Spaces of Graphs

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Abstract

We apply Shape Theory and Order Theory to spaces of graphs. We concentrate on small examples which are minimal for exhibiting various nontrivialities.

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1 Introduction

Let

$$\mathfrak{G} \ = \ (\mathfrak{V}, \mathfrak{E}) \tag{1}$$

be a *graph* [11, 15, 13, 20]: a collection of vertices forming the *vertex set* $\mathfrak{V}$ some of which are joined by edges forming the *edge set* $\mathfrak{E}$. It is of *order*

$$V \ := \ |\,\mathfrak{V}\,| \ = \ N \tag{2}$$

and *size*

$$E \ := \ |\,\mathfrak{E}\,| \ . \tag{3}$$

We restrict ourselves to one edge per distinct-vertex pair.[1] Then for each N, the number of possible edges runs from 0 to

$$E_{\text{max}} \ = \ \binom{N}{2} \ = \ \frac{N\,(N \ - \ 1)}{2} \ . \tag{4}$$

The current note concerns the *space of graphs of fixed order N*

$$\mathfrak{G}\text{raph}(N) \ , \tag{5}$$

and the *space of graphs up to fixed order N*,

$$\mathfrak{G}\text{raph}[N] \ = \ \coprod_{n=0}^{N} \mathfrak{G}\text{raph}(n) \ . \tag{6}$$

The latter is furthermore a first approximation to the *space of finite graphs*

$$\mathfrak{G}\text{raph} \ = \ \coprod_{n \,\in\, \mathbb{N}_0} \mathfrak{G}\text{raph}(n) \ . \tag{7}$$

'Space' is here taken to mean more than just 'set'; various levels of structure for it are investigated here and elsewhere.

This study parallels Shape Theory in the sense of Kendall [4, 6, 9, 14, 23, 25, 26, 27, 31, 33]. See also [22] for a recent first-principles development and [21, 24, 32, 33] for dynamical and physical applications, some of which are relational along Leibnizian lines [1]. Shape Theory is at present mostly studied as part of Probability and Statistics, while Relationalism is of interest in Philosophy as well as in Fundamental, Theoretical and Foundational Physics. Graph Theory was conversely applied to Shape Theory in [25, 26, 27, 29, 30, 31].

Working up to $N \ = \ 4$ suits the purposes of the current note; extension to larger N is forthcoming.

In Sec 2, we catalogue the graphs on 0 to 4 vertices. We first classify these by edge number, connectivity and cycle structure. A key observation is the presence of *graph complementarity*: most graphs occur in complementary pairs, though a few are self-complementary. A necessary but not sufficient condition for self-complementarity is

$$E \ = \ E_{\text{crit}} \ = \ \frac{E_{\text{max}}}{2} \ = \ \frac{N\,(N \ - \ 1)}{4} \ . \tag{8}$$

From a Shape-Theoretic or Relational point of view, it is furthermore natural to

[1] I.e. the *simple graphs* as opposed to multigraphs and/or looped graphs [20].

1) recentre variables about the exhibited symmetric case: self-complementarity (Sec 3). This is now parametrized by the *edge-criticality discrepancy* variable, i.e. the outcome of taking the difference of the edge number and the critical edge number,

$$D \ := \ E \ - \ E_{\mathrm{crit}} \ . \tag{9}$$

2) To *quotient out* the exhibited symmetry – for us complementarity – so as to pass to a reduced configuration space (Sec 4).

This amounts to considering [*graphs*]

$$[\mathfrak{G}] \tag{10}$$

– graphs modulo complementation – and spaces thereof,

$$[\boldsymbol{\mathfrak{G}}\mathrm{raph}](N) \ , \tag{11}$$

$$[\mathfrak{G}\mathrm{raph}][N] \ = \ \coprod_{n=0}^{N} [\boldsymbol{\mathfrak{G}}\mathrm{raph}](n) \tag{12}$$

and

$$[\boldsymbol{\mathfrak{G}}\mathrm{raph}] \ = \ \coprod_{n \, \in \, \mathbb{N}_0} [\boldsymbol{\mathfrak{G}}\mathrm{raph}](n) \ . \tag{13}$$

Various ways of representing [graphs] are considered: from picking a representative in Sec 4 to passing to Ramsey's red-and-blue-edge representation of [graphs] themselves in Sec 5. Aside from Ramsey Theory [15, 28, 13], another reason to take [graphs] seriously is that 'graph automorphisms' [34] actually solely depend on [graph].

At this level, our now-unsigned key variable $|\,D\,|$ has become

$$(\text{`Ramsey imbalance'}) \ , \quad I_{\mathrm{R}} \ = \ |\,\#(\text{blue edges}) \ - \ \#(\text{red edges})\,| \ , \tag{14}$$

for which we provide truer naming in Sec 5.

The next step in our analysis is to recognize that the $\mathfrak{G}\mathrm{raph}(N)$ are posets, and indeed lattices (Sec 6).

For $N \ \geq \ 1$, their bottom and top elements are the N-vertex *discrete graph* D_N and *complete graph* K_N respectively.

$N \ = \ 2$ is then minimal for the top and bottom elements to be distinct.

$N \ = \ 3$ is minimal for there to be a nontrivial middle.

$N \ = \ 4$ is minimal for the middle to be a nontrivial poset, by which the overall lattice is more than just a chain. This is why $N \ = \ 4$ suffices for the current note.

The $[\mathfrak{G}\mathrm{raph}](N)$ are however only semi-lattices, out of possessing but a single terminus,

$$(\text{maximally Ramsey-imbalanced graph}) \ = \ [\text{monochromatic graph}] \ = \ [\mathrm{D}_N] \ = \ [\mathrm{K}_N] \ . \tag{15}$$

At the other end, for $N \geq 4$ one has in general multiple maximally-balanced dichromatic graphs (Sec 7).

$\mathfrak{G}\mathrm{raph}[N]$ and $\boldsymbol{\mathfrak{G}}\mathrm{raph}$ can be seen to follow from the $\mathfrak{G}\mathrm{raph}(N)$ by involving one further operation, 'add vertex' (Sec 8). $[\mathfrak{G}\mathrm{raph}][N]$ and $[\boldsymbol{\mathfrak{G}}\mathrm{raph}]$ follow from the $[\mathfrak{G}\mathrm{raph}](N)$ by involving instead the monochromatic coning operation (Sec 9). We finally return to $\mathfrak{G}\mathrm{raph}[N]$ and $\boldsymbol{\mathfrak{G}}\mathrm{raph}$ in Sec 10, now from a complementarity-preserving point of view.

2 Graphs on 0 to 4 vertices

Graphs on up to 4 vertices are exhaustively listed and named in Fig 1.

Remark 1 $N = 1$ is minimal to have vertices.

Remark 2 $N = 2$ is minimal to exhibit edges, thus distinguishing graphs from *point clouds* pt^N (Which, as graphs, we denote by D_N: *discrete graphs*).

Remark 3 We take $N = 0$ to be the *ungraph* (and *unpoint*) interpretation of the empty set.

Remark 4 $N = 2$ is also minimal for *discrete-complete distinction*

$$\mathrm{K}_N \neq \mathrm{D}_N \ . \tag{16}$$

Remark 5 $N = 3$ is minimal for *discrete-complete-non-exhaustion*: i.e. for graphs other than K_N and D_N to exist.

Remark 6 $N = 4$ is minimal for (V, E) to not uniquely characterize graphs. In particular, it allows for 2 graphs with each of 2 and 4 edges, and 3 with 3.

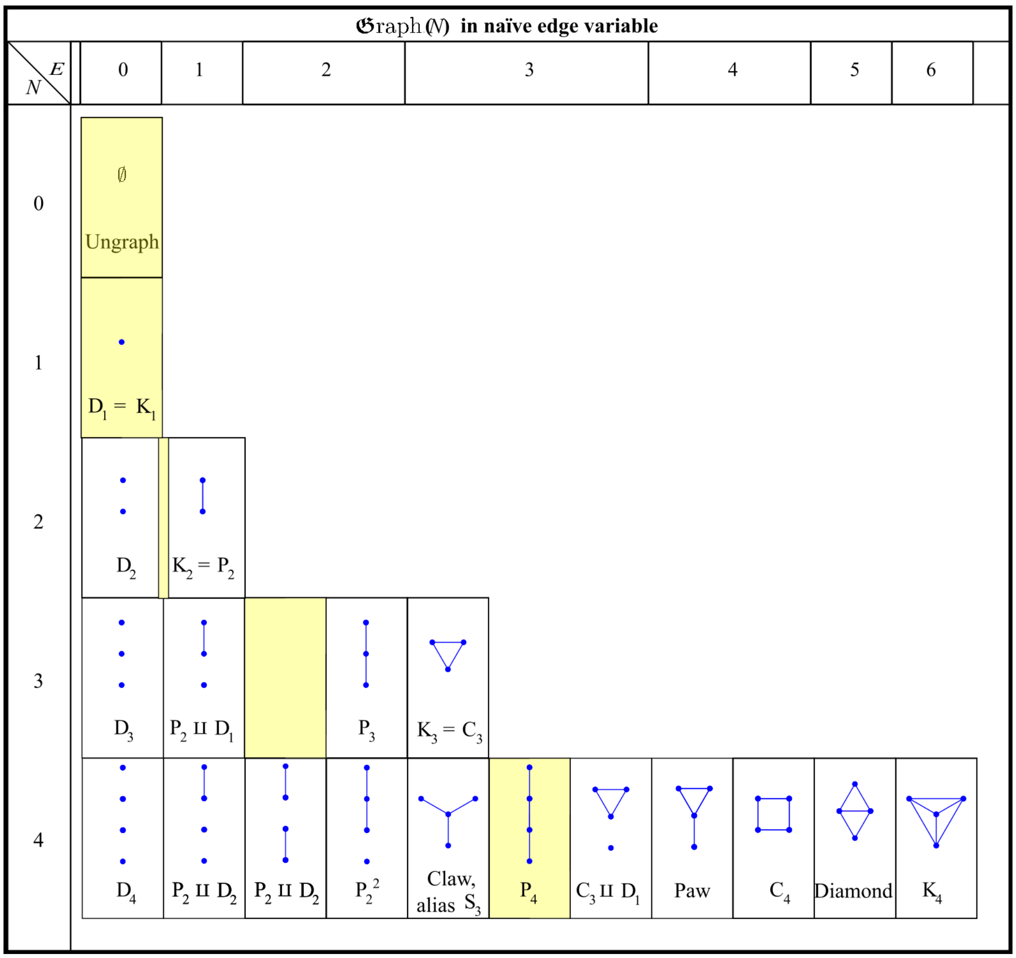

Figure 1:

Remark 7 A key observation is that most graphs come in *complementary pairs* under the operation of exchanging edges and non-edges. Fig 1 indicates these pairings by its first and last graphs for fixed N being complementary, its second and penultimate graphs likewise, and so on.

Remark 8 It is moreover possible for a graph to be *self-complementary*. A necessary condition for this is (8). This forces

$$N = 0 \text{ or } 1 \pmod 4) \ . \tag{17}$$

$N = 4$ suffices to demonstrate that this condition is not however in general sufficient. Our figure then leaves space for E_{crit} representatives to be self-complementary or not. We highlight the background squares of those which are in yellow.

Definition 1 Let us denote the *space of self-complementary graphs on* N *vertices* by

$$\mathfrak{SC}(N) \ , \tag{18}$$

the *space of self-complementary graphs on up to* N *vertices* by

$$\mathfrak{SC}[N] \tag{19}$$

and the *space of finite self-complementary graphs* by

$$\mathfrak{SC} \ . \tag{20}$$

Remark 9 At the level of configuration spaces, (17) becomes

$$\mathfrak{SC}(N) \ = \ \emptyset \ \ \text{for} \ \ N \ = \ 2, 3 \ \ (\text{Mod } 4) \ . \tag{21}$$

Thereby,

$$\mathfrak{SC} \ = \ \coprod_{N \, \in \, \mathbb{N}_0} \mathfrak{SC}(N) \ = \ \coprod_{M \, \in \, \mathbb{N}_0} \mathfrak{SC}(4\,M) \ \amalg \ \coprod_{M \, \in \, \mathbb{N}_0} \mathfrak{SC}(4\,M \ + \ 1) \ . \tag{22}$$

3 $\mathfrak{G}\mathrm{raph}(N)$ relationally characterized

Remark 1 Let us now jointly centre the most symmetric cases (at present in the sense of self-complementarity). Exploiting this is a first small instalment of Shape Theory [31]. We exhibit this in Fig 2, now using yellow shading to form a mirror line throughout, whether or not the N in question supports any self-complementary graphs. This is to be contrasted with Fig 1's square grid in the ab initio simplest variables N and E.

Structure 1 The dependent variable that we now switch to is the *edge-criticality-discrepancy variable* D of (9).

Remark 2 This can be thought of as a *relative* variable (compare relative separation vectors in Flat Geometry, and in Dynamics thereupon).

Remark 3 It is furthermore not just any relative variable – a difference – but specifically *relative to an intrinsically-significant value*. This has a loose parallel in *barycentric* relative separation vectors, alias *centre-of-mass* coordinates.

Remark 4 We furthermore now depict this with uniform width, vertically stacking equal-edge-number graphs. We however maintain the exception of stacking $D = 0$ in three columns. This is so as to keep self-complementary graphs separate from critical-but-not-self-complementary pairs (2 columns not 1) while fully exhibiting the self-complementarity symmetry (3 columns not 2).[2]

Remark 5 Fig 2.a) ranks graphs lexicographically by size, and number, of cycles present. In contrast, Fig 2.b) ranks graphs lexicographically by number, and size, of connected components. $N = 4$ is minimal for these to give different answers.

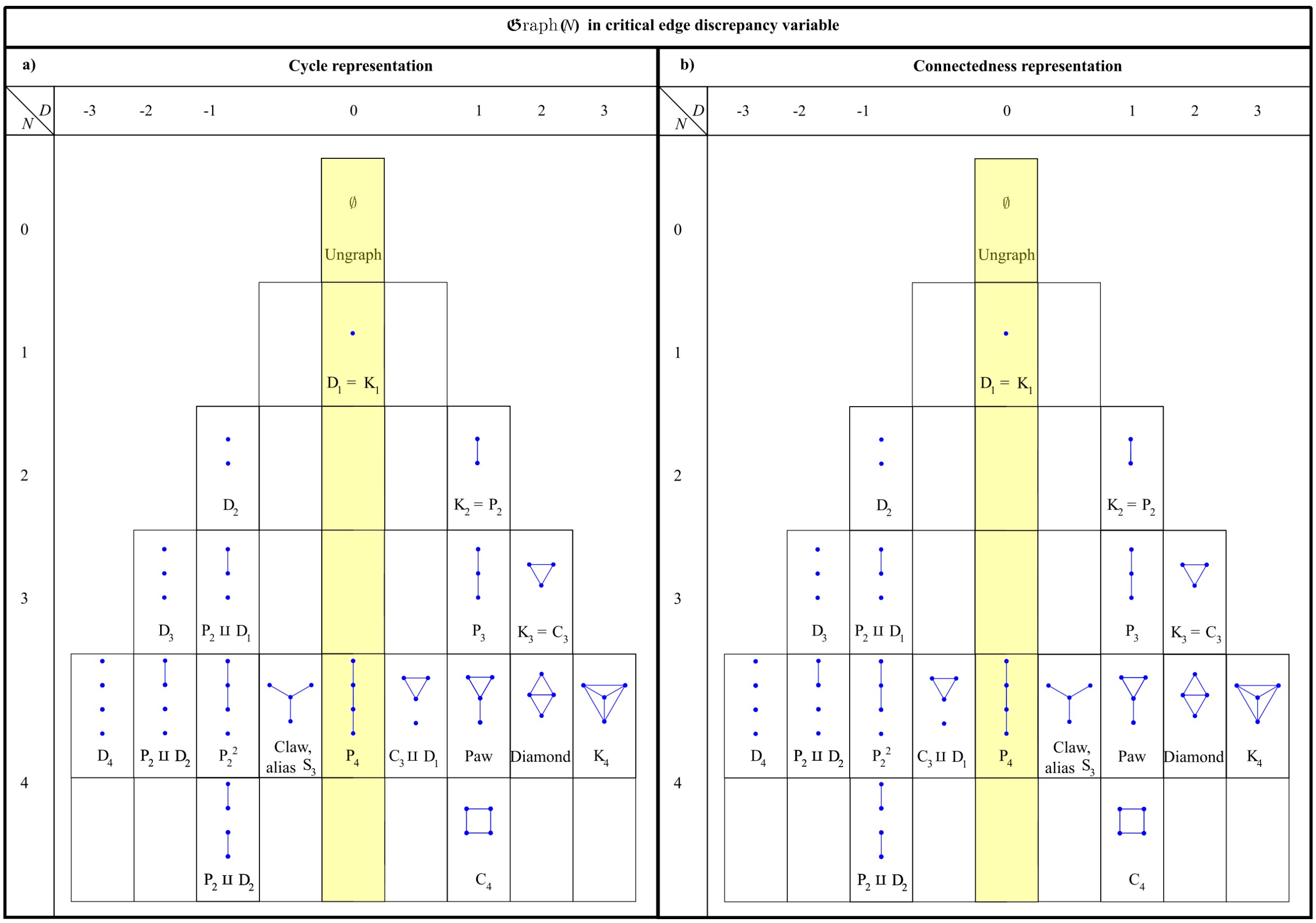

Figure 2:

[2] We could still represent this uniformly by tripling the width of the $D \neq 0$ boxes. Since this uses up space, we confine exhibiting this to a longer online version [36].)

4 [Graphs] on 0 to 4 vertices

Structure 1 As a second instalment of Shape Theory, we superpose all self-complementary pairs so as to treat each pair as a single indiscernible [1] entity (Fig 3). This amounts to passing to an *unsigned departure from criticality variable*

$$U \ := \ |\, D \,| \ = \ |\, E \ - \ E_{\mathrm{crit}} \,| \tag{23}$$

in the role of dependent variable. At the level of space of graphs, this amounts to discarding one unshaded wing of the previous Figure.

Remark 1 For $N \ = \ 4$, this can be phrased as a choice of a) *connected* or b) *cycle-free* – alias *forest* – reps. As we shall see elsewhere [36], these criteria break down for $N \ \geq \ 5$ by their boundaries ceasing to coincide with $\mathfrak{SC}(N)$. Another way of characterizing the given a) and b) is that they are (choices of) dense and sparse reps respectively.

Here, *dense* means with

$$E \ \geq \ E_{\mathrm{crit}} \tag{24}$$

while *sparse* means with

$$E \ \leq \ E_{\mathrm{crit}} \ . \tag{25}$$

Dense reps are useful in visually recognizing small [graphs]. However, as [graphs] become larger, sparse reps become far easier to recognize, draw, name and handle.

a)

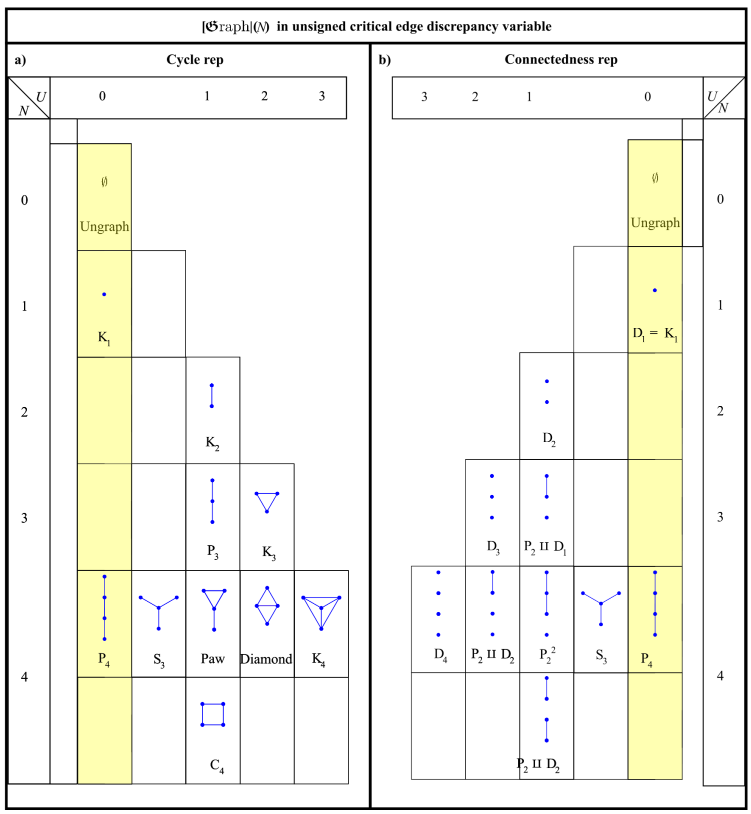

Figure 3:

Remark 2 One idea is to only name graphs if neither they nor their complements can be named as unions of already-named components. For instance, up to $N \ = \ 4$, the only primary names we need for [graphs] are K_N, P_3 and P_4. Nontrivial 'naming primes' [36] do subsequently appear however. For instance, the familiar C_5 and Bull self-complementary graphs on 5 vertices (Fig 4) are 'naming primes'.

Remark 3 $N \ = \ 5$ is thus also minimal for $\mathfrak{SC}(N)$ to be more than a point.

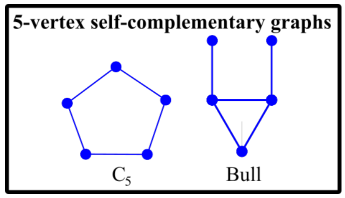

Figure 4:

5 'Ramsey imbalance'

Remark 1 To the Physicist,[3] the previous section's representatives are unsatisfactory, in the sense of being gauge choices (for $\mathbb{Z}_2$ gauge group, realized as complementation). This unsatisfactoriness extends to the Relationalist and to the Shape Theorist, who would frame the matter in terms of passing to a more reduced configuration space. What would be desirable is a *gauge-independent* or *relative-space* representation.

Remark 2 A trick [15] standard since Ramsey [2] is moreover available. This can be phrased as expanding from using blue for related edges to using red for unrelated edges as well. We exhibit this in Fig 5. The difference between versions a) and b) is specific to $N = 4$. For in $N \geq 5$, planar Ramsey representations of [graphs] cease to be possible. In $N = 5$, this is by K_5 being nonplanar. Whereas for $N \geq 6$, this is e.g. by K_N containing K_5 as a subgraph, this being one of the two forbidden subgraphs for planarity.

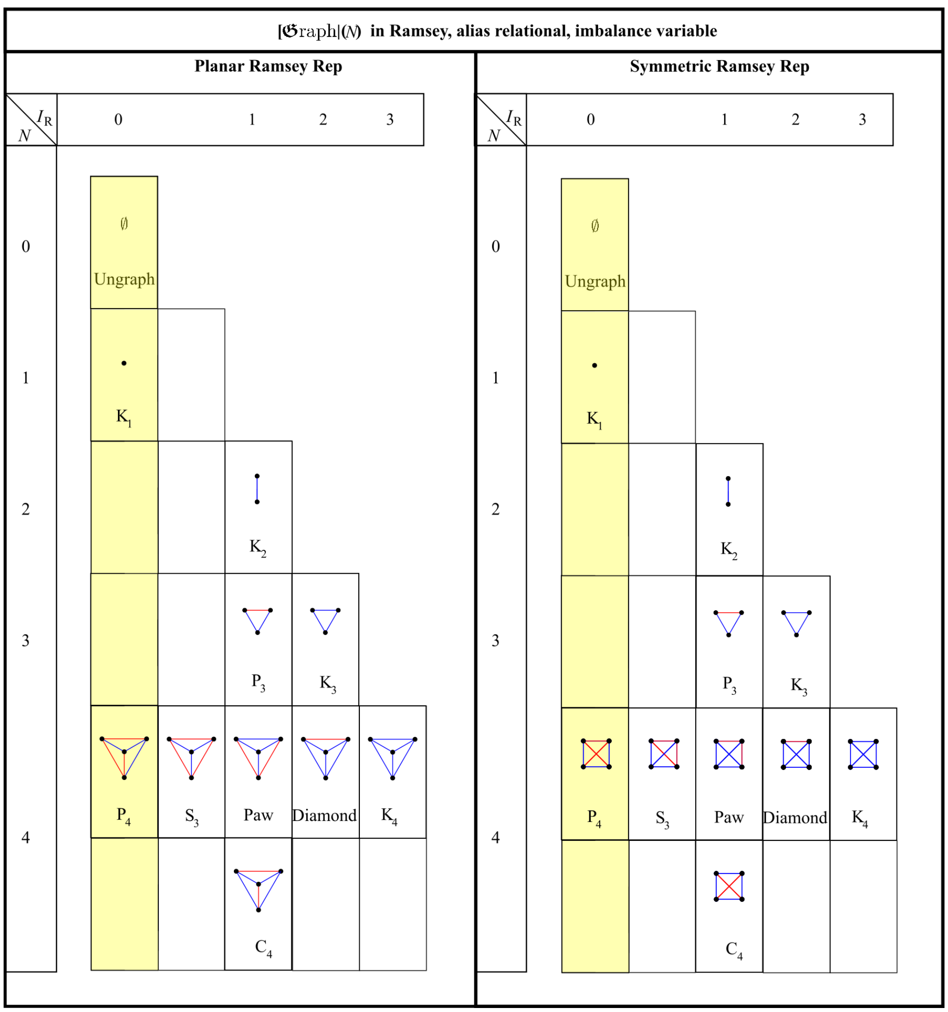

Figure 5:

Naming Remark 1 This amounts to reconceiving of our U as *Ramsey imbalance* I_{R}, as per (14, 26). The intercon-

[3]This being a Seminar given at an Applied Combinatorics for Physicists Summer School, it shall contain some commentary directed at Physicists.

version proceeds via E_{crit} having zero Ramsey imbalance.

$$I_{\text{R}} \; = \; | \, E \; - \; E_{\text{crit}} \, | \tag{26}$$

thus amounts to readjusting edge count to Ramsey imbalance.

Naming Remark 2 A truer name for Ramsey imbalance is *relational imbalance*, in the sense of

$$I_{\text{R}} \; = \; \sum_{i = \_ \; , \; -} \text{relata}(i) \; . \tag{27}$$

Here relata contribute positively for $i \; = \; \_$, corresponding to the answer 'yes' to the question 'is this vertex-pair related' (and thus modelled by a blue edge)? But the $i \; = \; -$ – written as a long overline of negation – correspond to the answer 'no' as modelled by a red edge.

Remark 3

$$\#\text{positives} \; - \; \#\text{negatives} \tag{28}$$

is moreover a basic type of *index* [7],

$$\Delta n \; = \; n_{+} \; - \; n_{-} \tag{29}$$

(c.f. *spectral index*, and also *signature* in basic Linear Algebra and Geometry). It is moreover an *unsigned* index: only its absolute value counts.

Remark 4 Being unsigned models that blue and red are *meaningless* labels. This is in the sense that while they are distinguishable from each other, neither label carries any further properties. Physically-familiar examples of such include Quantum Chromodynamics' 'red', 'blue' and 'green' colour labels. It is moreover standard to more deeply model meaningless labels by quotienting (see e.g. [18]).

Remark 5 Using Fig 3.b)'s orientation, we reflect our variable to '

$$S_{\text{R}} \; = \; E_{\text{crit}} \; - \; I_{\text{R}} \; = \; E_{\text{crit}} \; - \; |E_{\text{crit}} \; - \; E| \; , \tag{30}$$

This runs from 0 to E_{crit}. This $\mathbb{N}_0$-valued function is a *candidate entropy*, always increasing away from a zero-valued minimum at a unique distinguished point: the monochromatic [graph]. We leave testing this for further properties that a bona fide entropy would have for a future occasion. I_{R} is correspondingly a candidate *notion of information*.

Naming Remark 3 All, in all, the symbol I_{R} is well-chosen by standing for all of 'relational or Ramsey' 'imbalance, index or information'.

6 $\mathfrak{Graph}(N)$ as a lattice

We exhibit $\langle \mathfrak{Graph}(N), < \rangle$, for $<$ ordering the number of edges, in Fig 6.

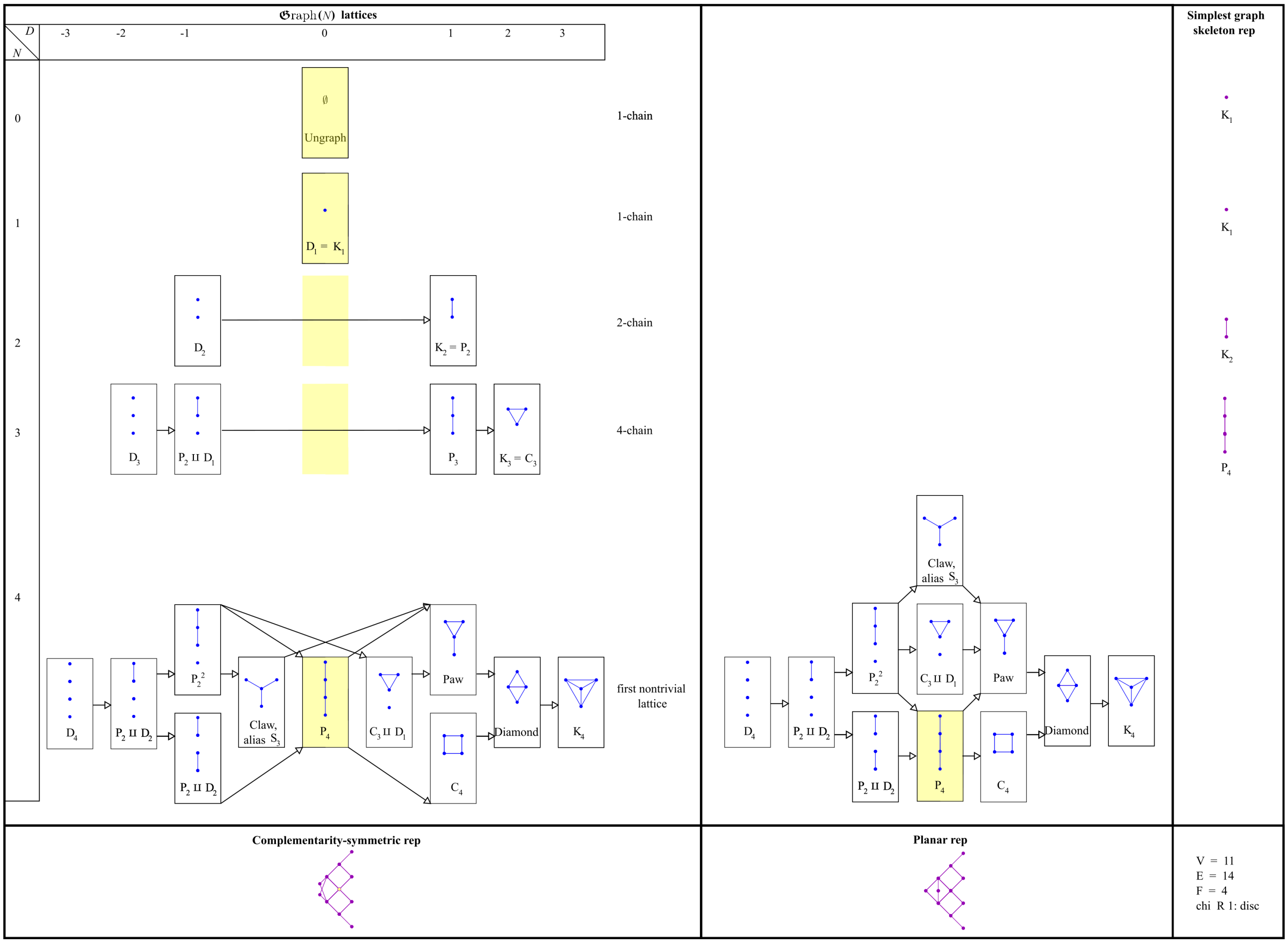

Figure 6:

Remark 1 This satisfies the poset axioms [12, 19]. At the level of graphs, these imply triangles being forbidden. This feature persists for arbitrary N. For one cannot get the same graph by, down one path adding 2 edges while down the other adding just 1.

Remark 2 The Introduction's list of minimalities are manifest.

Remark 3 Being connected and with unique terminal elements, this is furthermore a lattice [12, 19].

As a such, its dual order is just 'remove an edge'.

Its lattice operations are 'form union of edges' and 'form intersection of edges'. This is only valid on graphs $\mathfrak{G}$, $\mathfrak{G}'$ which have both $V = V'$ and $E = E'$.

Remark 4 The graph skeleton of a poset or lattice is what remains if its ordering structure is thrown away. Posets and lattices are additionally digraphs – for now represented using white arrows – and with some embedding distinction as well [35].

Remark 5 F is the number of faces in a planar graph, while χ is Euler's topologically-significant $F - E + V$. When we find a new or unnamed graph, we list its V and E, and its F and χ as well when lucid. We represent it planarly when possible, elsewise proving that it is nonplanar.

7 $[\mathfrak{Graph}](N)$ as a semi-lattice

We exhibit $\langle[\mathfrak{Graph}](N), \ < \ \rangle$, for $\ < \ $ ordering by value of relational imbalance I_{R}, in Fig 7.

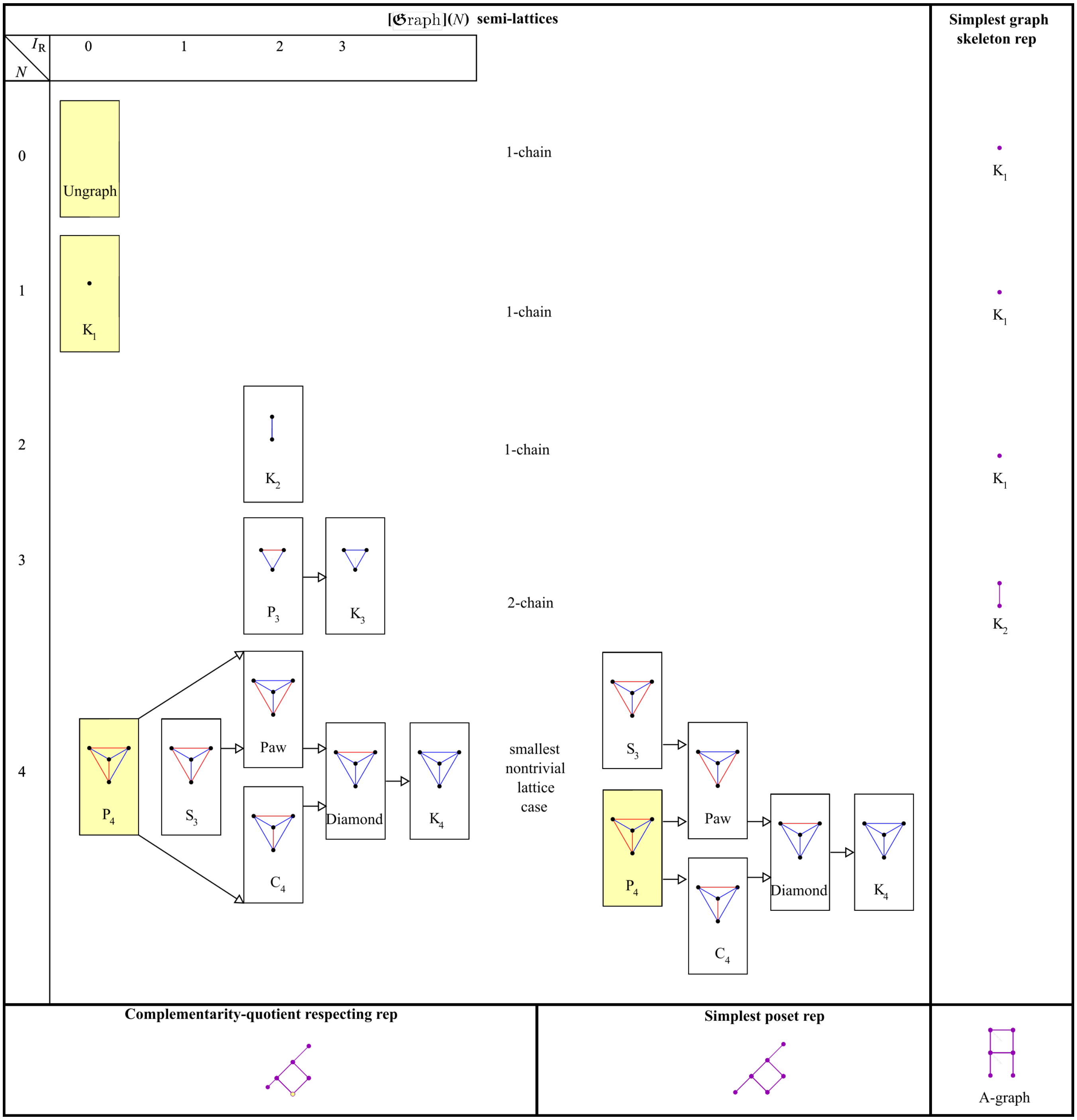

Figure 7:

Remark 1 The operations are now 'form the union of one colour and yet the intersection of the other'. While such ordering operations can be posited in both directions, in the balancing direction, one eventually finds pairs of elements that do not share a common bound. $N = 4$ is minimal for this to occur. Thus in general only a semi-lattice's single operation is realized.

8 A simple view on $\mathfrak{G}\text{raph}[N]$

Structure 1 We exhibit this as linked up by the add-vertex operation in Fig 8.

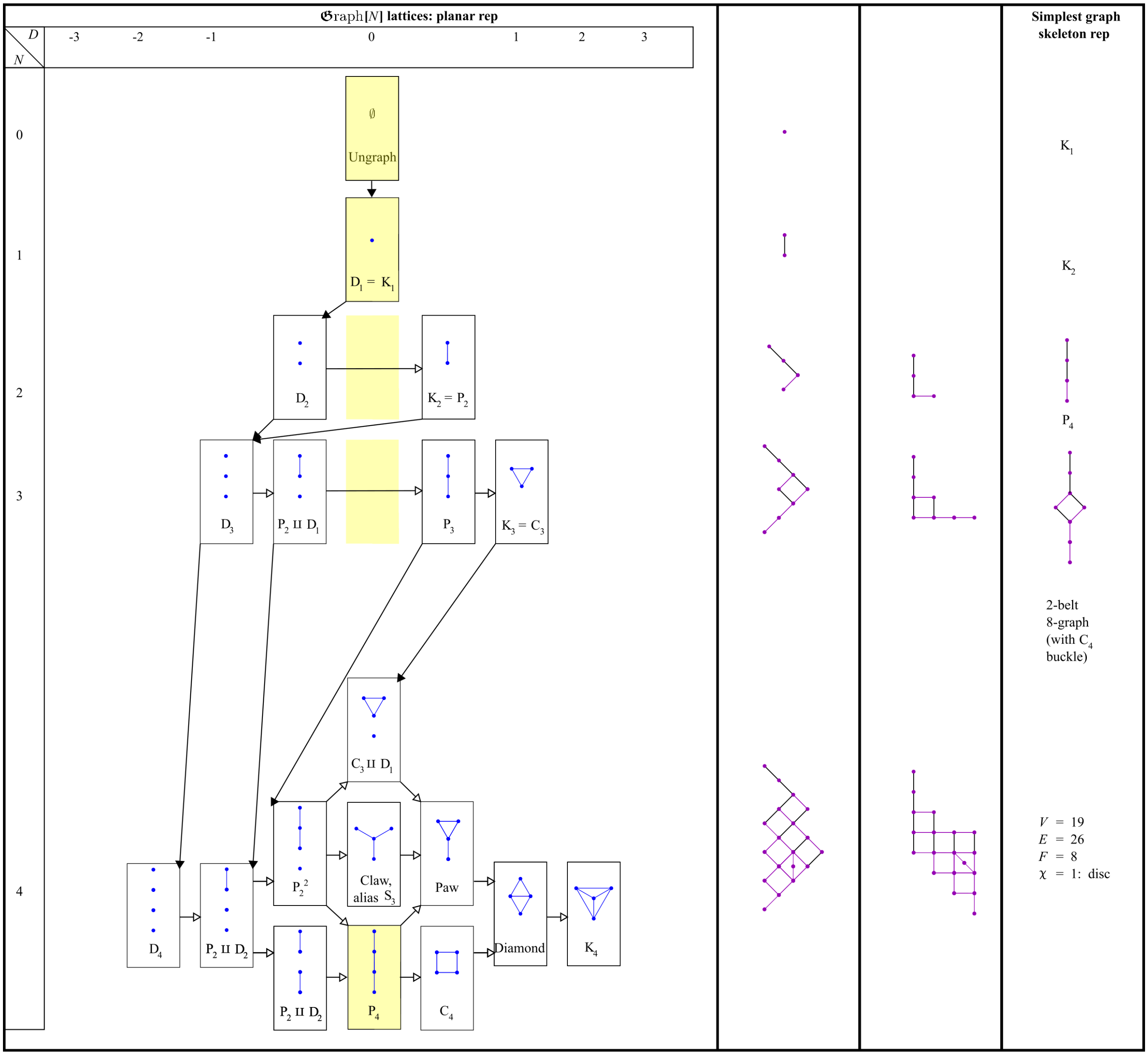

Figure 8:

9 [$\mathfrak{G}$raph][N]

Impasse 1 We cannot use vertex addition here, since this does not preserve graph complementarity.

Structure 1 We thus introduce instead the the *monochromatic coning operation.* I.e. given a graph,

1) *cone* it: add a single vertex that is edge-connected to each vertex.

2) *Colour* all these new edges monochromatically, either all in red or all in blue.

In the red case, this recovers our previous operation of adding an unconnected vertex. Our new operation moreover preserves complementation, thus constituting our desired extension to the previous section's N-traversing operation.

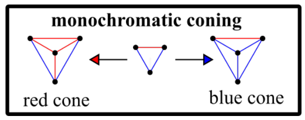

Figure 9:

Remark 1 As Physicists [5, 16, 17, 8, 3, 10] and Shape Theorists [23, 33] know well, working in a more reduced configuration space often comes with complications stemming from such configuration spaces' greater mathematical complexity. Needing a new operation – monochromatic coning operation – is the price we pay for working in a more reduced configuration space.

Remark 2 This installs chain struts between the [$\mathfrak{G}$raph](N) lattices on each floor. These and the reversed white arrows merge to form Fig 10's structures. Up to this point, one has the poset, lattice, modular lattice and distributive lattices specialization of structure.

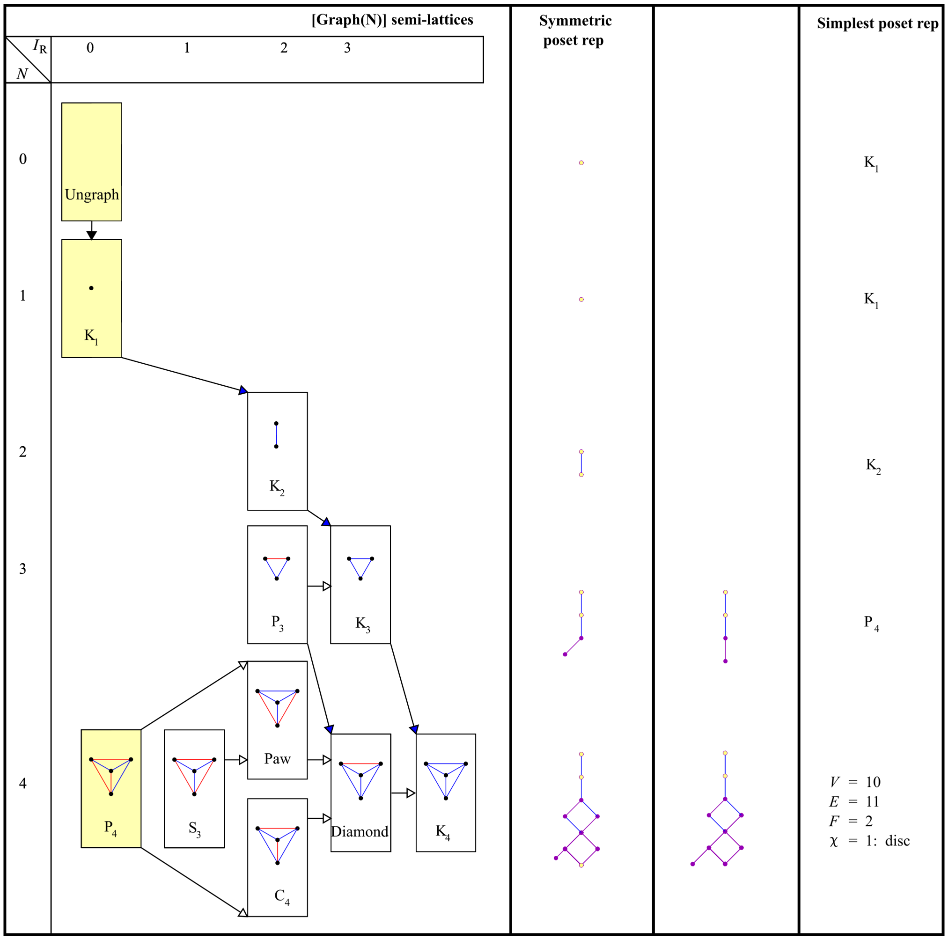

Figure 10:

10 A complementation-symmetric view of $\mathfrak{Graph}[N]$

Structure 1 We now use the monochromatic coning operation on the lattices of unreduced graphs (Fig 11). This yields the current note's largest skeleton graph, as indicated.

Remark 1 Considering all arrows at once, $\mathfrak{Graph}[2]$ is already clearly not a poset by being Paw and thus manifesting a triangle. $\mathfrak{Graph}[N]$ defined in this way is thus no more than a digraph.

11 Further directions

Future directions include the following [36].

0) Further ratio variables insights from Shape Theory applied to Graph Theory.

1) Spaces of graphs for larger N.

2) Subspaces of particular types of graph.

3) Envisaging suitable topological notions for [graphs].
4) Spaces of looped graphs, multigraphs, digraphs, hypergraphs (...) [20].

5) Spaces of complexes, of which point clouds and graphs are the two lowest-dimensional cases.

6) Spaces of further Order-Theoretic structures: posets, lattices...

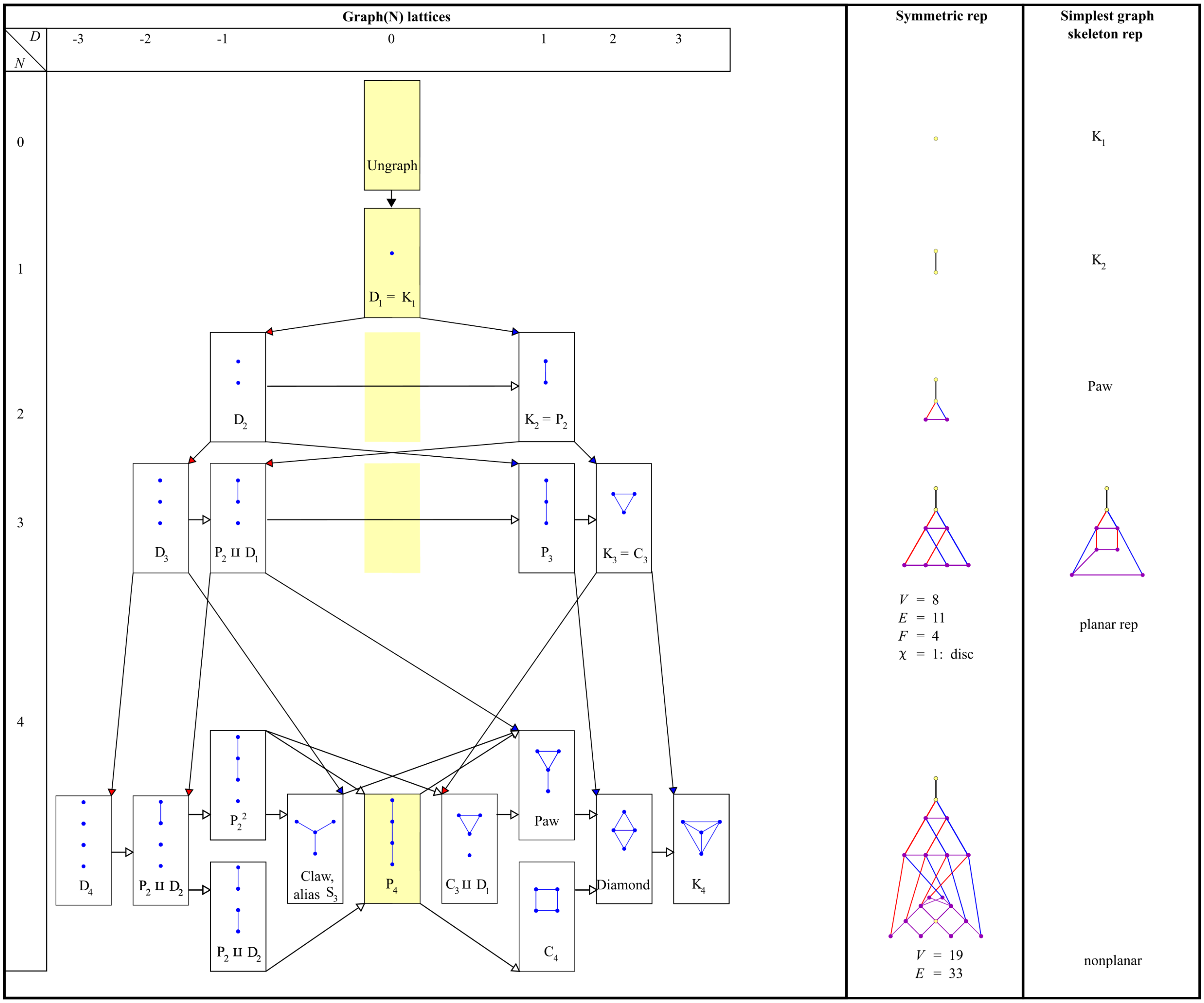

Figure 11:

Acknowledgments

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