{G}\text{raph} \cong \mathfrak{D}\text{i}\mathfrak{O}\text{ri}\mathfrak{G}\text{raph} .$$

So we are left with 6 of our 8 cases to consider. For each N, $\mathfrak{P}\text{oint}\mathfrak{C}\text{loud}$ is just a point. $N = 3$ is minimal to realize generic arrow-graphs (in the form of a single arrow-reversal pair). $N = 3$ is also minimal for oriented edges to add extra diversity. With both $\mathfrak{O}\text{ri}\mathfrak{G}\text{raph}$ being realized distinctly from $\mathfrak{G}\text{raph} \cong \mathfrak{D}\text{i}\mathfrak{G}\text{raph}$, and $\mathfrak{S}\text{imple}\mathfrak{O}\text{ri}\mathfrak{G}\text{raph} \cong \mathfrak{D}\text{i}\mathfrak{O}\text{ri}\mathfrak{G}\text{raph}$ distinctly from $\mathfrak{S}\text{imple}\mathfrak{D}\text{i}\mathfrak{G}\text{raph}$. With us only going as far as $N = 3$, our partial plots are thus to consist of 4 distinct nontrivial arenas and 2 loose points.



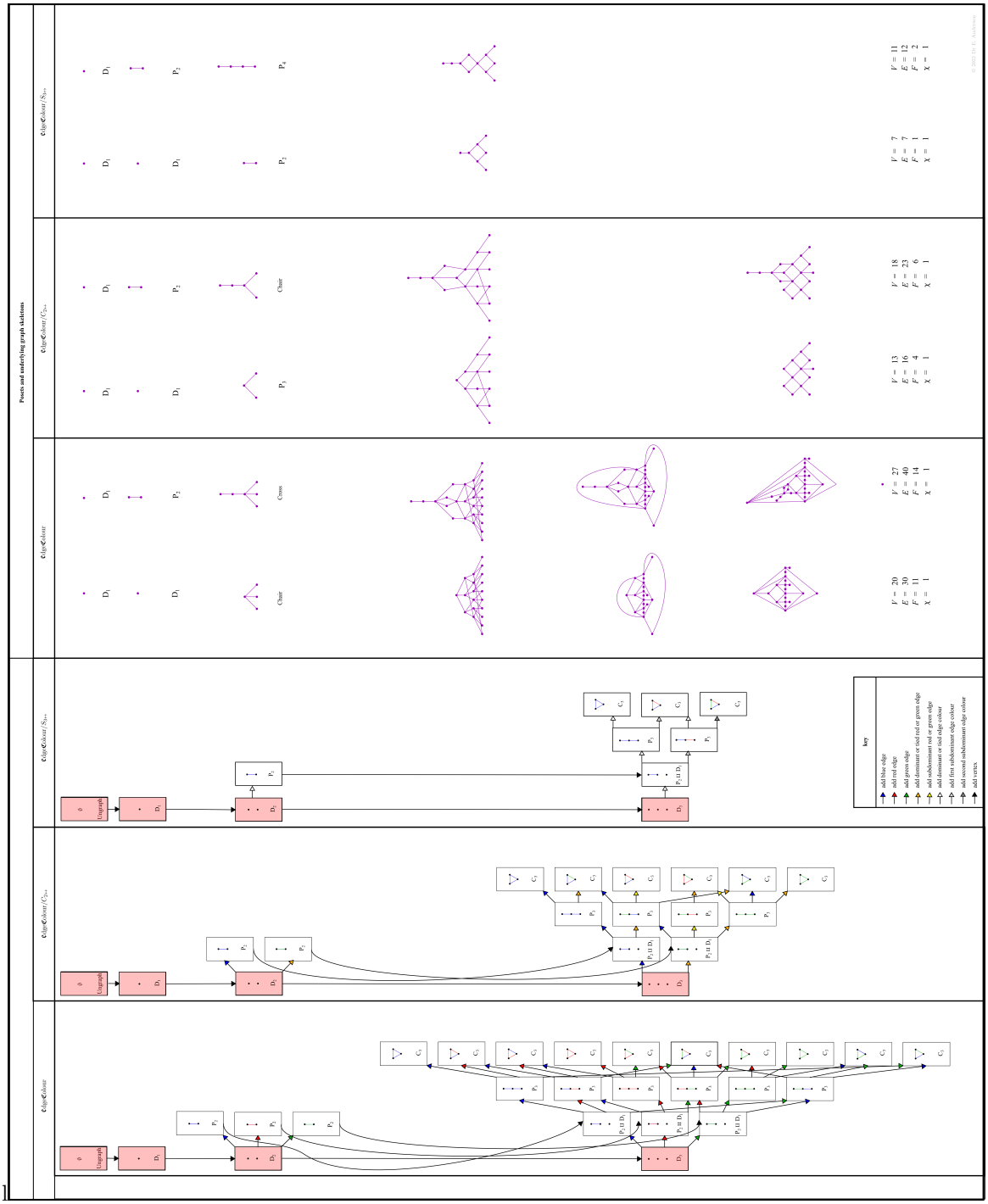


Figure 63.2:

63.3 Interlude: edge-colour arenas

Remark 1 This arena (Fig 63.2.a) turns out to be a simpler model sharing many features with 𝔄rrow𝔊raph. One is here in effect distinguishing between three types of edge without one of them being able to carry extra orientation structure. This fits the rubric of Fig 63.3: a close parallel of the conceptual figure 63.1.

Remark 2 This arena has an S_3 colour-meaninglessness symmetry; quotienting this out gives the greatly simplified Fig 63.2.c). This corresponds to $\mathfrak{EdgeColour}$ exhibiting a triangular symmetry (recall that $D_3 \cong S_3$). This is more complicated than the midline reflection symmetry structure in the arenas the current Volume has so far considered.

Remark 3 It is however the partly-meaningless case also supported – one pair of meaningless colours and a single intrinsically-meaningful colour – that we particularly need for the current Chapter's main study (Fig 63.2.b). Our meaningful colour is here an over-simplified model for oriented edges, while our meaningless pair are simple- and di-graph edges. This represents a symmetry breaking of our triangular symmetry: the 'isosceles' version of $\mathfrak{EdgeColour}$. [The original Fig 63.2.a) survives in the fully broken 'scalene' version of $\mathfrak{EdgeColour}$, in which all three colours are meaningful.]

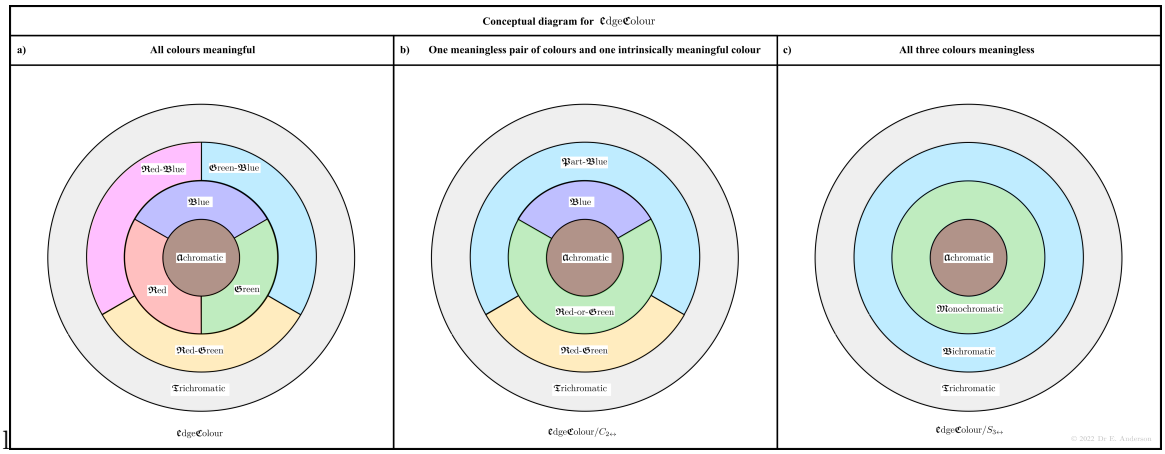


Figure 63.3:



63.4 Partial plots for the arrow-graph arenas themselves

Structure 1 We already presented the $\mathfrak{Graph}$ case in Fig 59.1; this includes $\mathfrak{PointCloud}$ as an extreme case.

Exercise 1 Work out the $\mathfrak{OriGraph}$ case (this again has again with $\mathfrak{PointCloud}$ as an extreme case).

Exercise 2 Extend both of the previous to include the full $\mathfrak{SimpleOriGraph}$ case.

Exercise 3 Work out the $\mathfrak{SimpleDiGraph}$ extension of the edge-type-exchange-symmetric $\mathfrak{Graph}$ and $\mathfrak{DiGraph}$.

[In doing Exercises 2 and 3, indicate in particular where the single proper pair of $N = 3$ generic arrow-graphs attach. Also include the corresponding quotients under arrow reversal.]

63.5 Unreduced arrow-graph arenas

Worked Example 1 What is the full arrowspace arena on 0 to 3 vertices?

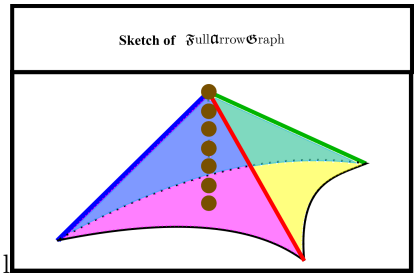


Figure 63.4:

Answer 1 Sec 63.4's Exercises' subarenas fit together in the form of Fig 63.4. The single-edge arenas form a Claw $=$ 3-S emanating from a point-cloud 'starpoint'. With two-edge-type arenas spanning between their corresponding single-edge-arenas. This builds structural detail into the conceptual sketch of Fig 63.1. A 'median-ribbed claw' triangle closely paralleling 63.2 becomes apparent.

63.6 Reduced arenas of arrow-graphs

Remark 1 Simple-graph-to-di-graph equivalence of arenas means that it is an 'isosceles' version of our arena triangle that is realized. This corresponds to an $S_2 = C_2$ subgroup of the $S_3 = D_3$. There is now additionally an'internal' symmetry in the oriented edges' structure: arrow reversal symmetry $C_{2\,\mathrm{rev}}$. We thus have overall a $V_4 = C_2 \times C_2$ group, acting as

$$C_{2\,\mathrm{rev}} \times C_{2\,\mathrm{simple}\leftrightarrow\mathrm{di}} \ .$$

Remark 2 Quotienting out $C_{2\,\mathrm{rev}}$ is always possible.

Quotienting out $C_{2\,\mathrm{plain}\leftrightarrow\mathrm{di}}$ as well quite closely parallels the middle quotient of Fig 63.2.b). This is why the previous section is placed in this interluding position. What we also need to do now is allow for orientation's structure to distort one colouring's sector (and subsequent multi-colourings that involve it). We thus arrive at Fig Arrowspace-Lattice-0-3.c).

Remark 3 This analysis reveals that such as $\mathfrak{G}$raph's centring coordinates are merely local within enlarged arenas, resulting from labelling at least some edges with arrows. It is now the common spine formed by the point clouds that forms a central edge; we mark this with a dark red background.

Remark 4 Arrowed graph arenas are in general more than 3-d in parametrization. Though up to our $N = 3$ case, there never is any orange under any white, so a 2-d bird's eye view suffices.

Remark 5 We do not present the arena of tournaments here since this has but antichain structure under the 'add arrowed edge' ordering operation.

63.7 Further study of the $\widetilde{\mathfrak{arrowGraph}}$ graph

For the cycle system in our largest such (63.6.a)

$$Z = \frac{9 \times 2 + 8}{2} = 13 \ .$$

This odd outcome prohibits a Hamiltonian cycle in such a purely even-cycle graph. The largest available cycle to invert the whole cycle structure into is rather of length 10; we use this in Fig 63.6.b).

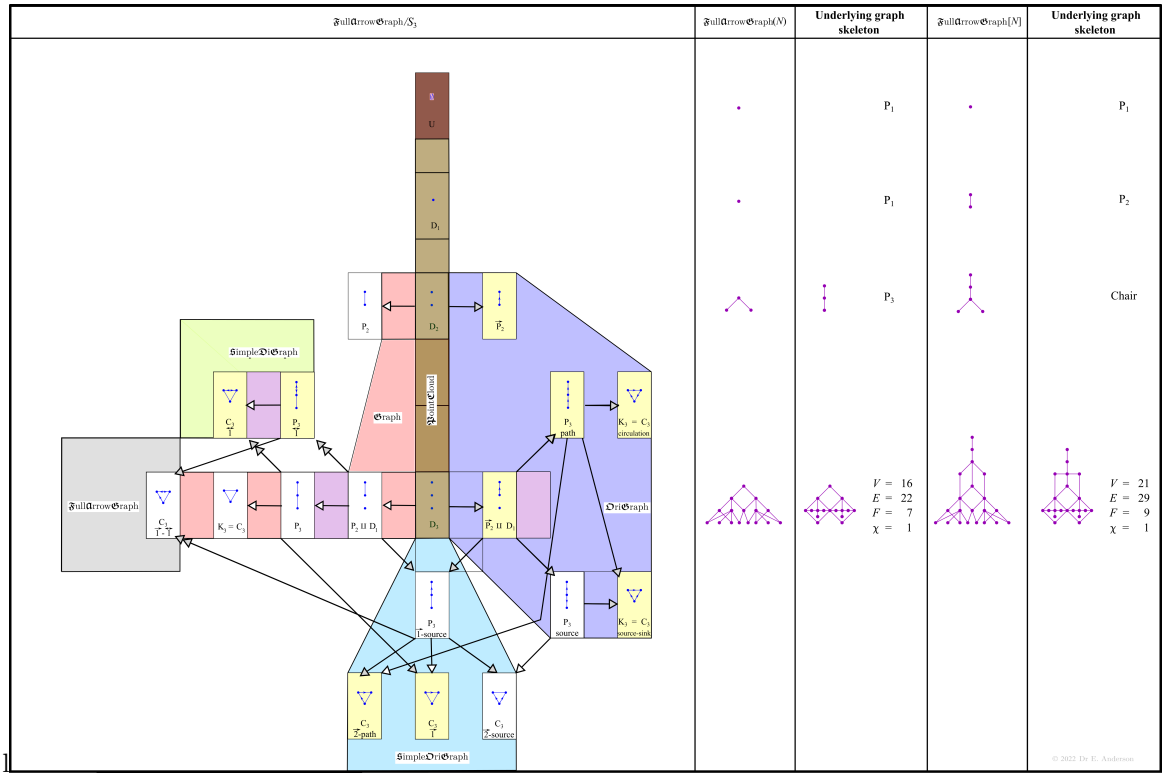


Figure 63.5:

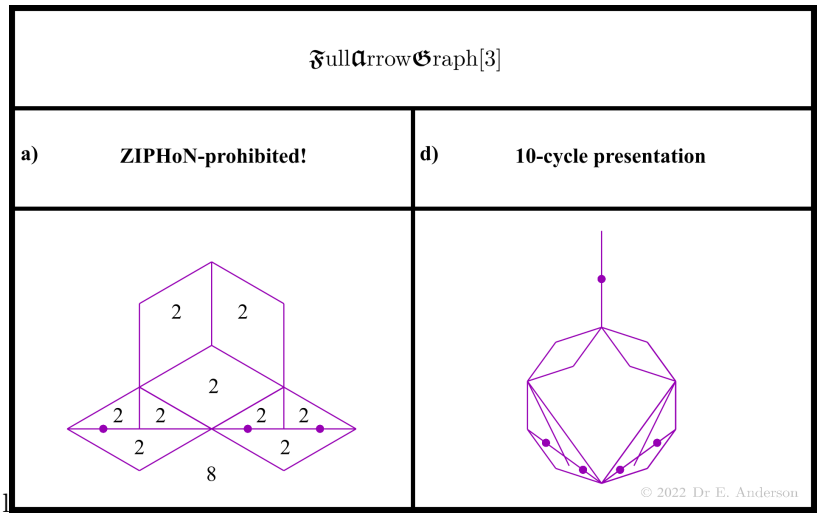


Figure 63.6:

63.8 Further Exercises

Exercise 4 Is EdgeColour(3)'s underlying graph skeleton planar? Is FullArrowGraph(3)'s?

Exercise 5 $\longrightarrow$ Graph affords a further option: to quotient out $C_{2\,\text{plain}\leftrightarrow\text{di}}$ but not $C_{2\,\text{rev}}$.

a) What are the counts for this on $N = 1$ to 3 vertices?

b) What is the ensuing lattice?

c) Is its underlying simple graph skeleton planar?

Exercise 6 Give an orbit–stabilizer analysis for both $\mathfrak{E}$dge$\mathfrak{C}$olour$\mathfrak{G}$raph(3) and $\mathfrak{A}$rrow$\mathfrak{G}$raph(3).

Exercise 7 Which arrow-graphs in Fig 62.6 are homeomorph irreducibles? What order structure do these form under the add-edge and add-loose-vertex pair of operations?

Exercise 8$^+$ Extend the analysis of $\mathfrak{E}$dge$\mathfrak{C}$olour to $N = 4$ with each of 2, 3 and 4 edge colours.

Exercise 9 Pick out the arena of licitly edge-coloured graphs as a subarena within $\mathfrak{E}$dge$\mathfrak{C}$olour on ≤ 4 vertices.

Exercise 10$^+$ Investigate the orders formed by the arrowed version of the (1, 2)-loop-multi-graphs on 0 to 2 vertices, and the underlying graph skeletons as well.

Exercise 11$^+$ Characterize the corresponding homeomorph-irreducible arena.

Exercise 12$^+$ Work out the corresponding topological-metric split arena.



Chapter 64

Order arenas

64.1 Poset arenas

Definition 1 We denote the arena of posets by

$$\mathfrak{P}\text{oset} .$$

The arena of ⟨posets⟩ – posets modulo order reversal – by

$$\langle\mathfrak{P}\text{oset}\rangle .$$

And the arena of homoreversible posets by

$$\mathfrak{HR}\text{-}\mathfrak{P}\text{oset} .$$

Structure 0 The ordering operation that we use in the current section is 'foliate or add edge'. In a manner that preserves the pre-existing order structure. I.e., firstly, that each target produced remains a licit poset. Secondly, that each arrow keeps all of each source's arrowing and up-down information in the corresponding target. Thirdly, each newly-added vertex is permitted to be linked in by either an upwards or a downwards edge.

Worked Example 1 What are the arenas of posets on 0 to 4 vertices?

Answer 1 See Fig 64.1. The first seven entries are just points or chains. The eighth entry – $\mathfrak{HR}$-$\mathfrak{P}$oset[3] – is a nontrivial upper semi-lattice. The ninth and tenth entries are merely posets due to not managing to be connected. This is because there is no way of generating C_4 from smaller homocompelementary posets.

$\mathfrak{HR}$-$\mathfrak{P}$oset[4] is a planar graph with a cycle system that, as depicted, consists of 4 internal 4-cycles. Its external cycle is of length 8. It also contains 2 sidechains and a loose vertex.

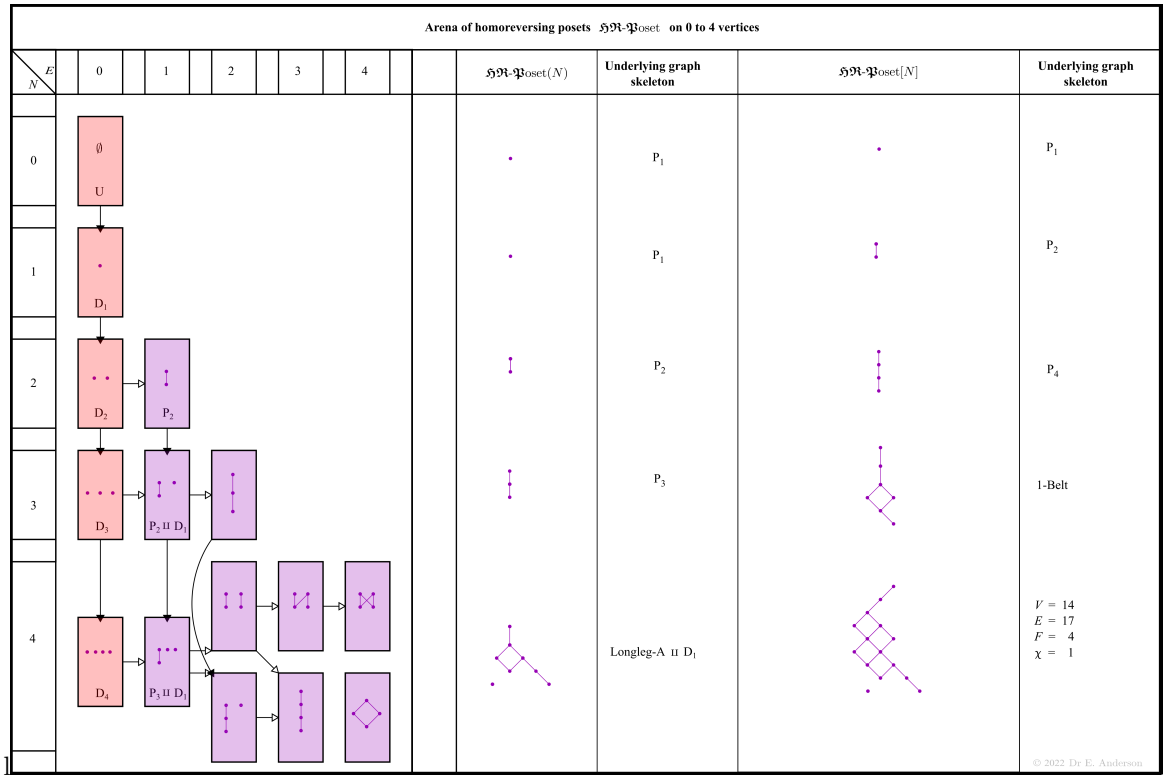


Figure 64.1:

Worked Example 2 What are the arenas of ⟨posets⟩ on 0 to 4 vertices?

Answer 2 See Fig 64.2. The first six entries are just points or chains. The seventh and eighth entries are simple upper semi-lattices. The ninth and tenth entries are just posets due to containing a C_4-nonconvex:

$$\{ {P_2}^2, P_3\text{-v}; P_4\text{-hockey}, P_4\text{-N} \} .$$

In more detail, $\mathfrak{hr}$-$\mathfrak{P}$oset(4)'s underlying graph skeleton is a planar graph with a cycle system which is a quadrangulation with 5 internal cycles. It also contains 3 leaves.

$\mathfrak{hr}$-$\mathfrak{P}$oset[4] is likewise just a poset. Its underlying graph skeleton is a more complicated and possibly novel planar graph. It consists of 9 fused cycles with excess cycle lengths as indicated, with 3 leaves and 1 2-path sidechain. Exercise 31.14 can be interpreted as establishing that each of these last two are planar.

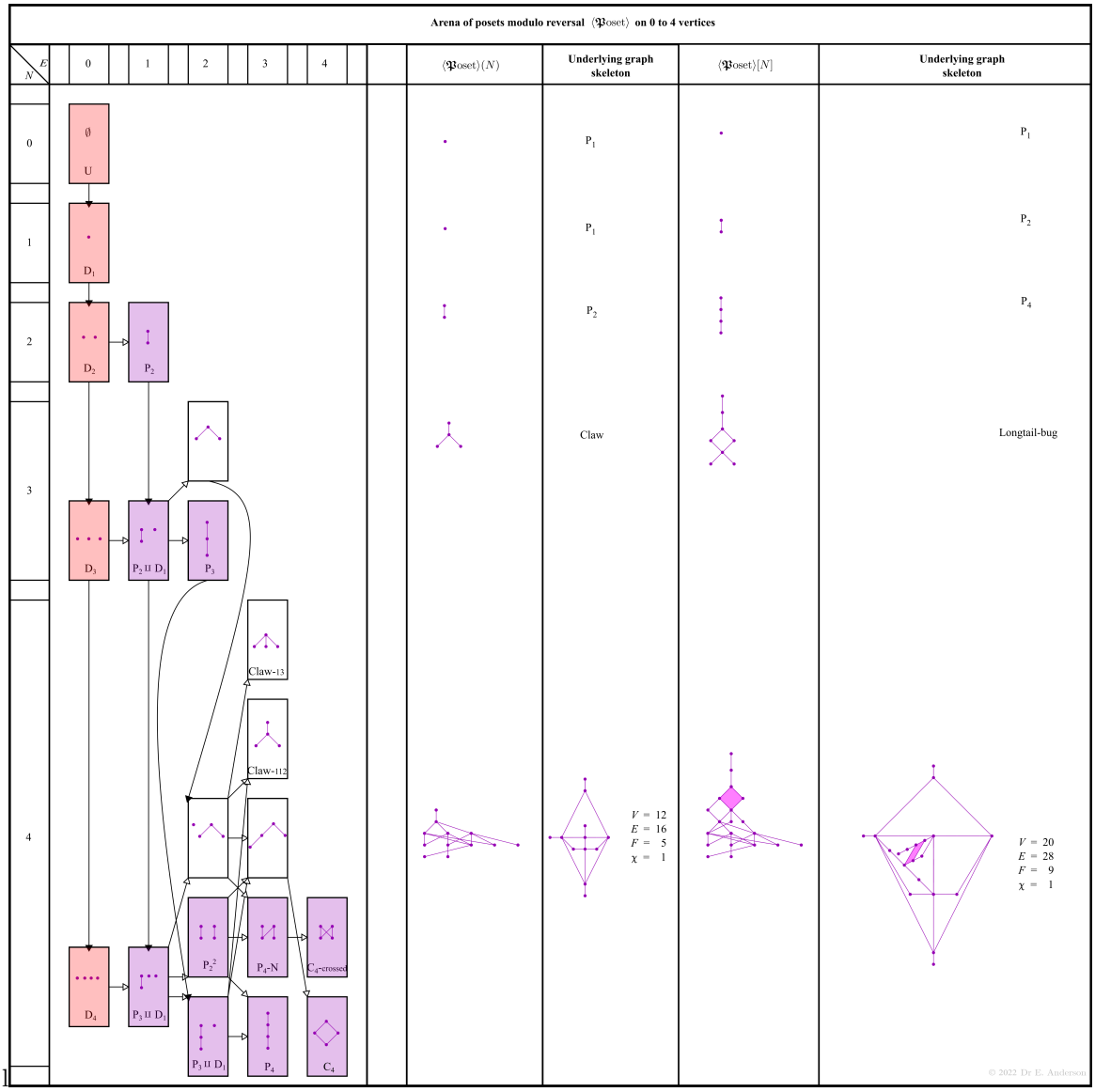


Figure 64.2:

Worked Example 3 What are the arenas of posets on 0 to 4 vertices? Are these in general lattices?

Answer 3 The first paragraph of Structure 2 applies again. The previous Example's C_4-nonconvex is uplifted to our larger current Example, so $\mathfrak{P}$oset[4] is not a lattice either.

Exercise 1$^+$ $\mathfrak{P}$oset(4) and $\mathfrak{P}$oset[4] are now more complicated, with underlying graph skeletons that are quite probably not known to occur elsewhere.

a) Find $\mathfrak{P}$oset(4)'s crossing number, and that show that $\mathfrak{P}$oset[4] adds just 1 to the crossing number.

Exercise 2^{++} Analyze the extent to which $\mathfrak{HR}$-$\mathfrak{P}$oset(5) and ⟨$\mathfrak{P}$oset⟩(5) are nonplanar.

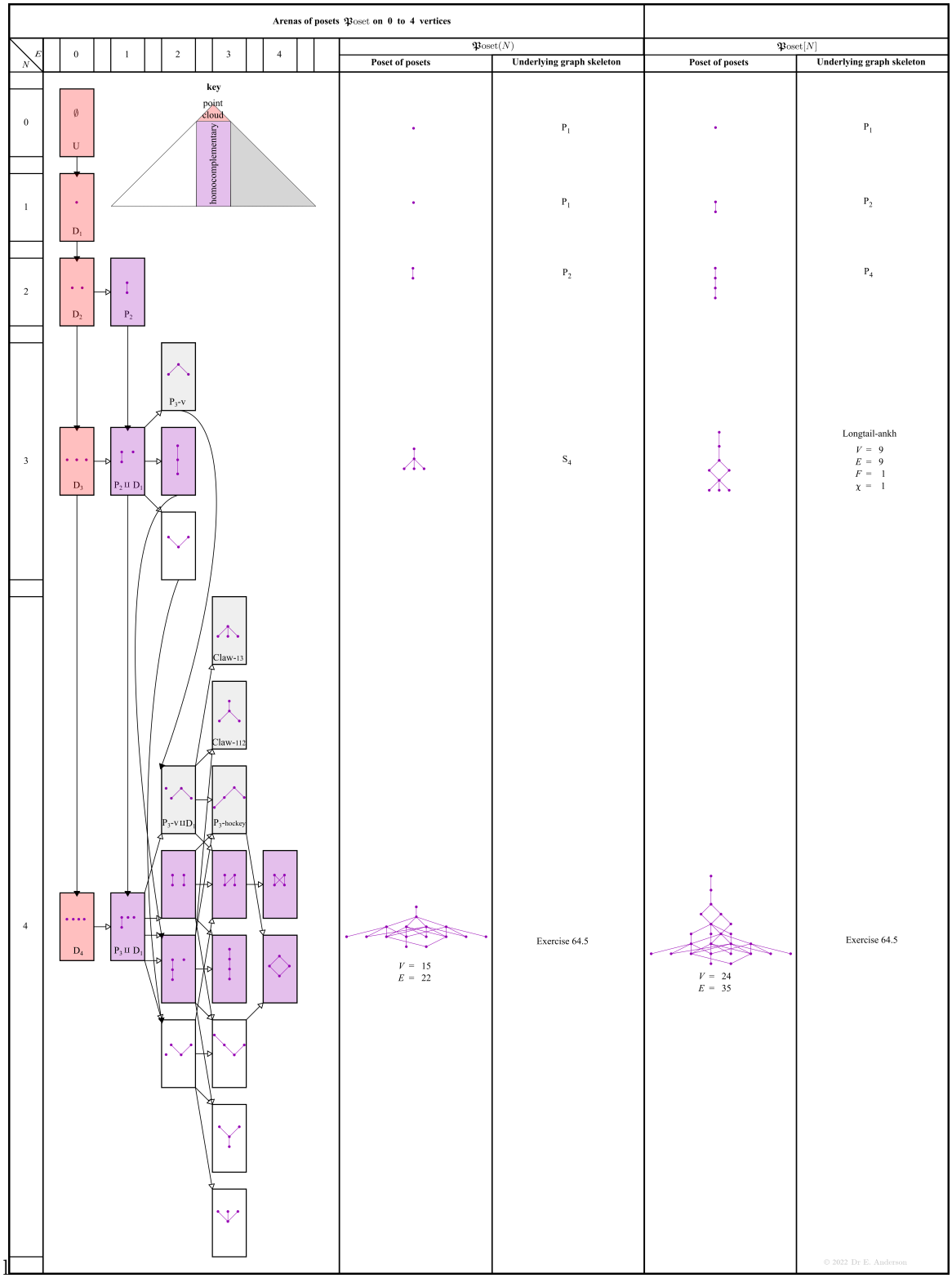


Figure 64.3:

64.2 Orders for lattice arenas

Order 1) Since lattices are special cases of poset, arenas of lattices could inherit the order used above in the arena of posets. For small lattices, however, this order structure turns out to be

considerably more sparse.

Order 2) An alternative is to drop preservation of arrow and up-down ordering structure. This can be viewed as ordering lattices' underlying graph skeletons. It is more weakly-related to the lattices albeit richer in its own order structure.

Remark 1 While order 1) is in some ways more theoretically-significant, it is simple enough for us to leave working it out to the Exercises. The rest of this Chapter studies order 2).

64.3 Distributive lattice arenas

Definition 1 We denote the arena of distributive lattices by

$$\mathfrak{Distrib}\mathfrak{Lattice} \ .$$

The arena of ⟨distributive lattices⟩ by

$$\langle \mathfrak{Distrib}\mathfrak{Lattice} \rangle \ .$$

And the arena of homodual distributive lattices by

$$\mathfrak{HD}\text{-}\mathfrak{Distrib}\mathfrak{Lattice} \ .$$

Worked Example 4 What are the arenas of homodual distributive lattices on 0 to 6 vertices?

Answer 4 See Fig 64.4. The first 11 entries are points or chains. The 12th – $\mathfrak{HD}$-$\mathfrak{Distrib}\mathfrak{Lattice}[5]$ – is a simple semi-lattice. The 13th and 14th are just posets, due to failing to be connected. For there is no way of reaching Buckle or Domino from within the homodual distributive lattices.

Remark 1 In reading the rest of this Chapter, bear in mind that lattice, ⟨lattice⟩ and homodual lattices only start to differ for $N = 5$. As do plain, modular and distributive differences. To have more than 1 point's worth of difference between the latter triple of situations, we need to go as far as $N = 6$.

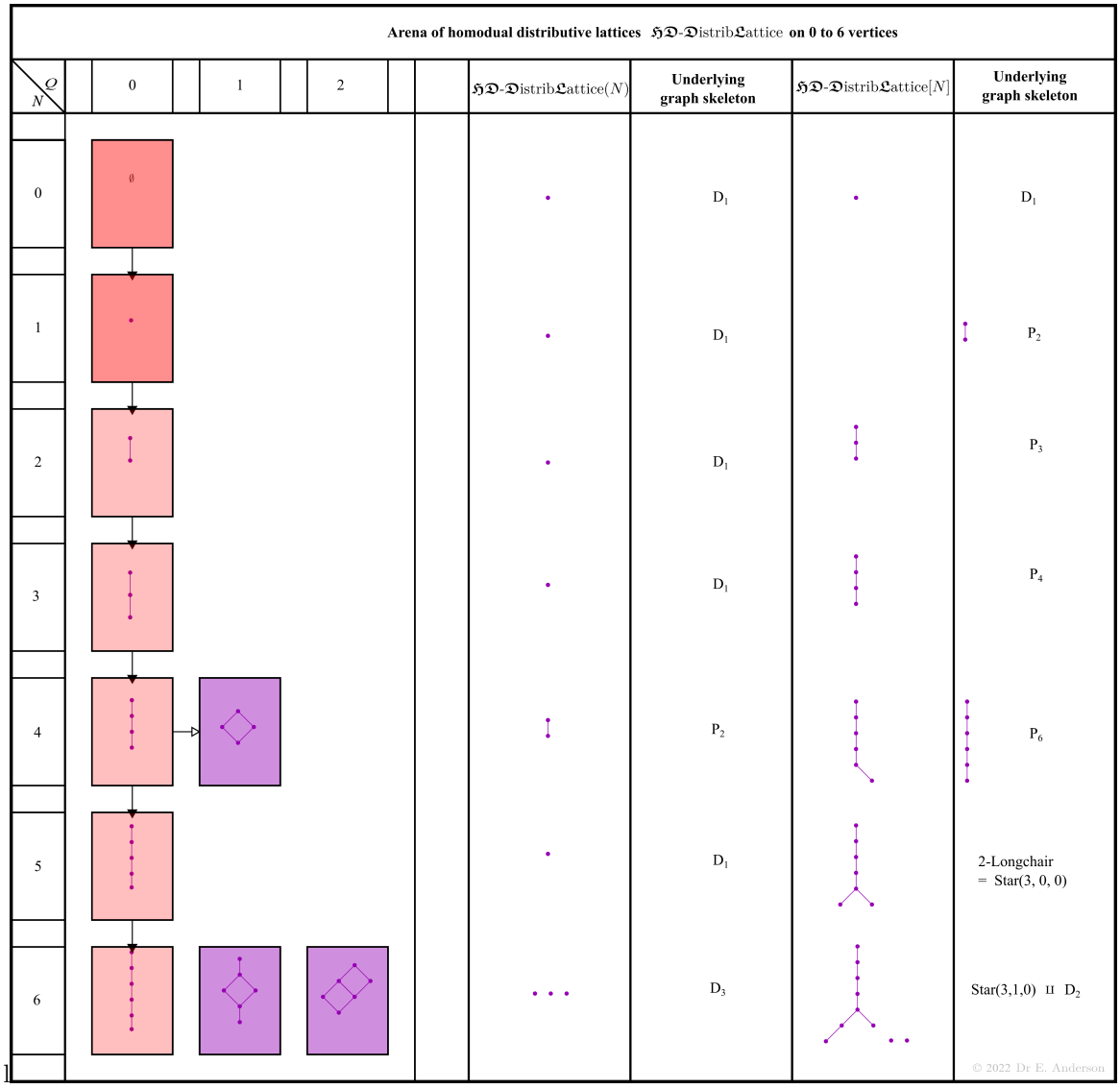

Arena of homodual distributive lattices ℌ𝔇-𝔇istrib𝔏attice on 0 to 6 vertices
Q
N
0
1
2
ℌ𝔇-𝔇istrib𝔏attice(N)
Underlying graph skeleton
ℌ𝔇-𝔇istrib𝔏attice[N]
Underlying graph skeleton
∅
D1
D1
P2
D1
P3
D1
P4
P2
P6
D1
2-Longchair
= Star(3, 0, 0)
D3
Star(3,1,0) ⊔ D2
© 2022 Dr E. Anderson

Figure 64.4:

Worked Example 5 What are the arenas of ⟨distributive lattices⟩ on 0 to 6 vertices?

Answer 5 See Fig 64.5. The first 11 entries are again points or chains. The 12th – ⟨$\mathfrak{D}$istrib$\mathfrak{L}$attice⟩[5] – is a simple semi-lattice. The 13th is just a poset, due to failing to be connected. This is because there still is no way of reaching Buckle within the $N = 6$ ⟨distributive lattices⟩. The 14th is a semi-lattice: Domino with 2 leaves and a P_4 sidechain.

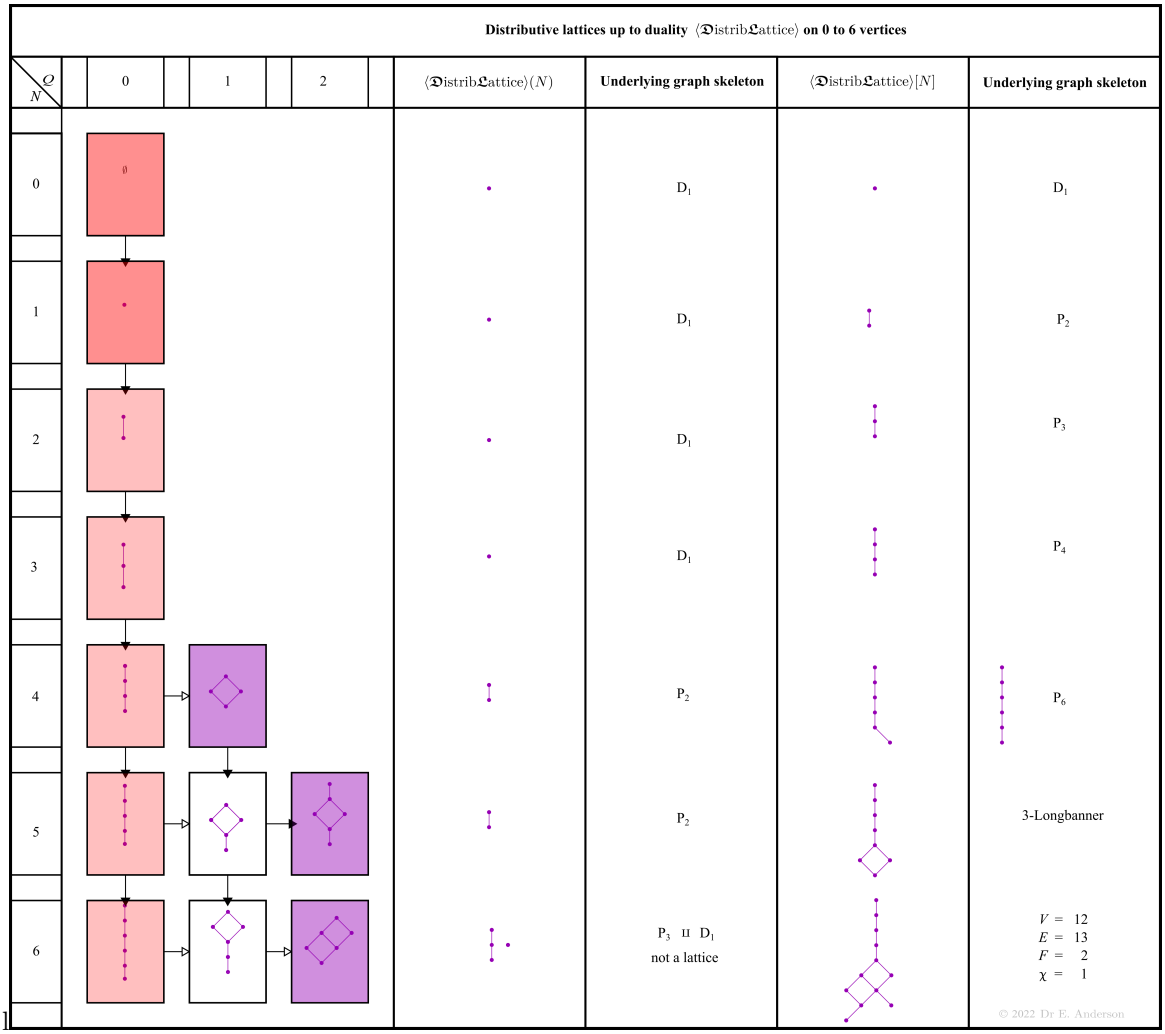


Figure 64.5:

Worked Example 6 What are the full arenas of lattices on 0 to 6 vertices?

Answer 6 See Fig 64.6. The first 11 entries are yet again points or chains. The 12th is just a simple poset since it contains C_4-nonconvex:

$$\{ C_4, P_5; \text{Banner-up}; \text{Banner-down} \} .$$

The 13th is also just a simple poset, since it fails to be conected, yet again due to Buckle not being reachable.

The 14th gives us our first new underlying graph skeleton for a arena of lattices. This consists of a cycle system with, as presented, 5 4-cycles and a 6-cycle. Its external cycle is also of length 4. Thus

$$Y = 7 \text{ and } Z = \frac{6 \times 2 + 4}{2} = 8 .$$

It finally contains a P_4 sidechain.

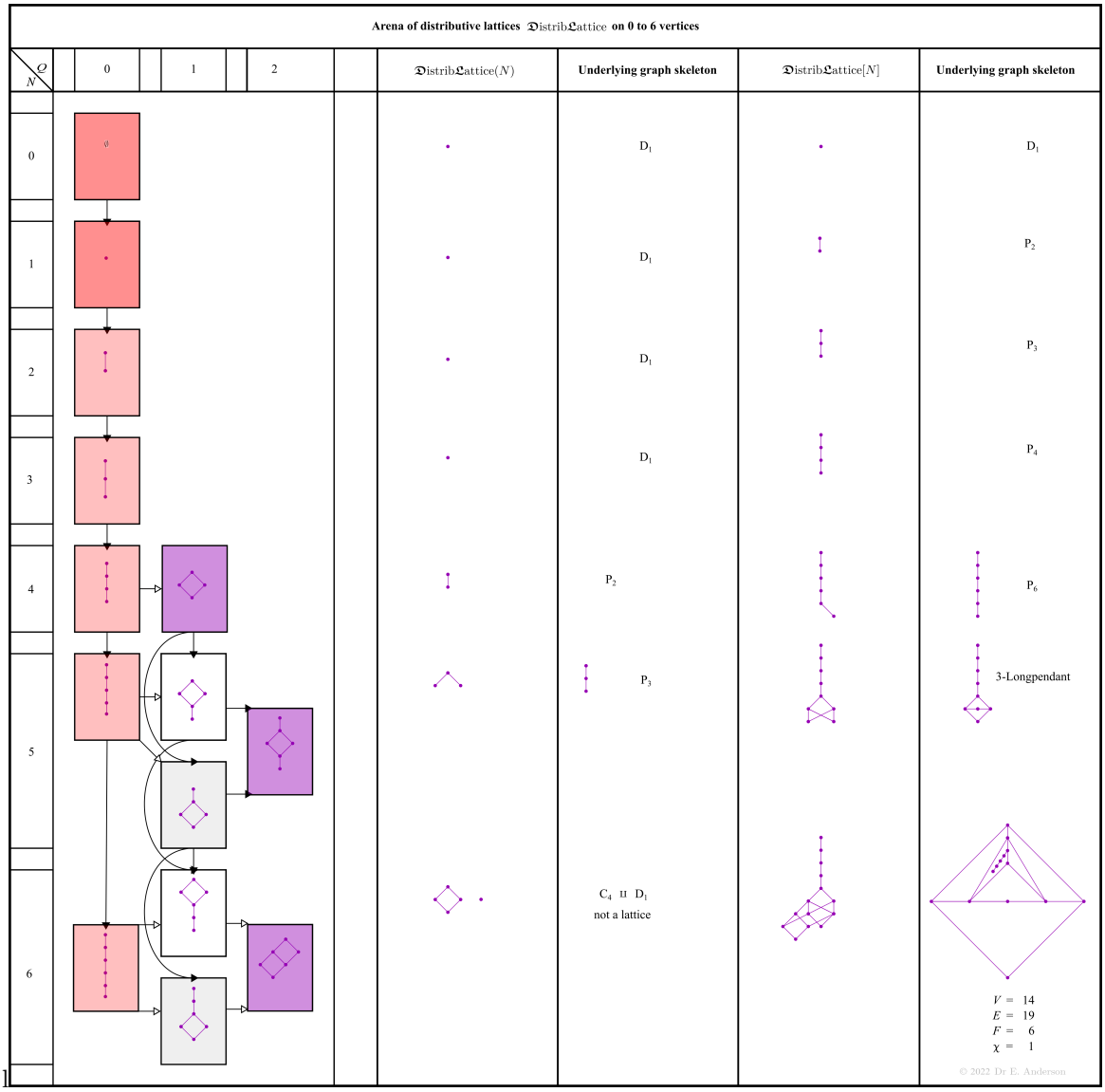


Figure 64.6:

64.4 Modular lattice arenas

Definition 1 We denote the arena of modular lattices by

$$\mathfrak{M}\text{odular}\mathfrak{L}\text{attice} .$$

The arena of ⟨modular lattices⟩ by

$$\langle\mathfrak{M}\text{odular}\mathfrak{L}\text{attice}\rangle .$$

And the arena of homodual modular lattices by

$$\mathfrak{HD}\text{-}\mathfrak{M}\text{odular}\mathfrak{L}\text{attice} .$$

Worked Example 7 What are the arenas of homodual modular lattices on 0 to 6 vertices?

Answer 7 See Fig 64.7. The first 10 cases are just points or chains. The other 4 cases are posets that are not semi-lattices, due to failing to be connected. This is because there is no way of reaching whichever of $K_{3,2}$, Buckle, Domino and $K_{4,2}$ are present.

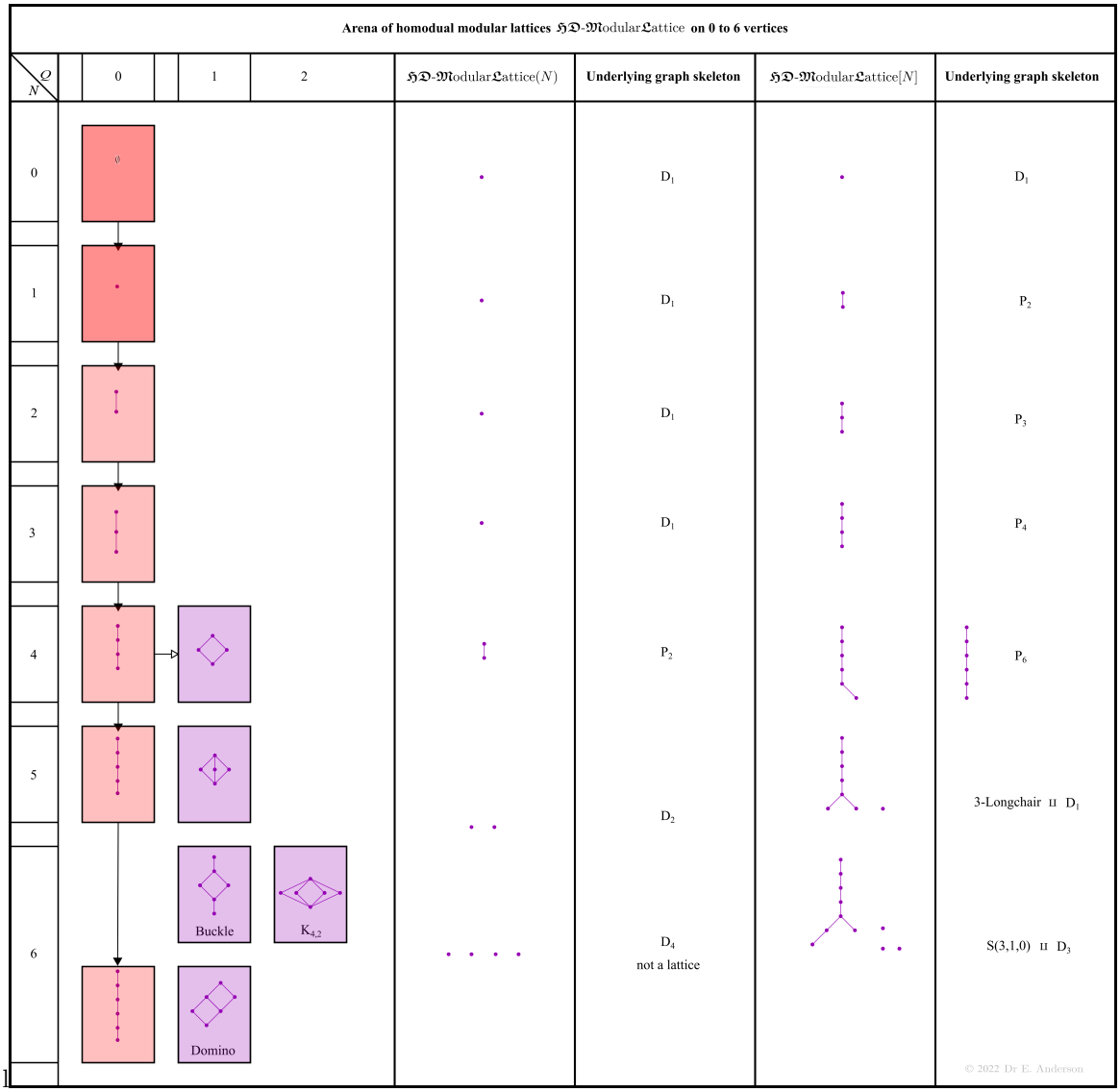


Figure 64.7:

Worked Example 8 What are the arenas of ⟨modular lattices⟩ on 0 to 6 vertices?

Answer 8 See Fig 64.8. The first 11 cases are again just points or chains. The 12th is a simple lattice. The 13th is a poset that is not a semi-lattice since it fails to be connected. For there is no way of linking up the

$$\{\,\text{Buckle, Mace, } K_{4,2}\,\}\ \text{ 3-chain}$$

and the

$$\{\,P_6\text{, Longbanner, Domino}\,\}\ \text{ 3-chain .}$$

The 14th gives a new underlying graph skeleton, involving Domino intersecting at a central vertex with a further C_4 alongside 2 leaves and a P_4 sidechain.

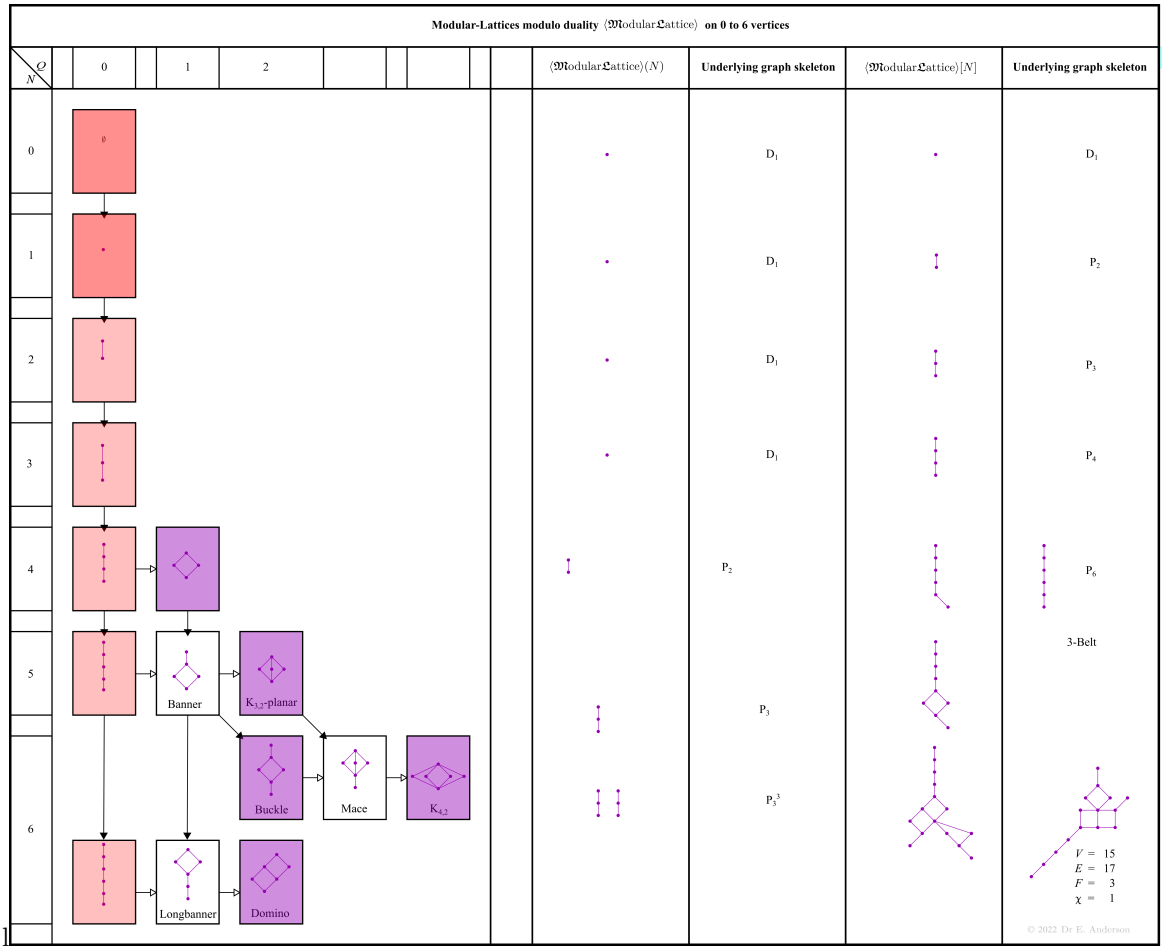


Figure 64.8:

Worked Example 9 What are the full arenas of modular lattices on 0 to 6 vertices?

Answer 9 We finally consider the full arena of modular lattices in Fig 64.9. The first 10 cases are just points or chains. The 11th is a simple lattice. The 12th is a poset that is not a semi-lattice due to containing the same C_4-nonconvex as in Worked Example 6. We recognize it as a $K_{3,3} - e$ subposet – highlighted in lime – with a P_4 sidechain. The 13th is a poset that is not a semi-lattice due to failing to be connected. For there is no way of linking between the

$$\{ K_{3,2}, \text{Mace}, K_{4,2} \} \ \text{4-cycle}$$

and the

$$\{ P_6, \text{Longbanner}, \text{Domino} \} \ \text{4-cycle} .$$

The 14th gives a sizeble new graph. This has the good fortune of being planar, as you worked out in Exercise 46.4. This graph contains a cycle system which, as depicted, consists of 9 4-cycles and a 6-cycle. The external cycle is of length 4. Thus

$$Y = 11 \ \text{and} \ Z = \frac{10 \times 2 + 4}{2} = 12 .$$

It finally contains a P_4 sidechain.

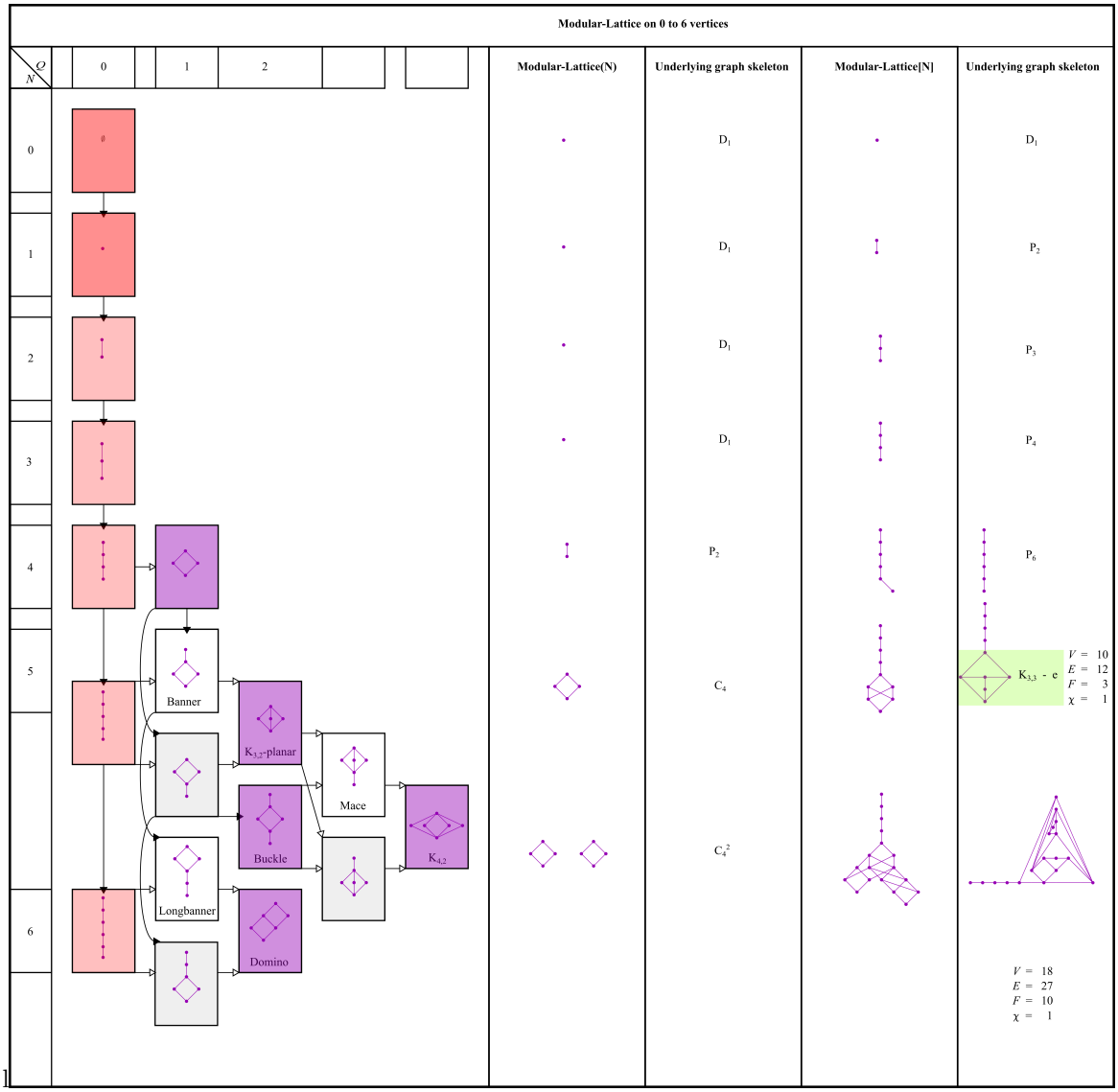

Modular-Lattice on 0 to 6 vertices
Q
N
0
1
2
Modular-Lattice(N)
Underlying graph skeleton
Modular-Lattice[N]
Underlying graph skeleton
0
1
2
3
4
5
6
Banner
Longbanner
Buckle
Domino
Mace
D1
D1
D1
D1
P2
C4
C4 2
D1
P2
P3
P4
P6
K3,3 - e
V = 10
E = 12
F = 3
χ = 1
V = 18
E = 27
F = 10
χ = 1

Figure 64.9:

64.5 Full lattice arenas

Definition 1 We denote the arena of lattices by

$$\mathfrak{L}\text{attice} .$$

The arena of ⟨lattices⟩ by

$$\langle \mathfrak{L}\text{attice} \rangle .$$

And the arena of homodual lattices by

$$\mathfrak{HR}\text{-}\mathfrak{P}\text{oset} .$$

Worked Example 10 What are the arenas of homodual lattices on 0 to 6 vertices?

Answer 10 These are built up in Fig 64.10. The first 10 cases are just points or chains. The last 4 are posets that are not semi-lattices on account of not being connected. For the 11th and 12th, this is because the graph $K_{3,2}$ – key to lattice distributivity – cannot be reached in any way. For the 13th and 14th, this is because neither $K_{3,2}$ nor the

$$\{ \text{Buckle, 1-}K_{3,2} \} \text{ 2-chain}$$

can be reached. [1-$K_{3,2}$ is the first homeomorph of $K_{3,2}$.] The 14th case's largest component is C_4 with a Quadruped$(0, 3)$ sidetree.

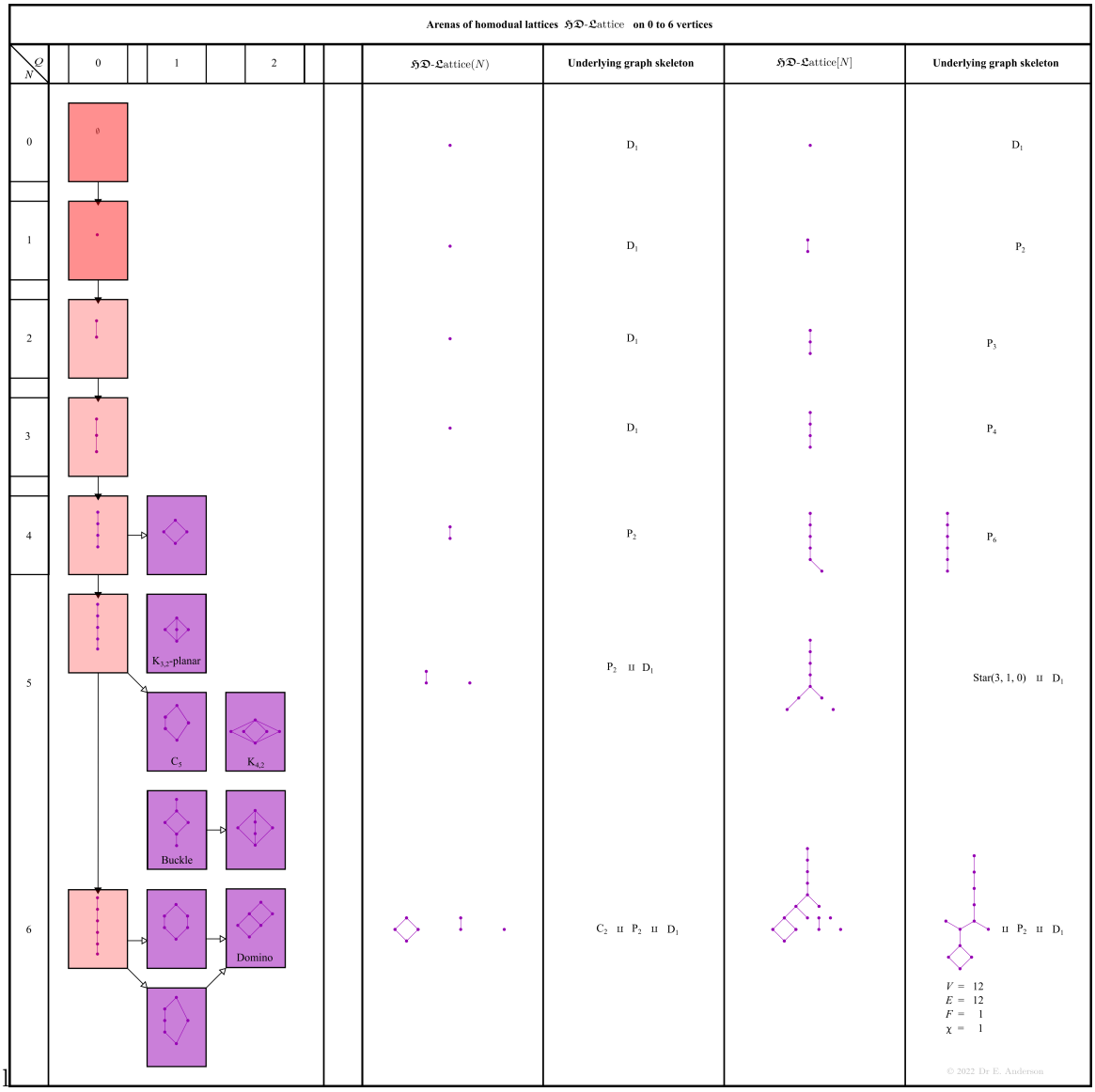


Figure 64.10:

Worked Example 11 What are the arenas of ⟨lattices⟩ on 0 to 6 vertices?

Answer 11 These are presented in Fig 64.11. The first 11 cases are just points or chains. The 12th is a semi-bounded lattice: C_4 with 2 leaves and a P_4 sidechain.

The 13th gives 2 $K_{3,2}$ cycle systems joined at a point with a P_2-sidechain. We established this to be planar in Exercise 46.4.

The 14th gives a more sizeable new graph. We established this to have crossing number 1 in Exercise 48.7.

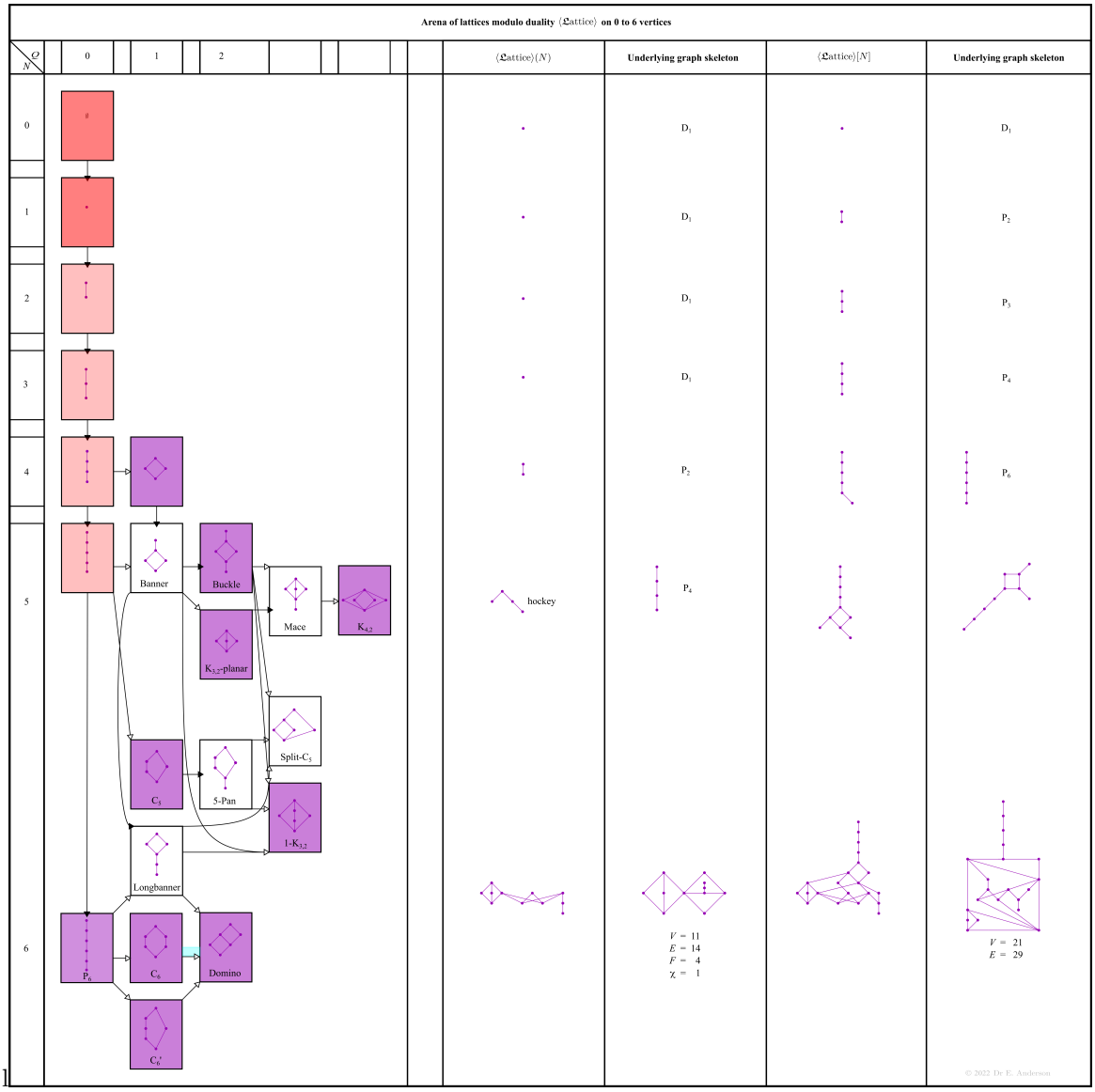


Figure 64.11:

Worked Example 12 What are the full arenas of lattices on 0 to 6 vertices?

Answer 12 See Fig 64.12. The first 10 cases are just points or chains. The 11th is a simple semi-lattice. The 12th is a poset that is not a semi-lattice, due to containing C_4-nonconvex, as uplifted from previous subcases. 64.12 also contains a $K_{4,2}$ subposet,

$$\{ P_6,\ C_6,\ C'_6,\ \text{Longbanner-up, Longbanner-down; Domino} \}\ .$$

Any one of whose four constituent $K_{3,2}$ subposets suffices to establish non-modularity.

$\mathfrak{L}$attice[3]'s underlying graph skeleton is $K_{3,3} - e$ with a leaf as well as a P_4 sidechain. The 13th and 14th subfigures' underlying graph skeletons can be thought of in terms of interlinking respectively 3 and 4 fairly dense subgraph groupings. They are the subject of the following two Exercises.

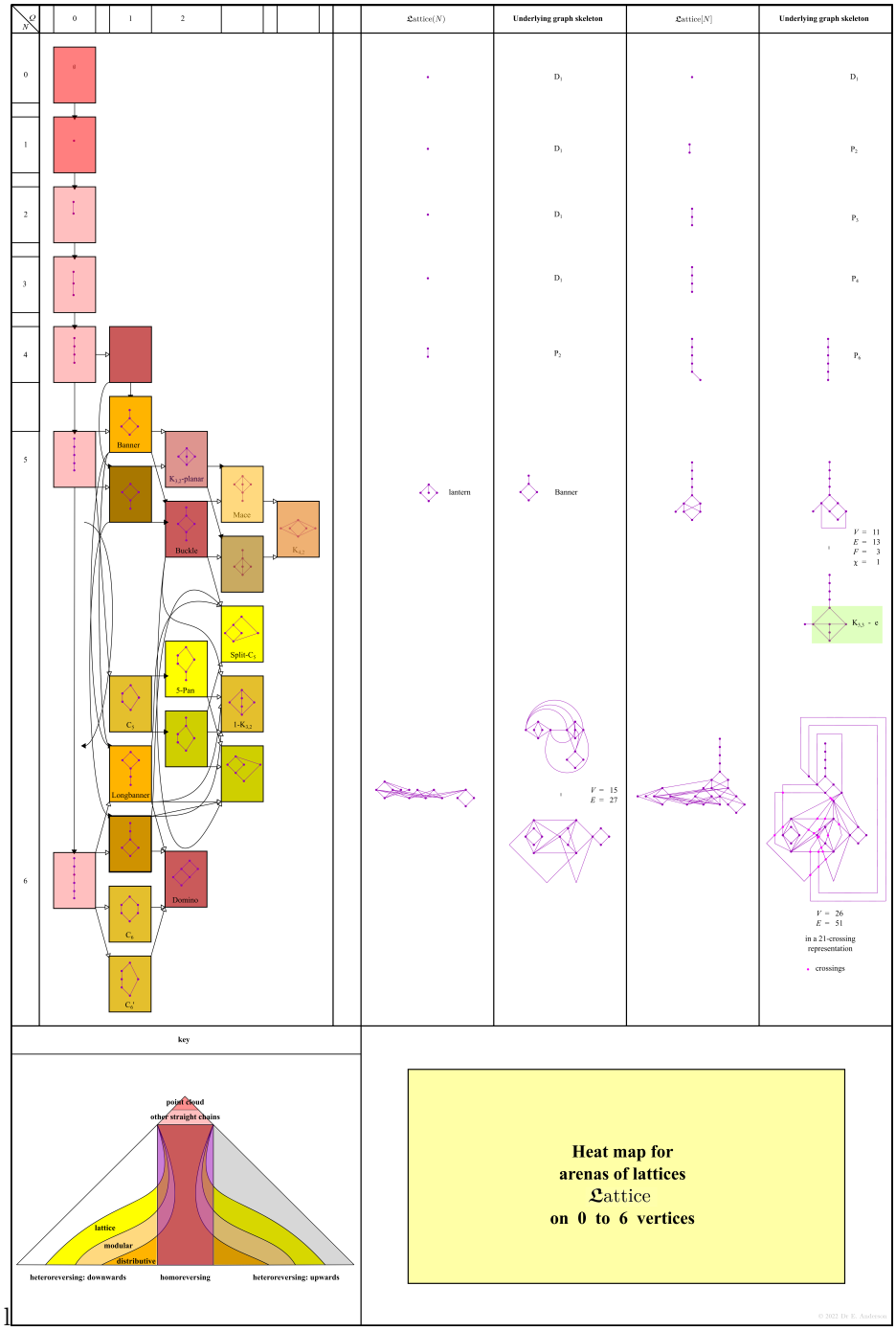


Figure 64.12:

Remark 1 For $\mathfrak{D}$istrib$\mathfrak{L}$attice[6], $Z = 8$, which is not prohibited, and indeed we exhibit a Hamiltonian cycle in Fig 64.13.a).

Remark 2 For $\mathfrak{M}$odular$\mathfrak{L}$attice[6], we give a forced cycle in Fig 64.13.b), so the corresponding graph cannot possibly be Hamiltonian. This can also be done for $\langle\mathfrak{L}$attice$\rangle$[6] (Subfigure c) and $\mathfrak{P}$oset[4] (Subfigure d).

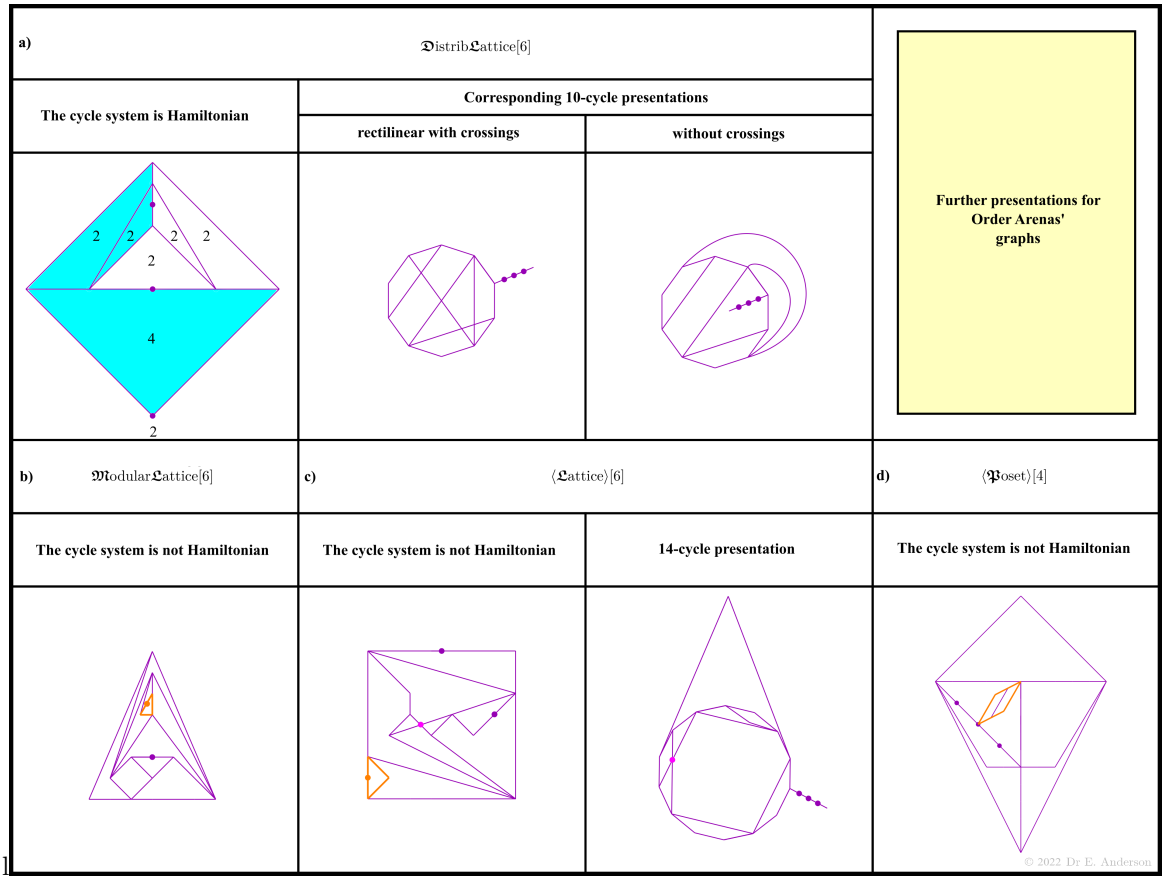


Figure 64.13:

64.6 Exercises and Research Projects

Exercise 3$^+$ Show that the three $V = 15$, $E = 27$ graphs in the bottom row of Fig 64.12 are isomorphic. Find the crossing number of this graph.

Exercise 4$^+$ Work out order 1's analogue of the previous 9 Figures. How many connected components does each order structure have? Deduce from this that most of the counterparts are posets that are not even semi-lattices. Compare order 1 and order 2's edge densities – as a measure of sparseness – for each of our 9 cases.

Exercise 5$^+$ With reference to Fig 64.3, show that $\mathfrak{P}$oset(4) is of thickness 2, and determine its crossing number. Also show that $\mathfrak{P}$oset[4] still has thickness 2 and just 1 further crossing.

Exercise 6$^+$+ Show that the two $V = 26$, $E = 51$ graphs in the bottom row of Fig 64.12 are isomorphic, if you have not already. Can you beat the second presentation's 21 crossings?

Exercise 7 Investigate the arenas of homocomplementary-excised graphs on 0 to 5 vertices at the level of orders and underlying graph skeletons.

Exercise 8 Investigate the arenas of homoconjugate partitions on 0 to 7 objects at the level of orders and underlying graph skeletons.

Exercise 9 Investigate the arena of RL rooted trees with RL-symmetric cases excised, on 0 to 4

vertices, at the level of orders and underlying graph skeletons.

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Research Project 1 What other orders might one consider placing on arenas of posets?

Research Project 2 The natural analogue of homeomorphs for posets are *downwards-preserving homeomorphs*, $\downarrow Homeomorph$. Investigate the poset arena modulo these,

$$\mathfrak{Top}\text{-}\mathfrak{Poset} \;:=\; \frac{\mathfrak{Poset}}{\downarrow Homeomorph}$$

for $N \leq 5$.

Let us also introduce some further variants of homeomorph with the following Exercises.

Exercise 10$^+$ The natural analogue of homeomorphs for rooted trees are *root-preserving homeomorphs*, $Homeomorph_{\mathrm{r}}$. Investigate the rooted-tree arena modulo these,

$$\mathfrak{Top}\text{-}\mathfrak{RootedTree} \;:=\; \frac{\mathfrak{RootedTree}}{Homeomorph_{\mathrm{r}}}$$

for $N \leq 5$. Also show that for posets, downwards-preserving already implies root-preserving.

Exercise 11 a) Which of the binary, ternary and quad trees in Figs 60.6 up to order 11, 60.7 and 60.8 are homeomorph-irreducible?

b) Which are root-preserving homeomorph-irreducible?

c) Which of the corresponding posets are downwards-preserving homeomorph-irreducible? What operation does this correspond to at the level of the original graphs?

Exercise 12$^+$ Design a suitable notion of homeomorph for the RL-distinguished rooted trees. Which RL-distinguished at-most binary trees for $N \leq 4$ are correspondingly irreducible? Order these by an RL-distinguishing single-leaf addition. Combine with your notion of homeomorph to produce a topological-metric split version of Fig 60.11.

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Remark 1 With reference to Research Project 2, in any case, we have seen that a partial order can be imposed on arenas of posets, so 'posets can be taken to be a fixed point of the Arenize map'. This prompts the following further question.

Research Project 3 Is there any reasonably natural order for which arenas of lattices are always lattices themselves?

End-Invitation Be welcome to contact the Author if you obtain results for any of these three projects.

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Chapter 65

Conclusion and Outlook

65.1 Overview

We have introduced Combinatorics, Group Theory, and Graph Theory, as well as Order Theory at the level of posets and lattices. Including applying these four topics to each other. Our zeroth emphasis is on systematically Naming each notion we define, each object type we consider and each Theorem that enters our account. We also formulate many notions and results in terms of relational quantities.

Since minimal counterexamples are useful, we have rather systematically given (and Named!) smallest examples of each structure and type of Combinatorial object encountered. Such Naming is a practice that is already common for simple graphs. But we have extended this to arrow-graphs, loop-multi-graphs, families, topologies, partitions, posets and lattices as well, most of which had not previously been Named.

The relational variables part of our account of Combinatorics, Graph Theory and Order Theory is also highly novel. In part, this amounts to using rational graph functions in place of cardinal graph functions. We also some indication of limitations starting to appear in expressing more complicated inequalities in relational terms. One underlying cause of this is that not all graph cardinals are (equally-)extensive.

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Ori-graphs are often included in first courses on Graph Theory [167, 190, 195, 183], while loop- and multi-graphs are not. But many applications require these, so we have developed these as well, with extra specific of notions cardinals and minimal examples.

A more general treatment than is usual for multi-regular graphs (degree alias valency restrictions) has also been provided, since this turns out to give a unified ground-level treatment of the graphs in many basic Science applications. Quantum Physics' Feynman diagrams are 4-sub-regular. Knot and link shadows, and carbon, are 4-regular. Molecules are rather generally 6-sub-regular. (At most) n-ary trees also exhibit sub-regularity. Cubic graphs – 3-regular – play a big part in Graph Theory, and describe Geometry's polytopes. Various further regularities are marked on pointers below. All in all (sub-)regularity turns out to be widely present in applications whose graph models are sparse.

We covered posets, lattices, modular lattices and distributive lattices in Part VII. In this zeroth Volume, only get as far as the Dilworth–Mirsky dual pair of Theorems and some diagnostics for the underlying graph skeletons. Comprehensively Naming the smallest such, and introducing relational variables for them are two of Part VII's innovations. Their application to Part VIII's arenas is a rather more significant contribution.

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These Series' main emphasis is on the arenas formed by each kind of object considered. These have order structure and underlying graph skeleton descriptions. Quotient arenas are the most interesting. Setting these up involves both Group Theory and relational quantities, justifying this Book's particular combination of topics. I.e. these topics are not only a very useful package of basics but also precisely what is needed for our novel aim at a Book on combinatorial arenas.

The specific arenas that we consider are of subsets, of families of subsets, and of cover, topology, and partition specializations of these. Of selection arenas, recurrence arenas and lattices of subgroups. Of graphs and their specialization to trees, with harder homeomorph quotients as well as complementation quotients incompatible therewith. With various subcases nested within the connected graphs and mostly Exercises on loop-multi-graph counterparts. Of arrow-graphs, which, like families, admit a somewhat larger quotient than just one of the commonly-encountered C_2 quotients. And the nested cases of poset, lattice, modular lattice and distributive lattice arenas.

65.2 Further reading in Sets, Functions, Numbers and basic Combinatorics

To expand your understanding of Parts I and II's topics, see e.g. [75] for some practice with simple notions about sets and functions between sets. Some slightly more advanced practice with these that is geared toward Topological spaces can be found at the start of [268, 242, 260]. See e.g. [27] and [227] for a subsequent layer of useful concepts about sets and functions.

See [46, 57] for more about number systems. For a bit of Number Theory, see [57] for a conceptual introduction and [64] for some further problems to solve. More advanced reading can be found within [22].

[67] provides an excellent introduction to Combinatorics as a whole, while [49] presents some very valuable more advanced material. These and [76] come with a lot of Exercises, and the last two of these contain very comprehensive lists of references as well. See e.g. [60, 42, 55, 53] for more on selections, and e.g. [42, 41, 23, 32, 66] for more on Combinatorial special functions. [22, 55, 76, 49] for more on generating series. Finally see the corresponding Chapter in [22] and the start of [50] for a next layer of concepts and results about partitions.

Asymptotic Combinatorics is a widely useful topic dealing with the large-N limit 'opposite' of our smallest and minimal examples emphasis, that we barely scratched (beyond stating Stirling's approximation!) [65] provides an excellent introduction and review of this topic. The remaining middling regime between the small and the large-N limit is much harder; at the level of counting types of object, consult [78] for the present state of progress in any one problem.

65.3 Further reading in Group Theory (expanding on parts III and IV)

To expand your understanding of Parts III and IV's topics, [105, 110] are useful first conceptual ports of call. [112] gives a playful introduction, while [113] gives a visual one. Basic Exercises can be found in [96], and more advanced reading in e.g. [288, 109, 106, 92]. Pólya's enumeration Theorems are further covered in [76, 42] and extended in e.g. [95, 36]. See [111] for modern computational Group Theory. See [114] for a first conceptual account of Lie Theory (suitable for second-year undergraduates upwards), this being a particularly useful topic for Physics and for 'continuum' Applied Mathematics more generally.

65.4 Graph Theory: further reading and applications

To expand your understanding of Parts V and VI's topics, [190, 195, 176] are excellent next ports of call; the last of these contains many Worked Examples. At the graduate level, see [196] and especially [178]; while part of [183] covers more basic material, the rest of it is comparably hard. More on what we call arrow-graphs can be found in [167]. Many further results on graphs (without proofs but with references) can be found in [193, 194].

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The current Volume's and Series' second main aim is treating widely-useful applications in a combinatorialized manner. We listed many of the current Volume's applications above under regularity; some more are as follows.

Book 1 [441]'s *matroid* abstraction of notions of independence makes use of graphs. As does its 'birdtrack' abstract index notation for contractions of strings of tensors, and its painting of mapping spaces as a simple type of hypergraph. And its account of electric circuits (see also Chapter 2 of [178], coupled springs and other analogues. This includes decouplings of simultaneous linear equation systems and large-N but highly-regular systems. Of flow balances with the Theorems of Ford–Fulkerson, Menger and König [183, 176]. Including a further proof of Hall's Pairs-Matching Theorem, and interconnection between all of these, and with the current Volume's Dilworth's Theorem for posets.

Many Computer Science applications involve trees, whether rooted, n-ary or at-most-n-ary. In particular, these model databases and sorting and searching therein; see [36] and Volume 2. Large networks also play a large part in Computer Science, for instance the internet and social media networks can be modelled thus in Graph-Theoretic terms. Book 2's introduction to these is expanded upon from topological and structural points of view in Volume 1 of Series 1; see also [186, 193]. Studies of network structure quantifiers [186, 197, 193, 450] – notions of centrality, of average path distance, of dissimilarity, of overcrowding, clustering coefficients... – is another area benefitting from Relational or Shape-Theoretic [396, 433] reformulations. Random graphs [150] play an even more significant role in the theory of network structure.

Euclidean Geometry's point-and-line(-and-circle) diagrams are considered at the topological level as graphs in Volume 3 [443], as well as at the more usual geometrical level. Projective Geometry subcases of such graphs [174, 179] are particularly significant, while Discrete Geometry remains a young field of study [436]. Graphs corresponding to various tessellations are also given here. See also [172, 161, 189] for more situations in which graphs can be applied in Flat Geometry.

Cayley graphs and cycle graphs for groups are considered in Volume 5 [444]. A Graph-Theoretic proof of the Nielsen–Schreier Theorem that free groups are hereditary is also offered there. As well as a proof of Frucht's Theorem [137, 193] that the current Volume evoked in establishing abstract–symmetry group equivalence. [This Volume also applies partitions' Young diagrams to the Representation Theory of the symmetric groups].

Book 5 [445] on Differential Geometry is our next port of call as regards graph colouring and triangulations. Root diagrams and Dynkin diagrams for Lie Algebras are further applications of graphs in Volume 7 [446], which also uses labelled Young diagrams – tableaux – for the Representation Theory of some simple Lie groups. Topological-level treatment of N-body problem configuration spaces can be found in Books 3, 5 and 9. All four Books mentioned in this paragraph are furthermore major clients for the above-mentioned birdtracks.

Volume 11 on Applied Geometry shall furthermore increase our repertoire of applications of graphs to the following. A) Polytopes (cubic!) B) Contact and packing configurations such as how to cover as much of a square or disc as possible using equal-sized non-overlapping discs (6-sub-regular!)

C) Fractal structures. Books 3, 5, 8 and Series 1 additionally build up a Geometrically, Graph-Theoretically and Topologically rooted theory of Computer Graphics, Image Analysis and Computer Vision.

65.5 Order Theory: conclusion further reading and applications

To expand your understanding of Part VII's topics, [224] and the Chapters on posets in [44, 49] are excellent next ports of call. Aside from the current Book's application to arenas, running along the current Series some further applications are as follows.

In Linear Algebra, Order Theory shows up in choices of basis and in flags. Order Theory also underlies many more advanced topics in Group Theory ([288, 221] and Volume 5). For instance derived and central series, and Fitting and Frattini subgroups. There are furthermore ring and Lie Theory counterparts of these examples [288].

Some of the current Book's subgroup lattice examples also have counterparts for more general structures: algebras. These results enter Theoretical Physics, for instance as classical-and-quantum constraint algebras ([440]'s reworking of [351, 382]). And as observables algebras and quantum-level operator algebras. Books 6 and 9 cover the classical case of this. In turn, Gravitational Theory necessitates further generalization to [351, 438] an algebroid version of Lie Theory. Geometry also has its own version of observables, and of flags (Books 5 and 6).

Lexicographic orders play a basic part in Computer Science (Book 2), with posets playing more advanced roles (see already [36]).

Order Theory is moreover the natural home for coarse-graining operations. While all Physicists know some examples of coarse-graining, especially in phase space and in Statistical Mechanics, comparatively few are aware of the following. a) Configuration space coarse-graining. b) That coarse-graining can furthermore be applied to a very wide range of mathematical structures. In particular, it can be applied to zooming out on the detail in modelling a region of space, surface, topological space, urban landscape, database... And also to peeling away layers of mathematical structure one by one.

Formal, if basic, Combinatorial reappraisal of other subjects is itself not yet widespread, other than in Algebra and in Topology, as of when I finished writing the first edition of this Volume (2022),



65.6 Topological concepts introduced in this Volume

Our third main emphasis was introducing more widely useful topological notions wherever possible.

0) We introduced *topological spaces*, *neighbourhoods* and *covers*, alongside first mention of *(para)compactne* applications of the last of these.

Also after seeing various examples, we gave brief conceptual discussions of the following notions.
1) *Notions of connectivity* (Chapters 35 and 37), including path-connectedness, connectedness, components, vertex-, edge- and ν-connectivity of graphs, and strong and weak connectivity of arrow-graphs. Vertex and edge cut-sets, independent sets, matchings and dominating sets.

2) *Notions of traversability* (Chapter 42), including in particular Eulerian and Hamiltonian graphs, which are respectively notions of edge- and vertex-traversability.

3) *Bases* and *first* and *second countability* in Sec 17.4.

4) *Notions of separation* in Sec 45.18, including in particular independent sets again and Hausdorffness.

5) *Homeomorphs* of graphs, planar graphs, departures from planar graphs as characterized by thickness and crossing number, and graph colouring also turn out to be highly Topological subjects.

5) Topological manifolds and knots were also touched upon, while Part VIII's combinatorial arenas are also a highly topological subject. First contact was also made with the Euler characteristic, homology, topological indices and Rigidity Theorems. For all that these remain very partial sketches which further Books in these Series will considerably expand upon.

65.7 Further reading on Topology

For further introductory reading on topological spaces and covers, see [268, 250, 260]. Connectedness, compactness, Hausdorffness and first and second countability are well-covered between [268, 250, 260]. Book 0 of Series 1 [449] shall also be comparatively introductory on these topics.

See [244, 270, 332] for more about manifolds. Some parts of this require a second wave of topological spaces material [260, 332, 453].

See e.g. [266, 267, 250] and at a more advanced level [261] for more on graphs in Topology. As regards the alternative topological aspects of Graph Theory inter-disciplinarity, see in particular [183] for planar graphs, [183, 194] for departures from planarity and [190, 183, 178] for graph colouring. [183, 176, 196] for Hamiltonianness, and [193] for notions of traversability more generaly. The current Volume itself serves to fill the gap as regards providing an elementary introduction to graph homeomorphs.

See e.g. [250, 443, 451] for more on the Euler characteristic. And [250, 260, 257] and then the more advanced [261] for more on (co-)homology. [318, 328] for conceptual and applied introduction to Index Theorems. [254] for a first conceptual account of knots, with [247, 250, 255] providing some useful supporting concepts, visualization and Exercises. Graph colouring and Knot Theory turn out to be related at the level of assigning invariant polynomials; we first consider these in Volume 1.

65.8 Topological applications of this Volume. i. Foundations

Concentrating on cardinal functions for graphs in this Volume is longstanding in the literature [For all that some of the cardinals mentioned are new or so far rarely considered.] Most first and second courses on topological spaces do not consider cardinal functions. However Engelking already approached topological spaces in via cardinal functions in his 1989 book [253]. This in substantial part requires some understanding of (non-naïve) Set Theory, placing it firmly in our Series 1.

In Topology, it is quite standard to use graphs and manifolds as two easier but rather different families of examples of topological spaces.

As a third arrow for our bow, simplicial complexes: as bigger than graphs and also as discrete approximations to manifolds. Higher-d complexes are of considerable practical modelling use as well as being structures and tools for Algebraic Topology. In studying complexes in subsequent Volumes (starting with 5 and 1.4) however, we shall also see that there are ways of reducing $n \geq 2$ to $n = 1$.

So there is a sense in which neither realisticness of modelling nor genericity are compromised by considering just graphs. Volumes 6 and 1.4 will already be making use of such reduction moves.

65.9 ii. Arenas

In asking what is the space formed by some type of mathematical object, it is quite often relevant to ask if it has an order structure or a geometry. Moreover, given some area of STEM's mathematical objects O, it is *always* a valid and revealing question to ask what topology their arena naturally realizes. Or at least which topolog*ies* would be natural for it. For as Part VIII illustrated, there can be more than one. Some may also carry whichever of a natural metric and a natural measure.

The rest of these Series has detailed treatment of many further arenas: the world's first large dedicated collection. Book 1 presents vector spaces and mapping spaces from this point of view, and Volume 2 likewise for message-or-code and data spaces Further Geometrical and Dynamical examples of arenas are covered in Volumes 3, 6 and 9 (and also in [396, 428, 431, 433, 434, 437]). Book 11 will cover many examples of arenas from Applied Geometry The current Volume's main repertoire of arenas of graphs, covers, partitions, topologies and orders will be further developed as Volume 3 of Series 1 [452].

65.10 iii. Applicable Modern Topology

For each item here, we give the indication promised in the Introduction of how far along the current two Series one would need to work through before these applications become accessible.

Pointer 1 *Genericity* and *robustness* determine which features of models are highly likely to be present when we do not know enough about the situation being modelled to have a model in exact correspondence with reality.

As an example of robustness, *initial data* for a physical differential equation is often not in practice exactly knowable. And for some (systems of) differential equations, this will very substantially alter predictions (*sensitivity to the initial conditions*). Be this the familiar breakdown of the weather forecast too far to the future (a simple model for which is the Lorenz system [400]). Or even a large instantaneous breakdown of the character of the solution, as is possible in ill-posed problems [342, 393]. These matters will be covered in more detail in Volume 1, though it will take until halfway through Series 1 to develop a state-of-the-art understanding.

As an example of genericity, if nothing precludes a geometrical surface or manifold from having small bumps, then these will generically wipe out all symmetries [306] of the simplified model. One will then no longer be able to cling to all that one can derive from having exact symmetries (which in Geometry and in Theoretical Physics [387, 368] is *most of* what is exactly-calculable [437]). Or at least to only be able to do so via more ambiguous conceptualization of *approximate symmetries*. This will be covered in more detail in Volumes 5 and 6, though it will take until halfway through Series 1 to have an in-depth comprehension.

One can however already conceive of robustness and genericity within Volume 0 and Volume 1's Graph Theory and Linear Mathematics topics. For instance, dense graphs and asymmetric graphs take steps in this direction. Also discrete equations are a counterpart to differential equations, with each case approximately modelling the other. One can furthermore ask which features render a linear space generic, and similarly for matrices or for the linear maps that matrices represent with respect to a given basis.

In Differential Topology [244, 332], *transversality* – extent of intersection of spaces – is a major feature of qualitative significance. A Linear Algebra precursor of this is given in Volume 1, to the

geometrical case being sketched in Volume 6 and analyzed in the middle of Series 1. *Topological analysis of differential equations*, or of *dynamical systems* more generally, is largely a *qualitative*-level affair. The point being that this offers further insights currently outside the reach of quantitative-level analysis. *General-position arguments* also play a major role in Differential Topology (say in the *Morse Theory* of critical points [321]); we have however already highlighted a few of these for Graph Theory in the current Volume.

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Pointer 2 *Homology and co-homology.* These 'theories of holes' are, respectively, a generalization of the current Volume's Euler Characteristic and a more advanced formulation of Vector Calculus' div and curl operations (Volume 1). At this new level of generality, these notions are usually first encountered for triangulations and simplicial complexes [250, 257] and somewhat more sophisticated models of complexes [261]. Or alternatively for manifolds [270, 332] and other differential-geometric structures. We deal with (co-)homology in these settings a quarter of the way along Series 1. (Co-)homology can however be set up for any branch of Mathematics, which we present half way along Series 1. The answers to many structural questions and the substance of many Existence Theorems involve whether co-homology groups are nontrivial or just *id.* Suitable Group Theory for understanding these groups is developed in Volume 5.

All of the above can be seen as a 1930s to 1960s '*Algebraic Topology*' [250, 261] development of some of Poincaré's ideas [236]. With the matter of application to any branch of mathematics coming from intermeshing with *Category Theory* [227, 259] some 10 years along.

On the one hand, one can develop homology and co-homology for any area of mathematics of particular interest. For which *Lie Theory* is the obvious target in these two Series. [Volumes 7 and 9 with local Geometry and Theoretical Physics applications and then the middle of Series 1 for the global counterparts.]

On the other hand, modern theorization is finding particular use for *persistence homology.* This keeps track of which features persist as one refines the accuracy of one's model, say from its large-scale shape to a more locally-detailed model. The study of *barcodes* – and indeed of *arenas of barcodes* – contains particular applications of this in to Computer Science and Technology. *Big Data* is a somewhat less specific target. See in particular [274, 278] (and eventually some of the middle of Series 1) for first outlines of this paragraph's topics.

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Pointer 3 *Sheaves* provide an especially useful way of attaching mathematical structures to points with particular means both to localize the global and to globalize the local. Each of the following provides infrastructure for sheaves. Topology, Category Theory and restriction maps (accounting for our current emphasis on the last of these). See the brief sections on sheaves in [274] and [431] for a first outline of sheaves. Sheaves are a particularly strong tool in designing and comprehending data structures and models of signalling [274, 271, 273].

Pointer 4 Suppose that one asks the current Series' main question for sheaves: what is the arena of sheaves? Then, shockingly, the answer that one encounters is another aspect of the Great Elephant of Topos Theory! This is in fact the setting in which Topos Theory was first arrived at by Grothendieck. Sec 52.13's context-dependent logic involves generalizing truth sets away from binary logic's 2-element set (true-false; c.f. Sec 6.3). See [225] for a particularly accessible outline of this approach to Topos Theory.

All in all, the main expected Readership of the current Volume will be able to work through works on some of the simpler kinds of sheaves in several years' time. Such as [336] in the context of differentiable manifolds (a good first choice for fledgling Theoretical Physicists). Or [271] for cellular

sheaves (a good first choice for budding Data Scientists). Or [362] in the Algebraic Geometry setting (but only if this is your main intended subject of study). Or Chapters 0.3 and 5.3 of [364] for a brief introduction to sheaves if your focus is Complex Analysis. In each case, perhaps a year or two later, you could read an intermediate-level book such as [374] or even [380] on some parts of the Great Elephant that is Topos Theory. [262] itself lies well beyond this scope.

Our general advice is to first gain a wide understanding of Topology: Series 1. This should serve you well in upgrading the theory of STEM in one or more areas of your own choice. Then after several years of experience with Topology, one may find oneself in a reasonable position to contemplate sheaves. And then asking which arena corresponds to sheaves points one though to an aspect of the Great Elephant.

For those Readers who have already embarked on a PhD or Fellowship on one of the above four topics, if you are thriving, keep on going. But if you are stuck, you may well find 'missing basics' by referring to our Series 1.

Sheaves and topoi are perhaps next ports of call for mathematical structures to be systematically applied to all of the Theory of STEM at some point between now and when topologization has run its full course by circa 2050. It remains to be seen whether Topos Theory will further our understanding of arenas of concepts themselves, for which a comparatively-simple lattice model is already available [224].

Acknowledgments

To those dear to me, including to the memory of my Father. Who taught me many of the topics in Parts I and II when I was young.

This Volume is however principally dedicated to my wisest friend. Who taught me of the extent to which many beautiful situations lack free parameters. Asked me what I thought the Space of Concepts would consist of, and then eventually proposed Order Theory and Topos Theory as candidates for a formal study of this. And caused of me to appreciate that understanding Order Theory adds understanding to practically any other field of study. By which the target of the current Volume became to set up a simple overview of Order Theory, and then to use this, on top of Graph Theory, to gain understanding of the many arenas of objects afforded by Combinatorics.

It was somewhat fortunate in these discussions that I already had some understanding of (some of the parts of the Great Elephant that is) Topos Theory, from my interactions with Chris Isham. And that I recognized this as having 'Many Routes' significance from my prior study of General Relativity, in particular along foundational lines developed by John Archibald Wheeler's school. My previous publicly-released works having mostly been on conceptually clarifying and thus eventually locally-classically resolving the Problem of Time in Quantum Gravity. In part via mechanical models whose key arenas turned out to be identical to those in Kendall's previous work in Statistics. In these earlier parts of my life, I thank in particular Malcolm MacCallum, Don Page and Niall ó Murchadha for discussions and support. Julian Barbour and Jimmy York Jr for certain specifics, and Timothy Gowers, Peter Johnstone, Karel Kuchař, Robert MacKay and John Stewart for some particularly inspiring lectures.

I also thank participants at the Applied Topology Discussion Group, the Applied Combinatorics Discussion Group, and the Global and Combinatorial Methods for Fundamental Physics Summer School 2021. Finally, this Volume is also dedicated to Gen-Z People currently experiencing many hardships.

Appendix A

Relational quantities

Remark 1 In the current Volume, we apply a simple version of Relational Theory [339, 356, 365, 370, 371, 392, 389, 390, 396, 398, 399, 414, 421, 422, 428, 431, 438, 437] throughout basic Combinatorics (and basic Group Theory). 'Simple' here refers to relative variables (App A.1) ratio variables (App A.2), and dilatational variables (Sec A.3). These correspond to

1) Assuming that the arena or particular space in question is $\mathbb{R}^p$ (which holds in many basic models).

2) Quotienting out, respectively, dilations Dil, translations $Tr(p)$, and both at once. More general Relational Theory quotients out larger geometrical isomorphism groups [370, 392, 396, 425, 428]. Its scale-free case is Shape Theory [370, 389, 396, 428, 433, 434, 437]. Another name for the dilatational case is Kendall's *preshape space*. The below account is however just a placeholder for a topic which will be developed in more detail in Volume 3 on Flat Geometry.

Remark 2 Using ratios is helpful in comparing (say discrete) structures of different sizes. This indicates the reason for the current Series including rational fractions as well as the more habitually used cardinal functions. This Appendix's considerations are subsequently useful in further developing the theory of structure of networks: a very relevant modern topic [186, 197, 193, 450].

A.1 A first outline of relative quantities

Definition 1 A set of physical quantities $\{q_I \,|I \;=\; 1 \;\text{ to }\; Q\}$ is *equidimensional* if each has the same physical dimension (in the sense of equal powers of length, time, mass...)

Definition 2 A physical or mathematical structural quantity is *intensive* if it is independent of the size of the structure, N. It is *extensive* if it is proportional to the size of the structure,

$$q_I \;\propto\; N \;.$$

It is *sub-extensive* if its dependence on the size of the structure is weaker than proportional, but *super-extensive* if stronger than proportional.

Remark 1 In the case of power-law dependencies (not exhaustive!)

$$q_I \;\propto\; N^{\alpha}$$

with $\alpha \;\in\; (0,\, 1)$ sub-extensive and $\alpha \;>\; 1$ super-extensive.

Definition 3 Given a set of quantities q_I the *difference variables* are

$$r_{IJ} \;:=\; q_I \;-\; q_J \;. \tag{A.1}$$

Remark 2 Relative position in Geometry and Physics is clearly a notion of this kind, opening our eyes to the significance and widespread use of relative quantities.

Remark 3 For difference variables to be meaningful, the set of quantities are to belong to the same space that possesses an invertible addition operation, and are to be equidimensional. If they are being used to inter-relate different-sized structures, their widespread usefulness is contingent on their being comparably-extensive. In the current Book, the sets of quantities are some cardinal functions for a given type of objects. More generally, meaningfulness requires the quantities' values to belong to a common space. The motivational example of relative separations illustrates furthermore the possibility that our objects are equal-component-number vectors (Volume 1 will further generalize this aspect).

♠ ◇ ♣

Structure 1 Let us next pick out maximal and minimal quantities $q_{\max}$, $q_{\min}$ within our set of quantities. We can then define the following.

Definition 4 The *upper-shifted relative quantity*

$$q_I^{\mathrm{u}} \ := \ q_{\max} \ - \ q_I \ . \tag{A.2}$$

And the *lower-shifted relative quantity*

$$q_I^{\mathrm{l}} \ := \ q_I \ - \ q_{\min} \ . \tag{A.3}$$

The *spread*

$$\Delta q \ := \ q_{\max} \ - \ q_{\min} \ . \tag{A.4}$$

Remark 4 The above quantities obey the following dependency.

$$q_I^u \ + \ q_I^l \ = \ q_{\max} \ - \ q_I \ + \ q_I \ - \ q_{\min} \ = \ q_{\max} \ - \ q_{\min} \ = \ \Delta q \ . \tag{A.5}$$

♠ ◇ ♣

Structure 2 Let us next introduce

$$S \ := \ \sum_{I \ = \ 1}^{Q} \ q_I \ : \tag{A.6}$$

the *sum* of all of our quantities. It is also a *first total moment*.

Definition 5 The *average* quantity,

$$\mu \ := \ \langle \, q \, \rangle \ := \ \frac{1}{N} \ \sum_{I \ = \ 1}^{Q} \ q_I \ . \tag{A.7}$$

Remark 5 We can also now consider further relative quantities such as

$$q_I \ - \ \sum_{I \ = \ 1}^{Q} \ q_I \ .$$

Though it makes more sense to consider the *fluctuation*

$$q_I \ - \ \langle \, q_I \, \rangle \ . \tag{A.8}$$

One reason for this is matching extensiveness. Another is having a matching number of plus terms and minus terms in our difference.

Exercise 1$^-$ Explain why variance

$$Var \ := \ \langle \, q^2 \, \rangle \ - \ \langle \, q \, \rangle^2 \tag{A.9}$$

also has matching extensiveness.

A.2 A first account of the theory of ratios

Structure 0 Suppose that we are given independent scalar quantities[1]

$$q_I \ , \quad I \ = \ 1 \ \text{ to } \ Q \ . \tag{A.10}$$

Then $R \ := \ Q \ - \ 1$ independent ratios are supported [351].

Structure 1 The *simple ratios* presentation for these uses

$$\mathcal{R}_i \ := \ \frac{q_i}{q_Q} \ , \quad i \ = \ 1 \ \text{ to } \ R \ . \tag{A.11}$$

Structure 2 The *square-normalized ratios* presentation for these uses

$$\mathcal{N}_I \ := \ \frac{q_I}{\sqrt{\sum_{I=0}^{Q} {q_i}^2}} \ : \tag{A.12}$$

These obey

$$\sum_{I=0}^{Q} {\mathcal{N}_I}^2 \ = \ 1 \ . \tag{A.13}$$

Structure 3 The *sum-normalized ratios* presentation for these uses

$$\mathcal{P}_I \ = \ \frac{q_i}{\sum_{I=0}^{Q} q_I} \ . \tag{A.14}$$

These obey

$$\sum_{I=0}^{Q} \mathcal{P}_1 \ = \ 1 \tag{A.15}$$

If the quantities in question can be negative, then there is a further distinction to make between the sum of absolute values of the quantities and the sum of the signed quantities themselves. For sure, many further presentations are possible.

♠ ◇ ♣

Example 1 $\sin\theta$ and $\cos\theta$ are normalized ratios. For 2 quantities, (A.13) can be interpreted as the on-circle condition (Fig A.1.a).

Example 2 Volumes 1 and 3 consider $\mathbb{S}^n$ generalizations of this, with Volume 3 also considering $\mathbb{S}^n$ and Projective Geometry use of simple ratios.

Example 3 Thermodynamics' *partial pressures* [432] are sum-normalized ratios. Fig A.1.b) illustrates the 2-term version of this.

Example 4 As are the dimensionless version of Mechanics' partial moments of inertia [433].

Example 5 Any fraction plus its complement (denoted by a c-superscript) gives unity:

$$X \ + \ X^{\mathrm{c}} \ = \ 1 \ . \tag{A.16}$$

This is also the 2-term version of (A.15).

♠ ◇ ♣

[1] Volume 1 details the theory of independence.

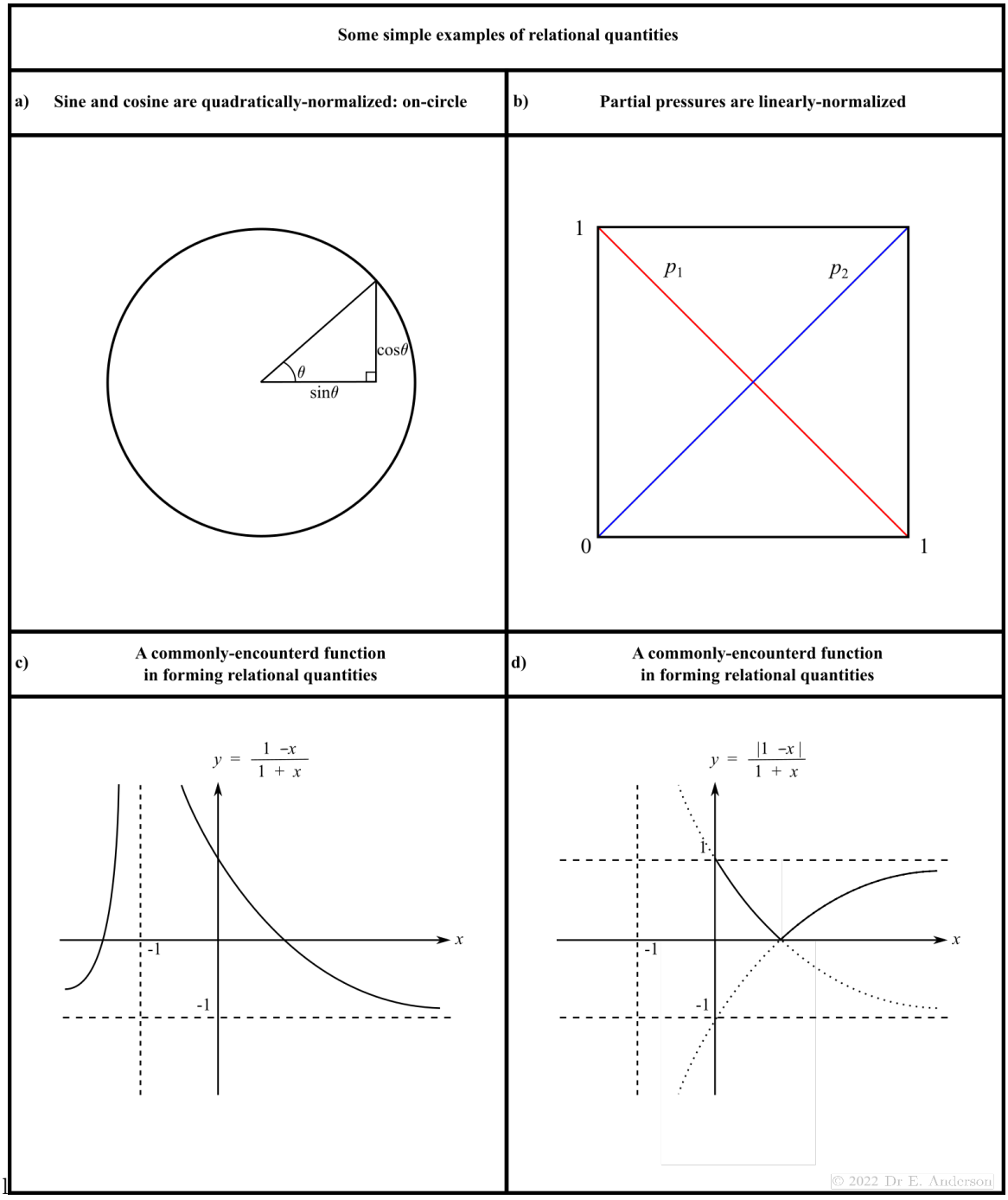


Figure A.1:

Remark 1 Suppose that the q_I all have matching physical dimension d and extensiveness N^α. Then the simple ratios formed from them are dimensionless and intensive:

$$1 \sim \frac{N^\alpha}{N^\alpha}\frac{q_I}{q_J} \sim \frac{d}{d} \sim 1 \, .$$

Remark 2 Suppose instead that two q_i have physical dimensions which are powers of each other,

$$q_1 \sim d \, , \quad q_2 \sim d^\alpha \, .$$

Then one can still form dimensionless ratios out of them:

$$\frac{q_2}{q_1^\alpha} \sim \frac{d^\alpha}{d^\alpha} \sim 1$$

and functions thereof. This heralds the start of the physical theory of *dimensionless groupings*.

Remark 3 Suppose instead that two q_i have extensivenesses which are powers of each other,

$$q_1 \sim N^\beta , \quad q_2 \sim N^\gamma .$$

Then one can still form intensive ratios out of them:

$$\frac{q_2^\beta}{q_1^\gamma} \sim \frac{N^{\beta\,\gamma}}{N^{\gamma\,\beta}} \sim 1$$

and functions thereof. This heralds instead a start on a less frequently contemplated structural theory of *intensive groupings*.

Remark 4 These two conditions will not however in general be concurrently satifiable, since this would require

$$\alpha = \frac{\gamma}{\beta} . \tag{A.17}$$

Remark 5 On the one hand, in the Physical Sciences – where physical dimensions play a nontrivial role – one has a choice between these two options.

Remark 6 On the other hand, in considering mathematical structures in the absence of physical dimensions, one is left with the physically less often chosen alternative.

Remark 7 In the current Book, it is observed that the theory of cardinal functions for most basic Combinatorics objects is straightforward in this sense, with everything being extensive or intensive. For cases involving at least Graph Theory's level of structural complexity, however, firstly basic ambiguities arise due to extensiveness splitting into such as vertex-extensiveness and edge-extensiveness. Secondly, further sub- and super-extensivenesses are observed, for now in particular in the intricately-topological topics of graph colouring and departures from planarity.

Remark 8 There are Physics counterparts to these phenomena, albeit quite seldom encountered in undergraduate courses. The basic case is that some problems can exhibit not spatial but planar or linear extensiveness due to 'thin' modelling conditions A simpler conception of such occurring concurrently within a single model is then indicated by material surface coatings. or by fluids with nontrivial surface layers. Or by electronic circuits whose components are partly thick ('resistors'), partly approximately planar ('chips' or 'motherboards') and part linear ('wires'). A more nuanced line of examples are how nonlinear systems can exhibit non-extensive entropy [375, 420].

Remark 9 These matters render it quite natural for communities with both Physics and Combinatorics expertise to investigate (simple, loop-, multi-, arrow-) graph, poset, simplex ... cardinals from the point of view of extensiveness, and intensive-grouping ratios.

Exercise 2$^-$ Explain why the ratio

$$\frac{\mu}{Var} \tag{A.18}$$

is of limited use. What ratio should instead be formed out of these quantities?

A.3 A first outline of relational – for now dilatational – quantities

Remark 1 Relative separations are translation-invariant whereas ratios are dilation- alias scale-invariant. Since the *dilatational group Dilatat*(d) results from joint consideration of translations Tr and dilations Dil quantities invariant under both translations and dilations are called *dilatational*. As we shall show in Volumes 3 and 7, ratios of relative separations are dilatation invariants. These can thus be formed by combining structures from the previous two sections. Some examples are then as follows.

Example 1 The ratios of relative separations

$$Q_{IJKL} \; := \; \frac{r_{IJ}}{r_{KL}} \; = \; \frac{q_I \; - \; q_J}{q_K \; - \; q_L} \; . \tag{A.19}$$

Example 2 *Normalized shifted variables*

$$n^{\mathrm{u}} \; := \; \frac{q_{\max} \; - \; q}{q_{\max}} \; , \quad n^{\mathrm{l}} \; := \; \frac{q \; - \; q_{\min}}{q_{\min}} \; . \tag{A.20}$$

Example 3 A *normalized* notion of *fluctuation*,

$$f_n \; := \; \frac{q_I \; - \; \langle \, q \, \rangle}{\langle \, q \, \rangle} \; = \; 1 \; - \; \hat{q}_i \; . \tag{A.21}$$

Example 4 A *normalized* notion of *spread*,

$$q_{\mathrm{c}} \; = \; \frac{\Delta \rho}{\rho} \; . \tag{A.22}$$

♠ ◇ ♣

Remark 1 Some more particular examples that the current Volume makes us of are as follows.

Example 5 Suppose that q runs from A to B, both finite. Then

$$q' \; := \; \frac{q \; - \; A}{B \; - \; A} \tag{A.23}$$

runs from 0 to 1. This is one way of obtaining standardized-range. If one or both of A and B were not finite, compactification can be attained using certain functions, for instance tan.

Example 6 Another is applying the mod sign to quantities running from -1 to 1.

Example 7 The function

$$f(x) \; = \; \frac{1 \; - \; x}{1 \; + \; x} \; = \; 1 \; - \; \frac{2 \, x}{1 \; + \; x} \; = \; 1 \; - \; 2 \, (\, 1 \; + \; x^{-1} \,)^{-1} \tag{A.24}$$

(Fig A.1.c) is needed for partitions, graphs and subsequent Volumes. If ranging over $(\, N^{-1} \, , \quad N \,)$, the linearly-related function taking values from 0 to 1 is

$$g(x) \; = \; \frac{1}{2} \left(\frac{1 \; + \; N}{1 \; - \; N} \frac{1 \; - \; x}{1 \; + \; x} \; + \; 1 \right) \; . \tag{A.25}$$

Example 8 The function

$$u(x) \; = \; \frac{| \, 1 \; - \; x \, |}{1 \; + \; x} \; = \; | \, 1 \; - \; 2 \, (\, 1 \; + \; x^{-1} \,)^{-1} \, | \tag{A.26}$$

(Fig A.1.d) is similarly also needed. The last equality here holds for $x > 0$, commensurate with our (N^{-1} , N) range. Having a linearly-related function range from 0 to 1 single-valuedly is not possible. But double-valuedly for $x \neq 1$,

$$v(x) = \frac{1 + N}{1 - N} \frac{1 - x}{1 + x}$$

will do.

Exercise 3^{-} Derive the last sentence of Example 7 and the last two of Example 8.

♠ ◇ ♣

Exercise 4 Generalize the Book's sum-based relational quantities to weighted versions.

Pointer 1 Volume 1 shall furthermore generalize them to higher moments.

Appendix B

Metric spaces

B.1 The Euclidean metric

We outline here a notion of distance – modelled on properties enjoyed by Euclidean distance,

$$\left|\left|\underline{\boldsymbol{x}} - \underline{\boldsymbol{y}}\right|\right| = \sqrt{(\underline{\boldsymbol{x}} - \underline{\boldsymbol{y}}) \cdot (\underline{\boldsymbol{x}} - \underline{\boldsymbol{y}})} \,, \tag{B.1}$$

while making no reference to Euclidean space's linear – vector space – level of structure.

Structure 1 The properties we insist on here are as follows.

i) *Non-negativity*

$$\left|\left|\underline{\boldsymbol{x}} - \underline{\boldsymbol{y}}\right|\right| \geq 0 \,. \tag{B.2}$$

ii) *Separability*

$$\left|\left|\underline{\boldsymbol{x}} - \underline{\boldsymbol{y}}\right|\right| = 0 \text{ iff } \underline{\boldsymbol{x}} - \underline{\boldsymbol{y}} = 0 \text{ iff } \underline{\boldsymbol{x}} = \underline{\boldsymbol{y}} \,. \tag{B.3}$$

iii) *Symmetry*

$$\left|\left|\underline{\boldsymbol{x}} - \underline{\boldsymbol{y}}\right|\right| = \left|\left|\underline{\boldsymbol{y}} - \underline{\boldsymbol{x}}\right|\right| \,. \tag{B.4}$$

iv) The *triangle inequality*

$$||\underline{\boldsymbol{x}} - \underline{\boldsymbol{z}}|| \geq \left|\left|\underline{\boldsymbol{x}} - \underline{\boldsymbol{y}}\right|\right| + \left|\left|\underline{\boldsymbol{y}} - \underline{\boldsymbol{z}}\right|\right| \,. \tag{B.5}$$

Exercise 1 Prove that these properties hold.

Remark 1 These properties mean respectively that Euclidean distance is non-negative. That two points at zero distance to each other have to be the same point. That the distance from $\underline{\boldsymbol{x}}$ to $\underline{\boldsymbol{y}}$ is the same as the distance from $\underline{\boldsymbol{y}}$ to $\underline{\boldsymbol{x}}$. That going between two points $\underline{\boldsymbol{x}}$ and $\underline{\boldsymbol{y}}$ via a third point $\underline{\boldsymbol{z}}$ entails covering at least as much distance as going straight between $\underline{\boldsymbol{x}}$ and $\underline{\boldsymbol{y}}$.

B.2 The general metric

Remark 1 We now axiomatize the above properties as follows.

Definition 1 A set $\mathfrak{X}$ equipped with a function

$$\mathrm{d} : \mathfrak{X} \times \mathfrak{X} \longrightarrow \mathfrak{X} \tag{B.6}$$

is a *metric space* if d obeys the following properties by which it is itself a metric.

i) Fig B.1.a)'s *non-negativity*

$$\mathrm{d}(x,\, y) \;\geq\; 0 \;. \tag{B.7}$$

ii) Fig B.1.b)'s *separability*

$$\mathrm{d}(x,\, y) \;=\; 0 \;\text{ iff }\; x \;=\; y \;. \tag{B.8}$$

iii) Fig B.1.c)'s *symmetry*

$$\mathrm{d}(x,\, y) \;\geq\; \mathrm{d}(y,\, x) \;. \tag{B.9}$$

iv) Fig B.1.d)'s *triangle inequality*

$$\mathrm{d}(x,\, y) \;\geq\; \mathrm{d}(x,\, z) \;+\; \mathrm{d}(z,\, y) \;. \tag{B.10}$$

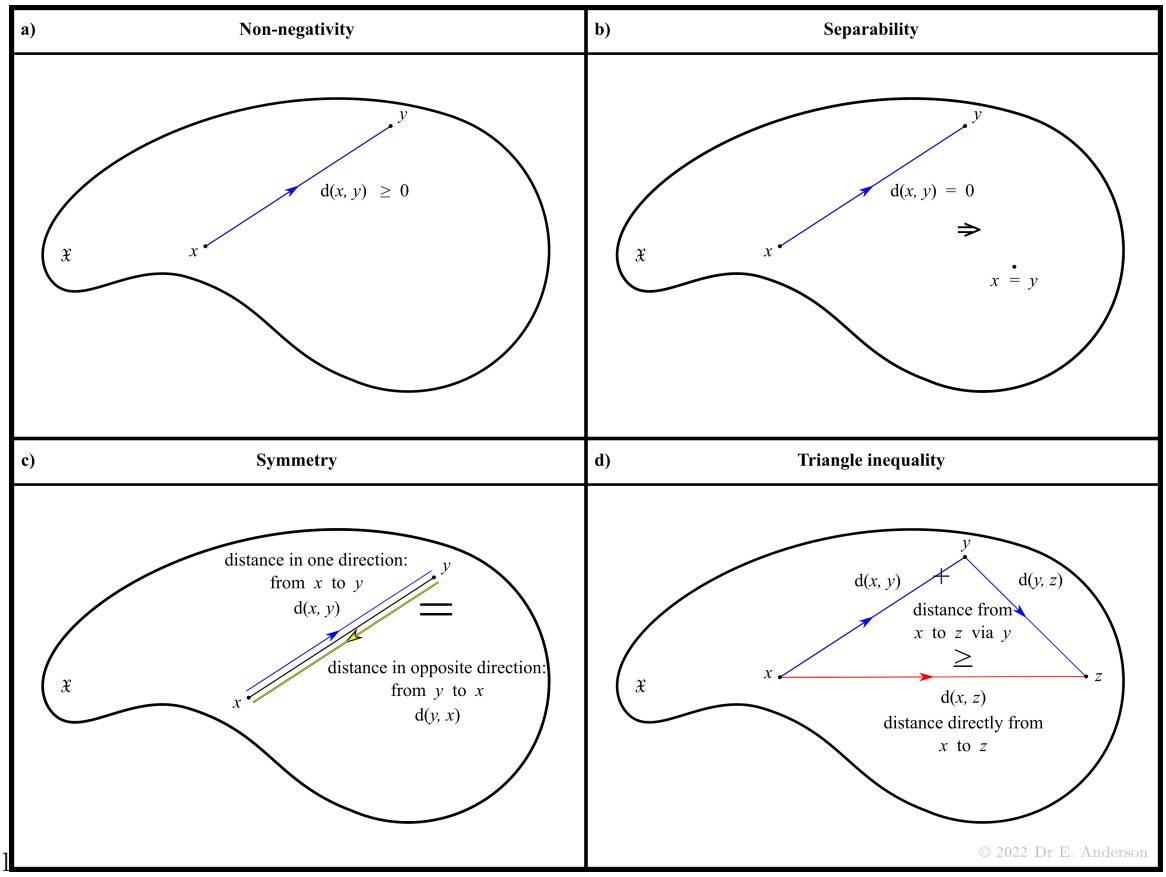


Figure B.1:

Definition 2 The *arena of metric spaces* is

$$\mathfrak{Met} \;.$$

This could be on a fixed topological space $\mathfrak{X}$, on any $\mathfrak{X}$, or the approximation to this on sets up to cardinality n.

B.3 Further examples of metrics

Structure 1 The metric d's *open ball* of radius ρ and centre x is

$$\mathbb{B}(x,\, \rho) \;:=\; \{\, y \,|\, d(x,\, y) \;<\; \rho \,\} \;. \tag{B.11}$$

The metric d's *closed ball* of radius ρ and centre x is

$$\overline{\mathbb{B}}(x,\,\rho) \;:=\; \{\, y \,|\, d(x,\,y) \;\leq\; \rho \,\}\;. \tag{B.12}$$

Exercise 2 Show that each of the following is indeed a metric, also explaining what form its balls take.

a) The *indiscrete metric*,

$$d_{\mathrm{i}}(x,\,y) \;=\; 0 \;\;\forall\, x, y \;\;.$$

b) The *discrete metric*,

$$d_{\mathrm{d}}(x,\,y) \;=\; \begin{cases} 0 & \\ 1 & x \;=\; yx \;\neq\; y \end{cases} \;\;.$$

c) The *taxicab product metric*

$$d_{\mathrm{T}}\left(\,(\,x_1,\,x_2\,)\;,\;\;(\,y_1, y_2\,)\,\right) \;=\; d_{\mathfrak{X}}\,(\,x_1,\,x_2\,) \;+\; d_{\mathfrak{Y}}\,(\,y_1,\,y_2\,)\;.$$

where $d_{\mathfrak{X}}$ and $d_{\mathfrak{Y}}$ are metrics on the metric spaces $\mathfrak{X}$ and $\mathfrak{Y}$ respectively.

d) The *Euclidean-type product metric*

$$d_{\mathrm{E}}\left(\,(\,x_1,\,x_2\,)\right)\;,\;\;(\,y_1,\,y_2\,)\,\right) \;=\; \sqrt{d_{\mathfrak{X}}\,(\,x_1,\,x_2\,)^2 \;+\; d_{\mathfrak{Y}}\,(\,y_1,\,y_2\,)^2}\;,$$

e) The *Hamming metric*,

$$d(x,\,y) \;=\; \sum_{i\,=\,1}^{N} |\,x_i \;-\; y_i\,|\;.$$

for x_i, y_i N-term sequences of 0's and 1's, alias N-term binary codes.

f) The *graph metric* in Chapter 38 .

g) Its generalization the *path metric*

$$d(x,\,y) \;:=\; \inf_{\Gamma\,:\,x\,\to\,y}\left(\int_{\Gamma} d(x,\,y)\mathrm{d}s\right)$$

for arc-length s.

Exercise 3 Use balls to justify how the separability property of metric spaces is indeed a notion of separation. And to establish that metric spaces are Hausdorff. Are they second-countable?

Definition 1 The *diameter* of a subset $\subset$ of a metric space $\mathfrak{M}$ is

$$diam(\mathfrak{a}) \;:=\; \sup_{a,a'\,\in\,\mathfrak{a}}\,(\,\mathrm{d}(a,a')\,)\;\;. \tag{B.13}$$

Definition 2 The *Hausdorff distance* between two subsets $\mathfrak{A}$ and $\mathfrak{B}$ of the same metric space $\mathfrak{M}$ is

$$\mathrm{d}^{\mathrm{H}}(\mathfrak{A}\,,\mathfrak{B}) \;:=\; \inf_{a\,\in\,\mathfrak{A},\,b\,\in\,\mathfrak{B}}\,(\,\mathrm{d}(a,\,b)\,)\;\;. \tag{B.14}$$

Structure 3 Isomorphisms ϕ between metric spaces $\langle\,\mathfrak{X}\;,\;\;\mathrm{d}_{\mathfrak{X}}\,\rangle$ $\langle\,\mathfrak{Y}\;,\;\;\mathrm{d}_{\mathfrak{Y}}\,\rangle$ are called *isometries*. These obey

$$\mathrm{d}_{\mathfrak{X}}(x,\,x') \;=\; \mathrm{d}_{\mathfrak{Y}}\,(\,\phi(x),\,\phi\,(x')\,)\;\;. \tag{B.15}$$

Structure 4 The *Gromov–Hausdorff distance* between metric spaces $\mathfrak{X},\mathfrak{Y}$ is

$$\mathrm{d}^{\mathrm{GH}}(\mathfrak{X},\mathfrak{Y}) \;=\; \inf_{\text{all metric spaces }\mathfrak{Z}\text{that }\mathfrak{X}',\mathfrak{Y}'\text{can be embedded into}}\inf_{\mathfrak{X}',\,\mathfrak{Y}'} d_{\mathfrak{Z}}^{\mathrm{H}}\,(\mathfrak{X}',\mathfrak{Y}')\;\;. \tag{B.16}$$

The primed objects here denote all the subspaces isomorphic to the unprimed version of the corresponding object. By showing that this is indeed a metric, we establish the following proposition (rewording a result of Gromov's [248]).

Proposition 1 The arena of metric spaces, $\mathfrak{Met}$, is itself a metric space. I.e. metric spaces constitute a fixed point of *Arenize*.

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though P. Stein [37] suggests that this kind of thinking may be untraceably old.

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Key:

I: Introductory recommendation.

C: Conceptual recommendation.

P: Problem-solving recommendation.

E: Expert recommendation.

⁻ This problem book mostly contains problems that you should have been able to do 1 or 2 years ago. It may be useful if stuck with all other problem sources.

⁺ This expert text is several years away from being fully understandable by our expeceted principal Readership.

⁺⁺ This expert text is *many* years away from being fully understandable by our expeceted principal Readership.

B. Group Theory with simple Combinatorial Applications (Parts III and IV)

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