Lie Theory suffices for Local Classical Resolution of the Problem of Time. 2. Observables as Function Space Algebras of Lie Bracket Commutants.

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Abstract

The Problem of Time is due to conceptual gaps between General Relativity and the other observationally-confirmed theories of Physics; it is a major foundational issue in Quantum Gravity. We already set this up at the classical level in Article 0's introductory and Relational treatment, and found algebra to start to dominate at the level of Closure in Article 1. We now classically resolve the further observables part of the Problem of Time at the local level. Observables are to be viewed as suitably-smooth functions over state spaces. In the presence of Lie algebra generators (including constraints in the canonical case), observables are furthermrore Lie brackets commutants with these. Observables form function spaces which are themselves Lie algebraic structures The Lie bracket commutant conditions can furthermore be recast as first-order PDE systems, to which the flow method can be applied. More specifically, such PDEs constitute a slight variant of Lie's intergal approach to geometrical invariants. Observables come in various strengths, firstly as regards how large a subalgebra of generators they commute with, and secondly as regards whether the commutation is strong or weak, in a sense following Dirac's. The first of these is lattice-valued whereas the second amounts to adding a particular integral term to one's PDE system. We illustrate observables with geometrical, canonical and spacetime examples, and also give technical reasons why field theories are harder than finite ones.

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1 Introduction

Having implemented the Relationalism [16, 17, 30, 59, 64, 70, 78, 79, 82, 83] and Closure [9, 13, 20, 38, 70, 80, 84, 86] part of Background Independence [19, 21, 54, 70, 77, 87] in [88, 89] – thus locally resolving five facets of the Problem of Time [22, 23, 20, 28, 37, 39, 40, 47, 58, 59, 60, 63, 73, 70, 78, 79, 80, 81, 82, 83, 84] at the classical level – we can now turn to the semi-decoupled Observables part [8, 14, 38, 39, 40, 42, 51, 52, 63, 70, 74, 76, 80, 85]. Observables are useful objects due to their physical content.

The purest form taken by the **Assignment of Observables** part of Background Independence is [63, 70, 80], given a state space $\mathfrak{S}$, **Taking Function Spaces Thereover**. These functions necessitate some minimum standard of smoothness, so as to be able to subsequently perform all physically necessary calculations. General, spacetime [86], geometrical [74] and canonical [76, 85] cases of unrestricted Assignment of Observables are outlined in Sec 2. While this looks trivial at the classical level, its quantum counterpart is nontrivial [33, 40, 50].

Above, 'purest form' and 'unrestricted' refer to lack of interplay with other aspects of Background Independence (alias lack of interference from other facets of the Problem of Time). In the presence of generators $\mathcal{g}$ (including constraints in the canonical case, and amounting to Closure interfering), a second source of nontriviality appears (Sec 3). This is *zero-commutant Lie brackets equations* [8, 42, 70, 80, 85, 86] already mentioned in [88]: now observables $\mathbf{O}$ furthermore obey

$$[\mathcal{g}, \mathbf{O}] \text{ `=' } 0 \,. \tag{1}$$

In the presence of generators, a Jacobi identity gives moreover that Assignment of Observables only makes sense *after* Closure is attained. This is what we mean above by 'semi-decoupled', and is indicated by the rightwards arrow only in the Lie claw digraph (Fig 1.3.d). This is the first great decoupling of the Problem of Time (and of its Background Independence implementing local resolution). By this, Relationalism–and–Closure can be implemented first without reference to observables. This can then be followed up by a further piece of mathematics – Assignment of Observables – in which a function space is found over the preceding implementation's state space. 'Taking Function Spaces Thereover' moreover continues to serve as a complete characterization in the presence of Lie generators. This is by the brackets equation (1) amounting to a restriction to the preceding Closure's reduced state space. The above two paragraphs *correct* the earlier suggestion [39, 40, 42] of (1) being observables' primary content. This is by identifying (1) to be, rather, a major interplay or interference between Closure and the simpler primary notion of Observables indicated above.

Observables furthermore come in various strengths, firstly as regards whether the commutation is strong or weak [63, 80, 85]. Indeed, our '=' symbol in 1 is a portmanteau of strong equality's = and weak equality's ≈. The latter broadens Dirac's sense of weak [20] from 'up to a linear function of constraints' to the generalized Lie setting's [86] 'up to a linear function of generators'. Secondly, as regards how large a subalgebra of generators they commute with. This ranges from unrestricted observables' absense of generators to the full case of all generators (of which

Dirac observables are the canonical case). There are furthermore 'middling' notions of observables between these two extremes [42, 67, 69, 70, 80]. These correspond to all the other consistent generator subalgebraic structures supported by a given theory-and-formulation. All in all, generator subalgebraic structures form a bounded lattice, with no generators as bottom and all generators as top. The notions of observables supported by that theory-and-formulation then constitute the dual lattice to that [69, 70, 76, 80]. Here, unrestricted observables are top: the largest and yet now simplest space, whereas maximally restricted observables are bottom: the smallest and yet most complicated space. This combination of smallest size and yet greatest complication is of course to be expected given that a mathematical reduction procedure is at work.

Sec 4 then converts from the brackets equation formulation to the corresponding explicit PDE system perspective. More specifically, (1) can be recast as [67, 74, 76, 85] an explicit first-order linear PDE systems, to which the Flow Method [5, 18, 29, 43, 61, 74, 76] is well-suited. Such flows are furthermore usefully envisaged as *congruences of integral curves* in the field of Differential Geometry [35, 61], and as *1-parameter subgroups* in the study of Lie groups [56, 35, 65]. Our particular application of the Flow Method to Assignment of Observables is, in further detail, a slight extension of [74, 76, 85] Lie's *Integral Approach to Geometrical Invariants* [5, 18, 43, 62]. The required outputs are moreover not individual observables functions that solve these brackets equations or corresponding PDE systems. They are, rather whole *algebras of observables* [74, 75, 76], which are the alluded-to function spaces and also constitute entire solution spaces for each observables PDE system.

Sec 5 provides specific examples from Flat Geometry [74] and Relational Particle Mechanics [76, 85]. These examples are finite (in the sense of finitely generated). Field Theory counterparts (infinitely generated) are given in Sec 6, including the situation for GR. Now one has not PDEs but FDEs – functional differential equations – as observables equations. Nonetheless, similar Flow Methods carry over.

In Sec 7, we consider the further difference between Assignment of Observables and the more practical matter of Expression in Terms of Observables. The latter involves expressing a sufficient set of physical quantities as functions of observables alone, 'sufficient' meaning for the purpose of answering all physical questions. Namely, the ones by which each physical quantity is to end up depending on the 'basis set' of observables alone. Suppose obtaining a sufficient set of observables to do Physics is in practice blocked: a common occurrence in Gravitational Theory. Then one has a **Problem of Observables** [39, 40, 63, 70] (Sec 8).

2 Unrestricted observables

Principle 1 Given a state space $\mathfrak{S}(\mathfrak{S})$ for a system $\mathfrak{S}$, the most primary notion of observables is *Taking a Function Space Thereover*,

$$\mathfrak{F}\text{unction}(\mathfrak{S}(\mathfrak{S})) \ . \tag{2}$$

Structure 1 $\mathfrak{S}$ has base objects B, for instance space coordinates, spacetime coordinates, configuration coordinates or phase space coordinates. We can then write

$$\mathsf{U}(\mathsf{B}) \tag{3}$$

as a general functional form for the corresponding notion of unrestricted observables.

Structure 2 A simple choice of smoothness we make in the current non-global Article is

$$\mathfrak{S} \ \longrightarrow \ \mathfrak{C}^{\infty}(\mathfrak{S}(\mathfrak{S})) \ , \tag{4}$$

for $\mathfrak{C}^{\infty}$ the smooth functions in the sense of infinite differentiablility. Some rougher choices are also practicable; Sec 4.3 gives a reason for $\mathfrak{C}^1$ – once continuously differentiable functions – constituting a minimal standard of smoothness for observables. This amounts to saying that the *space of unrestricted observables* can be mathematically identified with the above space:

$$\mathfrak{U}\text{nres-}\mathfrak{O}\text{bs}(\mathfrak{S}) \ = \ \mathfrak{C}^{\infty}(\mathfrak{S}(\mathfrak{S})) \ . \tag{5}$$

Remark 1 At the classical level, this is a trivial extension of the theoretical framework. At the quantum level, however, self-adjointness and Kinematical Quantization already impinge at this preliminary stage [33, 70].

Remark 2 In this Series, we use sans serif font to pick out notions of observables. We denote the corresponding spaces of observables by $\mathfrak{O}$bs with suitably descriptive prefixes.

Example 0 Unrestricted N-point observables for Flat Geometry are

$$\mathfrak{U}\text{nres-}\mathfrak{O}\text{bs}\left(\mathfrak{q}\left(\mathbb{R}^3, N\right)\right) \ = \ \mathfrak{C}^{\infty}\left(\mathfrak{q}\left(\mathbb{R}^3, N\right)\right) \ = \ \mathfrak{C}^{\infty}\left(\times_{i=1}^{N}\mathbb{R}^3\right) \ = \ \mathfrak{C}^{\infty}\left(\mathbb{R}^{3\,N}\right) \ . \tag{6}$$

Example 1 Unrestricted N-point spacetime observables for SR are

$$\mathfrak{Unres}\text{-}\mathfrak{Obs}\left(\mathfrak{S}\left(\mathbb{M}^4, N\right)\right) = \mathfrak{C}^\infty\left(\mathfrak{S}(\mathbb{M}^4, N)\right) = \mathfrak{C}^\infty\left(\times_{i=1}^{N}\mathbb{M}^4\right) . \tag{7}$$

Example 2 Unrestricted spacetime observables for GR are however a much taller order,

$$\mathfrak{Unres}\text{-}\mathfrak{Obs}(\mathfrak{PRiem}(\mathfrak{M})) = \mathfrak{C}^\infty(\mathfrak{PRiem}(\mathfrak{M})) . \tag{8}$$

Example 3 Unrestricted spatial observables for GR are

$$\mathfrak{Unres}\text{-}\mathfrak{Obs}(\mathfrak{Riem}(\boldsymbol{\Sigma})) = \mathfrak{C}^\infty(\mathfrak{Riem}(\boldsymbol{\Sigma})) , \tag{9}$$

which at least reside on a topologically better behaved space.

Example 4 Unrestricted N-point observables for the canonical formulation of Mechanics are

$$\mathfrak{Unres}\text{-}\mathfrak{Obs}\left(\mathfrak{Phase}\left(\mathbb{R}^3, N\right)\right) = \mathfrak{C}^\infty(\mathfrak{Phase}(\mathbb{R}^3, N)) = \mathfrak{C}^\infty\left(\times_{i=1}^{2N}\mathbb{R}^3\right) = \mathfrak{C}^\infty\left(\mathbb{R}^{6\,N}\right) ; \tag{10}$$

the restriction of these to N-point spatial observables coincides with Example 0.

Example 5 Unrestricted canonical observables for GR are

$$\mathfrak{Unres}\text{-}\mathfrak{Obs}(\mathfrak{Phase}(\boldsymbol{\Sigma})) = \mathfrak{C}^\infty(\mathfrak{Phase}(\boldsymbol{\Sigma})) \tag{11}$$

3 Restricted observables at the Lie brackets level of structure

Bracket (1) holds here. This zero Lie commutant equation is motivated as follows. In a theory with a redundancy group acting with the state space, quantities restricted to commute with these generators make for more useful modelling than just any function(al)s of the B. This is due to these quantities containing somewhere between more and exclusively just non-redundant modelling information.

Lemma 1 The *Jacobi identity* [2, 27] applied to two generators and one observable [63, 70]

$$\mathbf{J}(\boldsymbol{\mathcal{G}}, \boldsymbol{\mathcal{G}}, \mathsf{O}) = |[\boldsymbol{\mathcal{G}}, |[\boldsymbol{\mathcal{G}}, \mathsf{O}]|]| + \text{cycles} = 0 \tag{12}$$

signifies that finding Lie-brackets-commuting notions of observables is only consistent if the $\boldsymbol{\mathcal{G}}$ in question are already known to close. Thereby, in the presence of generators the Problem of Time semi-decouples into the Relationalism-and-Closure subproblem which has to be solved prior to Assignment of Observables. This explains Fig 1.3.d)'s bottom 1-way arrow. This is the first great decoupling of Problem of Time facets (and correspondingly of its local resolution's Background Independence aspects). This partly resolves the facet-ordering problem [39, 40, 42].

Lemma 2 The Jacobi identity on, reciprocally, one generator and two observables [8] is

$$\mathbf{J}(\boldsymbol{\mathcal{G}}, \mathsf{O}, \mathsf{O}) = 0 . \tag{13}$$

This enforces that observables cannot be treated piecemeal. Rather, they need to be treated as entire algebraic structures. (These are concurrently function spaces and algebraic structures.) The observables form an associated algebraic structure with respect to the same bracket operation as the preceding constraints algebraic structure,

Definition 1 Let us write our observables algebraic structure as

$$|[\,\underline{\mathsf{O}}, \underline{\mathsf{O}}'\,]| = \underline{\underline{O}} \cdot \underline{\mathsf{O}}'' , \tag{14}$$

for *observables algebra structure constants* $\boldsymbol{O}$.

Analogous Example 0 When the commutants are formed from the generators themselves, they are known as *Casimirs* [65]. These play a prominent role in Representation Theory, with $SU(2)$'s total angular momentum operator J^2 constituting the best-known such.

Remark 1 Observables algebras $\mathfrak{Obs}$, like constraint algebraic structures $\mathfrak{C}$, are comparable to configuration spaces $\mathfrak{q}$ and phase spaces $\mathfrak{Phase}$. This is as regards the study of the the nature of Physical Law, and whose detailed structure is needed to understand any given theory. This refers in particular to the topological, differential and higher-level geometric structures observables algebraic structures support. Function space and algebraic levels of structure are now additionally relevant.

Remark 2 We consequently need to focus on observables algebraic structures' own Tensor Calculus. This justifies our keeping observables tensors visibly distinct from generator ones and base object ones, by use of undertildes. In fact, my notation incorporates the increased abstraction of each type of tensor: no turns for base objects, one turn for generators and two turns for observables.

Remark 3 Everything we present in the current section applies in particular to geometrical observables, spacetime observables and canonical observables.

3.1 Strong and weak observables

Definition 1 Let us now furthermore distinguish between *strong observables*, obeying

$$|[\mathcal{g}, \mathbf{G}]| \ = \ 0 \ , \tag{15}$$

and *weak observables*, obeying

$$|[\underset{\sqcup}{\mathcal{g}}, \underset{\sim}{\mathbf{G}}]| \ = \ \underset{\underset{\sqcup}{\sim}}{\underset{\sqcup}{\boldsymbol{W}}} \cdot \underset{\sqcup}{\mathcal{g}} \ . \tag{16}$$

This is the Lie-generator-weak generalization [86] of Dirac's constraint-weak notion [9]. The $\boldsymbol{W}$ are 'weak observables' defining relation's structure constants'. Undertilde is our coordinate-free notation for an index running over (some type of) observables.

Remark 1 It should already be clear that what the weak observables equation is doing is adding an inhomogeneous term to a linear equation. By this, *properly weak observables* (i.e. those that are not also strong observables) can be interpreted as a particular integral (PI) to strong observables' complementary function (CF).

3.2 Fully restricted observables

Structure 1 The opposite extreme to imposing no restrictions is to impose all of a modelling situation's first-class generators. This returns the *full observables* $\mathtt{F}$ obeying

$$|[\mathcal{F}, \mathtt{F}]| \ \text{`='} \ 0 \ . \tag{17}$$

Structure 2 The *space of full observables* is

$$\mathfrak{F}\text{ull-}\mathfrak{O}\text{bs}(\mathfrak{S}) \ , \tag{18}$$

or, more explicitly, $\mathfrak{F}\text{ull-}\mathfrak{O}\text{bs}(\mathfrak{S}(\mathfrak{S}), \varepsilon)$.

Example 1 In the canonical setting, imposing all the first-class constraints gives full observables, which are here better known as *Dirac observables* [8]. $\mathfrak{D}\text{irac-}\mathfrak{O}\text{bs}(\mathfrak{p}\text{hase}, \mathcal{H})$ is then an alias for $\mathfrak{F}\text{ull-}\mathfrak{O}\text{bs}(\mathfrak{p}\text{hase}, \mathcal{H})$.

Example 2 One can place functions over spacetime which are $Diff(\mathfrak{m})$-invariant. I.e. Lie-brackets commutants with $Diff(\mathfrak{m})$'s generators,

$$[\,(\,\boldsymbol{\mathcal{D}}\,|\,\mathbf{Y}\,)\,,(\,\mathbf{S}\,|\,\mathbf{Z}\,)\,]_{\mathrm{S}} \ = \ 0 \tag{19}$$

for spacetime smearing variables $\mathbf{Y}$ and $\mathbf{Z}$, or perhaps the generator-weak equality extension of this equation.

Structure 2 These $\mathbf{S}$ form an infinite-d Lie algebra

$$\mathfrak{D}\text{iff-}\mathfrak{O}\text{bs}(\mathfrak{M}) \ . \tag{20}$$

Notions of spacetime observables more generally form a dual lattice to that of spacetime generator subalgebraic structures.

Remark 1 The unrestricted and full notions of observables are universal, in the sense that all models possess them.

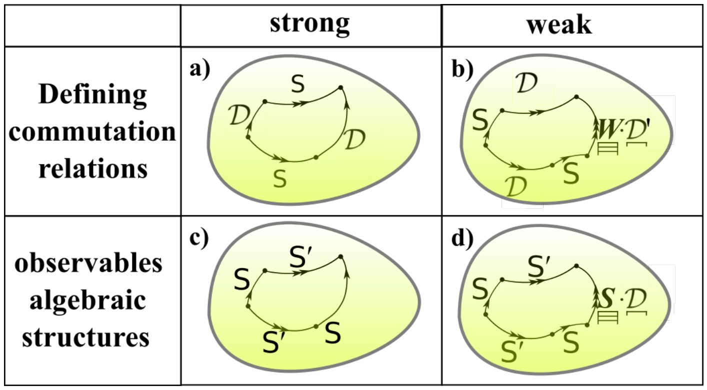

Figure 1: Observables' defining commutation relations and algebras.

3.3 Middling observables

Structure 1 Some physical theories moreover support further intermediate notions of classical observables. These correspond to each closed subalgebraic structure that the generator algebraic structure possesses. For instance, in the canonical case, we have the following.

Definition 1 *Kuchař observables* [42] are quantities which form zero classical brackets with all of a given theory's first-class linear constraints,

$$\{\, \mathcal{F}\mathbf{lin}, \mathsf{K} \,\} \ \text{‘='} \ 0 \ . \tag{21}$$

Example 1 These cover GR-as-Geometrodynamics, for which the $\underline{\mathcal{M}}$ close by (1.78). These moreover consist of Lie-brackets commutants with $Diff(\mathbf{\Sigma})$'s generators,

$$\{\, (\, \boldsymbol{\mathcal{M}} \,|\, \mathbf{L} \,) \,, (\, \boldsymbol{K} \,|\, \mathbf{F} \,) \,\} \ = \ 0 \ . \tag{22}$$

L and **F** are spatial smearing variables. Or perhaps the weak equality extension of eq. (22). Note the parallel with the preceding subsection's Example 2.

Remark 1 Whereas $\mathcal{F}$**lin** $=$ $\mathcal{G}$**auge** in the more commonly encountered cases, [38, 70, 84]'s counter-examples imply need for the following further notion.

Definition 2 $\mathfrak{g}$-observables alias *gauge-invariant quantities*

$$\mathsf{G} \ \ \text{indexed by} \ \ \mathsf{J} \tag{23}$$

obeying

$$\{\, \mathcal{G}\mathbf{auge}, \mathsf{G} \,\} \ \text{‘='} \ 0 \ . \tag{24}$$

Example 0 For unconstrained theories,

$$\mathsf{D} \ = \ \mathsf{K} \ = \ \mathsf{G} \ = \ \mathsf{U} \ . \tag{25}$$

Definition 3 In cases in which $\mathcal{C}$hronos self-closes, a notion of Chronos observables

$$\mathsf{C} \ \ \text{indexed by} \ \ \mathsf{H} \tag{26}$$

becomes meaningful. These obey

$$\{\, \mathcal{C}\text{hronos}, \mathsf{C} \,\} \ \text{‘='} \ 0 \ . \tag{27}$$

Example 2 For Minisuperspace and Spatially-Absolute Mechanics, the

$$\mathsf{K} \ = \ \mathsf{G} \ = \ \mathsf{U} \tag{28}$$

are also trivially any quantities of the theory since these theories have no linear constraints at all. The

$$\mathsf{D} \ = \ \mathsf{C} \ , \tag{29}$$

however, are nontrivial due to the presence of the constraint $\mathcal{C}$hronos.

Example 3 For RPMs, both $\mathcal{F}\mathbf{lin} = (\underline{\mathcal{P}}, \underline{\mathcal{L}})$ and $\mathcal{C}$hronos $= \mathcal{E}$ close separately. So both notions of Kuchař and Chronos observables are supported. In contrast, $\mathcal{C}$hronos $= \mathcal{H}$ does not self-close for inhomogeneous GR. Hance only Kuchař observables are supported in this case.

Structure 2 In each case closure permitting (supported by some theories but not others), the *space of classical Kuchař observables* is

$$\mathfrak{K}\text{uchar-}\mathfrak{O}\text{bs}(\mathfrak{S}) \ . \tag{30}$$

The *space of classical gauge observables* is

$$\mathfrak{G}\text{auge-}\mathfrak{O}\text{bs}(\mathfrak{S}) \ , \tag{31}$$

and *the space of Chronos observables* is

$$\mathfrak{C}\text{hronos-}\mathfrak{O}\text{bs}(\mathfrak{S}) \ . \tag{32}$$

Each of these spaces, when supported, furthermore constitutes an algebra.

Structure 3 The totality of notions of observables form a

$$\text{bounded lattice } \mathfrak{L}_{\mathsf{O}} \text{ dual to that of constraint algebras , } \mathfrak{L}_{\mathcal{C}} \ . \tag{33}$$

Its top and bottom elements are the Dirac D and unrestricted U notions of observables. The middle elements are *A observables*, $\mathcal{A}$ with each type of such a theory supports indexed by Z. Thus G comprises U, A_{Z} and D, arranged to form $\mathfrak{L}_{\mathsf{O}}$.

Remark 2 Kuchař observables are a middle element. The form taken by the general middle is more generally algebraically determined. It takes the form of [63, 69, 70, 74, 75] a bounded lattice [46] with unrestricted and full observables as its top and bottom. Indeed, for some theories, the $\mathcal{F}\mathbf{lin}$ do not close. E.g. it does not for Supergravity. So this is a theory for which Kuchař observables are ill-defined (Example 4). Kuchař observables are thus in general supplanted by a collection of notions *A observables* [63, 70, 80] that form our lattice's middle.

Structure 4 The totality of observables algebras consequently also form a

$$\text{bounded lattice } \mathfrak{L}_{\mathfrak{O}} \text{ dual to that of constraint algebraic structures , } \mathfrak{L}_{\mathfrak{C}} \ . \tag{34}$$

The algebra of unrestricted observables is the top alias unit element, and the algebra of Dirac observables is the bottom alias zero element. All other elements are middle elements: the *A observables algebraic structures*, denoted by

$$\mathfrak{A}\text{-}\mathfrak{O}\text{bs}(\mathfrak{S}) \text{ with each type indexed by } \mathsf{Z} \ . \tag{35}$$

Thus $\boldsymbol{\mathfrak{O}}\text{bs}(\boldsymbol{\mathfrak{S}})$ comprises $\mathfrak{U}\text{nres-}\mathfrak{O}\text{bs}(\mathfrak{S})$, $\mathfrak{A}\text{-}\mathfrak{O}\text{bs}(\mathfrak{S})_{\mathsf{Z}}$ and $\boldsymbol{\mathfrak{D}}\text{irac-}\mathfrak{O}\text{bs}(\mathfrak{S})$, arranged to form $\mathfrak{L}_{\mathfrak{O}}$.

Remark 3 The sizes of the spaces run in the dual lattice pair run in opposition. I.e. the bigger a constraint algebraic structure, the smaller the corresponding space of observables is. This is clear enough from constraints acting as restrictions, adding PDEs that the observables must satisfy...

Remark 4 We have noted Background Independence's algebraic centrality. It then becomes rather clear that *not theories* but lattice of constraints subalgebraic structures are where key determining theoretical content lies. One can moreover equivalently take the dual lattice of observables subalgebraic structures as key content. I.e. on the one hand, theories whose constraints-or-observables subalgebraic structures differ can manifest interestingly different realizations of Background Independence. On the other hand, ones in the same such class do not. [90] adds the deformations and pentagon diagram posing the generalization of Refoliation Invariance to the key features by which theories can realize Background Independence in interestingly different ways. Some of these additionas are now moreover *obstructions* to fully realizing Background Independence.

Remark 5 See [85] for consideration of joint constraints-and-observables algebraic structures.

Remark 6 In detailed calculations, both multiple types of generator and of observable may need to be kept distinct in calculations involving multiple thereof. I personally do this using a multicoloured Penrose birdtrack presentation of coordinate-free Multitensor Calculus; readers will see this explicitly in v4 of this Series' Articles.

4 Observables PDE systems

Each notion of observables' defining zero-commutation relations can be recast as [67, 70, 74, 76, 85] an explicit first-order linear PDE system. (For unrestricted observables, this is just the empty system.) In the canonical case, this is by expanding out the definition of Poisson brackets [67]. In the spacetime case, the same kind of equations arise instead from representing the generators as first-order differential operators [74].

4.1 First-order PDE systems

Remark 1 Lie's own work [5] is foremost on differential equations. He considered in particular first-order PDEs. For a single such PDE

$$\sum_{A=1}^{N} a^A(x^B, \phi)\partial_A\phi \;=\; b(x^B, \phi) \,, \tag{36}$$

Lie made use of locally straightening out [7, 11, 35] with respect to one variable, by which integration can be carried out. For a system of $A \;=\; 1$ to M such PDEs,

$$\sum_{B=1}^{N} a^{AB}(x^C, \phi)\partial_B\phi \;=\; b^A(x^C, \phi) \tag{37}$$

straightening out can moreover be done one equation at a time.

Six structural developments branch off at this point.

1) Each such PDE system $\mathscr{P}$ admits an ODE formulation by use of *Lagrange's Method of Characteristics* [1, 15, 29]. This moreover admits a more modern and Differential-Geometrical '*flow*' reinterpretation [5, 29, 35, 62, 61, 85]. Via '**local Lie dragging**' – a notion subsequently formalized as the Lie derivative [11, 24, 88] – *integral curve* flowlines ensue.

Definition 1 An *integral curve* (see e.g. [35]) of a vector field $\mathbf{V}$ in a manifold $\mathfrak{m}$ is a curve $\gamma(\nu)$ such that the tangent vector is $\mathbf{V}_{\mathrm{p}}$ at each p on γ (Fig 2.a).

Remark 2 Local existence-and-uniqueness follows [61] from standard ODE theory.

Remark 3 A set of complete integral curves corresponding to a non-vanishing vector field is called a *congruence*. This 'fills' a manifold or region therein upon which the vector field is non-vanishing.

There is relation between '**infinitesimal transformation groups**' $\mathfrak{g}_{\mathrm{T}}$ and such PDE systems $\mathscr{P}$, as follows.

2)**Forward route** Differentiation and elimination provides a $\mathscr{P}$ for each $\mathfrak{g}_{\mathrm{T}}$ [5].

3) **Second backward route 1** The above integral method sends such $\mathscr{P}$ back to the $\mathfrak{g}_{\mathrm{T}}$ of solutions for it.

4) **Second backward route 2** Given a geometry, a specific geometrical $\mathscr{P}$ – the generalized Killing equation [6, 11, 24] – can moreover be constructed whose solutions form the corresponding geometrical $\mathfrak{g}_{\mathrm{T}}$.

Remark 4 The 1-parameter subgroup's generator for a flow $\gamma(\nu)$ is realized as the tangent vector $\gamma'(0)$.

Remark 5 Proceeding along two local congruences of integral curves in either order (Fig 2.b) produces, to leading order, the *(Differential-Geometric) commutator*

$$x^i_{\mathrm{v}} \;-\; x^i_{\mathrm{u}} \;=\; [\,\boldsymbol{X}, \boldsymbol{Y}\,]\, \mathrm{d}\mu\, \mathrm{d}\nu \;+\; O(\mathrm{d}^3) \;. \tag{38}$$

5) Our $\mathfrak{g}_{\mathrm{T}}$'s can be built up from its 1-parameter subgroups – themselves a further interpretation for flows – by checking how each's generators commutators work out.

6) A more modern presentation of 'straightening out' is in terms of the **exponential map** [61] applied to each of our 1-parameter subgroups.

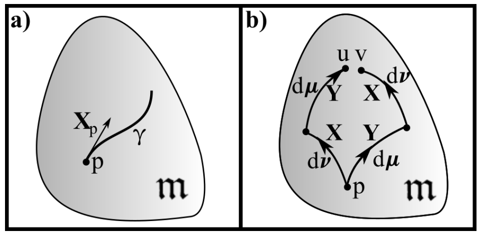

Figure 2: a) Integral curve on a manifold. b) Commutator corresponding to proceeding along two local congruences of integral curves in either order.

4.2 Strong observables PDEs

Strong observables' defining commutation relation gives

$$0 \;=\; \{\boldsymbol{c}, \mathsf{O}\} \;=\; \frac{\partial \boldsymbol{c}}{\partial \boldsymbol{Q}}\frac{\partial \mathsf{O}}{\partial \boldsymbol{P}} \;-\; \frac{\partial \boldsymbol{c}}{\partial \boldsymbol{P}}\frac{\partial \mathsf{O}}{\partial \boldsymbol{Q}} \;. \tag{39}$$

We can moreover take

$$\frac{\partial \boldsymbol{c}}{\partial \boldsymbol{Q}} \;\text{ and }\; \frac{\partial \boldsymbol{c}}{\partial \boldsymbol{P}} \;\text{ to be knowns .} \tag{40}$$

This leaves us with a homogeneous-linear first-order PDE system,

$$\sum_A a^A(x^B, \phi)\partial_A\phi \;=\; 0 \;: \tag{41}$$

as a a subcase of (37). First order their containing first-order partial derivatives $\partial_\alpha\phi$ and no higher. Linear refers to the unknown variables O.

Remark 1 As the strong case involves a homogeneous equation,

$$\mathsf{O} \;=\; \text{const} \;, \tag{42}$$

is always a solution. We refer to this as the *trivial solution*. We call all other solutions of first-order homogeneous quasilinear PDEs *proper solutions*: a nontrivial kernel condition.

4.3 Single strong observables PDE

The Flow Method [29, 61] immediately applies here, giving a corresponding ODE system of form

$$\dot{x}^A \;=\; a^A(x) \;, \tag{43}$$

$$\dot{\mathsf{O}} \;=\; 0 \;. \tag{44}$$

Here, the dot denotes

$$\frac{\mathrm{d}}{\mathrm{d}\nu} \;, \tag{45}$$

for ν a fiducial variable to be eliminated, rather than necessarily carrying any temporal (or other geometrical or physical) significance. Our ODE system is moreover *autonomous*, i.e. none of the functions therein depend on ν.

Remark 1 In the geometrical setting, the first equation here is that of *Lie's Integral Approach to Geometrical Invariants* [5, 18, 29, 61, 62]. There is semi-decoupling, so this equation can be solved prior to the last equation.

Remark 2 The second equation then acts to uplift this to Taking the Function Space Thereover. This is by feeding in the 'characteristic' solution u of the first equation into the last equation by eliminating ν. This u then ends up featuring as the 'functional form' that the observables depend on $\boldsymbol{Q}$ and $\boldsymbol{P}$ via

$$\mathsf{O} \;=\; \mathsf{O}(u) \;. \tag{46}$$

This simple outcome follows from the triviality of the last equation in the strong setting. The function space in question needs to be at least $\mathfrak{C}^1$ to satisfy the observables PDE system. The observables are thus suitably-smooth but elsewise-arbitrary functions of Lie's invariants. (This is literally so in the case of Geometry of space or spacetime, or of their phase space counterparts in the canonical case.)

Remark 3 Solving for such arbitrary functions is to be contrasted with obtaining a single function by prescribing a specific boundary condition. *Not* prescribing such a boundary condition, on the one hand, amounts to implementing Taking a Function Space Thereover. On the other hand, its more general technical name is *free* alias *natural* [10] characteristic problem. The Characteristic Problem formulation for a single linear (or quasilinear) flow PDE is a standard prescription [10, 29].

Remark 4 At the geometrical level, our procedure is as follows. We are given a constraint subalgebra $\boldsymbol{\mathcal{C}}$. Then, firstly, the observables equation $[\boldsymbol{\mathcal{C}}, \mathbf{O}] = 0$ determines a characteristic surface

$$\chi = \chi(\boldsymbol{Q}, \boldsymbol{P}) . \tag{47}$$

This follows from solving all but the last equation in the equivalent flow ODE system. Secondly, the last equation in the flow ODE system is a trivial ODE solved by any suitably-smooth function thereover. This gives the observables algebra as

$$\mathfrak{C}\text{an-}\mathfrak{O}\text{bs}(\mathfrak{S}) = \mathcal{C}^{\infty}(\chi) = \{ \text{ phase space functions whose restrictions to } \chi , \quad \mathrm{f}|_{\chi} , \quad \text{are smooth}\} . \tag{48}$$

N-point geometrical = purely configurational physical observables This subproblem has been covered in e.g. [4, 5, 18, 74].

Definition 1 *classical geometrical observables* are

$$\mathbf{O}(\boldsymbol{Q}) . \tag{49}$$

Structure 1 In the purely-geometrical setting, the a priori free functions $\mathbf{O}$ are subject to

$$[\boldsymbol{\mathcal{S}}, \mathbf{O}] = 0 . \tag{50}$$

Here, $\boldsymbol{\mathcal{S}}$ is the *sum-over-N-points* [18, 74] $\underline{q}^I$, $I = 1$ to N of each particular generator, with components

$$\mathcal{S}_G := \sum_{I=1}^{N} G_G{}^b(q^{cI}) \frac{\partial}{\partial q^{bI}} . \tag{51}$$

Remark 5 As detailed in [76], this is the pure-geometry analogue of $\boldsymbol{\mathcal{C}}$.

Remark 6 The Lie bracket equation (50) can furthermore be written out as an explicit PDE system. It should by now be clear that this PDE is additionally a subcase of that for canonical observables in Theoretical Physics,

$$\frac{\partial \boldsymbol{\mathcal{C}}}{\partial \boldsymbol{P}} \frac{\partial \mathbf{O}}{\partial \boldsymbol{Q}} = 0 , \tag{52}$$

$$\text{with } \frac{\partial \boldsymbol{\mathcal{C}}}{\partial \boldsymbol{P}} \text{ treated as knowns} . \tag{53}$$

In particular, N-point geometrical observables coincide with the purely configuration space restriction of physical observables [76]; see Sec 5 for examples.

Pure-momentum physical observables These

$$\mathbf{O}(\boldsymbol{P}) . \tag{54}$$

solve the *pure-momentum observables PDE system*,

$$\frac{\partial \boldsymbol{\mathcal{C}}}{\partial \boldsymbol{Q}} \frac{\partial \mathbf{O}}{\partial \boldsymbol{P}} = 0 , \tag{55}$$

$$\text{for } \frac{\partial \boldsymbol{\mathcal{C}}}{\partial \boldsymbol{Q}} \text{ treated as knowns} . \tag{56}$$

Remark 7 Configuration and momentum observables each readily represent a *restriction* of functions over $\mathfrak{p}$hase, to just over $\mathfrak{q}$, and to just over the space of momenta $\mathfrak{P}$ respectively. These are, more specifically, *polarization restrictions* [36] since they precisely halve the number of variables. This applies at least in the quadratic theories we consider in the current Series. These are by far the simplest and most standard form for bosonic theories in Physics.

4.4 Nontrivial system case: determinedness and integrability

Remark 1 For nontrivial systems, multiple sequential uses of the Flow Method may apply. What needs to be checked first is determinedness [15], and, if over-determinedness occurs, integrability.

Remark 2 In Geometry, we have $g := \dim(\mathfrak{g})$ constraints, and thus g observables equations. The observables carry an index O that has no a priori dependence on $\mathfrak{g}$. Thus, a priori, any of under-, well- or over-determinedness can occur. This conclusion transcends to the canonical approach to Physics as well. Here Temporal Relationalism and/or Constraint Closure can contribute further first-class constraints. g is thus replaced by a more general count $f := \dim(\mathfrak{F})$ of functionally-independent first-class constraints.

On the one hand, generalized Killing equations' integrability conditions are not met generically [11]. This signifies that there are only any proper generalized Killing vectors at all in a zero-measure subset of $\langle\, \mathfrak{M},\, \boldsymbol{\sigma}\, \rangle$. This corresponds to the generic manifold admitting no (generalized) symmetries.

On the other hand, preserved equations moreover always succeed in meeting integrability, by the following Theorem.

Theorem 1 Classical canonical observables equations are integrable.

Remark 3 Consequently, classical observables *always* exist (subject to the following caveats).

Caveat 1 The current Series, consider only *local* existence.

Caveat 2 Sufficiently large point number N is required in the case of finite point-particle theories. This is clear from the examples in [74, 75, 76]. It corresponds to zero-dimensional reduced spaces having no coordinates left to support thereover any functions of coordinates.

Remark 4 This Theorem is proven in [76], resting on the following vaguely modern version of Frobenius' Theorem.

Theorem 2 (A version of Frobenius' Theorem at the level of differentiable manifolds) [31, 61]. A collection $\mathfrak{C}$ of subspaces of a tangent space possesses integral submanifolds iff[1]

$$\forall\ \underline{X}\,,\ \underline{Y}\ \in\ \mathfrak{C}\,,\quad |[\underline{X}, \underline{Y}]|\ \in\ \mathfrak{C}\,. \tag{57}$$

4.5 Sequence of free characteristic problems for the strong observables system

We now have a more extensive ODE system of the form (43, 44).

Remark 1 Regardless of the single-equation to system distinction, corresponding observables ODEs are additionally *autonomous*: none of the functions therein depend on ν).

Remark 2 We now have a first block rather than a first equation.

Remark 3 The system version is not a standard prescription. But study of strong observables PDE systems gets past this by the above guarantee of integrability.

Sequential Approach. Suppose we have two equations. Solve one for its characteristics u_1, say. Then substitute $\mathcal{Q} = \mathcal{Q}(u_1)$ into the second equation to find which functional restrictions on the first solution's characteristics are enforced by the second equation. This procedure can moreover be applied inductively. It is independent of the choice of ordering in which the restrictions are applied by the nature of restrictions corresponding to geometrical intersections.

The Free Characteristic Problem posed above moreover leads to consideration of intersections of characteristic surfaces. This can moreover be conceived of in terms of restriction maps.

Theorem 3 Suppose

$$\mathbf{V}\ \text{ such that }\ [c_{\mathsf{V}}, \mathbf{V}]\ =\ 0\ \text{ forms characteristic surface }\ \chi_{\mathbf{V}} \tag{58}$$

and

$$\mathbf{W}\ \text{ such that }\ [c_{\mathsf{W}}, \mathbf{W}]\ =\ 0\ \text{ forms characteristic surface }\ \chi_{\mathbf{W}} \tag{59}$$

[1] $|[\ ,\ ]|$ is here a general Lie bracket.

for constraint subalgebraic structures c_{V} and c_{W}. Then

$$\mathsf{O} \ \text{ such that } \ [c_{\mathsf{V} \,\cup\, \mathsf{W}}, \mathsf{O}] \ = \ 0 \ \text{ forms the characteristic surface of Fig 3 .}$$

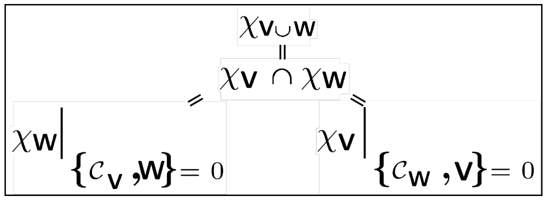

Figure 3: Characteristic surface resulting from commutation with two constraint subalgebraic structures.

Remark 4 This approach extends inductively to a finite number of equations in our flow ODE system.

Remark 5 See e.g. [74, 75, 76] for examples of its use.

Remark 6 The integrated form of the first m equations is used to eliminate t. The other $m - 1$ provide a basis of characteristic coordinates $\boldsymbol{u}$ arising as constants of integration. In the geometrical case, this can still be considered to be Lie's Method of Geometric Invariants. After all, essentially all the most familiar geometries involve more than one independent condition on their integral invariants.

Remark 7 To elevate this to a determination of the system's observables, we then substitute these characteristic coordinates into the last equation. We thus obtain the general – and thus free alias natural problem-solving – characteristic solution.

Remark 8 Our last equation remains a trivial ODE. It is thus solved by an suitably-smooth but elsewise arbitrary function of these characteristic coordinates, $\boldsymbol{u}$: $\mathsf{O} \ = \ \mathsf{O}(\boldsymbol{u})$.

Remark 9 Moving up the lattice moreover amounts to successive restrictions of the unreduced problem's 'solution manifold' function space.

Remark 10 The current Series just considers a local rather than global treatment of observables equations.

4.6 Weak observables PDEs

Structure 1 Weak observables' defining brackets relation gives

$$\underset{\sim}{\boldsymbol{W}} \cdot \underset{\sim}{\boldsymbol{c}} \ = \ \{\boldsymbol{c}, \mathsf{O}\} \ = \ \frac{\partial \boldsymbol{c}}{\partial \mathbf{Q}} \frac{\partial \mathsf{O}}{\partial \mathbf{P}} \ - \ \frac{\partial \boldsymbol{c}}{\partial \mathbf{P}} \frac{\partial \mathsf{O}}{\partial \mathbf{Q}} \ . \tag{60}$$

We can again

$$\text{take } \ \frac{\partial \boldsymbol{c}}{\partial \boldsymbol{Q}} \ \text{ and } \ \frac{\partial \boldsymbol{c}}{\partial \boldsymbol{P}} \ \text{ to be knowns ,} \tag{61}$$

now alongside $\boldsymbol{c}$ being a known as well and $\boldsymbol{W}$ taking some prescribed value. This leaves us with an inhomogeneous-linear first-order PDE system. As already alluded to in Sec 3.1, their general solution thus splits into CF plus PI. I.e. strong and nontrivially-weak observables respectively.

Remark 1 The pure-geometry case cannot however support any properly weak observables. This is because we are in a context in which constraints have to depend on momenta

$$\boldsymbol{c} \ = \ f(\boldsymbol{Q}, \boldsymbol{P}) \ \ \textit{with specific } \boldsymbol{P} \textit{ dependence} \ \ . \tag{62}$$

A fortiori, first-class linear constraints refer specifically to being linear in their momenta $\boldsymbol{P}$, By this, the weak configurational observables equation would have a momentum-dependent inhomogeneous term right-hand side. But this is inconsistent with admitting a solution with purely $\boldsymbol{Q}$-dependent right-hand-side.

4.7 Single weak observables equation

The weak observables PDE consists of a single inhomogeneous-linear equation. The corresponding ODE system is now of the form

$$\dot{x} \;=\; a(x, \phi) \;, \tag{63}$$

$$\dot{\phi} \;=\; b(x, \phi) \;=\; W\,c_1 \;. \tag{64}$$

Remark 1 The first block is the same as before. Lie's integral invariants (in Geometry or their canonical Physics generalizations) thus still enter our expressions for observables.

Remark 2 The inhomogeneous term in the last equation, however, means that one has further PI work to do in this weak case.

4.8 Nontrivial weak observables system

The corresponding ODE system now has inhomogeneous term $b_{\mathsf{O}} \;=\; W^{\mathsf{B}}{}_{\mathsf{AO}}\,c_{\mathsf{B}}$.

Remark 1 Determinedness and integrability considerations carry over. So does sequential use of free characteristic problems on the first block.

Remark 2 Solving the last equation in the system is then conceptually the same as in the previous subsection.

Remark 3 We leave a systematic Green's function approach to weak observables equations for another occasion.

Remark 4 If we reduce all constraints out, the reduced formulation has strong $\widetilde{\mathbf{U}}$ being all the observables there can be.

Remark 5 Suppose we reduce all constraints bar $\mathcal{C}$hronos out. Then the following applies since finite theory's case of this is a single constraint equation.

Corollary 1 [85] Suppose there is only one constraint. Then there is only one independent proper weak observable, and the proper weak observables algebra is abelian.

4.9 Killing and observables PDEs compared (including genericity)

Remark 1 For PDEs that are homogeneous in the derivatives, the constant always solves. For PDEs which are homogeneous linear in the variable to be solved for, zero always solves. These do not however constitute legitimate generalized Killing vectors or observables. *Proper solutions* of these require nontrivial kernel.

Remark 2 From a PDE point of view, systems are more generic than single equations. There is moreover no general treatment for Characteristic Problems for PDE systems of this general kind. So as to proceed, further details of the PDE system in question need to be considered, in particular whether integrability is guaranteed.

Remark 3 In Fixed-background Finite Theories of Geometry (or Physics), it is *geometrically generic* to have no (generalized) Killing vectors. Then there are 0 or 1 observables PDEs (1 for theories supporting $\mathcal{C}$hronos).

Remark 4 For Finite Theories, PDE system genericity is in general obscured by geometrical genericity. Having one Killing vector is moreover of secondary genericity between having no, and multiple, Killing vectors. From this point of view, unconstrained observables are most generic (no observables PDEs at all), single observables PDEs are next most generic, and the more involved case of multiple observables equations is only tertiary in significance.

Remark 5 Geometrical observables equation systems consist of of $G \;:=\; \dim(Aut(\mathfrak{M},\,\boldsymbol{\sigma}))$ equations for a single unknown observables vector. By this, the position at the outset is one of over-determination [15]. This leads to no solutions (or only the trivial solutions, when guaranteed by homogeneity as per Remark 1).

Remark 6 The way such a lack of (nontrivial) solutions might occur is via integrabilities.[2] On the one hand, generalized Killing equations (as feature in Relationalism [88]) do not generically meet integrability conditions [11]. This signifies that there are only any proper generalized Killing vectors at all in a zero-measure subset of $\langle\,\mathfrak{M},\,\boldsymbol{\sigma}\,\rangle$.

[2]While there is a conceptual counter-acting under-determination from the characteristicness and the freeness, it is the integrability that we point out that guarantees that things work out.

This corresponds to the generic manifold admitting no (generalized) symmetries. On the other hand, preserved equations moreover always succeed in meeting integrability, by the following Theorem.

Theorem 1 Observables equations are integrable.

(This rests [74] on an updated version [31, 61] of Frobenius' Theorem [3].)

Remark 7 So, while proper generalized Killing vectors generically do not exist, observables *always* do. At least in this Series' local sense, and for sufficiently large point number N. This last caveat is clear from the examples below. It corresponds to zero-dimensional reduced spaces having no coordinates left to support thereover any functions of coordinates. In particular, Frobenius' Theorem guarantees applicability of the above sequential method.

Remark 8 There is moreover a greater generality to consider: generalized Killing vector nonexistence means there are no generators to commute with. In this case, the most primitive element of Assigning Observables – Taking Function Spaces Thereover – is manifested in a particularly simple form. Namely, taking the free functions over the state space $\mathfrak{S}$.

Within the secondmost-generic case – possessing a single generalized Killing vector – we get the single-PDE version of the problem.

The $\geq$ 2-compatible generalized Killing vectors case is only the next most typical, it being here that the systematic method above does not apply.

4.10 Cartan's Differential Approach to Invariants

Cartan [12, 18, 62] gave an alternative way of finding geometric invariants. This is based on differentiation and Linear Algebra within the theory of mobile frames.

5 Explicit solutions for observables and their algebras

5.1 Nontrivial geometrical examples of Kuchař, gauge- and A-observables

Simplification 1 Suppose the constraints under consideration depend at most linearly on the momenta. This holds for $Eucl(d)$. Then in the observables equations,

$$\text{the cofactor of } \frac{\partial \mathsf{G}}{\partial \boldsymbol{Q}} \text{ – i.e. } \frac{\partial \boldsymbol{c}}{\partial \boldsymbol{P}} \text{ – is independent of } \boldsymbol{P} \, . \tag{65}$$

Simplification 2 Restrict attention to purely configurational such observables $\mathsf{K}(\boldsymbol{Q}) = \mathsf{G}(\boldsymbol{Q})$. This gives the particularly simple PDE [following on from Fig 0.4]

$$\mathrm{T}^{\mathsf{A}}{}_{\mathsf{N}}(\boldsymbol{Q} \text{ alone}) \frac{\partial \mathsf{G}}{\partial Q^{\mathsf{A}}} = 0 \, ; \tag{66}$$

Simplification 3 A few of the below examples involve just one observables equation (i.e. the N-index takes a single value). This is then amenable to the standard Flow Method.

Simplification 4 For $Eucl(d)$, passage to the centre of mass frame is available to take translations out of contention. Upon doing this, mass-weighted relative Jacobi coordinates [34] furthermore simplify the remaining equations [59, 71].

Notation 1 We additionally provided a concise notation for the outcome of solving the preserved equations for $Eucl(d)$: that in row 1 of Fig 4

Example 1) For translation-invariant geometry – $\mathfrak{g} = Tr(d)$ – the observables PDE is

$$\sum\nolimits_{I=1}^{N} \frac{\partial \mathsf{G}}{\partial \underline{q}^I} = 0 \, . \tag{67}$$

The solutions of this are, immediately, the relative interparticle separation vectors. These can be reformulated as linear combinations thereof. Among these, the relative Jacobi coordinates $\boldsymbol{\rho}$ turn out to be particularly convenient by the Jacobi map. Our solutions form

$$\mathfrak{Geom}\text{-}\mathfrak{Gauge}\text{-}\mathfrak{Obs}(d, N; Tr(d)) = \mathcal{C}^{\infty}(\mathbb{R}^{n\,d}) \, : \tag{68}$$

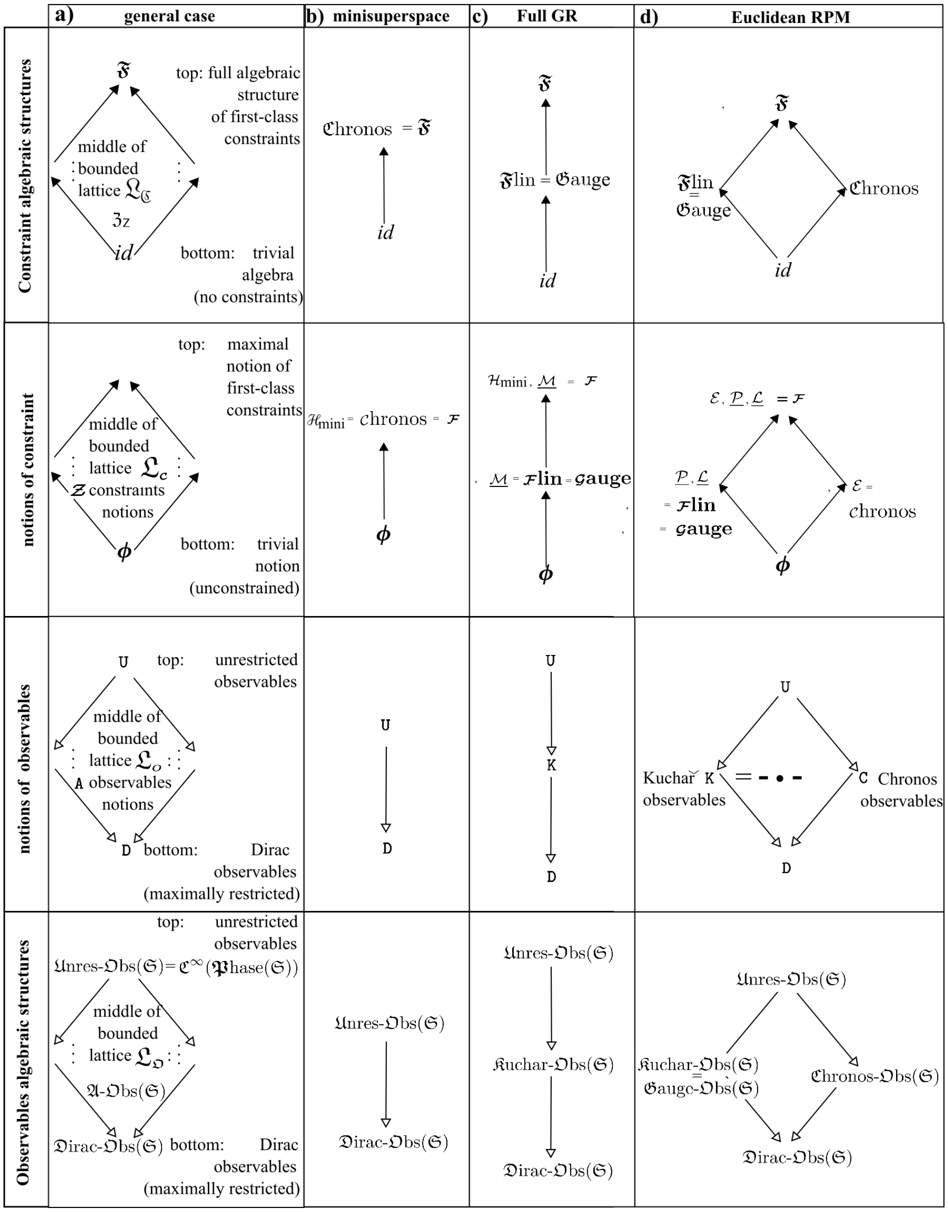

Figure 4: From closed subalgebraid structures to associated notions of observables. $- \cdot -$ denotes functions of dot products of differences.

the smooth functions over relative space.

Example 2 For rotationally-invariant geometry – $\mathfrak{g} = Rot(d)$ – the observables PDE is

$$\sum\nolimits_{I=1}^{N} \underline{q}^I \times \frac{\partial \, \mathbf{G}}{\partial \, \underline{q}^I} = 0 \ . \qquad (69)$$

This is solved by the dot products $\underline{q}^I \cdot \underline{q}^J$. Norms and angles are moreover particular cases of functionals of the above, which are straightforwardly an allowed extension [70].

Example 3 In Euclidean geometry – $\mathfrak{g} = Eucl(d) = Tr(d) \rtimes Rot(d)$ – the observables PDEs are both (67) and (69). Sequential use of the Flow Method gives that the solutions are dots of differences of position vectors. This can be worked into the form of dots of relative Jacobi vectors. These are additionally Euclidean RPM's [30, 59] configurational Kuchař $=$ gauge observables [67].

Each of Examples 1 and 2 also constitute nontrivially-$\mathbf{A}$ examples (in the present context $\mathbf{A}$ that are not also $\mathbf{K} = \mathbf{G}$) for Euclidean geometry.

5.2 Physically nontrivial examples of strong K, G, A and C observables

Example 1 Translationally-invariant RPM's pure-momentum observables are freely specifiable. However, since the total centre of mass position is meaningless in this problem, its momentum is meaningless as well. This leaves us with

$$\mathtt{G} = \mathtt{G}\left(\underline{p}_i - \underline{p}_N\right) = \mathtt{G}\left(\underline{\pi}_i\right) \tag{70}$$

for $\underline{\pi}_i$ the conjugate momenta to $\underline{\rho}^i$. These form

$$\mathfrak{Mom}\text{-}\mathfrak{Gauge}\text{-}\mathfrak{Obs}(d,\, N;\, Tr) = \mathcal{C}^\infty(\mathbb{R}^{n\, d}) \,, \tag{71}$$

i.e. the smooth functions over relative momentum space.

The corresponding general observables are

$$\mathtt{G} = \mathtt{G}\left(\underline{q}^i - \underline{q}^N, \underline{p}_i - \underline{p}_N\right) = \mathtt{G}\left(\underline{\rho}^i, \underline{\pi}_i\right) \,. \tag{72}$$

These form

$$\mathfrak{Can}\text{-}\mathfrak{Gauge}\text{-}\mathfrak{Obs}(d,\, N;\, Tr(d)) = \mathcal{C}^\infty(\mathbb{R}^{2\, n\, d}) \ : \tag{73}$$

the smooth functions over relative phase space.

Example 2 Rotationally-invariant RPM's pure-momentum observables PDE system is also the $\underline{q}^I \leftrightarrow \underline{p}^I$ of the corresponding geometrical observables PDE system. This is thus is solved by suitably smooth functions of the dot product,

$$\mathbf{G} = \mathbf{G}(\underline{p}^I \cdot \underline{p}^J) \,. \tag{74}$$

In 2-d, these form

$$\mathfrak{Mom}\text{-}\mathfrak{Gauge}\text{-}\mathfrak{Obs}(d, N; Rot(2)) = \mathcal{C}^\infty(C(\mathbb{CP}^N)) \,. \tag{75}$$

The corresponding general observables PDE system is

$$\sum\nolimits_{I=1}^{N} \left\{ \frac{\partial\, \mathbf{G}}{\partial\, \underline{p}^I} \times \underline{p}_I + \frac{\partial\, \mathbf{G}}{\partial\, \underline{q}^I} \times \underline{q}^I \right\} = 0 \,. \tag{76}$$

This is solved by suitably-smooth functions of

$$\cdot\mathrm{s} := \underline{q}^I \cdot \underline{p}^J + \underline{p}^I \cdot \underline{q}^J \ : \tag{77}$$

phase space symmetrized dot products. These are the outcome of applying the product rule to $\underline{q}^I \cdot \underline{q}^J$.

Example 3 Euclidean RPM combines the above translational and rotational equations. The Euclidean momentum observables solutions are suitably smooth functions

$$\mathbf{G} = \mathbf{G}(\underline{\pi}^i \cdot \underline{\pi}^j) \,. \tag{78}$$

The general observable solutions are suitably smooth functions

$$\mathbf{G} = \mathbf{G}(\underline{\rho}^i \cdot \underline{\pi}^j + \underline{\pi}^i \cdot \underline{\rho}^j) = \mathbf{G}(-\cdot\mathrm{s}\,-) \,, \tag{79}$$

i.e. relative phase space symmetrized dot products.

Example 7 Chronos observables $\mathbf{C}$ for the general (N, d) Euclidean RPM solve

$$\boldsymbol{p} \cdot \frac{\partial\, \mathbf{C}}{\partial\, \boldsymbol{q}} + \frac{\partial\, V}{\partial\, \boldsymbol{q}} \frac{\partial\, \mathbf{C}}{\partial\, \boldsymbol{q}} = 0 \,. \tag{80}$$

In the case of constant potential, this simplifies to

$$\boldsymbol{p} \cdot \frac{\partial\, \mathbf{C}}{\partial\, \boldsymbol{q}} = 0 \tag{81}$$

which is solved by

$$\mathtt{C} = \mathtt{C}\left(q_\Lambda p_{N\, d} - p_\Lambda q^{N\, d}, {p_\Lambda}^2 - {p_{N\, d}}^2\right) \,. \tag{82}$$

Treating the $N\, d$ component differently is an arbitrary choice; Λ then runs over all the other values, 1 to $N\, d - 1$. For RPMs with any Configurational Relationalism, these are a further species of **A** observable that is not a **K** or **G** observable.

Remark 1 That $\mathfrak{Chronos}$-$\mathfrak{Obs}$ (and $\mathfrak{Dirac}$-$\mathfrak{Obs}$) are potential-dependent, constitutes a massive complication at the computational level.

5.3 Examples of strong nontrivially-Dirac observables

Example 1 For RPMs involving whichever combination of translations and rotations, the additional PDE to obtain strong Dirac observables is

$$\sum\nolimits_{I=1}^{N} \left\{ \frac{\partial V}{\partial \underline{q}^I} \cdot \frac{\partial \mathsf{D}}{\partial \underline{p}_I} \;-\; \underline{p}_I \cdot \frac{\partial \mathsf{D}}{\partial \underline{q}^I} \right\} \text{ `='} \; 0 \; . \tag{83}$$

If dilations are involved as well, one needs to divide the kinetic term contribution by the total moment of inertia to have a ratio form.

The above are moreover mathematically equivalent to the corresponding $(d,\, n)$ Chronos observables problems, so e.g. in the $Eucl(d)$ case

$$\widetilde{\mathtt{D}} \;=\; \widetilde{\mathtt{D}}\left(\rho^{\tau}\pi_N \;-\; \rho^N\pi_\sigma,\, {\pi_\tau}^2 - {\pi_N}^2\right) \;, \tag{84}$$

with the true space index τ running over 1 to $n\,d - 1$.

Example 2 *Minisuperspace* (spatially homogeneous GR) only has a $\mathcal{H}$. A single finite constraint oversimplifies the diversity of notions of observables. The sole strong observables brackets equation is here

$$\{\,\mathcal{H}, \mathsf{D}\,\} \;=\; 0 \; . \tag{85}$$

This gives the observables PDE

$$\frac{\partial\,\mathcal{H}}{\partial\,\boldsymbol{Q}}\frac{\partial\,\mathsf{D}}{\partial\,\boldsymbol{P}} \;-\; \frac{\partial\,\mathcal{H}}{\partial\,\boldsymbol{P}}\frac{\partial\,\mathsf{D}}{\partial\,\boldsymbol{Q}} \;=\; 0 \tag{86}$$

$$\text{for } \frac{\partial\,\mathcal{H}}{\partial\,\boldsymbol{Q}} \text{ and } \frac{\partial\,\mathcal{H}}{\partial\,\boldsymbol{P}} \text{ knowns .} \tag{87}$$

There being just one such equation gives that $\mathsf{U} \;=\; \mathsf{G} \;=\; \mathsf{K} \;\neq\; \mathsf{C} \;=\; \mathsf{D}$. See e.g. [41] for direct construction of classical Dirac observables for Minisuperspace.

5.4 Nontrivially weak observables

Reduced versus indirect makes a clear difference here, since the indirect case has a longer string of constraints in its PI. Only at least partly indirectly formulated case has any space for nontrivially weak such: if all constraints are reduced out, no PI is left. See [85] for explicit examples.

6 Observables in Field Theory

In the general finite–field portmanteau case, the observables brackets equation is to be interpreted as a $\boldsymbol{\partial}$DE system. I.e. a portmanteau of a PDE (Sec 4) for Finite Theory and an FDE (functional differential equation) for Field Theory [3]

$$\int \mathrm{d}^n z \sum\nolimits_{\mathsf{A}} \left\{ \frac{\delta(\boldsymbol{c}|\partial\,\boldsymbol{\xi})}{\delta\,\mathrm{Q}^{\mathsf{A}}(z)}\frac{\delta(\underset{\sim}{\mathbf{Q}}|\partial\,\underset{\sim}{\boldsymbol{\xi}})}{\delta\,\mathrm{P}_{\mathsf{A}}(z)} \;-\; \frac{\delta(\boldsymbol{c}|\partial\,\boldsymbol{\xi})}{\delta\,\mathrm{P}_{\mathsf{A}}(z)}\frac{\delta(\underset{\sim}{\mathbf{Q}}|\partial\,\underset{\sim}{\boldsymbol{\xi}})}{\delta\,\mathrm{Q}^{\mathsf{A}}(z)} \right\} \text{ `='} \; 0 \; . \tag{88}$$

Remark 1 Taking

$$\frac{\delta\,\boldsymbol{c}}{\delta\,\mathbf{Q}} \text{ and } \frac{\delta\,\boldsymbol{c}}{\delta\,\mathbf{P}} \text{ to be knowns} \tag{89}$$

we have a homogeneous-linear first-order FDE system. We would also fix particular smearing functions (or not formally smear) in locally posing and solving our FDE system.

Remark 2 The general form is

$$\mathrm{a}^A\,[\underline{x},\, y_B(\underline{x}),\, \Phi[y_B(\underline{x})]]\, \mathcal{D}_{y_A}\Phi \;=\; \mathrm{b}^A(\underline{x},\, y_B(\underline{x}),\, \Phi) \; . \tag{90}$$

This covers both strong observables (homogeneous: $\mathbf{b} \;=\; 0$) and weak observables (*inhomogeneous* : $\mathbf{b} \;\neq\; 0$). For conventional Gauge Theory, the observables equation imposes gauge invariance at the configurational level. Both of the below examples have $\mathsf{D} \;=\; \mathsf{K} \;=\; \mathsf{G} \;\neq\; \mathsf{U}$.

Example 1) Electromagnetism has the abelian algebra of constraints (1.55), so its observables obey the brackets equation

$$\{\,(\,\mathscr{G}\,|\,\partial\,\xi\,), (\,\underset{\sim}{\mathsf{K}}\,|\,\partial\,\underset{\sim}{\chi}\,)\,\} \text{ `='} \; 0 \;\;\Rightarrow\;\; \text{the FDE } \;\; \underline{\partial}\cdot\frac{\delta\mathsf{K}}{\delta\underline{\mathrm{A}}} \text{ `='} \; 0 \; . \tag{91}$$

[3] Here $\partial\,\boldsymbol{\xi}$ and $\partial\underset{\sim}{\boldsymbol{\xi}}$ are smearing functions; see Appendix A for details.

This is solved by the electric and magnetic fields, $\underline{E}$ and $\underline{B} = \underline{\partial} \times \underline{A}$, and thus also (using Remark 7.1) by a functional $\mathcal{F}[\underline{B}, \underline{E}]$. $\underline{E}$ and $\underline{B}$ do not however constitute a *conjugate* pair. This looking to be a common phenomenon, we introduce the term 'associated momenta' to describe it. $\mathcal{F}[\underline{B}, \underline{E}]$ furthermore admits an integrated formulation in terms of electric fluxes $\Phi^{\mathrm{E}}_{\mathrm{S}}$ and loop variables

$$W_{\mathrm{A}}(\gamma) := \exp\left(i \oint_{\gamma} \mathrm{d}\underline{x} \cdot \underline{A}(x) \right) \quad : \tag{92}$$

$$\mathcal{F}\left[\iint_{\mathrm{S}} \underline{B} \cdot \mathrm{d}\underline{S} \, , \quad \iint_{\mathrm{S}} \underline{E} \cdot \mathrm{d}\underline{S} \right] = \mathcal{F}[W(\gamma), \Phi^{\mathrm{E}}_{\mathrm{S}}] \tag{93}$$

This reformulation proceeds via Stokes' Theorem with $\gamma := \partial\,\mathrm{S}$ and subsequent insertion of the exponential function subcase of Remark 7.1. This construct is also tied to the geometrical notion of holonomy. These loops are however well-known to form an *over*-complete set, due to *Mandelstam identities* therebetween [45].

Example 2) Its Yang–Mills generalization has the Lie algebra of constraints (1.57) Observables here solve the brackets equation

$$\{ (\mathscr{G}_I \,|\, \xi^I), (\underset{\sim}{\mathsf{K}} \,|\, \underset{\sim}{\chi}) \} \approx 0 \;\Rightarrow\; \text{the FDE} \quad \underset{\sqcup}{\underline{D}} \cdot \frac{\delta \mathsf{K}}{\delta \underset{\sqcup}{\underline{A}}} \approx 0 \, . \tag{94}$$

This is solved by Yang–Mills Theory's generalized $\underset{\sqcup}{\underline{E}}$ and $\underset{\sqcup}{\underline{B}}$; $\mathcal{F}[\underset{\sqcup}{\underline{E}}, \underset{\sqcup}{\underline{B}}]$ is thus also a solution. Once again, this can be rewritten as $\mathcal{F}[W(\gamma), \Phi^{\mathrm{E}}_{\mathrm{S}}]$, now for $\mathfrak{g}$-loop variable

$$W_{\mathrm{A}}(\gamma) := \mathrm{Tr}\left(P \exp\left(i\, g \oint_{\gamma} \mathrm{d}\underline{x}\; \underset{\sqcup}{\underline{A}}(x)\, \underset{\sqcup}{\mathrm{g}}(x) \right) \right) \, . \tag{95}$$

g are here group generators of $\mathfrak{g}_{\mathrm{YM}}$, g is the coupling constant, and P is the path-ordering symbol.

Example 3 The *cause célèbre* of canonical treatment of observables is GR. Now one has to contend with both the spatial 3-diffeomorphisms' $Diff(\mathbf{\Sigma})$ momentum constraint $\underline{\mathcal{M}}$ – linear in its momenta – and the Hamiltonian constraint $\mathcal{H}$ – quadratic in its momenta.

The K for GR as Geometrodynamics obey the brackets equation

$$\{ (\underline{\mathcal{M}} \,|\, \partial \underline{\mathrm{L}}), (\underset{\sim}{\mathsf{K}} \,|\, \partial \underset{\sim}{\chi}) \} \text{ `='} \ 0 \;\Rightarrow\; \text{the FDE} \quad \left(\left\{ \pounds_{\partial \underline{\mathrm{L}}} \mathbf{h} \bullet \frac{\delta}{\delta \mathbf{h}} + \pounds_{\partial \underline{\mathrm{L}}} \mathbf{p} \bullet \frac{\delta}{\delta \mathbf{p}} \right\} \underset{\sim}{\mathsf{K}} \,\middle|\, \partial \underset{\sim}{\chi} \right) \text{ `='} \ 0 \, . \tag{96}$$

This corresponds to the unsmeared FDE

$$2\, \mathrm{h}_{jk} \mathcal{D}_i \frac{\delta \mathsf{K}}{\delta \mathrm{h}_{ij}} + \{ \mathcal{D}_i \mathrm{p}^{lj} - 2\, \delta^j{}_i \{ \mathcal{D}_e \mathrm{p}^{le} + \mathrm{p}^{le} \mathcal{D}_e \} \} \frac{\delta \mathsf{K}}{\delta \mathrm{p}^{lj}} \text{ `='} \ 0 \, . \tag{97}$$

Up to weakly vanishing terms, we can furthermore discard the penultimate term. The $\mathsf{K}(\mathbf{Q})$ subcase solves

$$2\, \mathrm{h}_{jk} \mathcal{D}_i \frac{\delta \mathsf{K}}{\delta \mathrm{h}_{ij}} \text{ `='} \ 0 \, . \tag{98}$$

These are, formally, 3-geometry quantities '$\mathfrak{G}^{(3)}$' by (98) emulating (and moreover logically preceding) the 'momenta to the right' ordered quantum GR momentum constraint (0.13). The FDE for the $\mathsf{K}(\mathbf{P})$ (formally '$\Pi^{\mathfrak{G}^{(3)}}$') is

$$\{ \mathcal{D}_i \mathrm{p}^{lj} - 2\, \delta^j{}_i \{ \mathcal{D}_e \mathrm{p}^{le} + \mathrm{p}^{le} \mathcal{D}_e \} \} \frac{\delta \mathsf{K}}{\delta \mathrm{p}^{lj}} \text{ `='} \ 0 \, . \tag{99}$$

Geometrodynamics' D oby an extra FDE [67]

$$\{ (\mathcal{H} \,|\, \partial\, \mathrm{J}), (\underset{\sim}{\mathsf{D}} \,|\, \partial \underset{\sim}{\zeta}) \} \text{ `='} \ 0 \;\Rightarrow\; \text{the FDE} \tag{100}$$

$$\left(\left\{ \frac{\delta \underset{\sim}{\mathsf{D}}}{\delta \underline{\mathrm{p}}} \{ \underline{\mathrm{G}} - \underline{\underline{\mathrm{M}}} \mathcal{D}^2 \} - \frac{\delta \underset{\sim}{\mathsf{D}}}{\delta \underline{\mathrm{h}}} \underline{\underline{\mathrm{N}}}\, \underline{\mathrm{p}} \right\} \partial\, \mathrm{J} \,\middle|\, \partial \underset{\sim}{\zeta} \right) \text{ `='} \ 0 \, . \tag{101}$$

This features the *DeWitt vector quantities*

$$\underline{\mathrm{G}} := \tfrac{2}{\sqrt{\mathrm{h}}} \{ \mathrm{p}^{ia} \mathrm{p}_a{}^j - \tfrac{\mathrm{p}}{2} \mathrm{p}^{ij} \} - \tfrac{1}{2\sqrt{\mathrm{h}}} \{ \mathrm{p} \bullet \mathrm{p} - \tfrac{\mathrm{p}^2}{2} \} \mathrm{h}^{ij} - \tfrac{\sqrt{\mathrm{h}}}{2} \{ \mathrm{h}^{ij} \mathcal{R} - 2\, \mathcal{R}^{ij} \} + \sqrt{\mathrm{h}}\, \Lambda\, \mathrm{h}^{ij} \, . \tag{102}$$

These should already be familiar from the ADM equations of motion [16], and also $\underline{\underline{\mathcal{D}}}^2$ with components $\mathcal{D}^i\mathcal{D}^j$. In unsmeared form, (101) is the FDE

$$\{ \underline{\mathrm{G}} - \underline{\underline{\mathrm{M}}}\, \underline{\underline{\mathcal{D}}}^2 \} \frac{\delta \mathbf{D}}{\delta \underline{\mathrm{p}}} \text{ `='} \ 2\, \underline{\mathrm{p}}\, \underline{\underline{\mathrm{N}}} \frac{\delta \mathbf{D}}{\delta \underline{\mathrm{h}}} \ . \tag{103}$$

$\boldsymbol{M}$ and $\boldsymbol{N}$ are here the DeWitt supermetric and its inverse respectively.
Remark 1 As regards genericity in Field Theory, firstly for instance Electromagnetism has one observables equation, whereas Yang–Mills Theory has $g := \dim(\mathfrak{g}_{\mathrm{YM}})$ such. GR-as-Geometrodynamics has four observables equations: commutation with the 3 components of the momentum constraint and with the single Hamiltonian constraint. So, on the one hand, for Yang–Mills Theory and for GR, PDE system genericity can be relevant. On the other hand, for Finite Theories, PDE system genericity is in general obscured by geometrical genericity.

7 Expression in Terms of Observables

Structure 1 Suppose we are given a state space $\mathfrak{S}$. Then having an observables algebraic structure $\mathfrak{O}\mathrm{bs}(\mathfrak{S})$ as a Function Space Thereover does not yet mean being able to express each model-meaningful quantity in terms of observables. The further step of eliminating irrelevant variables in favour of observables is required. We attain this by use of one or both of Algebra and Calculus.

Remark 1 It is useful to point out that, from the definition and elementary Calculus, functions of observables are themselves observables. Only a smaller number of suitably-chosen observables moreover suffice for practical use: expression in terms of observables.

Structure 2 Spanning, independence and bases for observables algebras – which make sense by these being linear spaces – is also a significant part of the theory of observables. It is additionally convenient to pick a basis of observables. Or at least a spanning set of observables, in terms of which all (required) physically-meaningful quantities can be expressed.

Remark 2 A common case is for dim(reduced $\mathfrak{p}$hase) $= 2(k - g)$ basis observables (of type G) to be required. Here $k := \dim(\mathfrak{q})$ and $g := \dim(\mathfrak{g})$ is the total number of constraints involved, which are all gauge constraints. Some Euclidean RPM $\mathsf{K} = \mathsf{G}$ observables examples of this are as follows.

Example 1 3 particles in 1-d have the mass-weighted relative Jacobi separations ρ_1, ρ_2 as useful basis observables (in $\mathsf{K} = \mathsf{G}$ sense). This extends to N particles in 1-d having as basis observables a given clustering's ρ^i, $i = 1$ to $n := N - 1$.

Example 2 The 3 Hopf–Dragt coordinates of Sec V.5.5 are basis observables for the relational triangle [59].

Further Motivation Kinematical quantization uses many fewer classical observables than the totality of suitably smooth functions over $\mathfrak{p}$hase. Kinematical quantization uses, more specifically, a linear subspace thereof that the canonical group acts upon [33]. This linear subspace moreover has enough coordinates to locally characterize $\mathfrak{q}$, thus fitting within our looser conception of 'basis observables'. Kinematical quantization's linear subspace quantities do moreover *literally* form a basis for that linear space. In this way, they constitute 'more of a basis' for configuration space than the a priori concept of 'basis observables' do. This linearity does not however in general extend to kinematical quantization's corresponding momentum observables (angular momentum suffices to see this). All of our examples above are useful for kinematical quantization. Explicit 'Kuchař basis observables' are not however known in the case of GR-as-Geometrodynamics.

8 The Problem of Observables and strategies for it

8.1 The problem

The *Problem of Observables* contributes [80] one facet to each of the canonical ans spacetime perspectives on the Problem of Time. In part, it is that Assignment of Observables is hard. This is the case in particular for Gravitational Theory and for Background-Independent Theory more generally. More specifically, the canonical approach's full obsevables – Dirac observables – are hard to find. In particular, they are harder to find than unrestricted observables or partly restricted observables such as Kuchař or A observables. The quantum counterparts of each type of observables are furthermore even harder to find than classical ones (see Chapter 50 of [70]). Dirac observables are hard enough to find for full GR that Kuchař [42] likened strategies relying on having already obtained a full set of these to plans involving having already caught a Unicorn. The other part of the Problem of Observables consists of any further complications in attaining Expression in terms of Observables for a sufficient set of physical quantities for one's theory.

8.2 Strategies

The 'bottom' alias 'zero' and 'top' alias 'unit' strategies are as follows.

Strategy 0) *Use Unrestricted Observables*, **U** [51, 52], entailing no commutation conditions at all.

Strategy 1) *Insist on Constructing Full Observables*, which are Dirac observables **D** in the canonical case.

Remark 1 Strategy 1) has the conceptual and physical advantage of employing all the information in the final algebraic structure of a given formulation of a given theory's first-class generators, $\mathfrak{F}$. While Geometry equips us with some spacetime observables, finding Dirac observables is hard. The canonical equivalent of relying on Geometry just produces Kuchař observables, or other kinds of A observables that do not take into account the $\mathcal{C}$hronos constraint. The hardness of this in the context of Gravitational Theories is raised or discussed in e.g. [14, 25, 38, 39, 40, 42, 51, 52, 53, 55, 57, 63, 68, 70].

Remark 2 Strategy 0) is diametrically opposite in each of the above regards. It can moreover be used as first stepping stone toward strategy 1). The below can also be viewed as further stepping stones.

Strategy K) *Find Kuchař Observables* **K** [42]. This can entail treating $\mathcal{Q}$uad distinctly from the $\mathcal{F}$**lin**, some motivations for which were covered in [70, 84]. A further pragmatic reason is that the **K** are simpler to find than the **D**. This is due to, firstly, having one fewer observables PDE or FDE to solve. Secondly, the remaining equations sometimes being open to the above-mentioned geometrical insight.

Strategy G) *Find* $\mathfrak{g}$*-observables*, **G**. For, as observed before, **G** $\neq$ **K** in general.

Strategy C) *Find Chronos Observables*, C.

Remark 3: (non)universality Using **U** or **D** is always in principle possible. The first of these follows from no restrictions being imposed. The second follows from how any theory's full set of constraints can in principle be cast as a closed algebraic structure of first-class constraints. This is by use of the Dirac bracket, or the effective method, so as to remove any second-class constraints.

Remark 4 Strategy 1) moreover amounts to concurrently taking into account the unsplit totality of first-class constraints (Constraint Closure facet) and Taking Function Spaces Thereover. As a four-aspect venture (The last Fig in [85]), it is unsurprisingly harder than the single-aspect Strategy 0). Strategies G and C are, correspondingly, 3-aspect ventures corresponding to dropping one Temporal and Configurational Relationalisms respectively. (This is available for theories for which Closure dictates such a drop to be consistent.)

Strategy A) Let us finally introduce an additional universal strategy based on *using some kind of A-Observables*, $\mathbf{A}_\times$, *that a theory happens to possess*. These kinds correspond to the closed subalgebraic structures of constraints which are realized by a given formulation of a given theory. This is the universal 'middling' strategy replacement for strategies K, G, and C. The ultimate aim is to reach the top of the lattice. It is however often practically attainable to previously land somewhere in the middle of the lattice. From here, one is then to try to work one's way up to the top by solving further DEs (or, viewed geometrically, by further restricting constraint surfaces).

A second source of strategic diversity involves priorly reformulating one's theory by reduction.

Strategy 0′) The *unreduced approach*: working on the unreduced $\mathfrak{p}$hase($\mathfrak{S}$) with all the constraints.

Strategy 1′) The *true space approach* – working on $\mathfrak{T}$rue-$\mathfrak{p}$hase($\mathfrak{S}$) $=$ $\mathfrak{p}$hase($\mathfrak{S}$)/$\mathfrak{F}$ – is much harder. This is due to quotienting out $\mathcal{C}$hronos($\mathfrak{S}$) being harder and having to be done potential by potential. If $\mathfrak{T}$rue-$\mathfrak{p}$hase is known, classical observables are trivial.

Strategy G′) The *reduced approach*: working on $\widetilde{\mathfrak{p}\text{hase}}$($\mathfrak{S}$) $=$ $\mathfrak{p}$hase($\mathfrak{S}$)/$\mathfrak{g}$ with just $\mathcal{C}$hronos.

Strategy A′) The *partly-reduced approach*: working on $\mathfrak{p}$hase($\mathfrak{S}, \mathfrak{H}$) $=$ $\mathfrak{p}$hase($\mathfrak{S}$)/$\mathfrak{H}$ for some $id < \mathfrak{H} < \mathfrak{g}$. Note that these are not all the A's but rather just the subalgebraic structures supported by $\mathfrak{g}$.

Remark 5 The above primed family of strategies are moreover found to all coincide in output in the case of strong observables [76]. E.g. both unreduced and reduced approaches end up with

$$\mathfrak{C}\text{an-}\mathfrak{G}\text{auge-}\mathfrak{O}\text{bs}(\mathfrak{S}) \ = \ \mathcal{C}^{\infty}(\widetilde{\mathfrak{p}\text{hase}}(\mathfrak{S})) \tag{104}$$

or, in the purely geometrical case,

$$\mathfrak{G}\text{eom-}\mathfrak{G}\text{auge-}\mathfrak{O}\text{bs}(\mathfrak{S}) \ = \ \mathcal{C}^{\infty}(\widetilde{\mathfrak{q}}(\mathfrak{S})) \ . \tag{105}$$

Which weak observables are realized can differ between formulations [85].

9 Conclusion

9.1 Summary

Given a state space $\mathfrak{s}$, finding observables consists of Taking a Function Space Thereover: $\mathfrak{F}(\mathfrak{s})$. In the presence of generators, observables are furthermore to be [8, 86] zero Lie brackets commutants with these generators. $\mathbb{C}^1$ functions provide a minimum standard of smoothness for observables' functions. For observables to make sense in presence of generators, we must know beforehand that the generators close. The Jacobi identity based argument for this provides the first great decoupling of the Problem of Time. This is into a Relationalism–Closure problem which must precede the addressing the Problem of Observables. With [63, 70, 74, 76, 85] and the current Article, the days of finding individual or few observables are over. Solutions to the Problem of Observables are to involve, rather, a whole function space of solutions that is also an algebraic structure: observables algebraic structures.

The above Lie brackets zero commutant conditions can furthermore be recast as [5, 67, 70, 74, 76, 85, 86] explicit first-order systems of PDEs [5, 10] or FDEs [85]. Such DEs are amenable to the flow method [18, 29, 43, 48, 61, 62]. Observables DE systems widely manifest over-determinedness. Yet this is vanquished by [76] an integrability condition guaranteed by Frobenius' Theorem [3, 31, 61]. In Geometry's arena, treating the observables PDE systems in this way is a slight generalization [74, 76, 85] of *Lie's Integral Approach to Invariants* [5, 18]. This slight generalization involves the following. Firstly, uplifting from finding invariants to a *free* alias *natural* [10] characteristic problem for finding thereover 'suitably-smooth functions of the invariants': observables. Indeed, a free characteristic problem is the appropriate rendition of our aim to, given a state space $\mathfrak{s}$, we are Taking a Function Space Thereover: $\mathfrak{F}(\mathfrak{s})$. This consists of finding *all* functions solving our free problem, say withing a given smoothness category such as $\mathfrak{C}^{\infty}$, rather than a particular such corresponding to particular prescribed data. Secondly, our arena consists of not only Geometry but also of state spaces such as configuration space, phase space, spacetime and the space of spacetimes. For Field Theories, observables equations give instead linear first-order functional differential equation systems; similar Flow Methods carry over. This good fortune is rooted in the underlying benevolence of *Banach Calculus* [31, 44].[4]

There may be further complications as regards reaching Expression in terms of Observables for a sufficient set of physical quantities for one's theory. These revolve around Algebra and Calulus workings' attainability. On the one hand, this may involve a 'basis set' of observables that is substantially smaller than the space of observables. But, on the other hand it comes with the separate complication of Expression entailing elimination-type calculations. The reason for there being a Problem of Observables is that Assignment of Observables and Expression in terms of Observables are difficult to resolve for whichever of Gravitational Theories and Background Independent Theories. This is moreover even more of a problem at the quantum level. The lattice of notions of observables supported by a theory – one per subalgebraic structure of generators – provides the setting in which to discuss approaches to the Problem of Observables. These include stepping-stone approaches using e.g. geometrical tractability of a a consistent subsystem of observables equations.

Let us end by pointing to checks on the above treatment of observables' canonical branch being TRi-compatible [64, 70, 80, 85]. $\boldsymbol{Q}$, $\boldsymbol{P}$ and $\mathfrak{P}$hase are already-TRI. So are $\boldsymbol{c}$ and $\{\ ,\ \}$, and thus the definition of constrained observables as well. The current Article's smearing functions are given in TRi form. This is via first-class constraints are both trivially weak observables and TRi-smeared, which points to all observables requiring TRi-smearing as well. Observables algebraic structures are already-TRi in the Finite Theory case, or readily rendered TRi by adopting TRi-smearing in the Field Theory case. Splitting $\boldsymbol{c}$ into Temporal and Configurational Relationalism parts has the following knock-on effect [80]. Some notions of observables are just Configurationally Relational – $\mathfrak{g}$-observables –

[4] For now, we use [85] Banach Calculus. A more global treatment involving what is actually assumed in Mathematical Physics would instead involve e.g. tame Fréchet Calculus [32]. Both of these Calculi support Lie-theoretic combination of machinery required by our Local Resolution of the Problem of Time.

or just Temporally Relational – Chronos observables – in theories admitting such a split. GR permits $\mathfrak{g}$ = Kuchař observables but not Chronos observables, whereas RPM supports both. The last Fig of [85] gives Assignment of Observables' further Problem of Time facet interferences. Its scantness of interaction is testimony to the above-mentioned great decoupling of facets.

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