Global Problem of Time Sextet.

–1. Introduction and Notions of Globality

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Abstract

The Problem of Time is due to conceptual gaps between General Relativity and the other observationally-confirmed theories of Physics; it is a major foundational issue in Quantum Gravity. The Problem of Time's multiple facets were mostly pointed out over 50 years ago by Wheeler, DeWitt and Dirac. These facets were subsequently classified by Kuchař and Isham. They argued that the lion's share of the problem consists of interferences between facets. They also posed the question of in which order the facets should be approached. By further considering the nature of each facet at the local classical level, the Author showed the facets to be two copies of Lie Theory – spacetime and canonical – with a Wheelerian 2-way route therebetween. This solves the facet ordering question. The resulting mathematical framework turns out moreover to be consistent enough to smooth out all local classical facet interferences as well.

It would furthermore be preferable if all of the Background Independence aspects, resultant Problem of Time facets, and strategies to resolve these, were treated in a globally well-defined manner. The current article begins to address this by classifying what is meant by 'global'. Be this at the level of mathematical structure (Topology, Differential Geometry, Lie Theory, PDEs, Functional Analysis). At the level of which spaces the modelling actually requires (space, spacetime, configuration space, phase space, space of spacetimes...). Or at the level of each aspect of time, space or Background Independence more generally. We also include globalization strategies and justification of A Local Resolution of the Problem of Time being possible in the first place.

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1 Introduction

1.1 The Problem of Time

Each observationally-established physical paradigm has a distinct conceptualization of time. The resulting gap is principally between Newtonian Physics, Special Relativity (SR), Quantum Mechanics (QM), and Quantum Field Theory (QFT) on the one side, and General Relativity (GR) on the other.[1] Newtonian Physics and SR each have a different background notion of time. GR, however, has coordinate time, and upon assuming a canonical formulation, at least apparently no time. Both SR and GR admit spacetime geometrization. While SR spacetime continues to play a major role in QFT, spacetime may well be just emergent rather than primary for quantum GR.

This gap leads to the *Problem of Time* [172, 99, 101, 157], meant here a multi-faceted sense since multiple differences in conceptualization of time are involved. Most of these facets were first envisaged over 50 years ago by Wheeler [47], DeWitt [48], or Dirac [22, 23, 24, 28, 29, 40]. It took 25 further years for the Problem of Time's full conceptual content to be assembled into Kuchař's and Isham's [99, 101] classification of facets (also summarized in [151]). Numerous observations of attempting to extend one Problem of Time facets' resolution to include a second facet has a strong tendency to interfere with the first resolution. Due to this, Kuchař's and Isham argued for the lion's share of the Problem of Time to consist of facet interferences. It is thus worth according the notation (A, B) for pairwise interference between facets A and B, with obvious extension to n-tuples. In which order the facets should be approached has also been a longstanding problem [99, 101, 103].

1.2 ALRoPoT

A Local Resolution of the Problem of Time (ALRoPoT) has recently been given [178, 172, 186, 187, 188, 189, 190, 191, 192, 193, 194, 196, 195, 199, 200, 201, 202] , alongside reformulation as [172, 182, 184, 198] A Local Theory of Background Independence.[2] The classical part of this can be viewed as [197, 195] requiring just Lie's Mathematics [150, 8, 66, 161, 44, 61, 195], which for around a century has been entrenched in Topology [147, 76, 128, 155] and Differential Geometry [74, 161] while more recently being applied to the setting of contemporary Physics' state spaces [67, 97, 172, 177].

The classical Problem of Time moreover contains not one but two copies of Lie Mathematics. Namely, spacetime primality's [130, 79, 65] versus [172] spatial, configurational, dynamical, or canonical primality's [34, 40, 57, 99, 101, 106, 156, 143]. These occur alongside a Wheelerian [47, 57] two-way route between the two. Previous confusion in

[1]See Part I of [172] for details, and also for smaller differences in time and space concepts between the first three.

[2]See e.g. [38, 45, 142] for earlier accounts of Background Independence, and e.g. [1, 2, 7, 89, 109, 172] for previous 'absolute versus relational motion debate' considerations.

this field – especially facet ordering – also resulted from there being two copies, including failure to separate each copy's version of some concepts. Only facet orderings that make reference to *at least two* versions of each of Closure [8, 23, 28, 40, 97, 172, 188, 192, 200], Relationalism [2, 7, 72, 106, 109, 154, 172, 186, 187, 190, 191, 199] Observables [22, 99, 101, 103, 163, 172, 179, 180, 193, 201] and Constructability [47, 127, 160, 172, 194, 202]. [See Articles 1), 0), 2) and 3) respectively for what these mean, Fig 1 for the parts of Lie Theory correspond to, and Sec 20 for a first global overview.]

1.3 Graph-theoretical analysis: Closure's nexus, not Relationalism's root, is 'centre stage'

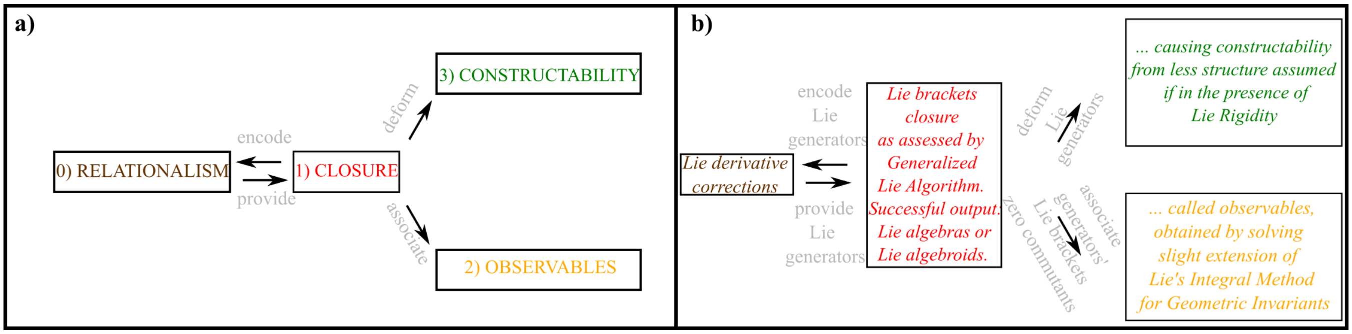

Figure 1: The Background Independent local resolution of the Problem of Time involves 2 copies of a)'s aspects, which receive b)'s Lie-Theoretic interpretation.

Lie's mathematics forms a nontrivial digraph between aspects (Backgrounf Independence's counterpart of the Problem of Time's facets: Fig 1), with two aspects' nodes playing prominent roles.

Relationalism, as provider, is the root of the Problem of Time digraph [195]. Closure is moreover the far more significant nexus thereof, connecting to all the other aspects. In the canonical setting, this nexus is powered by the Dirac Algorithm. In both canonical and spacetime settings, it is powered a fortiori by the Lie Algorithm. Relationalism can thus be taken to precede Closure at the procedural level. One can move in the reverse direction and Encode Constraints at the level of Principles of Dynamics actions. By this, the root *need not* be taken to procedurally come first.

Closure's nexus has the following further significances.

a) That *everything up to and including the nexus can be solved as a decoupled problem.*

b) Each facet emanating from the other side of the nexus can subsequently be solved as its own decoupled problem. Namely, the nexus – Closure – factorizes the Problem of Time into first having to solve a combined Relationalism–Closure Problem. Then, down one branch, there is a decoupled Problem of Observables, while, down the other branch, one enconters a Construction Problem. In this way, Closure is a powerful procedural nexus by acting as a divider. A fortiori, it is a divider whose machinery – the Lie Algorithm – either drives, or induces, each of the three factorized subproblems.

The power in resolving the Problem of Time derives from the nexus of Closure (as opposed to the root of Relationalism). Aside from connecting to all the other aspects, the Dirac and generalized Lie Algorrithms have the capacity to reject input combinations of generators, as well as to enlarge these combinations. Such Algorithms are thereby *selection principles.*

1.4 Outline of the current article's global considerations

Each facet resolution alias aspect formulation in [199, 200, 201, 202] is local (in the sense of a possible small neighbourhood) It would furthermore be preferable if all of the Background Independence aspects, resultant Problem of Time facets, and strategies to resolve these, were treated in a globally well-defined manner. We are however far from this goal, which involves many – and often distinct – uses of the word 'global'. Some of these have so far not even been explained in the literature.

Part I covers what we need as regards various basic levels of global structure. We begin with topological spaces [147, 128, 54, 92, 138] in Sec 2, topological manifolds [155] in Sec 3, LCHS and LCHP spaces [128, 161, 132, 164, 153] in Sec 4 and metric spaces [147, 128] in Sec 5. Differential Geometry [79, 95, 161, 26] including diffeomorphisms, flows and Lie derivatives is summarized in Sec 6, with further levels of geometrical structure [26, 37] in Sec 7. Lie brackets, Lie algebras and Lie groups [150, 66] are outlined in Sec 8, and Algebraic Topology [76, 77, 128, 131] alongside Differential Topology [62, 117, 122, 161] in Sec 9. Finally product spaces and bundles [122, 94, 107] are covered in Sec 10.

Part II considers various classifications of globality, firstly by qualitative extent of locality [131, 138, 140] in Sec 11. Secondly, as regards globality in which space, with space, time and spacetime distinctions [130, 57, 172] outlined in Sec 12, the notion of carrier spaces [21, 123, 176] in Sec 13, and the state objects of Sec 14's state spaces [67, 85, 199] in Sec 15. Natural Law requires a certain few 'Great Maps' which naturally generate more mathematics than is 'usually supposed' to occur in Physics. One of the toughest such is *Arenize* [47, 91, 96, 172]: the map from the mathematical type of an object to the mathematical type of the space formed by those objects. The classical fundamental dynamical laws [21, 67, 130, 57, 79, 113, 172] considered in addition to state spaces in the current Series are outlined in Sec 16. This is followed by viewing them as DE problems [146, 148, 169] in Sec 17, including a first supporting deployment of Functional Analysis [32, 74, 169], in particular $\mathfrak{C}^{\alpha}$ and Sobolev spaces. Function spaces required after *Arenize* are moreover generally distinct; (tame) Fréchet spaces [68, 75] will do (Sec 18).

Globality in the temporo-spatial context [99, 101, 173] is summarized in Sec 19, while Sec 20 spearheads each of the next 5 articles' considerations of globality in the more general context of classical Background Independence.

Sec 21 serves to justify A Local Resolution of the Problem of Time being possible in the first place, based on HP spaces [128, 161, 171] and their Shrinking Lemma [161, 138]. General kinds of Globalization Strategies are provided in Sec 22, with application to Background Independence. Over this series we will justify 'global Problems of Time' being highly plural. We conclude in Sec 23, including an outline of *Quantize* [82, 97, 121] and an overview of Functional Analysis involved in the modelling.

Part I

Global Levels of Mathematical Structure

2 Topological spaces

2.1 Subset-based definitions

Definition 1 A *topological space* [147, 128, 54, 92, 138] is a set $\mathfrak{X}$ together with a collection

$$\boldsymbol{\mathfrak{T}} = \{ \mathfrak{U}_i \subseteq \mathfrak{X} \}_{i \in \mathcal{I}} \tag{1}$$

(for $\mathcal{I}$ an arbitary index set) with the following properties.

i) $\mathfrak{X}, \emptyset \in \boldsymbol{\mathfrak{T}}$.

ii)

$$\bigcup_{j \in \mathcal{J}} \mathfrak{U}_j \in \boldsymbol{\mathfrak{T}} : \quad \text{closure under arbitrary unions} \quad . \tag{2}$$

iii)

$$\bigcap_{k = 1}^{K} \mathfrak{U}_k \in \boldsymbol{\mathfrak{T}} : \quad \text{closure under finite intersections} \quad . \tag{3}$$

Such sets are termed *open.* The complement of an open set is a *closed set.*

Remark 1 Subsets of a given set can be open, closed, both or neither.

Definition 2 The *subspace topology* on $\mathfrak{A} \subseteq \mathfrak{X}$ is $\boldsymbol{\mathfrak{T}}_{\mathrm{S}} := \{\mathfrak{A} \cap \mathfrak{U} | \mathfrak{U} \in \boldsymbol{\mathfrak{T}}\}$.

Definition 3 a is a *closure point* of $\mathfrak{J} \subseteq \mathfrak{X}$ if $\mathfrak{U} \cap \mathfrak{J} \neq \emptyset$ for every open sub$\overbrace{\text{set } a \in \mathfrak{U}} \subseteq \mathfrak{X}$. The *closure* $\mathrm{Clos}(\mathfrak{J})$ of $\mathfrak{J}$ is the set of closure points of $\mathfrak{J}$ in $\mathfrak{X}$.

A point a is an *interior point* of a subset $\mathfrak{J}$ in a topological space $\mathfrak{X}$ if $\exists$ op$\overbrace{\text{en } a \in \mathfrak{U}} \subseteq \mathfrak{X}$. The *interior* $\mathrm{Int}(\mathfrak{J})$ of $\mathfrak{J}$ is the set of interior points of $\mathfrak{J} \in \mathfrak{X}$.

The *frontier* of $\mathfrak{J}$ in $\mathfrak{X}$ is $\mathrm{Clos}(\mathfrak{J}) - \mathrm{Int}(\mathfrak{J})$

Definition 4 An *open cover* [147, 128] for $\mathfrak{X}$ is a collection of open sets $\boldsymbol{\mathfrak{C}}\mathrm{over} := \{\mathfrak{U}_l\}_{l \in \mathcal{L}}$ such that

$$\mathfrak{X} = \bigcup_{l \in \mathcal{L}} \mathfrak{U}_l \ . \tag{4}$$

Definition 5 A subcollection of an open cover that is itself an open cover is termed a *subcover*, $\{\mathfrak{U}_m\}_{m \in \mathcal{M}}$ for $\mathcal{M} \subseteq \mathcal{L}$.

Definition 6 An open cover $\{\mathfrak{V}_n\}_{n \in \mathcal{N}}$ is a *refinement* [128, 31] of $\{\mathfrak{U}_l\}_{l \in \mathcal{L}}$ if

$$\text{each } \mathfrak{V}_n \text{ has a } \mathfrak{U}_l \text{ such that } \mathfrak{V}_n \subset \mathfrak{U}_l \ . \tag{5}$$

$\{\mathfrak{V}_n\}$ is furthermore *locally finite* if

$$\text{each } x \in \mathfrak{X} \text{ has an open neighbourhood } \mathfrak{N}_x \text{ such that only finitely many } \mathfrak{V}_n \text{ obey } \mathfrak{N}_x \cup \mathfrak{V}_n \neq \emptyset \ . \tag{6}$$

Definition 7 A *base* [147, 128] for a topological space $\langle \mathfrak{X}, \boldsymbol{\mathfrak{T}} \rangle$ is a subcollection

$$\boldsymbol{\mathfrak{B}}\mathrm{ase} := \{\mathfrak{b}_p\}_{p \in \mathcal{P}} \subseteq \boldsymbol{\mathfrak{T}} \tag{7}$$

such that every open subset $\mathfrak{U} \in \mathfrak{X}$ is covered by elements of $\boldsymbol{\mathfrak{B}}$,

$$\mathfrak{U} = \bigcup_{p \in \mathcal{P}} \mathfrak{b}_p \ . \tag{8}$$

Definition 8 A *local base* [128] $\mathfrak{B}\text{ase}_x$ for a topological space $\langle \mathfrak{X}, \mathfrak{T} \rangle$ at point $x \in \mathfrak{X}$ is a collection of neighbourhoods of x such that every neighbourhood $x \in \mathfrak{U}$ contains some $\mathfrak{b} \in \mathfrak{B}\text{ase}_x$.

Definition 9 A *sub-base* [54] for a topological space $\langle \mathfrak{X}, \mathfrak{T} \rangle$ is a subcollection of its subsets,

$$\mathfrak{S}\text{ubbase} := \{\mathfrak{s}_s\}_{s \in \mathcal{S}} \subseteq \mathfrak{T} . \tag{9}$$

This is such that $\mathfrak{X}$ alongside the collection of all finite intersections of elements of $\mathfrak{S}$ubbase form a base.

2.2 Topological properties

Definition 10 Let $\langle \mathfrak{X}, \mathfrak{T} \rangle$ and $\langle \mathfrak{Y}, \mathfrak{S} \rangle$ be topological spaces. A map $\phi : \mathfrak{X} \longrightarrow \mathfrak{Y}$ is *continuous* if $\forall\, \mathfrak{U} \in \mathfrak{S}$, $\phi^{-1}(\mathfrak{U}) \in \mathfrak{T}$. A continuous ϕ is furthermore a *homeomorphism* if it is a bijection and in possession of a continuous inverse.

Definition 11 *Topological properties* [76, 54, 92, 138] are those attributes of a topological space that are homeomorphism-invariant.

Remark 2 Some topological properties that the current Article makes use of are as follows.

Definition 12 Suppose that $\mathfrak{U}$, $\mathfrak{V}$ are open sets such that

$$\mathfrak{U} \cap \mathfrak{V} = \emptyset , \ \ \mathfrak{U} \cup \mathfrak{V} = \mathfrak{X} \tag{10}$$

and neither $\mathfrak{U}$ nor $\mathfrak{V}$ are $\emptyset$. Then $\mathfrak{U}$ and $\mathfrak{V}$ are said to *disconnect* $\mathfrak{X}$. If $\mathfrak{X}$ is not disconnected by any two sets, then it is *connected* [76, 147, 155].

Remark 3 Connectedness is chiefly motivated by considering how far one of the foundational Theorems of Analysis – the Intermediate Value Theorem [135] – can be generalized.

Definition 13 $\mathfrak{X}$ is *path-connected* if for x, $y \in \mathfrak{X}$, $\exists$ a path γ from x to y.

Notions of *separation* are also topological properties, indeed involving separating two objects (points, certain kinds of subsets) by encasing each in a disjoint subset; a particular such is as follows; Article 0 contains two more. The following is a particular such.

Definition 14 A topological space is *Hausdorff* [147, 155, 54] if

$$\begin{gathered}\text{for } \ x, y \in \mathfrak{X} , \ \ x \neq y , \ \ \exists \ \text{ open sets } \ \mathfrak{U}_x , \ \ \mathfrak{U}_y \in \mathfrak{T} \\ \text{such that } \ x \in \mathfrak{U}_x , \ \ y \in \mathfrak{U}_y \ \text{ and } \ \mathfrak{U}_x \cap \mathfrak{U}_y = \emptyset . \end{gathered} \tag{11}$$

I.e. any pair of points can be separated by open sets.

Remark 4 Hausdorffness allows for each point to have a neighbourhood without stopping any other point from having one. This generalizes a property of $\mathbb{R}$ that much Analysis depends upon. Hausdorffness guarantees in particular in this way uniqueness for limits of sequences.

Definition 15 $\langle \mathfrak{X}, \mathfrak{T} \rangle$ is *first countable* [128, 54] if each $x \in \mathfrak{X}$ has a countable local base.

Definition 16 $\langle \mathfrak{X}, \mathfrak{T} \rangle$ is *second countable* [128, 54] if it admits a countable base.

Definition 17 $\langle \mathfrak{X}, \mathfrak{T} \rangle$ is *locally Euclidean (LE)* [155] if every point $x \in \mathfrak{X}$ has a neighbourhood $\mathfrak{N}_x$ homeomorphic to $\mathbb{R}^p$: Euclidean space.

Definition 18 $\langle \mathfrak{X}, \mathfrak{T} \rangle$ is *compact* [76, 147, 155, 54] if every open cover of $\mathfrak{X}$ has a finite subcover.

Remark 5 Compactness is useful e.g. through its generalizing continuous functions being closed and bounded on a closed interval of $\mathbb{R}$ (i.e. the Heine–Borel Theorem [147, 128]).

Remark 6 Second countability is a more stringent global condition to first countability's local condition. These countability criteria protect one's topology from containing 'too many' open subsets. On the one hand, first countability corresponds to topological spaces in which sequences suffice to detect most topological properties. On the other hand, second-countability has the additional useful feature of guaranteeing equivalence of compactness and sequential compactness [54].

Definition 19 *Local compactness* is that each point $\mathrm{x} \in \mathfrak{X}$ is contained in a compact neighbourhood.

Definition 20 $\langle \mathfrak{X}, \mathfrak{T} \rangle$ is *paracompact* [155, 128, 54] if every open cover of $\mathfrak{X}$ has a locally finite refinement.

3 Topological manifolds

Definition 1 *Topological manifolds* [155] are topological spaces $\mathfrak{M}$ possessing the three bastions of Hausdorffness, second-countability and local Euclideanness (LEHS).

Remark 1 Hausdorffness and second-countability form a useful combination in the manner of a balance of analytical tractability. I.e. the previous section's comments about use of sequences fit together to leave non-Hausdorff topological spaces with too few open sets for much of Analysis while non-second-countable ones have too many.

Remark 2

$$\text{LE} \Rightarrow \text{LC} . \tag{12}$$

Remark 3 Connectedness and path-connectedness coincide for manifolds.

Remark 4 Local Euclideanness moreover permits the following useful construct.

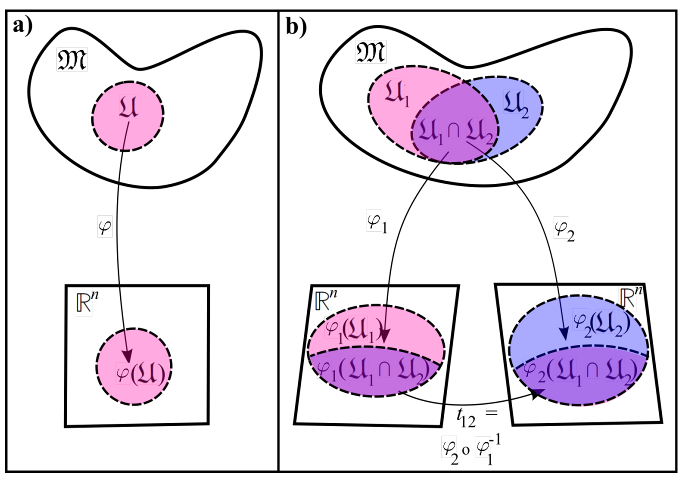

Figure 2: a) A chart. b) Overlapping charts and transition functions.

Definition 1 A *chart* alias *local coordinate system* for $\mathfrak{M}$ (Fig 2.a) is an injective map

$$\varphi \ : \ \mathfrak{U} \longrightarrow \varphi(\mathfrak{U}) \ \subset \ \mathbb{R}^n \tag{13}$$

for open $\mathfrak{U} \subseteq \mathfrak{M}$.

Remark 5 Each chart does not in general cover the whole manifold. One can get around this by considering a suitable collection of charts. Such a collection provides homeomorphisms which guarantee the locally Euclidean property.

Remark 6 When two charts overlap, it is furthermore beneficial to compare what that overlap looks like in each of the two charts. This is depicted in Fig 2.b), with

$$\varphi_1 \ : \ \mathfrak{U}_1 \longrightarrow \mathbb{R}^n \ , \quad \varphi_2 \ : \ \mathfrak{U}_2 \longrightarrow \mathbb{R}^n \tag{14}$$

which do indeed overlap:

$$\mathfrak{U}_1 \cup \mathfrak{U}_2 \neq 0 . \tag{15}$$

Definition 2 The composite map

$$t_{12} \ := \ \varphi_2 \ \circ \ \varphi_1^{-1} \tag{16}$$

which sends $\mathfrak{U}_1 \ \cup \ \mathfrak{U}_2$ to itself is a locally-defined map $\mathbb{R}^n \ \longrightarrow \ \mathbb{R}^n$ is termed a *transition function.*

Remark 7 This is a standard notion of local coordinate transformation.

Definition 3 A *topological atlas* $\mathfrak{A}$tlas is a collection of charts, such that between them they cover the whole manifold. If it additionally includes as many charts as possible, it is *maximal* [95].

4 LCHS and LCHP spaces

Remark 1 Local compactness is already a useful avenue to Analysis by being enough to support compactness techniques such as attaining bounds [135, 147] and compact generation [128].

Remark 2 CH spaces are similar to complete metric spaces [128].

Structure 1 Another class of furtherly 'analytically nice spaces' are the *LCHS spaces* [155, 161, 54]: locally-compact Hausdorff second-countable. By (12), manifolds are themselves LCHS. Yet nontrivially stratified LCHS spaces (Part III of Article 0) suffice to show that LCHS spaces are more general than manifolds. Furthermore [128, 54, 161]

$$\text{LCHS} \ \Rightarrow \ \text{P} \ : \ \ \text{paracompact} \ . \tag{17}$$

LCHS spaces additionally admit exhaustion by compact sets.

Structure 2 *LCHP spaces* [128] – locally compact Hausdorff paracompact – are another analytically well-behaved class. Clearly from (17),

$$\text{LCHS} \ \Rightarrow \ \text{LCHP} \ , \tag{18}$$

so LCHS generalizes LCHP.

Remark 5 LCHS and LCHP spaces are used in many further areas of Mathematics, such as the following.

i) Topological groups [66], of which Lie groups are a subcase (Sec 8.4), which are needed for Relationalism.

ii) Lie Groupoids [137], which are needed for Closure and for the study of foliations.

iii) Nice kinds of sheaves [165, 171, 132, 153], which are needed for Relationalism and possibly for Observables [205] as well.

iv) Random sets [63], which are needed for timeless and histories approaches [172].

Remark 6 This common mathematical base moreover gives a sense of *compatibility*. This is a significant feature in, firstly, composing Background Independence aspects, i.e. overcoming Problem of Time facet interferences. Secondly, it enters inter-relating different approaches to these, Shape Theory, or Foundations of Geometry.

5 Metric spaces

Definition 1 A *metric space* [135, 147] is a set $\mathfrak{X}$ equipped with a *metric function* $\text{Dist} \ : \ \ \mathfrak{X} \ \times \ \mathfrak{X} \ \longrightarrow \ \mathbb{R}$ satisfying the following properties.

i) $\text{Dist}(x, \, y) \ \geq \ 0 \ \ \forall \ x, \, y \ \in \ \mathfrak{X}$ (non-negativity).

ii) If $\text{Dist}(x, \, y) \ = \ 0$, then $x \ = \ y$ (separation).

iii) $\text{Dist}(x, \, y) \ = \ \text{Dist}(y, \, x)$ (symmetry).

iv) $\text{Dist}(x, \, y) \ \leq \ \text{Dist}(x, \, z) \ + \ \text{Dist}(z, \, y)$ (triangle inequality).

Remark 1 These properties encapsulate features of the Euclidean notion of distance, which are now applied to a wider range of settings. Dist – standing for 'distance between' – generalizes the Euclidean norm $|| \ ||$ of $\mathbb{R}^n$, and continues to support the concept of balls.

Remark 2 Metric spaces are thus Hausdorff (housing off using balls). They are also paracompact by Stone's Theorem [50], and so are HP.

Remark 3 Connectedness and path-connectedness also coincide for metric spaces.

Definition 2 In a metric space $\langle \mathfrak{X}, \mbox{Dist} \rangle$, a sequence $\{x_p\}_{p \, \in \, \mathbb{N}}$ is *Cauchy* if given any $\epsilon \ > \ 0$, $\exists \ n(\epsilon)$ such that $\mbox{Dist}(x_p, \, x_q) \ < \ \epsilon \ \forall \ p, \, q \ \geq \ n(\epsilon)$.

Definition 3 A metric space is *complete* [135] if every Cauchy sequence in it converges.

Remark 4 Completeness is *not* a topological property (Exercise: prove this by counterexaple)

Remark 5 Suppose the metric space possesses a '+' operation and $\mbox{Dist}(x \ + \ w, \, y \ + \ w) \ = \ \mbox{Dist}(x, \, y)$. Then Dist is said to be *translation invariant.*

Definition 4 A topological space is *metrizable* (M) if is is homeomorphic to a metric space.

$$\mbox{M} \ \Rightarrow \ \mbox{HP} \ . \tag{19}$$

in place of (17). Hausdorffness is clear, by use of balls to 'house off', whereas paracompactness is proven in e.g. [128].

Remark 6 Metrizability is necessary but not sufficient for Riemannian Geometry to be supported; piecewise LE is also required.

6 Differential Geometry

Differentiable manifolds are topological manifolds further equipped with differentiable structure.

6.1 Meshing conditions and differential atlases

Structure 1 Charts can furthermore allow for one to tap into the standard $\mathbb{R}^p \ \longrightarrow \ \mathbb{R}^q$ Calculus. This is supported by the Analysis that is rooted on manifolds being HS. This allows for manifolds to be equipped with *differentiable structure* [161, 37] in addition to topological structure. So far, the above allows for a local differentiable structure in each coordinate patch $\mathfrak{U}_i$.

Structure 2 One can a foritori also have a notion of global differentiable structure. This is via to the '*meshing condition*' on the coordinate patch overlaps (Fig 2.b). In this setting, the transition functions be interpreted as Jacobian matrices of derivatives for one local coordinate system $\boldsymbol{x}$ with respect to another $\bar{\boldsymbol{x}}$:

$$\mbox{J}^A{}_B \ = \ \frac{\partial x^A}{\partial \bar{x}^B} \ . \tag{20}$$

[We use capital Latin indices on the general manifold $\mathfrak{M}$.]

Definition 1 A *differentiable manifold* is a topological manifold that furthermore possesses a (global) differentiable structure.

Structure 3 The above topological manifold notion of atlas can also be equipped with differentiable structure. A collection of charts constituting an atlas is a more subtle question here. Our main interest here is however really in *equivalence classes of atlases.* Differentiable structure is often in practice approached using a convenient small atlas [94].

Remark 1 Having Calculus available throughout the manifold allows one moreover to study *differential equations.* These can in turn represent Physical Law in a conventional manner.

6.2 Partititions of unity and bump functions

Definition 1 Let $\{\mathfrak{U}_l\}_{l\,\in\,\mathcal{L}}$ be a cover of a topological space $\mathfrak{X}$ by open sets. A *partition of unity* dominated by this cover is a family of continuous functions

$$\phi_l \ : \ \mathfrak{X} \longrightarrow [0,1] \tag{21}$$

such that the following properties hold.

i) The support of ϕ_l,

$$\text{Supp}(\phi_l) \ \subseteq \ \mathfrak{U}_l \ \text{ for each } \ l \ \in \ \mathcal{L} \ . \tag{22}$$

[For f a function Supp(f) is the set of values y in f's domain such that $f(y) \ \neq \ 0$.]

ii) $\{\text{Supp}(\phi_l)\}_{l\,\in\,\mathcal{A}}$ is locally finite.

iii) The unity condition:

$$\sum_{n\,\in\,\mathcal{N}} \phi_n(x) \ = \ 1 \ \text{ for each } \ x \ \in \ X \tag{23}$$

for $\mathcal{N} \ := \ \{\mathcal{L}\,|\,\phi_l(x)$ is nonvanishing in a neighbourhood of $x\}$. [This $\mathcal{N}$ is finite by ii), by which our sum is indeed well-defined.]

Definition 2 A *bump function* is a smooth function that takes the value 0 outside of some region $\mathfrak{U}$ and 1 on another region $\mathfrak{V} \ \subset \ \mathfrak{U}$.

See Fig 3 for concrete examples on $\mathbb{R}$ and $\mathbb{R}^2$.

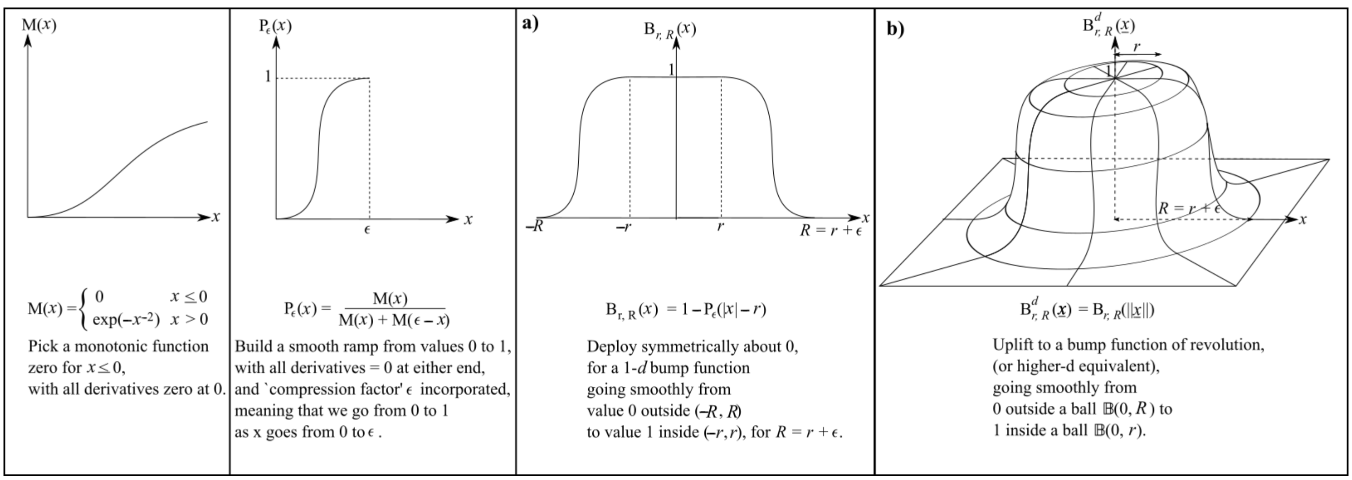

Figure 3: Bump functions on a) the interval $(-R, R)$ in 1-d, and b) the open ball $\mathbb{B}^d(0, R)$, using the same function as a radial profile for a surface of revolution bump function.

Remark 1 Given manifolds' paracompactness, the partitions of unity thus guaranteed also readily permits integral Calculus thereupon.

Remark 2 Partitions of unity carry over [161, 128, 138] to Hausdorff paracompact (HP) spaces.

6.3 Vectors and tensors

Structure 1 *Functions* on manifolds are defined as per Fig 4.a).

Structure 2 Let us next introduce *vectors* on manifolds [133, 95] as the tangents to *curves*, which are themselves mappings

$$\mathfrak{I} \longrightarrow \mathfrak{M} \tag{24}$$

for $\mathfrak{I} \ \subset \ \mathbb{R}$ a closed interval (as per Fig 4.b). The vectors themselves are maps ([95] and Fig 4.c)

$$\gamma'_{\text{p}} \ : \ \mathfrak{C}^\infty(\mathbb{R}) \ \longrightarrow \ \mathbb{R}$$

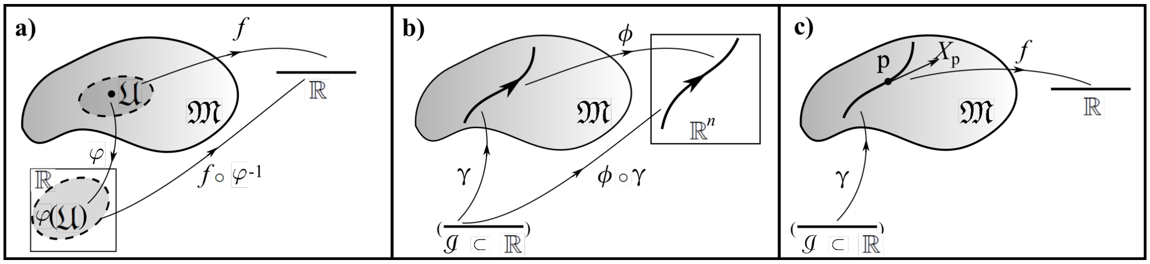

Figure 4: a) A function on a manifold.
b) The curve construct on a manifold.
c) A notion of vector on a manifold based on the curve construct and action on a fiducial function.

$$f \ : \quad \mapsto \quad \frac{\mathrm{d}}{\mathrm{d}\nu} f \circ \gamma \Big|_{\nu \,=\, 0} \quad . \tag{25}$$

Structure 3 The vectors thus defined at a given point p form the *tangent space* at p,

$$\mathfrak{T}_{\mathrm{p}}(\mathfrak{M}) \ . \tag{26}$$

Remark 1 One can furthermore compose curve and chart maps to make use of standard $\mathbb{R}^p \longrightarrow \mathbb{R}^q$ Calculus.

Remark 2 One can additionally straightforwardly show that all notions involved are chart-independent: a well-definedness criterion [95].

Remark 3 One can finally apply [37, 79, 95] the basic machinery of Linear Algebra to produce the following notions.

Structure 4 At a point p on the manifold, a *covector* [133, 95] is a linear map

$$\mathfrak{T}_{\mathrm{p}}(\mathfrak{M}) \ \longrightarrow \ \mathbb{R} \ . \tag{27}$$

Structure 5 The covectors at p form the *cotangent space*

$$\mathfrak{T}^*_{\mathrm{p}}(\mathfrak{M}) \ : \tag{28}$$

the Linear Algebra dual of the tangent space.

Structure 6 The *rank* $(k,\, l)$ *tensors* [95, 79] at p are multilinear maps

$$\times_{i\,=\,1}^{k} \mathfrak{T}^*_{\mathrm{p}}(\mathfrak{M}) \ \times \ \times_{j\,=\,1}^{l} \mathfrak{T}_{\mathrm{p}}(\mathfrak{M}) \ \longrightarrow \ \mathbb{R} \ . \tag{29}$$

Structure 7 A union of vectors, one at each p $\in$ $\mathfrak{M}$, constitutes a *vector field* over $\mathfrak{M}$; *tensor fields* are similarly defined. In terms of components, $(k,\, l)$-tensors transform according to

$$\mathrm{T}^{\bar{A}_1 \,...\, \bar{A}_k}{}_{\bar{B}_1 \,...\, \bar{B}_l} \ = \ \mathrm{L}^{\bar{A}_1}{}_{A_1} \,...\, \mathrm{L}^{\bar{A}_k}{}_{A_k} \mathrm{L}^{B_1}{}_{\bar{B}_1} \ ... \ \mathrm{L}^{B_l}{}_{\bar{B}_l} \mathrm{T}^{A_1 \,...\, A_k}{}_{B_1 \,...\, B_l} \tag{30}$$

in passing between plain and barred coordinate systems. Forms [36, 57, 74] are a subcase.

Structure 8 Let us use $\boldsymbol{\sigma}$ to denote further levels of geometrical structure: objects with given geometrically well-defined transformation laws for a wider range than just tensors, including also e.g. densities [79] and connections [37].

6.4 Diffeomorphisms

Definition 1 A *diffeomorphism* (see in particular [161]) is a bijection

$$\phi \ : \ \mathfrak{M} \ \longrightarrow \ \mathfrak{N} \tag{31}$$

between equidimensional manifolds $\mathfrak{M}$, $\mathfrak{N}$, that is $\mathfrak{C}^\infty$ (or e.g. $\mathfrak{C}^k$), and has an inverse map ϕ^{-1} of matching minimal standard of differentiability.

We often furthermore specialize to the $\mathfrak{M} \ = \ \mathfrak{N}$ subcase.

Structure 1 The map (31) induces a *push-forward* (Fig 5.b) on the tangent space

$$\phi_* \ : \ \mathfrak{T}_{\mathrm{p}}(\mathfrak{M}) \ \longrightarrow \ \mathfrak{T}_{\phi(\mathrm{p})}(\mathfrak{M}) \ . \tag{32}$$

This maps the tangent vector to a curve γ at p to that at the image of the curve $\phi(\gamma)$ at $\phi(\mathrm{p})$.

Structure 2 It also induces a *pull-back* (Fig 5.c) on the cotangent space

$$\phi^* \ : \ \mathfrak{T}^*_{\phi(\mathrm{p})}(\mathfrak{M}) \ \longrightarrow \ \mathfrak{T}^*_{\mathrm{p}}(\mathfrak{M}) \tag{33}$$

which maps 1-forms in the opposite direction.

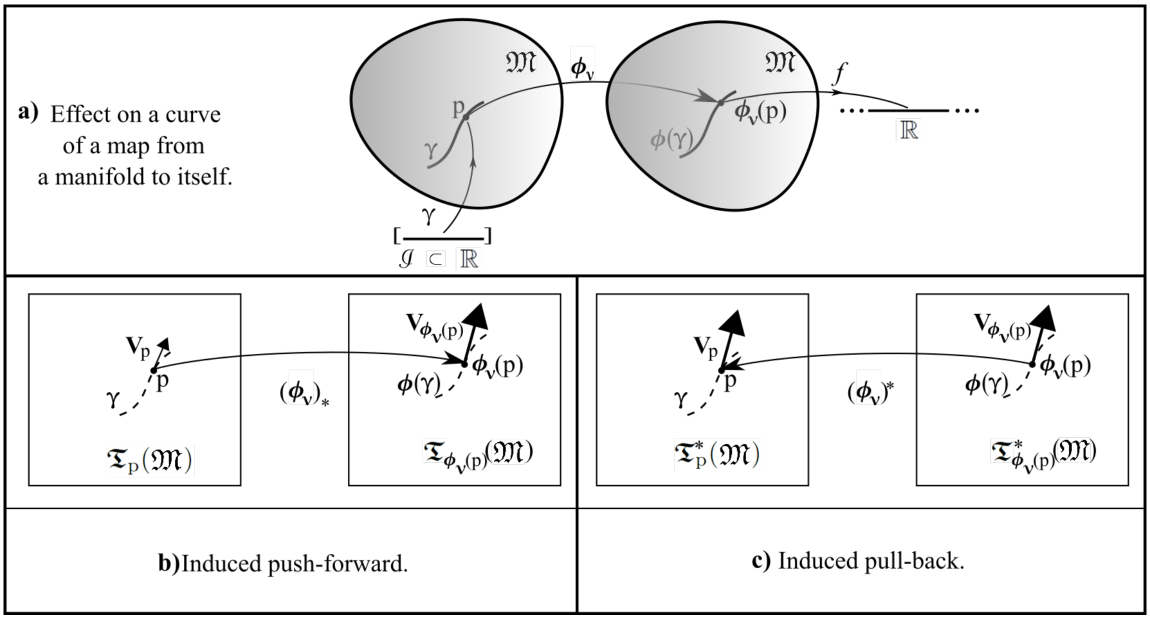

Figure 5: Curve, push and pull.

6.5 Subspaces of manifolds

Definition 1 A *topological embedding* $\mathcal{T} \ : \ \mathfrak{X} \ \longrightarrow \ \mathfrak{Y}$ is a homeomorphism onto its image $\mathcal{T}(\mathfrak{X}) \ \subseteq \ \mathfrak{Y}$ in the subspace topology $\mathfrak{T}_{\mathrm{S}}$.

Definition 2 A *smooth immersion* is a map $\mathcal{F} \ : \ \mathfrak{M} \ \longrightarrow \ \mathfrak{N}$ whose differential is injective at each point.

Definition 3 A *smooth embedding* $\mathcal{E} \ : \ \mathfrak{M} \ \longrightarrow \ \mathfrak{N}$ is a topological embedding that is also a smooth immersion [161].

Definition 4 An *embedded submanifold* [161] of a manifold $\mathfrak{M}$ is a subset $\mathfrak{s} \ \subseteq \ \mathfrak{M}$ that is

i) a (δ-)manifold in $\mathfrak{T}_{\mathrm{S}}$.

ii) Equipped with a differentiable structure with respect to which the inclusion map

$$\iota \ : \ \mathfrak{s} \ \longrightarrow \ \mathfrak{M} \tag{34}$$

is a smooth embedding.

6.6 Notions of derivative

Physics and Differential Geometry make plentiful use of *derivatives*. Such are not straightforward to set up in generally curved geometry. This is since the customary flat-space derivatives entail taking the limit of the difference between vectors at different points. In the context of differentiable manifolds, however, such vectors belong to different tangent spaces. In contrast to ℝ, where one can just move the vectors to the same point, there is no direct counterpart of this procedure on a general manifold (c.f. Fig 4.c). Not having the means of placing the two vectors in the same vector space (of which tangent spaces are an example), leaves the notion of subtraction needed for 'taking the difference' undefined.

Additionally, the usual partial derivation is in general undesirable in the Differential-Geometric setting. This is since it does not preserve *tensoriality*: the mapping of tensors to tensors. The sole exception is that partial derivation maps scalars to vectors. This is termed trivial action on scalars. We moreover expect our Differential Geometry notions of derivative to reduce to partial derivatives in this context. In fact, to construct a notion of derivative, it suffices for it to act trivially on scalars while also prescribing its action on vectors. This is because the Leibniz product rule then dictates how it acts on tensors of all other ranks [79, 95]. The *covariant derivative* [79, 95] is a well-known example of Differential-Geometric derivative, though not the principal one used in the current Series.

6.7 Flows and Integral curves

Definition 1 An *integral curve* (see e.g. [95]) of a vector field $\mathbf{V}$ in a manifold $\mathfrak{M}$ is a curve $\gamma(\nu)$ such that the tangent vector is $\mathbf{V}_{\mathrm{p}}$ at each p on γ (Fig 6.a).

Remark 1 These have local existence-and-uniqueness by standard ODE theory [161].

Definition 2 A set of complete integral curves corresponding to a non-vanishing vector field is called a *congruence*.

Remark 2 This 'fills' a manifold or region therein upon which the vector field is non-vanishing: the curves go through all points therein. A second interpretation of flow is as a congruence of integral curves.

Definition 3 A *flow* is a continuous action of $\mathbb{R}$ on a manifold $\mathfrak{M}$ [8, 73, 95, 159, 161, 193],

$$\theta \; : \; \mathbb{R} \times \mathfrak{M} \longrightarrow \mathfrak{M} \tag{35}$$

with

$$\theta_t \circ \theta_s(p) \; = \; \theta_{t\,+\,s}(p) \; . \tag{36}$$

and $\theta_0(p) \; = \; p$

Structure 1 For later reference, proceeding along two local congruences of integral curves in either order (Fig 6.b) produces, to leading order, the commutator

$$\mathbf{x}_{\mathrm{v}} \; - \; \mathbf{x}_{\mathrm{u}} \; = \; [\,\mathbf{X},\, \mathbf{Y}\,]\, \mathrm{d}\mu\, \mathrm{d}\nu \; + \; O(\mathrm{d}^3) \; . \tag{37}$$

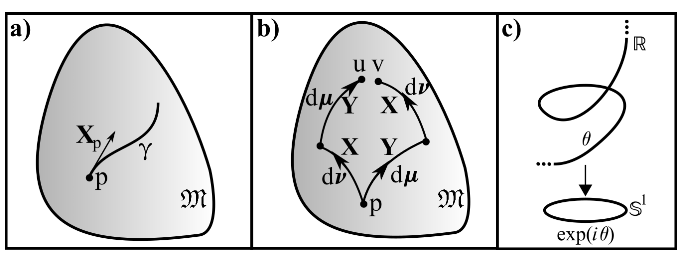

Figure 6: a) Integral curve on a manifold.
b) Commutator corresponding to proceeding along two local congruences of integral curves in either order.
c) Exponential map.

6.8 The exponential map

Definition 1 The *exponential map* is

$$\theta \;\longrightarrow\; \exp(i\,\theta) \; . \tag{38}$$

Remark 1 This is valid locally (Fig 6.c), i.e. for a sufficiently small interval of $\mathbb{R}$ ($<\; 2\,\pi$ in length).

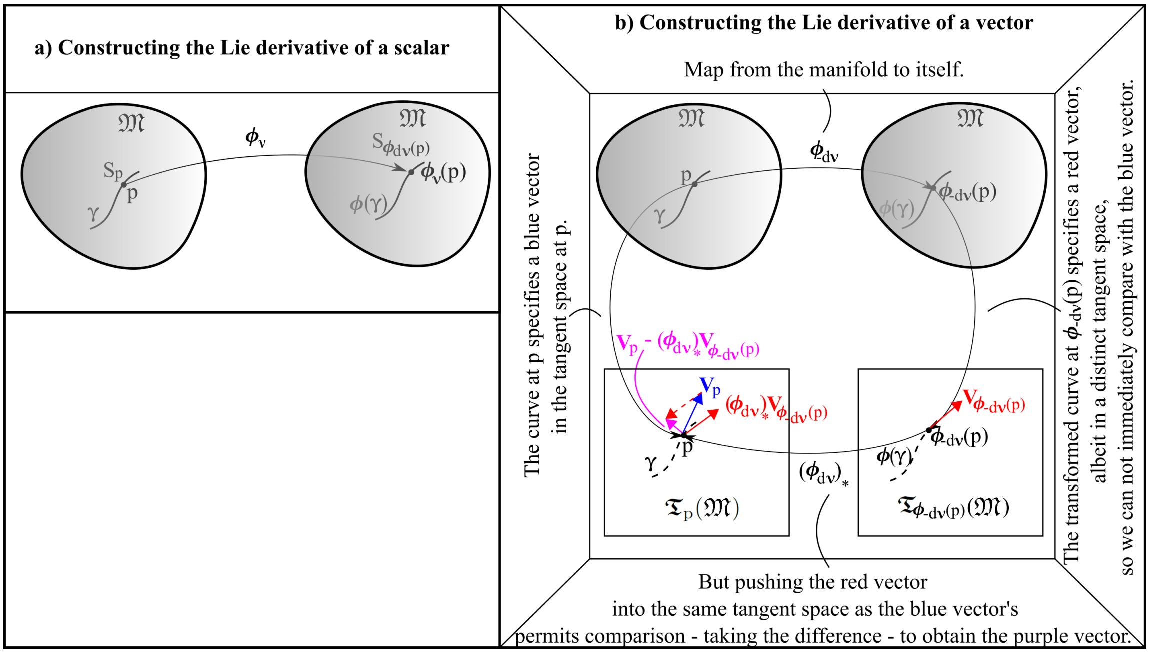

Figure 7: Decomposition of the first-principles construction of the Lie derivative of a) a scalar and b) a vector.

6.9 Lie derivatives

Structure 1 First-principles considerations using Fig 7.a)-b)'s constructs give the actions of the *Lie derivative* [12, 16, 18, 26, 52, 95, 161], on scalars and vectors as the first equalities below. For γ the integral curve of $\boldsymbol{\xi}$ through p inducing a 1-parameter group of transformations ϕ_ν with parameter ν, the Lie derivative with respect to $\boldsymbol{\xi}$ at p of a scalar S is

$$(\pounds_{\boldsymbol{\xi}}\mathrm{S})_{\mathrm{p}} \;=\; \lim_{\mathrm{d}\nu \to 0}\left(\frac{\mathrm{S}_{\phi_{\mathrm{d}\nu}(\mathrm{p})} \;-\; \mathrm{S}_{\mathrm{p}}}{\mathrm{d}\nu}\;.\right) \tag{39}$$

For a vector $\mathbf{V}$, it is

$$(\pounds_{\boldsymbol{\xi}}\mathbf{V})_{\mathrm{p}} \;=\; \lim_{\mathrm{d}\nu \to 0}\left(\frac{\mathbf{V}_{\mathrm{p}} - (\phi_{\mathrm{d}\nu})_*\mathbf{V}_{\phi_{-\mathrm{d}\nu}(\mathrm{p}))}}{\mathrm{d}\nu}\right)\;. \tag{40}$$

The 'Straightening Lemma' [161] (called 'useful Lemma' in [95] and already present in [8, 17, 26]) allows us to pass to the following'computational' forms in each case:

$$\pounds_{\boldsymbol{\xi}}\mathrm{S} \;=\; \boldsymbol{\xi}\cdot\boldsymbol{\partial}\,\mathrm{S}\;, \tag{41}$$

and

$$\pounds_{\boldsymbol{\xi}}\mathbf{V} \;=\; \boldsymbol{\xi}\cdot\boldsymbol{\partial}\,\mathbf{V} \;-\; \mathbf{V}\cdot\boldsymbol{\partial}\,\boldsymbol{\xi} \;=\; [\,\boldsymbol{\xi},\,\mathbf{V}\,]\;. \tag{42}$$

The latter gives the differential-geometric commutator. This can in turn be interpreted in terms of advancing along two different pairs of integral curves [95] as per Fig 6.b).

One can then readily obtain the Lie derivatives for tensors [26] of all the other ranks from these scalar and vector results by use of Leibniz's rule.

Remark 1 As a derivative, the Lie derivative is tensorial. It is also *directional* in the sense of involving an additional vector field $\boldsymbol{\xi}$ along which the tensors are dragged.

Remark 2 Lie derivatives generate the local infinitesimal version of the diffeomorphisms.

Remark 3 *Lie dragging* involves moving an object along a particular vector field's (or equivalently flow's) integral curves. This is by means of the Lie derivative with respect to the corresponding vectors.

7 More structured Differential Geometry

7.1 Levels of geometrical structure

Some further levels [26, 37, 52, 56] of geometrical structure $\boldsymbol{\sigma}$ supported by Differential Geometry are as follows.

Example 3 The affine structure $\boldsymbol{\Gamma}$ implied in the above mention of covariant derivatives [37].

Example 1 Metric structure, e.g. Riemannian metrics on space or semi-Riemannian metrics on spacetime $\mathbf{m}$ [87, 37].

Example 2 Similarity structure [30].

Example $\mathbf{t_C}$ Conformal structure [79, 37].

Example $\mathbf{t_P}$ Projective structure [37].

Example s Symplectic structure [170].

Remark 1 The next subsection gives a nice brief way of introducing these; the labelling used for these examples is explained in Sec 0.9.4.

7.2 Symmetry transformations

Structure 1 Let $\mathfrak{M}$ be a differentiable manifold and $\mathbf{T}$ be a well-defined object, say a tensor, thereupon. Then each diffeomorphism for which furthermore

$$\phi_*\mathbf{T} \ = \ \mathbf{T} \tag{43}$$

defines a *symmetry* of $\mathbf{T}$. We a fortiori cast geometrically significant objects (not necessarily tensors) in the role of $\mathbf{T}$ below. In particular, we consider each of the previous subsection's levels of geometrical structure $\boldsymbol{\sigma}$ in this role. These can be called $\boldsymbol{\sigma}$-omorphisms (though many of these have earlier, less systematic and yet still widespread names, of which several are given below).

Example 1 The *metromorphisms*, usually called *isomorphisms*, preserve $\mathbf{m}$.

Example 2 The *similaromorphisms*, usually called *similarities*, preserve $\mathbf{m}$ up to nonzero global-constant scalefactor k: $\mathbf{m} \ \longrightarrow \ \overline{\mathbf{m}} \ = \ k^2\,\mathbf{m}$.

Example $\mathbf{t_C}$ The *conformomorphisms* preserve $\mathbf{m}$ up to nowhere zero sufficiently-differentiable local-function scale-factor $\omega(\underline{y})$: $\mathbf{m} \ \longrightarrow \ \overline{\mathbf{m}} \ = \ \omega(\underline{y})^2\,\mathbf{m}$.

Example 3 The *affinomorphisms* preserve $\boldsymbol{\Gamma}$ in the obvious way.

Example $\mathbf{t_P}$ The *projectomorphisms* preserve $\boldsymbol{\Gamma}$ in a more general way involving choice of plane of projection [26].

Example s The *symplectomorphisms*, previously called *canonical transformations*, preserve the symplectic form $\boldsymbol{\eta}$.

7.3 From infinitesimal transformations to generalized Killing equations

Remark 1 We next consider in particular infinitesimal transformations

$$\underline{x} \ \longrightarrow \ \underline{x}' \ = \ \underline{x} \ + \ \epsilon\,\underline{\xi} \ . \tag{44}$$

For this to preserve our object $\boldsymbol{\sigma}$, substituting (44) into $\boldsymbol{\sigma}$'s transformation law and equating first-order terms in ϵ gives the following equation to first order.

Definition 1 The *generalized Killing equation (GKE)* [26, 52] is

$$\pounds_{\underline{\xi}}\boldsymbol{\sigma} \ = \ 0 \ . \tag{45}$$

Remark 1 This means that $\mathbf{T}$ is invariant under displacements along the integral curves of the corresponding vector field $\underline{\xi}$.

Remark 2 This is moreover to be regarded as a PDE to solve for $\underline{\xi}$: the *generalized Killing vectors (GKV).*

Example 1 In the most widely known case [10, 26, 52, 79, 136] of the (arbitrary-signature Riemannian) metric level of structure $\mathbf{m}$, the following arises. The infinitesimal transformations here take the form

$$\boldsymbol{\epsilon} \longrightarrow \boldsymbol{\epsilon} - \pounds_{\underline{\xi}}\mathbf{m} \, . \tag{46}$$

Definition 1 In this case, symmetries solve *Killing's equation*

$$\pounds_{\underline{\xi}}\mathbf{m} = 0 \, . \tag{47}$$

Remark 3 The above *defining form* of Killing's equation is in terms of the Lie derivative. It can however then be expanded out in first covariant and then partial derivatives as follows:

$$0 = 2\,\mathscr{D}_{(A}\xi_{B)} = \partial_A\xi_B + \partial_B\xi_A - 2\,\Gamma^C{}_{AB}\xi_C \, . \tag{48}$$

These forms are more useful for further Differential-Geometric and PDE considerations respectively.

Remark 4 Killing's equation is to be solved for the *Killing vectors* $\underline{\xi} = \underline{\boldsymbol{g}}$. These are individual infinitesimal isometries of $\langle \mathfrak{M}, \mathbf{m} \rangle$.

Example 2 For $\langle \mathfrak{M}, \mathbf{m}' \rangle$ a manifold equipped with metric structure modulo constant rescalings, $\mathbf{m}'$,

$$\pounds_{\underline{\xi}}\mathbf{m}' = 0 \tag{49}$$

is the *similarity Killing equation* [26, 136]. Its solutions $\underline{\xi}$ are *similarity Killing vectors*, consisting of isometries alongside 1 or 0 *proper similarities*, i.e. non-isometries.

Example $\mathbf{t_C}$ For $\langle \mathfrak{M}, \mathbf{m}^{\check{}} \, \rangle$ a manifold equipped with metric structure modulo local rescalings, $\mathbf{m}^{\check{}}$,

$$\pounds_{\underline{\xi}}\mathbf{m}^{\check{}} = 0 \tag{50}$$

is the *conformal Killing equation.* Its solutions $\underline{\xi}$ are *conformal Killing vectors*; for $d \geq 3$ these consist of similarities and special conformal transformations. For $\mathbb{R}^2$ or $\mathbb{C}$', however, we get an infinity of analytic functions by the flat conformal Killing equation collapsing in 2-d to the Cauchy–Riemann equations [74].

Example 3 For $\langle \mathfrak{M}, \boldsymbol{\Gamma} \rangle$ a manifold equipped with affine structure, $\boldsymbol{\Gamma}$

$$\pounds_{\underline{\xi}}\boldsymbol{\Gamma} = 0 \tag{51}$$

is the *affine Killing equation.* Its solutions $\underline{\xi}$ are *affine Killing vectors*, consisting of similarities alongside shears and squeezes alias Procrustes stretches [30].

Example $\mathbf{t_P}$ For $\left\langle \mathfrak{M}, \overset{P}{\boldsymbol{\Gamma}} \right\rangle$ a manifold equipped with projective structure [26],

$$\pounds_{\underline{\xi}}\overset{P}{\boldsymbol{\Gamma}} = 0 \tag{52}$$

is the *projective Killing equation.* Its solutions $\underline{\xi}$ are *projective Killing vectors*, consisting of affine transformations alongside 'special projective transformations'.

We turn out not to need to entertain the symplectic version.

7.4 Further theory of (generalized) Killing equations

Remark 1 GKEs are *homogeneous linear first-order systems* of PDEs.

Remark 2 GKEs are in general *over-determined* systems, lending themselves to having a lack of nontrivial solutions. Trivial solutions – the zero, or in some cases constant, vectors – are guaranteed by homogeneity. Only nontrivial solutions count as Killing vectors, however: a nontrivial kernel condition.

Remark 3 Whether over-determined PDE systems admit (nontrivial) solutions is characterized by whether they satisfy *integrability conditions*. For instance, Killing's Lemma [79] can be interpreted as an integrability condition for Killing's equation to be solvable.

Remark 4 The geometrically-generic $\langle \mathfrak{M}, \boldsymbol{\sigma} \rangle$ admits no nontrivial generalized Killing vectors. Many nontrivialities require at least two Killing vectors to be present (and non-commuting at that) [136]. These are *fairly highly* nongeneric manifold geometries $\langle \mathfrak{M}, \boldsymbol{\sigma} \rangle$. Each kind of generalized Killing equation has furthermore a manifold-dimension-dependent maximal number of independent generalized Killing vectors [26]. This is the most special, i.e. least geometrically-generic case. For Killing's equation itself, these are the maximally-symmetric spaces, which are required to be of constant curvature. (E.g. $\mathbb{R}^n$ and $\mathbb{S}^n$ are such.)

Global Issue -1.I Global identifications can lose (generalized) Killing vectors. This is already clear from the distinction between flat $\mathbb{R}^d$ and flat $\mathbb{T}^d$'s sets of Killing vectors.

8 Lie brackets, algebras and groups

8.1 Lie brackets

Definition 1 For $\mathfrak{g}$ for now a vector space, *Lie bracket* is a bilinear map

$$|[\ ,\ ]| \ : \ \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g} \tag{53}$$

that is antisymmetric

$$|[g\,, h]| \ = \ -|[h\,, g]| \ \ \forall \, g, h \ \in \ \mathfrak{g} \tag{54}$$

and obeys the *Jacobi identity*

$$0 \ = \ \mathbf{J}(g\,,\ h\,,\ k) \ := \ |[g\,, |[h\,, k\,]|\,]| \ + \ \text{cycles} \quad \forall \, g, h, k \ \in \ \mathfrak{g}\ . \tag{55}$$

Remark 1 Thus equipped, $\mathfrak{g}$ becomes a Lie algebra. The useful shorthand $\mathbf{J}$ here merits the name *Jacobiator*. This is a particular subcase of *associator*, i.e. measure of departure from associativity. Compare the notion of commutator as viewed as a measure of departure from commutativity. Aside from the statement that Lie algebras have zero Jacobiator, nonzero Jacobiator gives a measure of departure from having a Lie algebra.

8.2 Examples

Case 1 A geometry's symmetries carry Lie brackets structure.

Case 2 *Poisson brackets* $\{\ ,\ \}$ are Lie brackets. In their finite canonical realization, Poisson brackets of phase space functions $A(\boldsymbol{Q}, \boldsymbol{P})$ and $B(\boldsymbol{Q}, \boldsymbol{P})$ are given by

$$\{\, A, B\,\} \ := \ \frac{\partial A}{\partial \boldsymbol{Q}} \cdot \frac{\partial B}{\partial \boldsymbol{P}} \ - \ \frac{\partial B}{\partial \boldsymbol{Q}} \cdot \frac{\partial A}{\partial \boldsymbol{P}}\ . \tag{56}$$

For Field Theories, Poisson brackets of phase space functions $A(\mathbf{Q}, \mathbf{P})$ and $B(\mathbf{Q}, \mathbf{P})$ is given by

$$\{\, A, B\,\} \ := \ \int_{\boldsymbol{\Sigma}} \mathrm{d}\boldsymbol{\Sigma} \left\{ \frac{\delta A}{\delta \mathbf{Q}} \cdot \frac{\delta B}{\delta \mathbf{P}} \ - \ \frac{\delta A}{\delta \mathbf{P}} \cdot \frac{\delta B}{\delta \mathbf{Q}} \right\}\ . \tag{57}$$

In addition to (54, 55), Poisson brackets furthermore obey the *Leibniz* alias *product rule*,

$$\{\, A, B\,C\,\} \ = \ B\,\{\, A, C\,\} \ + \ \{\, A, B\,\}\,C\ , \tag{58}$$

by which they are also a *derivation*. Brackets which obey these three axioms can be viewed as Poisson algebras, even if they do not have the specific computational form of Poisson brackets. In this sense, quantum commutators are Poisson algebras. Indeed, one reason for Poisson brackets' significance is as a preliminary step toward quantization. Another is that they enable systematic treatment of constraints; both of these observations are due to Dirac [14, 23].

Remark 1 The fundamental Poisson bracket is

$$\{\, \underline{\mathbf{Q}}\,, \underline{\mathbf{P}}\,\} \ = \ \underline{\underline{\boldsymbol{\delta}}}\ . \tag{59}$$

$\mathbf{Q}$ and $\mathbf{P}$ are portmanteaux of the finite and field theoretic cases' configurations and momenta. $\boldsymbol{\delta}$ is the portmanteau of the finite Kronecker δ and the product of a field-species-wise such with a field-theoretic Dirac $\delta^{(d)}(\underline{x} \ - \ \underline{x}')$. This bracket being established for all the **Q** and **P** establishes the brackets of all once-differentiable quantities $A(\mathbf{Q}, \mathbf{P})$ as well. The entries into each slot of the Poisson brackets could also be functionals $\mathscr{A}$, $\mathscr{B}$ rather than just functions A, B.

Remark 2 If prephase space – the space of configurations and momenta – is equipped with the Poisson bracket, it becomes phase space, $\mathfrak{P}$hase. This can furthermore be rephrased in terms of equipping with a symplectic structure [67].

Remark 3 The Poisson bracket and phase space are already-TRi [199].

8.3 Lie algebras

Definition 1 A *Lie algebra* $\mathfrak{g}$ is a vector space equipped with a Lie bracket, such that the bracket of two elements in the vector space also lies in the vector space: *closure under the Lie algebra.*

Remark 1 As more general context, having an algebra amounts to having one more operation than vector spaces. Going beyond this gives bialgebras – $n = 4$ operations and so 2 in excess of a vector space – and multialgebras: $n \geq 3$ operations and so $n - 2$ in excess of a vector space.

Definition 2 As a vector space, a basis of elements can be picked therein. Such can be viewed as *generators* for our Lie algebra. As [199] already argued, we denote these by

$$\underset{\sqcup}{\mathcal{g}} \ , \quad \text{indexed by} \quad \mathsf{G} \ . \tag{60}$$

Remark 2 Given a basis of generators $\mathcal{g}$, computing

$$|[\, \mathcal{g}_G \, , \, \mathcal{g}_{G'} \,]| \; = \; G^{G''}{}_{GG'} \mathcal{g}_{G''} \ , \tag{61}$$

permits us to read off the *structure constants* $G^{G''}{}_{GG'}$ for the Lie algebra with respect to this basis. This amounts to formulating a Lie algebra as Lie brackets of generators which return solely linear combinations of generators. (These thus indeed lie within the original vector space.)

A coordinate-independent form for this is [188]

$$|[\, \underset{\sqcup}{\mathcal{g}} , \underset{\sqcup}{\mathcal{g}}{}' \,]| \; = \; \underset{\sqcup\sqcup\sqcup}{G} \cdot \underset{\sqcup}{\mathcal{g}}{}'' \ . \tag{62}$$

$\boldsymbol{G}$ are here *structure constant 3-arrays* or *trilinear maps*: a more succinct and coordinate-independent presentation. It readily follows from (61, 54) that the structure constants obey the antisymmetry property,

$$G^{G''}{}_{GG'} \; = \; - \, G^{G''}{}_{G'G} \ , \tag{63}$$

and, from the Jacobi identity [4], the homogeneous-quadratic restriction

$$G^{G}{}_{[G'G''}G^{G'''}{}_{G''''\,]G} \; = \; 0 \ . \tag{64}$$

Remark 3 In the canonical setting, the algebras can be taken to be Poisson algebras; see e.g. [108, 162] for introductions to these. Also, at least within a restricted range of formulations of a restricted range of theories, the generators can be taken to be constraints.

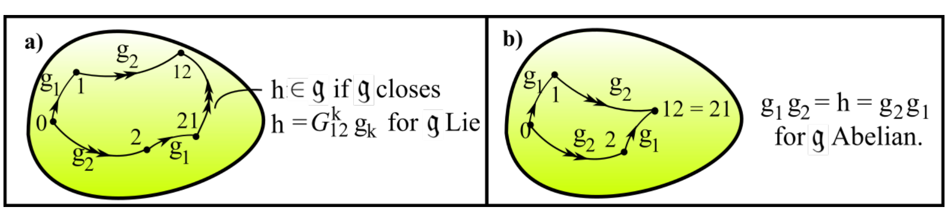

Figure 8: a) An algebra's commutator. This compares applying two transformations g_1, g_2 in either order to a common initial object 0. b) The even more straightforward commuting subcase, for which the final objects 12 and 21 coincide as well. Many instances of a) and b) occur in ALRoPoT, as picked out among the Series's figures by being depicted on lime-green egg-shaped spaces.

8.4 Lie groups

Definition 1 A *topological group* G [61, 42, 167] is a group that is concurrently a topological manifold. Additionally its composition and inverse operations are matchingly continuous.

Definition 2 A *Lie group* G [61, 42, 167] is a group that is concurrently differentiable manifolds. Additionally its composition and inverse operations are matchingly differentiable.

Remark 1 Finite transformations form Lie groups. Large transformations are included among these in the mixed case. See Sec 8.7 for examples.

Remark 2 In a further interpretation of flows, the 1-parameter subgroup's generator for a flow $\gamma(\nu)$ is moreover the tangent vector $\gamma'(0)$.

8.5 From Lie groups to Lie algebras

Looking at an infinitesimal neighbourhood of the identity of a Lie group G gives the corresponding Lie algebra $\mathfrak{g}$. I.e. the modern version of 'infinitesimal transformation group' G_{T}. Lie algebras and Lie groups have subsequently become a large field of study. See [150] if in need of an undergraduate-level introduction, [61] for a graduate-school Physics text, or [33, 42, 27, 139, 167] for more advanced texts.

Structure 1 We can view the line $\mathbb{R}$ realized as $i\,\theta$ in $\mathbb{C}$ as (Fig 8.c) the tangent to

$$\{z \in \mathbb{C}\,|\,|z| = 1\} = \mathbb{S}^1 = U(1) \ . \tag{65}$$

This corresponds to the usual exponential map (38) underpinning the above use of 'gives'.

Structure 2 A tangent space interpretation continues to apply [19] in the case of higher-dimensional Lie groups G (Fig 8.d). By this, the corresponding Lie algebras $\mathfrak{g}$ can be viewed as 'tangent space' near G's identity element. This can be set up by considering 1-parameter subgroups [17, 26, 95, 161] (a reinterpretation of integral curves) one at a time.

Remark 1 The Lie algebra is, on the one hand, more straightforward to handle than a Lie group. This is since it is a linear space (a vector space with an extra bracket product).

Remark 2 On the other hand, remarkably little information is lost in passing from a Lie group to the corresponding Lie algebra. For instance, the representations of $\mathfrak{g}$ determine those of G.

Remark 3 The Lie bracket arises from considering Lie group structure in the vicinity of the identity. This can can be seen for instance from restricting thereto the differentiation of conjugation,

$$\frac{\mathrm{d}}{\mathrm{d}\nu}\left(g\,H\,g^{-1}\right)\Big|_{\nu\,=\,0} = [\,G, H\,] \ , \tag{66}$$

for $G = g'(0)$ and using $g(0) = 1$.

8.6 From Lie algebras to Lie groups

Structure 1 Working in the opposite direction, the globalization move is

$$\text{solving } \exp(X) \ \exp(Y) = \exp(Z) \text{ for } Z \text{ when } X, Y \text{ do not necessarily commute .} \tag{67}$$

To connect with the literature, this is often also attributed to Baker for having posed it [150]. In fact, Hamilton [6] already intuited the first correction to be

$$\tfrac{1}{2}\,[\,X, Y\,] \ . \tag{68}$$

Schur [9] worked on subsequent correction terms. One remaining issue at that point was explicit formulae for these terms. Another was proving that these depend on X and Y through successive uses of commutators alone:

$$Z = Z(\,[\ ,\ ]\ \text{ alone}\,) \ . \tag{69}$$

It was Hausdorff [11] who first produced a complete proof of this, so let us term this *Hausdorff's Lie-Globalization Theorem.* Dynkin [20] subsequently tidied matters up by producing closed-form expressions for the general-n terms manifestly in terms of commutators alone. For junior readers who would like to see a proof they can readily follow, Stillwell's account [150] of Eichler's proof [49] – induction using just basic algebra – is recommended.

Remark 1 Hausdorff's Lie-globalization Theorem signifies that the global *group commutator*

$$g\,h\,g^{-1}\,h^{-1} \tag{70}$$

for a Lie group is totally controlled by the local Lie algebra's commutator (53). This is how the above-mentioned remarkably little loss of information comes to be.

Structure 2 The Jacobi identity carries over to the *Hall–Witt* alias *three subgroups identity* of Lie groups [42]

$$\mathbf{HW}(g, h, k) \;=\; [\,g, h^{-1}, k\,]^{h} \times (\text{cycles}) \;=\; 1 \;, \tag{71}$$

where the exponent h denotes conjugation by h. 'HW' stands here for '*HallWittator*' (c.f. commutator and especially Jacobiator).

Remark 2 Only the presence of other sectors supported by discrete transformations – giving further components not connected to the identity – is lost in the passage from Lie groups to Lie algebras. It is then not remembered in the reverse passage (Sec 8.5's local 'looking'). This amounts to loss of a small amount of information of a topological nature.

Global Problem -1.II The infinitely generated case can be blocked at this point. It is confirmed that this specifically occurs for unsplit diffeomorphisms, thus including the spacetime perspective on GR.

Remark 3 Globalizing passage from Lie algebras to Lie groups is afforded, as are some cases of such a passage from Lie algebroids to Lie groupoids [137].

8.7 Elementary examples

Example A The simple finitely-generated Lie groups are the series $O(n)$ (orthogonal), $U(n)$ (unitary) and $Sp(2n)$ (symplectic), alongside a very small number of exceptional Lie groups [139].

Remark 1 We furthermore need the following simple accidental relation and manifold result.

$$SO(2) \;=\; U(1) \;=\; \mathbb{S}^1 \tag{72}$$

where the last equality is as a manifold.

Example B The diffeomorphisms constitute a given differentiable manifold's automorphisms, forming the group

$$Aut(\mathfrak{M}) \;=\; Diff(\mathfrak{M}) \;. \tag{73}$$

This is an infinite-d example of Lie group. Infinitesimally, the corresponding Lie algebra $diff$ is infinitely-generated by the Lie derivatives with respect to arbitrary vector fields. This includes firstly the spatial diffeomorphisms

$$G \;=\; Diff(\boldsymbol{\Sigma}) \;: \tag{74}$$

for GR's configurations, and secondly, the spacetime diffeomorphisms

$$G \;=\; Diff(\mathfrak{m}) \;. \tag{75}$$

Examples C The GKVs solving a particular $\langle\,\mathfrak{M}, \boldsymbol{\sigma}\,\rangle$'s GKE moreover close [56] as a Lie algebra,

$$|[\,\underline{\xi}(\mathbf{x}), \underline{\xi}(\mathbf{x})\,]| \;=\; \underline{\underline{\underline{Z}}} \cdot \underline{\xi}(\mathbf{x}) \;. \tag{76}$$

$\boldsymbol{Z}$ are here the corresponding *structure constants*. As a Lie algebra, this corresponds to the continuous connected component of the identity part of the automorphism group,

$$Aut(\mathfrak{M}, \boldsymbol{\sigma}) \;, \tag{77}$$

which we denote by

$$aut(\mathfrak{M}, \boldsymbol{\sigma}) \;. \tag{78}$$

These are [17, 26] subalgebras

$$aut(\mathfrak{M}, \boldsymbol{\sigma}) \ \leq \ diff(\mathfrak{M}) \ . \tag{79}$$

They are furthermore usually finitely-generated, in the sense of usually being a finite count of independent generators.

Subexample 1 The continuous isometries alias Riemmetromorphisms – the totality of Killing vectors – form the *isometry group*

$$RiemMet(\mathfrak{M}, \mathbf{m}) \ = \ Isom(\mathfrak{M}, \mathbf{m}) \ . \tag{80}$$

In flat space,

$$Isom(\mathbb{R}^d) \ = \ Eucl(d) \ = \ Tr(d) \ \times \ Rot(d) \ = \ \mathbb{R}^d \ \times \ SO(d) \tag{81}$$

for translations $Tr(d)$, rotations $Rot(d)$ and semidirect product of groups $\times$ [126].

Subexample 2 The continuous similarities alias similaromorphisms – the totality of the similarity Killing vectors – form the *similarity group*

$$Sim\,(\mathfrak{M}, \mathbf{m}') \ ; \tag{82}$$

for $\langle\, \mathfrak{M}, \mathbf{m}\, \rangle$ possessing no proper similarities,

$$Sim\,(\mathfrak{M}, \mathbf{m}') \ = \ Isom\,(\mathfrak{M}, \mathbf{m}) \ . \tag{83}$$

In flat space,

$$Sim(d) \ = \ Tr(d) \ \times \ (Rot(d) \ \times \ Dil) \ = \ \mathbb{R}^d \ \times \ (SO(d) \ \times \ \mathbb{R}_+) \ , \tag{84}$$

for dilations Dil and direct product of groups $\times$.

Subexample $\mathbf{t_C}$ The continuous conformomorphisms – the totality of the conformal Killing vectors – form the *conformal group*

$$Conf(\mathfrak{M}, \check{\mathbf{m}}) \ . \tag{85}$$

$Conf(\mathbb{R}^2)$ provides a counterexample to these necessarily being finitely-generated.

Subexample 3 The continuous affinomorphisms – the totality of the affine Killing vectors – form the *affine group*

$$Aff(\mathfrak{M}, \boldsymbol{\Gamma}) \ . \tag{86}$$

Subexample $\mathbf{t_P}$ The projectomorphisms – the totality of the projective Killing vectors – form the *projective group*

$$Proj\left(\mathfrak{M}, \overset{P}{\boldsymbol{\Gamma}}\right) \ . \tag{87}$$

8.8 Clearer nomenclature

A truer name for 'generalized Killing equation' is *continuous automorphism Lie algebra finding equation* (CALAFE) [179, 200]. Such are solved to obtain the *continuous automorphism Lie algebra generators* (CALAG), hitherto referred to as 'generalized Killing vectors'. We shall see in Article 0 that these form posets (and bounded lattices, under more stringent circumstances).

8.9 Finite, continuous and mixed groups

We model this by finite groups and then by taking discrete quotients[3]

Combining these two procedures lets us furthermore deal with groups that are a mixture of continuous and discrete transformations. We need at least Lie groups to model this.

The notion of connected components is useful in this regard. Restricting attention to the component that is (path-)connected to the identity, mixed groups reduce to purely-continuous groups.

Example $O(1) \ = \ \mathbb{Z}_2$ models e.g. 1-d for reflections and for 1-d rotations (180-degree rotations and inversions more generally). This has two components. Nontrivial Lie groups G over $\mathbb{R}$ or $\mathbb{C}$ can be continuous or a mixture.

Global Issue -1.III $Aut(\mathfrak{M}, \boldsymbol{\sigma})$ has further global issues: whether or not to include large automorphisms. Such include reflections in the case of flat Geometry or Mechanics thereupon, and large diffeomorphisms in the case of GR [119].

[3]Finite and discrete are moreover not the same as adjectives applied to groups, e.g. $\mathbb{Z}$ is infinite and yet discrete

9 Algebraic Topology and Differential Topology

9.1 Homotopy

Definition 1 Maps f and g from $\mathfrak{X}$ to $\mathfrak{Y}$ are *homotopic* if there is a continuously changing 1-parameter family f_t , $t \in [0, 1]$ of maps that interpolates between f and g, i.e. such that $f_0 = f$ and $f_1 = g$.

Remark 1 The current work just [76, 128] touches upon this at the level of probing with loops, as encoded by the (fundamental group) $=$ (first homotopy group).

Definition 2 A space is *contractible* if it is homotopy-equivalent to a single point [76].

9.2 Simplicial complexes

Definition 1 A *k-simplex* σ_k is the k-dimensional generalization generalization of the point, line segment, triangle, tetrahaedron... formed by 0, 1, 2, 3, ... , k points. This comes with an obvious arbitrary-d generalization of the notion of face.

Definition 2 A *simplicial complex* $\boldsymbol{\mathfrak{K}}$ is a collection of simplices that 'fit together nicely' in the following sense [76].

i) If $\sigma_k \in \boldsymbol{\mathfrak{K}}$, then so are all of its faces.

ii) Simplices can solely intersect on a common face.

9.3 Homology

Definition 1 Given a topological space $\mathfrak{X}$, a *chain complex* C_n of Abelian groups can be associated with it. Successive members of this complex are to be related by *boundary operator* homomorphisms

$$\partial_n \ : \ C_n \longrightarrow C_{n-1} \ . \tag{88}$$

Definition 2 *Boundaries* are elements of

$$B_n(\mathfrak{X}) := \text{Im}(\partial_{n+1}) \ . \tag{89}$$

Definition 3 *Cycles* are elements of

$$Z_n(\mathfrak{X}) := \text{Ker}(\partial_n) \ . \tag{90}$$

Remark 1

$$\partial_n \partial_{n+1} = 0 \tag{91}$$

holds: 'the boundary of a boundary is zero'.

Remark 2 Since the C_n are Abelian groups, all their subgroups are normal. The following quotient group is thus well-defined.

Definition 4 The nth *homology group* [76, 131] is

$$H_n(\mathfrak{X}) := \frac{Z_n(\mathfrak{X})}{B_n(\mathfrak{X})} \ . \tag{92}$$

Remark 3 This quantifies the extent to which each image is a subset of the subsequent kernel (Fig 9.c). Homology is a means of constructing topological invariants from cellular arrays.

Example 1 Simplicial homology is the most obvious homology corresponding to a simplicial complex $\boldsymbol{\mathfrak{K}}$.

9.4 Differential Topology

Cohomology [117, 46, 131] ensues instead in applications in which the maps are taken to go in the opposite direction (9.c). Now one considers *cochains*

$$\delta_n \ : \ C_n \longrightarrow C_{n+1} \ , \tag{93}$$

coboundaries are elements of

$$\text{Im}(\delta_{n-1}) := B^n(\mathfrak{X}) \ , \tag{94}$$

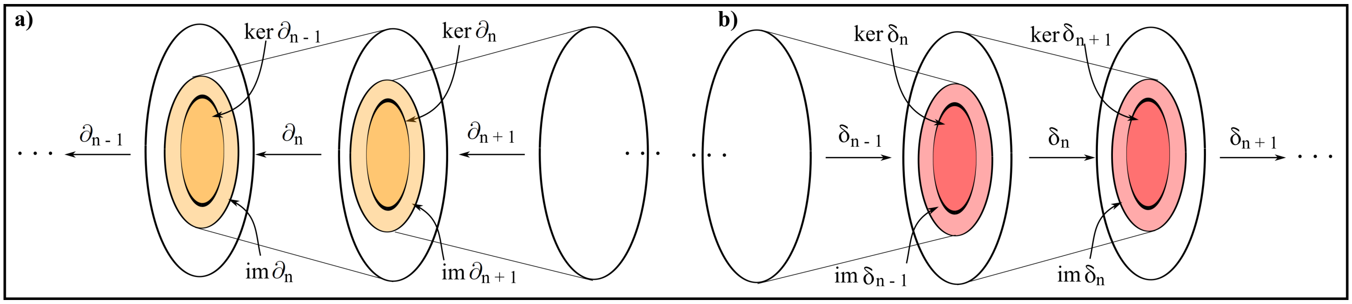

Figure 9: Spaces, maps, images and kernels in a) homology and b) cohomology.

and *cocycles* are elements of

$$\text{Ker}(\delta_n) \ := \ Z^n(\mathfrak{X}) \ . \tag{95}$$

Finally, the quotient

$$H^n(\mathfrak{X}) \ := \ \frac{Z^n(\mathfrak{X})}{B^n(\mathfrak{X})} \tag{96}$$

is the nth *cohomology group*.

Example 1) The most commonly encountered type of cohomology evoked in Theoretical Physics is *de Rham cohomology* [117], which is for a smooth differentiable manifold in the role of $\mathfrak{X}$. Here d is just the exterior derivative, so this example concerns closed and exact differential forms. I.e. respectively forms f for which df = 0 and those which can be written as $f = \text{d}g$. $\text{d}^2 = 0$ here means that all exact forms are closed.

Remark 1 (Co)homology is categorical, by which a wide diversity of mathematical structures possess such. It has more structure than homology. At least at a more basic level, this is the cohomological cup product. This gives further ways of computing out spaces' cohomology groups.

10 Products and bundles

10.1 Productive topological properties

Structure 1 The product of topological spaces can be equipped with the product topology τ_{P} is the set of all unions of open boxes made from 'box sides' that are open in each factor space's topology [128].

Remark 1 The simpler finite-product case of this suffices for the purposes of this Series.

Definition 2 A topological property is *productive* if it is preserved under products of topological spaces.

Proposition 1 Hausdorffness, second-countability, and local-Euclideanness are productive.

Corollary 1 Manifoldness is productive.

Proposition 2 Local compactness is productive but paracompactness is not [128].

Corollary 2 LCHS is productive.

10.2 General and fibre bundles

Some physical modelling involves not just $\mathfrak{M}$ but fibre bundles thereover as well. Indeed, such fibre bundles serve to encode some of $\mathfrak{M}$'s global properties. This involves a bigger topological space, and on some occasions a bigger smooth structure.

Structure 1 Consider first topological spaces which project down continuously onto lower-d topological spaces, $\pi \ : \ \mathfrak{E} \longrightarrow \mathfrak{B}$. Such can be viewed in reverse as higher-d bundle *total spaces* $\mathfrak{E}$. Each of these is built over a lower-d *base space* $\mathfrak{B}$; π is a *projection map*. This is the *general bundle* notion; see Fig 10.a)–c) and [122] for an outline and [107] for an advanced account.

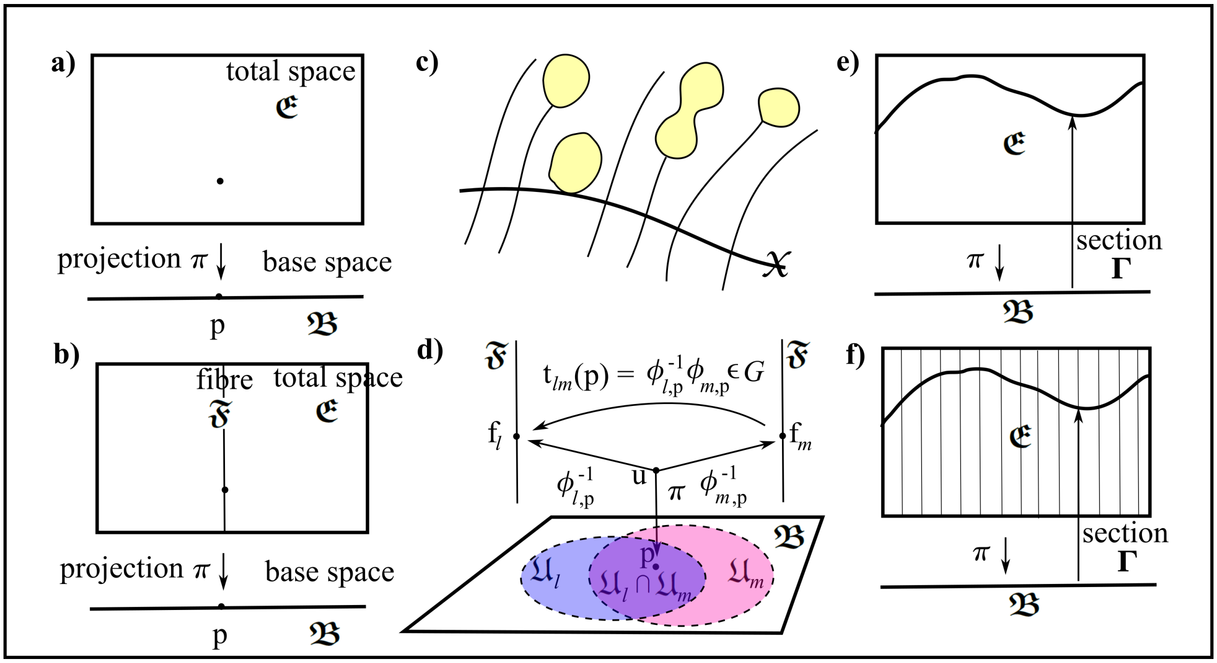

Figure 10: a) General bundle at the topological level.
b) For a fibre bundle, the total space $\mathfrak{E}$ consists of identical disjoint copies of a fibre manifold $\mathfrak{F}$, one at each point in the base space $\mathfrak{B}$.
c) The general bundle structure is a broader concept via permitting distinct manifolds to be attached to different parts of the base space.
d) The fibre bundle additionally involves transition functions (c.f. manifolds) and a structure group G. e) and f) illustrate the further key notion of (cross-)sections for general and fibre bundles.

Structure 2 Suppose that one further introduces a local product structure. Herein, the total space consists of identical copies of a *fibre space*, alias just *fibre*) $\mathfrak{F}$. Fibres are ab initio to be regarded as further topological manifolds. They might subsequently be accorded such as differential and metric structure as well. Then one has a topological-level *fibre bundle*. see Fig 10.b), d) and [122, 94, 74, 77] for introductions and [107, 145] for advanced accounts.

From a global perspective, fibre bundles are moreover typically 'twisted versions' of product spaces. In contrast, global product spaces are the trivial cases of fibre bundles. Figs 10.e)-f) are simple examples of these respectively. The inverse image $\pi^{-1}(\mathrm{p})$ is the fibre $\mathfrak{F}_\mathrm{p}$ at p (Fig 10.b). That all fibres are the same is mathematically encoded by $\mathfrak{F}_\mathrm{p}$ being homeomorphic to $\mathfrak{F}$. Extra isomorphic equivalence is included if and when required.

In fact, fibre bundles are furthermore taken to have a *structure group* G acting upon the fibres $\mathfrak{F}$, by which they are denoted $\langle\mathfrak{E}, \pi, \mathfrak{B}, \mathfrak{F}, G\rangle$.

Example 1) For the significant case of a *principal fibre bundle* $\mathfrak{p}(\mathfrak{M}, G)$ alias *G-bundle*, G and $\mathfrak{F}$ coincide. Thus now G just acts on itself.

Taking an open cover $\{\mathfrak{U}_l\}_{l \,\in\, \mathcal{L}}$ of $\mathfrak{B}$, each $\mathfrak{U}_l$ is equipped with a homeomorphism

$$\phi_l \ : \ \mathfrak{U}_l \ \times \ \mathfrak{F} \ \longrightarrow \ \pi^{-1}(\mathfrak{U}_l) \ . \tag{97}$$

This is such that $\pi\,\phi_l$ sends (p, f) – for f a point on $\mathfrak{F}_\mathrm{p}$ – down to p. ϕ_l is termed a *local trivialization*, since its inverse maps $\pi^{-1}(\mathfrak{U}_l)$ onto $\mathfrak{U}_l \ \times \ \mathfrak{F}$: a trivial product structure. *Local* triviality refers to *globally* nontrivial fibre bundles encoding information beyond that in the globally trivial product space. Our ongoing definition of fibre bundle can furthermore be shown to be independent of the choice of covering. We thus do not include this paragraph as part of the definition.

As a final structural input, consider $\mathfrak{U}_l$ and $\mathfrak{U}_m$. I.e. an arbitrarily chosen pair of open sets except that nontrivial overlap between them is guaranteed, $\mathfrak{U}_l \ \cup \ \mathfrak{U}_m \ \neq \ \emptyset$. Somewhat simplify the notation according to

$$\phi_{\mathsf{A}}(\mathrm{p}, \mathrm{f}) \ = \ \phi_{\mathsf{A},\,\mathrm{p}}(\mathrm{f}) \ , \quad \phi_{\mathsf{A},\,\mathrm{p}} \tag{98}$$

for the homeomorphism sending $\mathfrak{F}_\mathrm{p}$ to $\mathfrak{F}$. The *transition functions*

$$t_{im}(\mathrm{p}) \ := \ \phi^{-1}_{l,\,\mathrm{p}}\phi_{m,\,\mathrm{p}} \ : \ \mathfrak{F} \ \longrightarrow \ \mathfrak{F} \tag{99}$$

corresponding to the overlap region as per Fig 10.b) are then elements of G. ϕ_l and ϕ_m are moreover related by a continuous map

$$t_{lm} \ : \ \mathfrak{U}_l \cup \mathfrak{U}_m \longrightarrow G \tag{100}$$

according to

$$\phi_l(\text{p, f}) \ = \ \phi_l(\text{p}, t_{lm}(\text{p}^{\text{f}})) \tag{101}$$

as per Fig 10.d). N.B. the parallels between this and the meshing condition for topological manifolds of Fig 2.b).

Definition 1 Topological *fibre bundle morphisms* are continuous maps between fibre bundles, $\langle \mathfrak{E}_i, \pi_i, \mathfrak{B}_i, \mathfrak{F}_i, G_i \rangle$, $i = 1, 2$ that map each fibre $\mathfrak{F}_1$ onto a fibre $\mathfrak{F}_2$.

Definition 2 A *(cross-)section* of a topological fibre bundle is a continuous map in the opposite direction to π, $\Gamma : \mathfrak{B} \longrightarrow \mathfrak{E}$ such that $\pi(\Gamma(x)) = x \ \forall \ x \in \mathfrak{B}$.

Global obstruction -1.IV The section is to cut each fibre precisely once. Moreover, fibre bundles do not in general possess a global section.

The fibre bundle structure – and of the corresponding morphisms and sections – can furthermore be elevated to the differentiable manifold level of structure. These now have smooth maps in place of continuous maps and diffeomorphisms in place of homeomorphisms.

Example 2) Tangent space, cotangent space and the general space of tensors can also be thought of as *tangent*, *cotangent* and *tensor bundles* respectively.

Example 3) Gauge Theory can be formulated in terms of fibre bundles (using both principal and more general associated fibre bundles); see e.g. [122, 77, 94, 71, 74] for details. This requires considering *connections* on fibre bundles. One can now indeed interpret Gauge Theory's potential **A** as a connection. Notions of parallel transport and of covariant derivative **D** follow. Finally, the field strength 2-form **F** corresponding to the potential 1-form **A** indeed plays the corresponding role of curvature.

The above references and [97] show how the Gribov effect (Sec 0.14.3) monopoles (Sec 0.14.4), anomalies (Secs 1.4 and 1.7), and BRST Quantization (Sec 1.6-7) afford lucid treatment in terms of fibre bundles. So can spinors, in either flat [112] or curved [79, 95] spaces.

Remark 1 Let us finally return to the unqualified notion of bundle. Such can be viewed as a generalization in which there need no longer be a notion of identical fibre at each point of the base space. This is useful since assuming such identical fibres throughout turns out to be a significantly restrictive assumption in some kinds of modelling that Theoretical Physics requires.

Part II

Types of Global Background Independence

11 Classifying globality by i) qualitative extent of Locality

Global issues are legion, so a number of conceptual and technical classifications are needed to clearly specify what each is.

Classification by qualitative extent of locality Locality being an opposite notion to globality, extent of locality is also a qualification of globality.

(a) Not localized anywhere on a space at all.

(b) Localized at a point.

(c) Localized in a neighourhood of a point

(c^c) Localized almost anywhere in a space (complement of a neighbourhood of a point).

(b^c) Localized anywhere but at a point.

(a^c) Localized anywhere on the space (fully global).

Remark 1 [b] [i.e. (b) treated jointly with (b^c)] extends to a finite number of points, or even to larger cardinalities of points that are in some sense isolated from each other. (b^c) and these extensions of it are often referred to as *punctured* versions of the whole space. Potential Theory suffices to see that punctured and unpunctured are clearly well capable of being mathematically and physically distinct. The maximum extent that a single chart on $\mathbb{S}^2$ can cover also has a puncture. Puncturing a space in general alters its topological properties; Homotopy Theory [76, 128, 131] involves many well-known examples of this.

Remark 2 (c^c) is sometimes referred to as *quasilocality*, as in for instance 'quasilocal energy' [140].

Remark 3 [c] is capable of further infinitesimal- versus extended-neighbourhood distinction for metric spaces (Fig 11.a), but not in general for topological spaces.

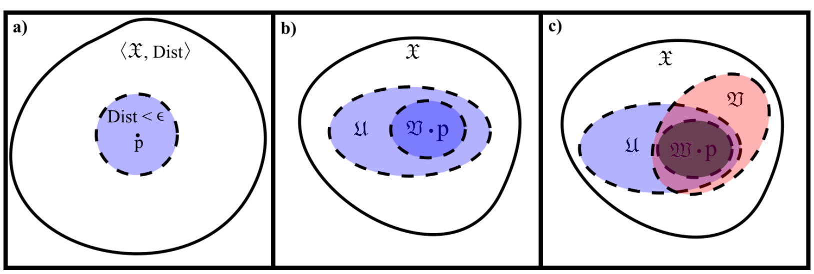

Figure 11: a) Infinitesimal neighbourhood in a metric space.
b) Localization: restrict from $\mathfrak{U}$ to $\mathfrak{V} \subset \mathfrak{U}$.
c) Localization to lie within an intersection of domains or 'neighbourhoods' $\mathfrak{W} \subset \mathfrak{U} \cap \mathfrak{V}$.

Remark 4 General topological spaces moreover retain intersectional information.

i) One can distinguish for instance between open sets and strict subsets thereof (Fig 11.b).

ii) One can also pass from two intersecting open sets $\mathfrak{U}$, $\mathfrak{V}$ to an open set contained in their intersection (Fig 11.c).

iii) One can furthermore distinguish between finiteness, *local finiteness* (each subset lying in at most a finite intersection of the topology's open sets) and *pointwise finiteness* (likewise for each point).

Example 1 Compactness enjoys [138] a further formulation in terms of refinements: 'every open cover has a finite refinement'. In contrast, paracompactness is that 'every open cover has a locally-finite refinement'. Finally, *metacompactness* is that 'every open cover has a pointwise-finite refinement'. Plenty of other gradings of definitions of topological properties by notions of extent have been formulated [138].

Remark 5 Absolute topological properties concern [c] while relative topological properties [131] concern [b] and [c], with relative (co)homology being an example.

12 ii) Globality in which space. a) Space, time and spacetime

Case 1 For conventional Mechanics, space is modelled by $\mathbb{R}^n$ (usually for $n \leq 3$) and time by $\mathbb{R}$ (or some interval thereof).

Case 2 In Special Relativity, spacetime is taken to be Minkowski's $\mathbb{M}^n$ with flat indefinite metric η. This is usually considered for $n = 4$ though now both $n < 4$ and $n > 4$ are considered for some models.

Case 3 In General Relativity, spacetime [79, 87] is a topological manifold $\mathfrak{M}$ equipped with a semi-Riemannian metric γ. A fixed spatial topology $\mathbf{\Sigma}$ is usually considered. This possesses a positive-definite spatial metric $\mathbf{h}$, and can be interpreted as a spatial slice of GR spacetime GR spacetime is usually considered in 4-d, splitting into 1 temporal and 3 spatial dimensions. While other spatial dimensions are sometimes considered, there are strong mathematical and physical arguments against temporal dimension being other than 1 [172]. We take $\mathbf{\Sigma}$ to be compact without boundary (CWB) and connected. 3-spheres $\mathbb{S}^3$ and 3-tori $\mathbb{T}^3$ are the most commonly considered specific such in the GR literature to date. GR has a spacetime manifold local coordinate notion of time, t. It may be interval-valued by the chart it is defined in not extending over all of spacetime.

Notation We use

$$\underline{x} \text{ with components } x^a \text{ to denote spatial coordinates }, \tag{102}$$

and

$$\vec{X} \text{ with components } X^\mu \text{ to denote spacetime coordinates }, \tag{103}$$

13 b) Carrier spaces

Structure 1 *Carrier space* [176]

$$\mathfrak{C}^d \tag{104}$$

is an at least incipient model for the structure of space, alias *absolute space* [1] in the context of physical modelling. Geometry was originally conceived of as occurring in physical space or objects embedded therein (parchments, the surface of the Earth...) It was however subsequently reconceived [3, 5] as occurring in abstract space. We thus say 'carrier space' rather than 'absolute space' in the purely geometrical context. Carrier space can moreover also be interpreted as a *sample space* in the context of Probability-and-Statistics, of *location data* [123].

Casting a fixed d-dimensional manifold in the role of carrier space,

$$\mathfrak{C}^d = \mathfrak{M}^d , \tag{105}$$

is quite a general possibility (and one that the current Series of Articles resides entirely within).

Example 1 For standard Mechanics carrier space is just

$$\mathfrak{C}^d = \mathbb{R}^d : \tag{106}$$

the 'most obvious' case: all flat and bereft of topological nontriviality.

Structure 2 Spacetime can also be viewed as a carrier space.

Remark 1 We use C superscripts for the group in the spatial/configurational/dynamical/split space-time case to S superscripts in the spacetime case. We use no superscripts for considerations that apply just as well in each case. Such joint conceptualization begins with the notion of state, covering both spacetime state and space state, in each case with internal extensions allowed.

14 State objects

14.1 Configurations

Structure 1 Both the Geometry and Probability-and-Statistics contexts involve studying *multiple points on* carrier space. In some physical applications, moreover, the corresponding points on absolute space furthermore model material particles (classical, and taken to be of negligible extent).

To cover both of these situations at once in our exposition, we use a points-or-particles portmanteau concept. So in all three of Geometry, Probability-and-Statistics, and Physics, one is to study N-point-or-particle *constellations*: a type of *configuration* [21, 67]

$$\mathbf{Q} \text{ with components } \underline{Q}^I , \tag{107}$$

where the underline denotes carrier space vector and the point-or-particle label I runs from 1 to N.

More generally, one's objects are now *configurations*: [21, 67]

$$\boldsymbol{Q} \text{ with components } Q^A : \tag{108}$$

instantaneous snapshots of the state of a system $\mathfrak{S}$.

14.2 Example 1) N-point-or-particle configurations

In the N points-or-particles setting, we denote the incipient configurations, alias *constellations*, by

$$\boldsymbol{q} \text{ with components } q^{iI} \tag{109}$$

for i a carrier space vector label and I a point-or-particle label running from 1 to N.

14.3 Example 2) Field configurations

Example 1 The single scalar field has $\phi(\underline{x})$.

Example 2 We also consider $\mathbf{A}(\underline{x})$ with components A^i for Electromagnetism or A^{iI} for Yang–Mills Theory.

14.4 Example 3) GR

For GR, the incipient configurations are

$$\text{Riemannian 3-metrics } \mathbf{h} \text{ with components } \mathrm{h}_{ab}(\underline{x}) \tag{110}$$

on a fixed $\mathbf{\Sigma}$ interpreted as a spatial slice of GR spacetime. Since the 3-metric $\mathbf{h}$ is a symmetric 3×3 matrix, it has 6 degrees of freedom per space point.

14.5 Dynamical variables

To this end, we can use for instance Lagrangian configuration-and-velocity variables $(\mathbf{Q}, \dot{\mathbf{Q}})$, Jacobi–Mach configuration-and-change variables $(\mathbf{Q}, \mathrm{d}\mathbf{Q})$, or Hamiltonian configuration-and-momentum variables $(\mathbf{Q}, \mathbf{P})$.

14.6 N-worldline states (and histories variants)

In the N worldlines setting,

$$\boldsymbol{W} \text{ with components } W^{\mu I}(\vec{X}) \tag{111}$$

for μ a spacetime vector label and I a particle label running from 1 to N.

14.7 N-event states

In the N events setting,

$$\boldsymbol{S} \text{ with components } S^{\mu I} \tag{112}$$

for μ a spacetime vector label and I an event label running from 1 to N. Events could for instance be worldline intersections ('meetings').

14.8 Field states in spacetime

The single scalar field now has $\phi(\vec{X})$

We also consider $\mathbf{A}(\vec{X})$, with components A^{μ} for Electromagnetism or $\mathrm{A}^{\mu I}$ for Yang–Mills Theory.

All the above examples are subcases of *(base) state objects*, which we denote by B (sans serif is our Finite–Field portmenteau font).

15 c) State spaces

Remark 1 We jointly refer to all the below cases as *state spaces*, denoted by $\mathfrak{S}$.

Great Map 1

$$\mathit{Arenize} \ : \ \ \text{(state)} \ \longrightarrow \ \text{(state spaces)} \ . \tag{113}$$

Naming 1 This derives from the word 'arena'.

15.1 Configuration spaces

Definition 1 *Configuration space* [21, 67, 123, 172, 177] $\mathfrak{q}(\mathfrak{S})$ is then the abstract space formed from the totality of possible values that a given system $\mathfrak{S}$'s configurations $\mathbf{Q}$ can take.

Notation 1 The dimension of configuration space is

$$k \ := \ \dim(\mathfrak{q}(\mathfrak{S})) \tag{114}$$

Notation 2 In the current Series, we use slanted font for finite-dimensional entities and straight font for field entities. This is so as to immediately avoid confusion between objects of a given type and the space of all such objects.

Remark 1 A global treatise is *not* to view configuration space as just a set, but rather as carrying a topology. Also, in all cases considered here, as a metric in the sense of Analysis, and a metric geometry (not necessarily associated with its analytic metric).

15.2 Example 1) Constellation space

Structure 1 Mechanics and Shape Theory's *Constellation spaces* [176] $\mathfrak{q}(\mathfrak{C}^d,\, N)$ are the configuration spaces of constellations for some fixed count N of points-or-particles. Constellation spaces are furthermore particularly simple examples of configuration spaces, being just *finite product spaces*

$$\mathfrak{q}(\mathfrak{C}^d,\, N) \ = \ \times_{i\,=\,1}^{N}\mathfrak{C}^d \ = \ \times_{i\,=\,1}^{N}\mathfrak{M}^d \ . \tag{115}$$

For $\mathfrak{C}^d \ = \ \mathbb{R}^d$, these further simplify as follows.

$$\mathfrak{q}(d, N) \ := \ \mathfrak{q}(\mathbb{R}^d,\, N) \ = \ \times_{I\,=\,1}^{N}\mathbb{R}^d \ = \ \mathbb{R}^{N\,d} \ . \tag{116}$$

Finally, by Corollary 1 of 10.1, if carrier space $\mathfrak{C}^d$ is a manifold, the corresponding constellation spaces $\mathfrak{q}(\mathfrak{C}^d,\, N)$ are manifolds.

Structure 2 Constellation space carries the kinetic matrix

$$\boldsymbol{M} \ \text{ with components } \ M_{ijIJ} \ := \ m_I \delta_{ij} \delta_{IJ} \tag{117}$$

for m_I the Ith particle's mass.

15.3 Example B) Field configuration spaces, including Gauge Theory's

Example 1 Scalar Field Theory's configuration space $\mathfrak{S}$ca is the space of scalar field values $\phi(\underline{x})$

Example 2 Electromagnetism's configuration space is the space $\mathbf{\Lambda^1}$ of 1-forms $\mathrm{A}_{\boldsymbol{i}}\boldsymbol{(\underline{x})}$. Yang–Mills Theory's configuration space is a larger space $\mathbf{\Lambda}^1_G$ of 1-forms $\mathrm{A}_i^P(\underline{x})$. [$\mathfrak{S}$ca and $\mathbf{\Lambda}^1$ have implicit dependence on the model of space in use.

15.4 Example C) Vacuum GR's $\mathfrak{Riem}(\boldsymbol{\Sigma})$

The space formed by the totality of the $\mathbf{h}$ on a fixed $\boldsymbol{\Sigma}$ is GR's incipient configuration space

$$\mathfrak{q}(\boldsymbol{\Sigma}) \ = \ \mathfrak{Riem}(\boldsymbol{\Sigma}) \ ; \tag{118}$$

Structure 1 The space of Riemannian geometries $\mathfrak{Riem}(\boldsymbol{\Sigma})$ can be modelled as an open positive convex cone[4] in the Fréchet space $\mathfrak{fr\acute{e}}_{\mathfrak{S}\mathrm{ym}(0,2)}(\mathfrak{C}^{\infty})$ for $\mathfrak{S}\mathrm{ym}(0,2)$ the symmetric rank-2 tensors.

Proposition 1 $\mathfrak{Riem}(\boldsymbol{\Sigma})$ can furthermore be equipped [53] with a metric space notion of metric. this can additionally be chosen to be preserved under $Diff(\boldsymbol{\Sigma})$. $\mathfrak{Riem}(\boldsymbol{\Sigma})$ is thus a metrizable topological space.

Corollary 1 $\mathfrak{Riem}(\boldsymbol{\Sigma})$ obeys all the separation axioms – including in particular Hausdorffness.

Proposition 2 $\mathfrak{Riem}(\boldsymbol{\Sigma})$ is also paracompact, and thus HP.

Proposition 3 $\mathfrak{Riem}(\boldsymbol{\Sigma})$ is additionally second-countable [149], and has an infinite-dimensional analogue of the locally-Euclidean property as well. Thus there is a $\mathfrak{C}^{\infty}$ sense in which it is an infinite-d manifold. So the single corresponding type of chart suffices in this case.

Structure 2 $\mathfrak{Riem}(\boldsymbol{\Sigma})$ is supplied with its own metric by GR's action in split space-time form [34]: the inverse DeWitt metric [48],

$$\mathbf{M} \ \text{ with components } \ \mathrm{M}^{abcd} \ := \ \sqrt{\mathrm{h}}\,(\,\mathrm{h}^{ac}\mathrm{h}^{bd} \ - \ \mathrm{h}^{ab}\mathrm{h}^{cd}\,) \ : \tag{119}$$

the inverse of the DeWitt supermetric

$$\mathbf{N} \ \text{ with components } \ \mathrm{N}_{abcd} \ := \ \frac{1}{\sqrt{\mathrm{h}}}\,(\,\mathrm{h}_{ac}\mathrm{h}_{bd} \ - \ \tfrac{1}{2}\,\mathrm{h}_{ab}\mathrm{h}_{cd}\,) \ . \tag{120}$$

$\mathbf{M}$ is the analogue of the $\mathbb{R}^{Nd}$ Euclidean metric on Shape Theory's underlying configuration space, or more generally of the product metric on the constellation space (116). While the inverse DeWitt supermetric is indefinite, so is Event Shape Theory's incipient product space metric [182].

15.5 Example D) GR with matter fields

We use the notation $\mathfrak{RIEM}(\boldsymbol{\Sigma}, \Phi)$ for this, where Φ encodes the matter fields under consideration.

15.6 Dynamical variables' state spaces

Configuration-and-velocity space and configuration-and-change counterparts are two representations of $\mathfrak{T}(\mathfrak{q})$.

Configuration-and-momentum space is prephase space, which becomes phase space $\mathfrak{P}$hase upon being equipped with a Poisson bracket.

15.7 Eventspaces

Definition 1 The corresponding *event constellation spaces* are

$$\mathfrak{q}(\mathbb{M}^{D}, N) \ := \ \times_{I\,=\,1}^{N}\mathbb{R}^{d,1} \ = \ \mathbb{R}^{N\,d,\,N} \ . \tag{121}$$

$\mathbb{R}^{a,b}$ here denotes the indefinite flat space with a + signs and b − signs.

15.8 Space of fields on spacetime

Example 1 Scalar Field Theory's spacetime state space $\mathfrak{S}\mathrm{ca}_{\mathrm{S}}$ is the space of scalar field values $\phi(\vec{X})$

Example 2 Electromagnetism's spacetime state space is the space $\boldsymbol{\Lambda}^{1}_{\mathrm{S}}$ of 1-forms $\mathrm{A}_{i}(\vec{X})$. Yang–Mills Theory's configuration space is a larger space $\boldsymbol{\Lambda}^{1}_{G\,\mathrm{S}}$ of 1-forms $\mathrm{A}_{i}^{P}(\vec{X})$.

[4]This is a Linear Algebra characterization of a space $\mathfrak{S}$ [53, 64], that is not itself linear but obeys $\mathfrak{S} + \mathfrak{S} \subset \mathfrak{S}$ and $m\,\mathfrak{S} \subset \mathfrak{S}$ for $m \in \mathbb{R}_{+}$. See [53] for more on this as well as for consideration of why Fréchet spaces are appropriate. Do not confuse this use of 'cone' with Article 0's topological and geometrical uses.

15.9 Space of spacetimes

Structure 1 The *space of spacetimes* on a given $\mathfrak{M}$ is denoted, following Isham [85], as

$$\mathfrak{P}\text{Riem}(\mathfrak{M}) \ . \tag{122}$$

This stands for 'space of pseudo-Riemannian metrics' on our given $\mathfrak{M}$;[5] and had its geometrical structure worked out in [81].

15.10 Space of spacetimes with matter fields thereupon

We use the notation $\mathfrak{P}\text{RIEM}(\mathbf{\Sigma}, \Phi)$ for this, where Φ encodes the matter fields under consideration.

15.11 Quotientative topological properties

r-state spaces upon some $G = Aut(\mathfrak{C}, \boldsymbol{\sigma})$ acting on the incipient state space, are then considered in Article 0. r-space refers to the confluence of reduced versus relational approach (see Article 0).

Great Map 2

$$Quotientize \ : \ \ (\text{space}) \ \longrightarrow \ (\text{quotient space}) \ . \tag{123}$$

A minimal discussion of quotient spaces that is useful at this point is as follows.

Remark 1 Nontrivial relational spaces are quotients of constellation spaces by automorphism groups. Mathematically, this is a subcase of [128, 155]

$$\frac{(\text{manifold})}{(\text{Lie group})} \ = \ \frac{\mathfrak{M}}{G} \ . \tag{124}$$

In this way, quotient spaces enter our study.

Structure 2 Quotienting a topological space by an equivalence relation, $\langle \mathfrak{X}, \mathfrak{T} \rangle / \tilde{\ }$, produces the corresponding *quotient topology* $\mathfrak{T}_{\text{q}}$ [128, 155].

Definition 1 A topological property is *quotientative* if it is preserved under quotients.

Remark 2 Unfortunately, this seldom occurs. In particular, we have the following result.

Proposition 3 [128, 161, 54] Hausdorffness, second-countability, and local-Euclideanness are not in general quotientive.

Corollary 4 Manifoldness is not in general quotientive.

Corollary 5 Constellation spaces $\mathfrak{q}(\mathfrak{C}^d, N)$ are manifolds. However, relational spaces – quotients of these by geometrical automorphism Lie groups – are not in general manifolds.

16 Main dynamical laws considered in this Series

These not only involve the above state spaces but also involve dynamical laws; these are second-order.

Example 1 Mechanics has Newton's Second Law ODE. We also consider a Background Independent counterpart: timeless differentials, alongside translation and rotation corrections. This covers both Relational Particle Mechanics (RPM) and Kendall's Shape Theory [80, 90, 123] (which plays a key role in RPM as a configuration space).

Example 2 SR Field Theory as per [130], including Electromagnetism, and also Yang–Mills Theory [113], which are Gauge Theories. These last two have homogeneous and inhomogeneous equations. The homogeneous equations just playing out as integrabilities guaranteeing the existence of the potentials in use. The remaining 4 inhomogeneous equations split into 1 Gauss equation (constraint) and 3 Ampère–Maxwell (evolution) equations.

Example 3 GR has 10 Einstein eield equations, splitting into the Hamiltonian-and-momentum constraint system of 4 equations, and 6 Einstein evolution equations.

[5] The current series does not go as far as considering topological-level Background Independence; see e.g. [96, 100, 175, 198] for work in this direction.

Example 4 GR with minimally-coupled matter, including each of the above 3 models. For a scalar field, this includes a seventh evolution equation. For the other two, a fifth constraint equation and a total of 9 evolution equations.

17 DE problems and supporting function spaces

17.1 Definition

Remark 1 We study not only the above spaces but also DEs defined thereupon. This is the conventional form that the Laws of Physics take. Function spaces for the DEs' coefficients, data, and solutions thus also enter.

Definition 1 A *DE problem* is a DE system alongside prescribed data, such as boundary conditions, or initial conditions.

Example 1 Prescription of values of fields and their velocities on an initial surface is *Cauchy data* for the *Cauchy problem* for a second-order PDE.

Definition 2 A PDE problem is *well-posed* if there exist unique solutions to it with *continuous dependence* on the initial data.[6]

Structure 1 Consider a general PDE system

$$F\left(\underline{x}, \partial^{(1)}\mathrm{u}, \dots, \partial^{(r)}\mathrm{u}\right) \;=\; 0\;, \tag{125}$$

where $\partial^{(i)}$ denotes the ith-order partial derivatives,

Definition 3 The *principal symbol* σ_{P} is obtained by taking the highest-order part and using $\underline{\xi}$ in place of $\underline{\partial} \;=\; \partial^{(1)}$.

Definition 4 An *elliptic system* is one for which σ_{P} is positive-definite and invertible.

Remark 2 This is but one of the simpler definitions of ellipticity (more precisely of *strong ellipticity*) [146].

Remark 3 On the other hand, *hyperbolicity* is more sensitive to lower order terms. This is taken into account for instance by *Leray hyperbolicity* (p 152-154 of [146]).

Global feature -1.V Elliptic PDEs are globally sensitive, e.g. on the whole space, or as regards boundary conditions throughout the boundary of an extensive piece of the space.

Example 1 In positive-definite spaces, the generalized Killing equation's ellipticity renders it globally sensitive. This is in contrast to Lie's elimination (the reverse differential route) being just a local affair.

Remark 5 At least in all the cases mentioned above in the spatial setting, the

Remark 4 Hyperbolic PDE problems require the following as a fourth well-posedness criterion.

Definition 6 The *future domain of dependence* is [79]

$$\mathrm{D}^{+}(\mathrm{R}) \;:=\; \{\,\mathrm{P} \;\in\; \mathfrak{M}\,|\,\text{every past inextendible causal curve through}\;\;\mathrm{P}\;\;\text{intersects}\;\;\mathrm{R}\,\}\;. \tag{126}$$

Remark 5 This enforces a sensible notion of causality, permitting compatibility with Relativity. This is by enforcing that a given region can only influence those other regions to which it is connected by a causal curve: one along which physical particles, and thus signals, could travel.

Global limitation -1.VI The Domain of Dependence property of hyperbolic PDEs limits the extent over a space on which the solution from local data can be worked out to within Fig 12.)'s 'sandcastle'. For Relativistic field equations, this is a Relativistically imposed (causal) limitation on extent of globalization. To be more global than that with extent of solution, one needs a global data prescription to begin with, i.e. data on the whole of a Cauchy surface.

[6] Without this last condition, an arbitrarily small change in the data could cause an arbitrarily large immediate, precluding any physical predictability. N.B. this really does mean *immediate* – see e.g. p 229 of [32] – rather than some issue of chaos or unwanted growing modes.

Global Problem -1.VII Symbol is in general only local, in the sense that it can vary from place to place within a space.

Example 2 *Tricomi's equation* [32]

$$\phi_{xx} \ + \ x\,\phi_{yy} \ = \ 0 \tag{127}$$

is elliptic in $x \ > \ 0$, hyperbolic in $x \ < \ 0$ (and parabolic at $x \ = \ 0$).

Remark 6 Single first-order equations are largely exempt from equation-type restrictions on function spaces. We additionally consider at most only second-order systems in the current Article. The cases this leaves are as follows.

Remark 7 Well-posedness criteria can quite often only be established locally.

Remark 8 A further consideration is whether singular solutions can be included in one's treatment.

17.2 $\mathfrak{C}^\alpha$ functions

Some further global issues next enter with choice of function space used in modelling the DE at hand.

Structure 1 Let us first consider $\mathfrak{C}^\alpha$ for $\alpha \ = \ \mathbb{N}_0 \cup \infty \cup \omega \ = \ 0,\, 1,\, 2,\, ...\, \infty\,,\, \omega$: the continuous, once-differentiable, twice-differentiable, ... smooth and analytic functions respectively.

Remark 1 Physics mostly does not care what the function spaces are, unless the function spaces are 'at either extreme'.

On the one hand, $\alpha \ = \ 0$'s continuous functions would preclude use of Calculus, and thus of DEs.

On the other hand, $\alpha \ = \ \infty$'s analytic functions admit *analytic continuation* as a globalizing method (Fig 12.a). This property is however incompatible [79, 146] with modelling Relativistic Physics' spacetime. This is by precluding independence of causally disconnected regions (not path-connected by causal curves: Fig 12.b).

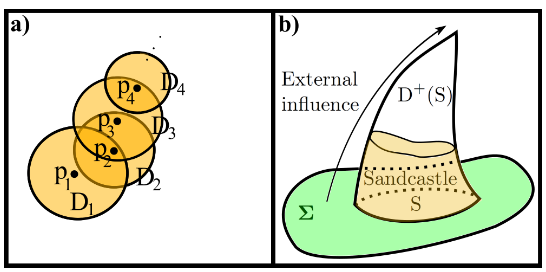

Figure 12: a) Analytic continuation, as exhibited by the analytic functions as follows. Firstly, expand in a power series valid in a disc $\mathbb{D}_1$ around a point p_1. Next expand in another power series about an off-centre point $\mathrm{p}_2 \in \mathbb{D}_1$ to form a new disc $\mathbb{D}_2$ partly outside $\mathbb{D}_1$, and so on.
b) Analytic continuation clashes with causality whenever it creeps outside of the domain of dependence of one's region.

Remark 2 For k-times differentiable functions $\mathfrak{C}^k$, firstly, k needs to be high enough for the PDEs in question to be defined. This often precludes $k \ = \ 1$, since Physics' fundamental-level DEs are conventionally at most second-order. Secondly, the DE's coefficients and data in general correspond to a solution correspond to different k, as may part or all of its data. In contrast, $\mathfrak{C}^\omega$ and $\mathfrak{C}^\infty$ are *closed* under differentiation and thus under PDE solution.

Remark 3 The combined requirements of being Relativistic and closed under differentiation and thus under DE operators narrow down α to precisely ∞: the smooth functions.

17.3 Further PDE-adapted function spaces

Remark 1 Some function spaces are well adapted to PDEs, including globally. A traditional take on this is that which function spaces are adapted to a PDE usually depends on the type of PDE.

On the one hand, *Hölder spaces* $\mathfrak{C}^{k,a}$ were developed for better treating elliptic PDEs [32].

On the other hand, *Sobolev spaces* $\mathfrak{S}\text{ob}^k$ were developed for better treating hyperbolic PDEs [79].

Remark 2 Nonlinear PDEs are harder to handle than linear ones; this applies to the second and third of the following examples.

Example 1 The Maxwell and Yang–Mills equations however each split into 1 elliptic constraint PDE (Gauss' Law) and 4 hyperbolizable evolution equations (Ampère–Maxwell).

Example 2 The Einstein field equations split into an elliptizable system of 4 constraints and a hyperbolizable system of 6 evolution equations.[7]

Remark 3 Subsequent work established use of Sobolev spaces for elliptic equations [169]. These permit rougher data than smooth functions while manitaining closedness under differentiation and thus under PDEs.

Remark 4 We thus have no qualms putting mixed elliptic–hyperbolic systems treated as Initial Value Problems (IVPs) [69, 111] followed by Cauchy Problems (CPs) [79] on the common footing [146, 148] of Sobolev spaces.

Remark 5 A more modern point of view is that Sobolev spaces are a *recategorization* of (or *designer category* replacement for) $\mathfrak{C}^\alpha$ that is adapted for study of PDEs. For PDEs do not preserve $\mathfrak{C}^k$ from (equation coefficients, data) to (solutions), whereas Sobolev spaces are designed to be stable in this way. They attain this by treating functions and their derivatives – up to some number of derivatives large enough for the PDE to cope – on an equal footing (c.f.the below definition). So using Sobolev spaces for all PDEs, not just hyperbolic ones or the EFEs, makes good conceptual sense from the postmodern recategorizing or 'designer categories' perspective.

Definition 1 *Sobolev spaces* [74, 148, 169] $\mathfrak{S}\text{ob}^k$ bear some similarity to $\mathfrak{L}^p$ spaces, but are now built out of the *Sobolev norm* which involves up to kth derivatives:[8]

$$||f||_{\mathfrak{H}^{k,p}} := \sum\nolimits_{|i| \, \leq \, k} ||\partial^{(i)} f||_{\mathfrak{L}^p} \ . \tag{128}$$

17.4 First-order systems

Motivation 1 These can always be obtained by decoupling higher-order systems.

Motivation 2 Dirac's equation for spin-$\frac{1}{2}$ fermions is first-order.

Motivation 3 Observables equations are first-order (see Article [205], or, while this is not yet in the public domain, [201]).

Structure 1 For first-order systems, the main underlying factors are under-, well-, or over-determinedness. Integrability is then key in the last of these.

Remark 1 The current Series' topic also requires handling first-order FDEs (functional differential equations) about which so far rather less is known [201].

17.5 Further spaces in which locality can occur

1) One can have locality in an underlying function space $\mathfrak{F}\text{un}$.

2) One can also have locality in a DE solution space $\mathfrak{S}\text{ol}$.

Global Problem -1.VIII A further matter with global inputs is whether simple given solutions to a nonlinear PDE possess *linear stability* [104].

17.6 Physics on r-spaces

If Hausdorff fails, moreover, we get 'bad' stratified manifolds: ones we can't do Analysis on and therefore can't formulate laws of nature as DE's upon. So knowing how to do PDEs on a manifold does not suffice to cover every Physics need.

[7] This is [148] hyperbolizable by a suitable choice of gauge in each case, and elliptizable in the case of GR by York's conformal approach [69] to GR's IVP.

[8] These are meant in a distributional sense to ensure completeness is maintained as well.

18 Function spaces after *Arenize*

Structure 1 For flat-space(time) Field Theory state spaces, the function space $\mathfrak{L}^2$ of square-integrable functions in the sense of Lebesgue [64] provide one starting point.

18.1 First two rungs of a more general ladder

Consider the following ladder of increasingly general topological vector spaces which are infinite-d function spaces [74, 68].

Definition 1 A *Hilbert space* $\mathfrak{H}$ilb is a complete inner product space.

Definition 2 A *Banach space* $\mathfrak{B}$an is a complete normed space.

Exercise Show that Sobolev spaces are subcases of the above.

18.2 A third rung: Fréchet spaces

Remark 1 If one values *Arenize*, one's natural recategorize gives one Fréchet spaces.

Definition 1 A topological vector space is *metrizable* if its topology can be induced by a metric space metric which is furthermore translation-invariant.

Remark 2 This qualification is required since for topological vector spaces, one uses a collection of neighbourhoods of the origin (vector space 0). From this, translation (by the vector space + operation) establishes the collection of neighbourhoods at each other point.

Definition 2 A *Hamel basis* itself is a maximal linearly-independent subset of $\mathfrak{v}$ (this is one of various notions of basis supported by infinite-dimensional spaces).

Definition 3 A *base* in a topological vector space $\mathfrak{v}$ is a linearly-independent subset $\mathfrak{b}$ such that $\mathfrak{v}$ is the closure of the linear subspace with Hamel basis $\mathfrak{b}$.

Definition 4 A subset $\mathfrak{Y}$ of a vector space $\mathfrak{v}$ is *convex* if

$$px + (1 - p)y \in \mathfrak{Y} \ \forall \, x, y \in \mathfrak{Y} , \ \ p \in [0,1] . \tag{129}$$

Definition 5 A topological vector space $\mathfrak{v}$ is *locally convex* [74] if it admits a base that consists of convex sets.

Definition 6 A *Fréchet space* $\mathfrak{f}$ré is a complete metrizable locally-convex topological vector space [75].

Remark 3 Fréchet spaces are moreover very naturally associated with $\mathfrak{C}^\infty$ smoothness [75].

Remark 4 Fréchet spaces are suitable for modelling state spaces corresponding to whichever of fields on curved spaces and GR.

Remark 5 Many substantial results in Functional Analysis furthermore carry over from Banach spaces to Fréchet spaces [75]. On the other hand, we caution that Fréchet spaces no longer in general [75] possess an Inverse Function Theorem [135, 51]. We get around this by restricting attention to the following case.

18.3 Tame Fréchet spaces

Definition 1 Let $\mathfrak{V}$ be a vector space over a field $\mathbb{F}$. The *seminorm* of $v \in \mathfrak{V}$ is a real number $||v||$ such that [68] $\forall \, v, w \in \mathfrak{V}$ and $k \in \mathbb{F}$,

i) $||v|| \geq 0$ (non-negativity),

ii) $||v + w|| \leq ||v|| + ||w||$ (triangle inequality),

iii) $||k\, v|| = |k| \, ||v||$ (scalar multiplication).

Definition 2 A *grading* on a Fréchet space is a collection of seminorms $\{|| \ ||_n\}_{n \in \mathbb{N}_0}$ which [75]

i) are increasing in strength: $||f||_p \ \leq \ ||f||_q$ for $p \ < \ q$, and

ii) define the topology in use.

Definition 3 A Fréchet space is *graded* if it is equipped with a grading.

Definition 4 For graded Fréchet spaces $\mathfrak{G}_1$ and $\mathfrak{G}_2$, a linear map $Lin \ : \ \mathfrak{G}_1 \longrightarrow \mathfrak{G}_2$ satisfies a *tame estimate* of degree d and base b if [75]

$$||Lin\, f||_n \ \leq \ C||f||_{n+d} \ \text{ for each } \ n \ \geq \ b \ , \tag{130}$$

for constant $C \ = \ C(n)$.

Definition 5 Lin is *tame* if it satisfies a tame estimate for some d and b.

Definition 6 For graded Fréchet spaces $\mathfrak{G}_1$ and $\mathfrak{G}_2$, $\mathfrak{G}_1$ is a *tame direct summand* of $\mathfrak{G}_2$ [75] if we can find tame linear maps Lin_i, $Lin_j \ : \ \mathfrak{G}_1 \longrightarrow \mathfrak{G}_2$ such that $Lin_j \circ Lin_i \ = \ id$.

Definition 7 A graded Fréchet space is *tame* if it is a tame direct summand of a space $\Sigma(\mathfrak{B}\text{an})$ of exponentially-decreasing sequences in some Banach space $\mathfrak{B}$an.

Remark 1 The main point of tame Fréchet spaces is that they possess the *Nash–Moser Theorem* [75], which serves as a replacement for the Inverse Function Theorem.

Remark 1 While [53] precedes tameness and [116] employs a distinct type of Functional Analysis, using tame Fréchet spaces for GR configuration spaces is established [158, 185]. This is relevant as regards the efficacity of the *Portmanteau Calculus* [154, 172, 191] that ALRoPoT employs. It is straightforward that [74] the portmanteau of standard and Banach Calculi works well. Yet GR state spaces require more than this, and the portmanteau of standard and tame Fréchet Calculus is more extensive [75] than that of standard and unqualified Fréchet Calculus.

18.4 Hilbert, Banach and Fréchet Manifolds

Structure 1 Topological manifolds' local Euclideanness and ensuing $\mathbb{R}^p$ charts extend well to infinite-d. There is diversity of modelling here, since the charts in question could for instance be Hilbert, Banach or Fréchet spaces. See e.g. [114, 74, 75] for accounts of Hilbert, Banach and Fréchet manifolds respectively.

Structure 2 Finite manifolds' incorporation of differentiable structure also has an analogue in each of the above cases. So e.g. one can consider differentiable functions and tangent vectors for each, and then apply multilinearity to set up versions for tensors of any other rank (p, q) and symmetry type S. In particular, applying this construction to a Fréchet manifold with tangent space $\mathfrak{fr}\acute{e}(\mathfrak{C}^{\infty})$ produces another Fréchet space $\mathfrak{fr}\acute{e}_{S(p,q)}(\mathfrak{C}^{\infty})$.

19 Globality in temporo-spatial context

19.1 Notions of times' globality

The next two sections and Sec 22 develop globality specifically for the classical Problem of Time and its Background Independence resolution.

Case 1 'Global Problem of Time' could for instance refer to the global standing of a notion of time itself.

Case 2 It could also refer to globality over space, such as the subset of points $t \ = $ const not being a (differential-geometrically path-)complete 3-d submanifold of the spacetime [101].

Further cases Other possibilities are globality over spacetime, or over any of the spaces in Secs 15.

Yet further cases Frames and transformations therebetween are also in general only local.

A distinct classification is into the following.

Kuchař's Embarrassment of Poverty [99] A timefunction could have a 'Global Problem of Time' in the sense of there being a global obstruction to there being any timefunction at all.

Kuchař's Embarrassment of Riches [99] A timefunction could have a 'Global Problem of Time' if global extensions exist but are (highly) nonunique. On each occasion this occurs, one might wish to dream up a Selection Principle that uniquely picks out which global extension to use.

Case A Some global issues with timefunctions are merely due to coordinate restrictions on manifolds, which are readily resolvable in Differential Geometry.

Case B Some global issues with timefunctions are however at the more involved and not generally resolved level of PDE solutions.

Case C If more than one of a problem's associated spaces occur, these might arrive formulated in terms of mismatched function spaces.

19.2 Global breakdown of entities or of properties

Aside from the global breakdown of a mathematical entity itself, e.g. a function blowing up, ceasing to be defined or going complex, we can have the following.

Global breakdown of properties P of our mathematical entity which are required for it to match its proposed physical role.

Examples Non-frozenness ($t \neq 0$), non-haltingness ($t \neq 0$ anywhere), monotonicity (t everywhere strictly increasing) of a time function might only hold locally for a candidate timefunction. In GR, a coordinate might only remain timelike within a finite region. Some properties of a candidate spaces and frames may also only hold locally.

20 Full Background Independence's notions of globality

Remark 1 Understanding ALRoPoT considerably sharpens understanding of what the Global Problems of Time consist of. The current series exploits this understanding to rebuild the classical Global Problems of Time from scratch.

Remark 2 Globality can furthermore affect each of the following five elementary parts of Background Independence.

20.1 0) Relationalism

Spacetime and Configurational Relationalisms involve physical content residing in the quotient state spaces mentioned in Sec 15.11. The topology and geometry of such spaces then plays a leading role [48, 53, 116, 123, 149, 172, 176, 177, 182, 183, 184]. Stratified spaces often ensue (see Article 0), including some which would permit little to no Analysis and thus PDE formulation of Natural Law.

Temporal Relationalism begins with Leibnizian timelessness for the universe as a whole at the primary level. But a mild recategorization [154, 172, 190, 191, 192, 193, 194] permits this to be incorporated into our local Lie Theory. One consequently needs to use changes in place of velocities, and specifically with Jacobi-type actions in place of Euler–Lagrange ones. Each such action is homogeneous-linear in change, implying at least one primary constraint [40], which we refer to as $\mathcal{C}$hronos (see [199, 186] or Article 0 for details).

In the spacetime case, and in a few split space-time cases, a direct formulation of the first paragraph's Relationalisms is possible at the level of the Principles of Dynamics action. In most split space-time cases, however, only an indirect formulation is known. The group G is here encoded by Lie derivative corrections to the change variables in the action. Variation with respect to these corrections' auxiliary G-variables $\mathbf{g}$ provides constraints $\boldsymbol{\mathcal{S}}$huffle. Solving $\boldsymbol{\mathcal{S}}$huffle for the $\mathbf{g}$ then removes G-dependence from the Physics. This procedure involves solving algebraic equations for Mechanics, or PDEs for GR (the so-called Thin Sandwich Problem [35, 99, 101, 102, 172]). As we shall see in Article 0, both of these cases encounter global obstructions.

Primary-level timelessness is finally locally reconciled with everyday semblance of time by Mach's time being abstracted from change via a rearrangement of $\mathcal{C}$hronos. There are however various 'patching' issues with passing to a global description of such an emergent time (see [172] and Article 0).

20.2 1) Closure

Given the constraints provided by Configuational and Temporal Relationalism, it remains to check these are consistent or otherwise. Dirac-type Algorithms [23, 28, 40, 97, 154, 160, 172, 188, 192, 200] perform this function. More generally, given a candidate set of generators, the Generalized Lie Algorithm [195, 200] assesses whether these are consistent or otherwise. Whether or not candidate spacetime generators close is another subcase of this. It is key that Dirac imbued such Algorithms with selection principle properties, hence all of this paragraph's useds of 'or'. Closure entering at this stage reflects that the Lie-theoretic fact that brackets relations are needed as well as generators, by which Relationalism and Closure constitute a single fused mathematical problem.

When successful, the algorithm's output is a Generator Algebraic Structure, covering Lie algebra or Lie algebroid. We both consider reduction and extension approaches in Article 1.

From Closure's centrality in Fig 1, it is this and not preliminary Relationalism that forms the core of this subject. This centrality also entails major participation in more nontrivial (Background Independence aspect interrelations) = (Problem of Time facet interferences), with each of 0) , 2) and 3) separately.

Global considerations concerning Closure are chiefly as follows (see Article 1 for details).

i) *Constraint surfaces* (or generator surfaces) [97].

ii) *Local differences in generator algebraic structure.*

iii) Dirac-type Algorithms and Generalized Lie Algorithms *branching into multiple paths* [97, 160, 181].

iv) *Anomalies* [113, 115, 84] arise, noting this *conventional* major problem with approaches to Quantum Gravity is now also to be viewed as the topological-level Queen of the most central part of the Problem of Time's Background Independence resolution.

The reduction approach corresponds to adding generators, by which a copresheaf [165] picture is suitable. The alternative extension approach [97] works well for conventional Yang–Mills Gauge Theory, but encounters obstacles once Gravitational Theory is involved (see Article 1 for details).

Having solved the combined Relationalism–Closure problem, two distinct follow-ups are given in each of the next two subsections.

20.3 2) Extent of validity of Observables

At the most primary level, given a state space the corresponding Problem of Observables is to find a suitable function space thereover [163].

In the presence of generators, this selection of a function space is to be of functions forming zero commutation relations with the generators [22]. These brackets relations can moreover be recast as [166, 172, 179, 180, 193] a first-order PDE system [32, 73, 161]. The Flow Method can be applied to this, giving a minor modification [180, 193] of Lie's Integral Approach to Geometrical Invariants [8].

Local versus global flow considerations [73, 161] are then pertinent.

With each zero commutation condition amounting to a restriction, a presheaf [171] formulation [180, 193] of classical observables looks to be natural.

20.4 3) Constructability

One here considers a more general ansatz [127, 160] for a Principles of Dynamics action, to see if making less structural assumptions makes no difference. Recovery of space from less structural assumptions made about space, ditto for spacetime and ditto for spacetime from space are three pertinent cases [194, 196, 202], by which the meaning of the name 'Constructability' is clear. Dirac-type Algorithms [160, 194, 202], and the Generalized Lie Algorithm more generally [195, 181, 194, 202] have selection principle properties that enable this to occur. This is often observed to occur hand in hand with the algorithm branching into multiple finite paths, by which a theory is recovered not by itself but as one among a few consistent possibilities.

A more modern view of [194] amounts to deforming generators, or deforming an action so as to produce deformed generators. Constructability then works when *rigidity* is encountered.

This also points to further interest in the space of deformations of a consistent set of generators, as well as the space of consistent generator algebraic structures resulting from deformations.

20.5 A) Foliations and Refoliation

Embeddings, the *space of embeddings*, *foliations* and *the space of foliations*, each for spaces into spacetimes, also play a role [101, 172].

Foliation Independence is a desirable Background Independent property, attained in GR itself by *Refoliation Invariance* [58] by the form of GR's Dirac Algebroid of constraints. More general interplay between foliations and a subset of Lie algebroids can be found in [137].

21 Justifying local resolutions in the first place

21.1 HP r-spaces as a technical selection principle

Technical Selection Principle [195].

$$\text{`r-spaces are to be HP (Hausdorff paracompact) spaces'} \ . \tag{131}$$

Remark 1 As argued above, these include manifolds, LCHS spaces and metric spaces.

Remark 2 This Selection Principle rests on *feasibility of adopting a local having a particularly strong justification* for such spaces, for the following reasons.

A) HP spaces support differential structures [164].

B) HP spaces admit *partitions of unity* [128], and thus

i) *bump functions* are well-adapted to HP spaces [128, 88], further enhancing Differential Geometry.

ii) The familiar kind of theory of integration [74, 161] is available for HP spaces.

C) A-B) ⇒ PDEs [93] and variational principles can be posed on HP spaces; PDE Theory is itself aided by bump functions. Variational principles and PDEs are in turn are what conventional continuum-like Theories of Physics require for substantial and conventional developability.

D) In HP spaces, we can *shrink* the local region if necessary. I.e. the following [128, 171] is available for HP spaces; it can be used to compatibilize multiple local criteria.

21.2 Shrinking Lemma(s)

Shrinking Lemma Let $\mathfrak{X}$ be an HP space, and $\{\mathfrak{U}_a\}_{a \,\in\, \mathfrak{A}}$ be a cover thereof by open sets. Then ∃ a locally-finite cover $\{\mathfrak{V}_b\}_{b \,\in\, \mathfrak{B}}$ such that for each a the closure $\text{Clos}(\mathfrak{V}_b) \ \subset \ \text{some} \ \ \mathfrak{U}_a$.

Remark 1 Shrinking applies in Fig 11.c)'s context.

Remark 2 For those readers whose specializations do make considerable use of topological spaces, we point out that this is not the only Shrinking Lemma. The further topological properties of *normality* [128] and *metacompactness* [138] support weakenings of, or alternatives to, the current Article's HP-space Selection Principle.

Remark 3 A) to B)'s needs [and Articles 0) and 1)'s follow-ups] are moreover collectively more stringent than the question of which topological spaces admit a Shrinking Lemma...

21.3 Background Independence and Problem of Time applications

Remark 1 Each of Articles 0) and 1) substantially build parts of Background Independence on HP space foundations, on some occasions more particularly for LCHP or LCHS $\Rightarrow$ P spaces.

Remark 2 Both Wheeler's superspace and Kendall's approach to similaritiy shape spaces (each explained in Article 0) proceed to establishing HP not by LCHS $\Rightarrow$ P but by M $\Rightarrow$ P.

Remark 3 See Sec 22.6 for a futher simplification.

Remark 4 Multiple localities and small localities were already part and parcel of Lie's own work [8]. This long preceded HP technology, however (Hausdorff by 2 decades and paracompactness by 7), Lie having used analytic function spaces instead. Many of the Global Problems of Time's own regions of validity are moreover in an initially bewilderingly diverse collection of mathematical spaces. These were also largely developed decades through to a century after Lie's own epoch [14, 21, 22, 23, 40, 48, 53, 67, 101, 116, 137, 172, 188, 192, 193].

22 Globalization Strategies

22.1 Five strategies

Some overall strategies for Global Problems of Time are as follows.

Strategy α) *Globalize by Extension.* Some structures used locally may happen to remain globally valid.

Strategy β) *Globalize by Restriction.* Some structures a priori assumed to be global can in fact be supplanted by locally restricted structures.

Strategy γ) *Globalize by Replacement.* Some structures used locally may not remain globally valid but can be replaced by ones which are.

Strategy δ) *Globalize by Discarding.* Some structures used locally may be globally meaningless, and thus require discarding entirely in a global treatment.

Strategy η) *Globalize by interrelation of local and global information.*

22.2 Examples of Globalize by Extension

Patching together is a subcase of Globalize by Extension.

Remark 1 Some such constructs, from Manifold Geometry or Fibre Bundle Theory are standard.

Example α.i) Extension by meshing charts together in the study of manifolds $\mathfrak{M}$. I.e. extending from pieces of flat space by using multiple such with smooth inter-relation on the overlaps.

Example α.ii) Extension via transition functions for fibre bundles ($\mathfrak{B}$, $\mathfrak{F}$, π, $\mathfrak{E}$).

Example α.iii) Extension via employing multiple local sections Γ for nontrivial bundles (Article 1).

Example α.iv) Extension via stacking cycles with cancelling cross-cuts by which a big region is covered by many elementary regions.

Remark 2 Yet these well-understood techniques does not in general carry over to the following.

Example α.v) Extension via patching together local solutions of PDEs holding on $\mathfrak{M}$. I.e. patching in a function space context is a further, detail-dependent and in general unresolved globalization.

Aside from patching together solutions to dynamical DEs of Physics, this also includes patching together solutions to the obsevables PDEs (Article 2).

Example α.vi) Extension via patching together approximation regimes (Article 0), of which *connection formulae for going between WKB regions* is a well-known prototype.

Example α.vii) Extension by adding variables to a state space.

Example α.viii) Extension by adding generators to an algebraic structure.

22.3 Examples of Globalize by Restriction

Example β.i) Replace a manifold by a local region whose entire image lies within a single chart.

Example β.i$'$) Replace a manifold by a quotient thereof by a group.

Example β.ii) Replace a fibre bundle by a local product space.

Example β.iii) Replace a global section by a local section.

Example β.iv) Deform a larger cycle to a smaller one, or a more extensive region to a less extensive one, in each case while continuing to succeed in capturing the problem at hand. Well-known cases of this include deforming contours to small discs and pieces of thin cross-cuts in Complex Analysis, and making do with relative (co)homology [131] in place of its global absolute counterpart. Making do with a Lie algebra in place of a Lie group can also be viewed in this way.

Example β.v) Restriction by removing variables from a state space.

Example β.vi) Restriction by adding generators to an algebraic structure.

22.4 Examples of Globalize by Replacement

Example γ.i) Pass from product spaces to fibre bundles.

Example γ.ii) Supplant the role played by a manifold $\mathfrak{M}$ by a singular manifold such as stratified manifold $\mathfrak{X}_{\mathrm{Strat}}$ (Article 0), as is often necessitated by *Quotientize*.

Remark 1 There is further increase in intractability in passing from singularities of solutions on a fixed background manifold versus singularities of manifolds themselves (e.g. GR singularities).

Example γ.iii) Pass from fibre bundles to generalized bundles, presheaves or sheaves [165, 171] (Article 0). This is often for instance a knock-on effect of Example γ.ii).

Example γ.iv) Pass from a Lie algebra $\mathfrak{g}$ to a Lie group G. This could furthermore be at the level of this structure acting on a second space $\mathfrak{M}$ or $\mathfrak{X}$.

Remark 2 Almost all the information is kept however [150]. This is via the Hausdorff Lie-Globalization Theorem and the Normal Subgroups–Ideals Correspondence Theorem [150]. The remaining information is topological, in particular.

A) whether one has a single or multiple cover.

B) How many connected components the group has.

C) Complications due to noncompactness effects, for instance when the action involved ceases to be *proper* (see [199] for a definition and e.g.[161] for futher discussion).

Example γ.v) Another way a Lie algebra $\mathfrak{g}$ can become less local is by passing to a Lie algebroid. This allows for a different action on each point of a manifold $\mathfrak{M}$, say.

Example γ.vi) We can now complete the square by either passing from a Lie group to a Lie groupoid or from a Lie algebra to a Lie algebroid.

Example γ.vii) Some choices of function space are moreover better-suited for whichever of one type of PDE or for global considerations. If not using such a function space at the outset, one can view reformulation in terms of a such as a case of 'Globalization by Replacement'.

Example γ.viii) Replace a single generator algebraic structure by multiple local generator algebraic structures.

Example γ.ix) Replace a single observables algebra by multiple local observables algebras.

Example γ.x) Replace a single deformed algebraic structure by multiple locally-dependent deformation algebraic structures.

Example γ.xi) Replace local embedding theorems (such as Janet–Cartan's [55] or Whitney's [83]) by global ones.

Example γ.xii) Pass from a slice chart to a foliation (each of these notions is explained in Article A).

Example γ.xiii) *Recategorize* [i) to vii) can be further viewed as simple subcases of this].

22.5 Examples of Discards due to global obstructions

Example δ.i) Obstruction by the zeros of a function.

Example δ.ii) Obstruction by presence of nontrivial GKVs.

Remark 1 The above two examples are jointly covered in Article 1, on account of occurring side by side in GR's Thin Sandwich Problem [102]. Obstruction by zeros might be resolvable, but the GKVs correspond to nontrivial stratification which carries both geometrical and topological implications.

Example δ.iii) Obstruction by only locally existing, only being locally unique, or only being a local maximum or minimum in the case of an extremum.

Example δ.iv) Cohomological obstructions. A first batch of cohomologies of note as regards the current Series is as follows.

a) *Poisson cohomology* [162, 121], of relevance to Closure.

b) *BRST cohomology* [97], also of relevance to Closure,

c) *Lie algebra mapping cohomology* is of relevance to Constructability: rigidity here is in turn is encoded by certain maps between Lie algebras' cohomology [41, 44].

d) Whether fibre bundles possess a global section can often be insightfully expressed in *cohomological* terms [107] (and Sec 9 for an outline) and gives rise to the theory of *characteristic classes* [94, 107].

e) The *Gribov effect* ([97] and Article 0) for non-existence and non-uniqueness in use of sections in Gauge Theory and the *Haefliger cocycle* reformulation of foliations ([137] and Article A) both feed into the standard de Rham cohomology.

Remark 2 Some deeper underlying No-Go Principle's presence can be symptomized by *persistent cohomology* [165]: some obstructions just persist under modelling upgrades.

Example δ.v) Algebraic structures can pick up *anomalies*, of relevance to both Classical and Quantum Closure.

Example δ.vi) *Strong vanishing* is by causing a complication to go away by judicious fixing of constant coefficients. On occasion, this can even remove anomalies. E.g. this is how String Theory acquires extra spacetime dimensions – by fixing coefficients – in a trade-off for anomalous terms dropping out.

22.6 Globalize by Local–Global Inter-relation

These include the following.

Example $\boldsymbol{\eta}$.i) *Morse Theory* [118] infers global information about a whole space from local information about critical points of functions thereupon.

Example $\boldsymbol{\eta}$.ii) Use of *Index Theorems* [94], in particular in the context of Closure's anomalies.

Example $\boldsymbol{\eta}$.iii) Use of sheaves [165, 171] is of relevance to Relationalism's r-spaces and possibly also to the study of observables. Further cohomologies to take not of at this point are as follows.

f) *Sheaf cohomology* [171, 86] then enters the growing list of cohomological applications.

g) Within the HP space remit of the current Series, sheaves are *soft* [171] (a particular sharpening of Remark 5.iii) of Sec 4's 'nice'). For these, sheaf cohomology reduces to just *Čech cohomology* (a simpler and older combinatorial type of cohomology outlined in Article 0).

Remark 1 With (topological space, sheaf pairs) under consideration, the following other pairs may be of some use; these are γ's not η's through not having extral local–global interplay.

Example $\boldsymbol{\gamma}$.xiv) Use of (topological space, general bundle pairs).

Example $\boldsymbol{\gamma}$.xv) Use of (topological space, presheaf pairs).

23 Summary and frontiers

23.1 Global summary symbol

Some strategies for Global Problems of Time are as follows. Clarity benefits from labelling each Global Problem by the space S it is on, the extent E of globality thereupon, the affected quantity Q and whether it is a specific property P of the quantity that is affected. Let us encompass this by the *global summary symbol* (S, E, Q, P), omitting the final entry if no property is involved.

23.2 Nature of Dynamical Law

It should also be clear we are not just describing a 'quaint problem from the 1960's as reviewed by Kuchař and Isham in the 1990s called the Problem of Time'. We are describing, rather, the Nature of Physical Law in a rather more advanced manner than has been possible before. This arising from the mistakes made from the 1960s through to 2017 on the matter of Problem of Time being -interesting enough- to point us in the right direction.

23.3 And the quantum?

Great Map 3

$$\textit{Quantize} \ : \ \ \text{(something like a Poisson algebraic structure)} \ \longrightarrow \ \text{(something like a } C^* \text{ algebraic structure)} \ . \tag{132}$$

In general, this is not even known to be functorial. There is much more control over the input space. 'Something like a C^* algebraic structure' [121] refers to a another branch of Functional Analysis. Furthermore, to reach the entirety of Quantum Theory's levels of structure, further maps and spaces are required. These can be envisaged as (something like) C^* algebra's analogue of the Lie claw. We present this in Article A; see [174] for some quantum-level global issues with Background Independence and the Problem of Time.

The idea that there is a small subset of Mathematics that suffices for doing Physics is destroyed by *Arenize*. For this quite simply does not preserve such a subset of mathematics. Knowing Topology, that is not enough, for the space of topologies on a fixed finite set (say) is a lattice. (This was mathematically known in the 1960s [43]. It was first mentioned in the theoretical physics literature by Isham in the 1980s [91]). That the space of metric spaces is itself a metric space (by carrying the Gromov–Hausdorff metric [125]) is the exception, not the rule. QM moreover unfolds on configuration space (or some other *half-polarization* mathematical-alias *Lagrangian-submanifold* portion of a symplectic space). *Arenize* means that more than the usual branches of Mathematics enter Quantum Physics at this point.

23.4 Functional Analysis round-up

More advanced Physics can be modelled by the following types of Functional Analysis.

1) Sobolev spaces [169, 146, 148] on spacetime or space.

2) Tame Fréchet spaces [75, 158, 185] on configuration and phase state spaces.

3) C^* algebras or similar [60, 121, 78, 105] (for use in the Quantum realm).

4) Sheaves [165, 171, 86] are a type of function space that can be furthermore profitably viewed, firstly, as a type of functor category. Secondly, as a mathematically powerful way of attaching data to each point on a given space.

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