Lie Theory suffices for Local Classical Resolution of Problem of Time. 1. Closure, as implemented by Lie brackets and Lie's Algorithm, is Central.

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Abstract

The Problem of Time is due to conceptual gaps between General Relativity and the other observationally-confirmed theories of Physics; it is a major foundational issue in Quantum Gravity. A key point in resolving the Problem of Time turns out to be that Algebra rapidly takes centre stage. The first algebraic aspect encountered is that constraints must close, as must spacetime generators. This Closure aspect is assessed by the generalized Lie Algorithm, of which Dirac's Algorithm is the constrained canonical perspective's subcase. Such algorithms, by having the capacity to shut down trial sets of generators for being inconsistent, constitute a selection principle. Those sets of generators which survive form Lie algebras or Lie algebroids, such as the Lie algebra of spacetime diffeomorphisms and the Dirac algebroid of constraints in GR. Around 3/4 of Problem of Time aspects revolve, moreover, around brackets algebraic structures, with Observables and Constructability joining Closure in this regard, while Relationalism is distinct.

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1 Introduction

A key point in resolving [70, 77, 78, 81, 82, 83, 84, 87] the Problem of Time [29, 28, 25, 48, 49, 67, 77] is that Algebra rapidly takes the centre stage away from philosophical considerations. This is through the most philosophically-rooted [1, 6, 53] aspect of Background Independence – Relationalism [44, 51, 68, 77, 81, 82, 85, 86, 93] – rapidly fusing with the first algebraic aspect of Background Independence: Closure ([14, 19, 25, 47, 83, 87] and the current article). This is most readily envisaged by noting that Relationalism is implemented by Lie derivatives, with the ensuing generators (constraints in the canonical case) then needing to close under *Lie brackets* (Sec 2) [26] to form a *Lie algebraic structure*. In some simpler cases, this is just a *Lie algebra* (Sec 3) [22, 26, 62], including the Lie algebra of spacetime diffeomorphisms (Sec 4). In other cases, however, a *Lie algebroid* [23, 52, 56, 57, 63, 65] ensues (Sec 6), such as the *Dirac algebroid* [15, 20, 25, 35, 69] of constraints in GR (Sec 8), or a further qualitatively distinct algebroid of constraints in Supergravity [40, 75, 77]. All subsequent accounts of the Problem of Time, or of Background Independent physics, would do well to take this fusion into account.

It should be clear that when there is a single type of Relationalism, in particular in the spacetime version, this involving a continuous group itself comes with guarantees of Closure. Both Spacetime and Spatial Relationalism's diffeomorphisms close in this manner. Similar applies for more structured geometries, for which a Generalized Killing equation [8, 11, 17, 31] controls the Relationalism; the output of this is a Lie algebra of automorphisms [32]. For approaches with two Relationalisms acting as generator providers – Configurational and Temporal Relationalism acting as constraint providers – the situation is less clear-cut. It is here natural to ask whether one has found all of the constraints. Quite a general approach to this question of *Constraint Closure* (Sec 10 with examples in Secs 7 and 8) follows from Dirac-type Algorithms [14, 19, 25, 47, 77, 83, 87]. If the answer is in the negative, one has a *Constraint Closure Problem* (Sec 10): a Problem of Time facet [48, 49, 70, 77]. In full Lie-Theoretic generality, however, one is to use a Generalized Lie Algorithm ([90] and Sec 6) of *Generator Closure*, of which Dirac-type Algorithms are a subcase. These algorithms have the capacity to shut down inconsistent sets of generators, thus constituting a selection principle. These algorithms are moreover how closure becomes inextricably fused with Relationalism. Those sets of generators which survive form Lie algebras or Lie algebroids.

In working with constraints, the combination of working in Hamiltonian variables ($\mathbf{Q}, \mathbf{P}$) and making use of the classical Poisson brackets has the following benefits. It turns out to allow for a *systematic* treatment of constraints: the *Dirac Algorithm* [14, 20, 25, 47, 77, 83, 87]. It also places the classical laws of nature into a framework which transcends into Quantum Theory [9]. The quantum-level version of closure is Functional Evolution Problem [48, 49]. Closure is the classical-or-quantum, canonical-or-spacetime, and finite-or-field generalization of the Functional Evolution Problem iii). 'Functional' here implicitly refers to Field Theory, whereas iii) is formulated quantum-mechanically and at the canonical level, for which the generators take the form of constraints.

In the Lie Background Independence program, Closure literally plays a central role: as Fig [93].1's central connecting vertex. By adjacency, this means that many facet interferences are pairwise (Closure, arbitrary). The remaining parts of the Problem of Time are moreover resolved by further pieces of brackets algebra:

A) observables algebraic structures [74, 88, 94]

B) Cohomological characterization of Lie algebraic structures' deformations [89, 90, 95].

C) A particular algebraic pentagon relation providing the general question that is answered in the affirmative for classical GR by its being Refoliation Invariant [90, 95].

2 Lie brackets

Definition 1 For $\mathfrak{g}$ for now a vector space, *Lie bracket* is a bilinear map

$$|[\ ,\ ]|\ :\ \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g} \tag{1}$$

that is antisymmetric

$$|[\, g \, , \, h \,]| \ = \ - \, |[\, h \, , \, g \,]| \ \ \forall \ g \, , \ h \ \in \ \mathfrak{g} \tag{2}$$

and obeys the *Jacobi identity*

$$0 \ = \ \mathbf{J}(g \, , \ h \, , \ k) \ := \ |[\, g \, , \, |[\, h \, , \, k \,]| \,]| \ + \ \text{cycles} \ \ \forall \ g \, , \ h \, , \ k \ \in \ \mathfrak{g} \ . \tag{3}$$

Remark 1 Thus equipped, $\mathfrak{g}$ becomes a Lie algebra. The useful shorthand $\mathbf{J}$ here merits the name *Jacobiator*. This is a particular subcase of *associator*, i.e. measure of departure from associativity. Compare the notion of commutator as viewed as a measure of departure from commutativity. Aside from the statement that Lie algebras have zero Jacobiator, nonzero Jacobiator gives a measure of departure from having a Lie algebra.

2.1 Instances of Lie brackets

Case 1 A geometry's symmetries carry Lie brackets structure. We already encountered various places where such feature in [93]: for spacetime $\mathfrak{M}$, space $\mathbf{\Sigma}$ and state spaces $\mathfrak{S}$ such as configuration space $\mathfrak{q}$, phase space $\mathfrak{p}$hase and the space of spacetimes. More specifically, both diffeomorphisms and solutions to the generalized Killing equation [17, 31] carry such a bracket.

Case 2 *Poisson brackets*

$$\{\ ,\ \} \tag{4}$$

are Lie brackets. In their finite canonical realization, Poisson brackets of phase space functions $A(\boldsymbol{Q}, \boldsymbol{P})$ and $B(\boldsymbol{Q}, \boldsymbol{P})$ are given by

$$\{\, A, B \,\} \ := \ \frac{\partial A}{\partial \boldsymbol{Q}} \cdot \frac{\partial B}{\partial \boldsymbol{P}} \ - \ \frac{\partial B}{\partial \boldsymbol{Q}} \cdot \frac{\partial A}{\partial \boldsymbol{P}} \ . \tag{5}$$

For Field Theories, Poisson brackets of phase space functions $A(\mathbf{Q}, \mathbf{P})$ and $B(\mathbf{Q}, \mathbf{P})$ is given by

$$\{\, A, B \,\} \ := \ \int_{\mathbf{\Sigma}} \mathrm{d}\mathbf{\Sigma} \left\{ \frac{\delta A}{\delta \mathbf{Q}} \cdot \frac{\delta B}{\delta \mathbf{P}} \ - \ \frac{\delta A}{\delta \mathbf{P}} \cdot \frac{\delta B}{\delta \mathbf{Q}} \right\} \ . \tag{6}$$

In addition to (2, 3), Poisson brackets furthermore obey the *Leibniz* alias *product rule*,

$$\{\, A, B\, C \,\} \ = \ B \, \{\, A, C \,\} \ + \ \{\, A, B \,\} \, C \ , \tag{7}$$

by which they are also a *derivation*. Brackets which obey these three axioms can be viewed as Poisson algebras, even if they do not have the specific computational form of Poisson brackets. In this sense, quantum commutators are Poisson algebras. Indeed, one reason for Poisson brackets' significance is as a preliminary step toward quantization. Another is that they enable systematic treatment of constraints; both of these observations are due to Dirac [9, 14].

Remark 1 The fundamental Poisson bracket is

$$\{\, \underline{\mathsf{Q}} \, , \, \underline{\mathsf{P}} \,\} \ = \ \underline{\underline{\boldsymbol{\delta}}} \, . \tag{8}$$

Q and P are portmanteaux of the finite and field theoretic cases' configurations and momenta, and $\boldsymbol{\delta}$ is the portmanteau of the finite Kronecker δ and the product of a field-species-wise such with a field-theoretic Dirac $\delta^{(d)}(\underline{x} \ - \ \underline{x}')$. This bracket being established for all the Q and P establishes the brackets of all once-differentiable quantities $\mathscr{F}\lfloor \mathsf{Q}, \, \mathsf{P} \rfloor$ as well. The entries into each slot of the Poisson brackets could also be functionals $\mathscr{F}$, $\mathscr{G}$ rather than just functions F, G.

Remark 2 If prephase space – the space of configurations and momenta – is equipped with the Poisson bracket, it becomes phase space, $\mathfrak{p}$hase. This can furthermore be rephrased in terms of equipping with a symplectic structure [41].

Remark 3 The Poisson bracket and phase space are already-TRi [93].

3 Lie algebras

Definition 1 A *Lie algebra*

$$\mathfrak{g} \tag{9}$$

is a vector space equipped with a Lie bracket, such that the bracket of two elements in the vector space also lies in the vector space: *closure under the Lie algebra*.

Remark 1 As more general context, having an algebra amounts to having one more operation than vector spaces.

Definition 2 As a vector space, a basis of elements can be picked therein. Such can be viewed as *generators* for our Lie algebra. As [93] already argued, we denote these by

$$\underline{\mathsf{g}} \ , \quad \text{indexed by} \quad \mathsf{G} \ . \tag{10}$$

Notation We use italic font for generators represented by finite quantities, such as angular momenta $\underline{L} = \underline{q} \times \underline{p}$. Upright font for generators represented by field quantities, which we additionally smear (see Sec 4 for the example of diffeomorphisms). Sans serif font for the portmanteau of the previous two.[1]

Remark 2 Our Finite–Field portmanteau notation for generators is

$$\mathsf{g} \ := \ \mathscr{F}\lfloor \mathbf{Q}, \mathbf{P} \rfloor \ : \ \mathrm{F}(\boldsymbol{Q}, \boldsymbol{P}) \ \ (\text{finite}) \quad \text{and} \quad \mathcal{F}(\underline{x}; \mathbf{Q}, \mathbf{P}] \ \ (\text{field}) \ . \tag{11}$$

Remark 3 Given a basis of generators g, computing

$$|[\, \mathsf{g}_{\mathsf{G}} \, , \, \mathsf{g}_{\mathsf{G}'} \,]| \ = \ G^{\mathsf{G}''}{}_{\mathsf{G}\mathsf{G}'} \mathsf{g}_{\mathsf{G}''} \ , \tag{12}$$

permits us to read off the *structure constants* $G^{G''}{}_{GG'}$ for the Lie algebra with respect to this basis. This amounts to formulating a Lie algebra as Lie brackets of generators which return solely linear combinations of generators (which thus indeed lie within the original vector space).

A coordinate-independent form for this is [83]

$$|[\, \underline{\mathsf{g}}, \underline{\mathsf{g}}' \,]| \ = \ \underline{\underline{\underline{G}}} \cdot \underline{\mathsf{g}}'' \ . \tag{13}$$

$\boldsymbol{G}$ are here *structure constant 3-arrays* or *trilinear maps*: a more succinct and coordinate-independent presentation. It readily follows from (12, 2) that the structure constants obey the antisymmetry property,

$$G^{G''}{}_{GG'} \ = \ - \, G^{G''}{}_{G'G} \ , \tag{14}$$

and, from the Jacobi identity [4], the homogeneous-quadratic restriction

$$G^{G}{}_{[G'G''} G^{G'''}{}_{G'''']G} \ = \ 0 \ . \tag{15}$$

Remark 4 In the canonical setting, the algebras can be taken to be Poisson algebras; see e.g. [52, 73] for introductions to these. Also, at least within a restricted range of formulations of a restricted range of theories, the generators can be taken to be constraints.

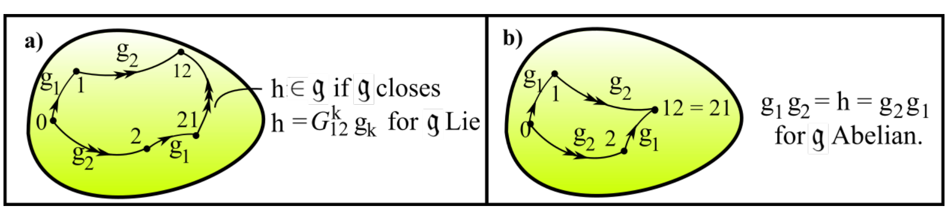

Figure 1: a) An algebra's commutator. This compares applying two transformations g_1, g_2 in either order to a common initial object 0. b) The even more straightforward commuting subcase, for which the final objects 12 and 21 coincide as well. Many instances of a) and b) occur in ALRoPoT, as picked out among the Series's figures by being depicted on lime-green egg-shaped spaces.

[1] This is well-defined at the level of Calculus and Principles of Dynamics for ii) modelled by Banach spaces or a certain subset of Fréchet spaces; further details are left for a separate global-level review.

Remark 6 For Poisson brackets, and also for differentially-represented Lie algebra generators [93],

$$\underset{\sqcup}{\mathcal{g}} \;=\; \underset{\sqcup}{\boldsymbol{T}} \cdot \underline{\nabla} \;, \tag{16}$$

the right hand side brackets are formed by at most linear algebra operations and differentiation, by which they are systematically evaluable.

Structure 1 For a particular $\langle\, \mathfrak{m}, \boldsymbol{\sigma} \,\rangle$ the generalized Killing vectors moreover close [32] as a Lie algebra,

$$[\, \underset{\sqcup}{\boldsymbol{\xi}}(\mathbf{x}), \underset{\sqcup}{\boldsymbol{\xi}}(\mathbf{x}) \,] \;=\; \underset{\sqcup\sqcup}{\boldsymbol{Z}} \cdot \underset{\sqcup}{\boldsymbol{\xi}}(\mathbf{x}) \;. \tag{17}$$

A, B, C are here multi-indexes comprising both the corresponding spatial index a, b, c, and G is a generator-basis index, and $\boldsymbol{Z}$ are the corresponding *structure constants*. As a Lie algebra, this corresponds to the continuous connected component of the identity part of the automorphism group,

$$Aut(\mathfrak{m}, \boldsymbol{\sigma}) \;. \tag{18}$$

We thus denote this by

$$aut(\mathfrak{m}, \boldsymbol{\sigma}) \;. \tag{19}$$

3.1 Weak equality

Definition 1 Let us use

$$\approx \tag{20}$$

to mean equality up to a linear function(al) of the generators: *Lie-* alias *generator-weak equality*. This is the general Lie arena's extension of Dirac's use of the same symbol to mean equality up to a linear function(al) of constraints: *Dirac-* alias *constraint-weak equality*

Remark 1 In contrast, strong equality

$$= \tag{21}$$

is just equality in the usual sense; this clearly does not require any 'constraint'/'generator' or 'Dirac'/'Lie' qualifications.

Definition 2 Let us finally introduce

$$\text{`='} \tag{22}$$

to denote *portmanteau equality*: strong or weak. Having already used this for *Dirac-* alias *constraint-portmanteau equality* in [77, 87], we now extend it to mean *Lie-* alias *generator-portmanteau equality*.

Remark 2 Closure as a Lie algebra is then of the schematic form

$$|[\, \mathcal{g} \,,\, \mathcal{g} \,]| \;\; \text{`='} \;\; 0 \;. \tag{23}$$

This is a portmanteau for the strong version

$$|[\, \mathcal{g} \,,\, \mathcal{g} \,]| \;=\; 0 \;\; : \tag{24}$$

– a commuting Lie algebra – and the weak version

$$|[\, \underset{\sqcup}{\mathcal{g}} \,,\, \underset{\sqcup}{\mathcal{g}} \,]| \;=\; \underset{\sqcup\sqcup}{\boldsymbol{G}} \cdot \underset{\sqcup}{\mathcal{g}} \;. \tag{25}$$

4 Spacetime Generator Closure

For now, we comment that the outcome of infinitestimal generators closing as a Lie algebra

$$|[\, \underset{\sqcup}{\mathcal{g}} \,,\, \underset{\sqcup}{\mathcal{g}}' \,]| \;=\; \underset{\sqcup\sqcup}{\boldsymbol{G}} \cdot \underset{\sqcup}{\mathcal{g}}'' \;, \tag{26}$$

for structure constants $\boldsymbol{G}$ straightforwardly suffices to cover the case of GR's spacetime diffeomorphisms,

$$\boldsymbol{\{}(\vec{\mathcal{S}}, \vec{\mathrm{E}})\boldsymbol{,}\, (\vec{\mathcal{S}}, \vec{\mathrm{F}})\boldsymbol{\}} \;=\; (\vec{\mathcal{S}},\, \overrightarrow{[\mathrm{E}, \mathrm{F}]}) \;\; . \tag{27}$$

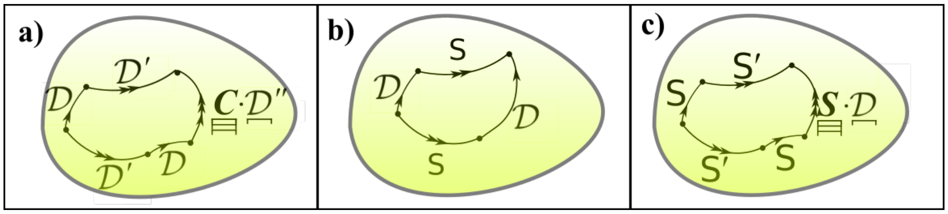

Figure 2: a) Spacetime diffeomorphisms close as a Lie algebra.

b) Strong spacetime observables close as an Abelian Lie algebra.

c) Weak spacetime observables themselves close as a Lie algebra.

This forms a Lie algebra,

$$\mathfrak{g}_{\mathrm{S}} \;=\; diff(\mathfrak{m}) \tag{28}$$

which is infinite in the sense of having an infinite number of generators. E, F here are smearing functions, with (A, B) denoting $\int \mathrm{d}^3 y\, \mathrm{A}(y)\, \mathrm{B}(y)$ and $[\ ,\ ]$ denoting differential-geometric commutator. This equation is a subcase of generator-weakly vanishing Generator Closure.

Example 1 The Poincaré algebra $Poin(3,1) \;=\; Isom(\mathfrak{M}^4)$ has commutation relations [54]

$$[P_\mu, P_\nu] \;=\; 0\ , \tag{29}$$

$$[M_{\mu\nu}, P_\rho] \;=\; 2\,\eta_{\rho[\mu}P_{\nu]}\ , \tag{30}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] \;=\; 2(\eta_{\rho[\mu}M_{\nu]\sigma} \;-\; \eta_{\sigma[\mu}M_{\nu]\sigma})\ . \tag{31}$$

5 Generalized Lie Algorithm

Given a candidate set of generators, one forms the Lie brackets between them. At the local level, five types of equation can result.

5.1 Outcome a) generators old and new

One possible outcome is zero, in which case the generators commute. A more general possibility is that they give a linear function of one's known generators; this was the only possibility considered by Clebsch [5], and returns a Lie algebra.

It is also possible to get *structure functions* in place of structure constants:

$$|[\,\underline{\mathcal{g}}\,,\,\underline{\mathcal{g}}'\,]| \;=\; \underline{\underline{\boldsymbol{G}}}(\mathbf{B}, \boldsymbol{c}) \cdot \underline{\mathcal{g}}''\ , \tag{32}$$

but with *structure functions* $\boldsymbol{G}(\boldsymbol{B}, \boldsymbol{c})$ instead of structure constants, we have a *Lie algebroid*. The $\boldsymbol{B}$ are base objects, whereas the $\boldsymbol{c}$ are constants. The possibility of structure functions was first noted by Cartan in 1904 [18]. We shall see below that a case of it also occurs in Dirac's work on GR's constraints. It has subsequently been formalized as the *Lie algebroid* [23, 52, 56, 57, 63, 65] generalization of Lie algebra. We furthermore term the portmanteau of Lie algebras and Lie algebroids a *Lie algebraic structure*.

Dirac furthermore considered being zero up to a linear function of constraints as a weak notion of zero in the Poisson brackets setting. We now take this to transcend to a linear function of generators in the general Lie brackets setting. By this, the current paragraph's possibilities jointly count as one type of outcome. Lie [7] and subsequently Dirac [14, 19, 25] furthermore allowed for the possibility that new generators $\boldsymbol{\mathcal{g}}'$ are discovered in the process. These are so-called *integrabilities* [71] of the candidate set of generators (or of constraints more specifically in Dirac's case).

Remark 1 Lie brackets are well-known to arise naturally in two ways (at least).

i) Its zeroness ensures that **second-order terms vanish for our infinitesimal transformations**.

ii) It arises in the **integrability condition**:

$$\text{if} \;\; X \;\; \text{and} \;\; Y \;\; \text{solve flow PDE system} \;\; \mathscr{P} \;\; , \;\; |[X, Y]| \;\; \text{also solves} \;\; \mathscr{P} \; . \tag{33}$$

In case i), it arises in this second way.

Remark 2 Dirac also formalized how discovering integrabilities in general requires proceeding recursively. So having found new generators $\boldsymbol{\mathcal{N}}$,

$$|[\boldsymbol{\mathcal{G}}, \boldsymbol{\mathcal{N}}]| \quad \text{and} \quad |[\boldsymbol{\mathcal{N}}, \boldsymbol{\mathcal{N}}]| \tag{34}$$

need to be investigated, and might themselves produce further generators.

5.2 Outcome b) identities

Both Lie and Dirac also envisaged the necessity of including the possibility of brackets producing identities, i.e. equations reducing to $0 = 0$. This is as many outcomes as Lie's Algorithm included (excluding algebroids).

5.3 Outcome c) inconsistencies

Dirac applied various further insights, albeit only to the narrower arena of Constrained Dynamics' Poisson brackets of constraints. The Author subsequently argued [91, 90] for these insights to carry over to the general Lie case, giving the Generalized Lie Algorithm.

Most significantly, Dirac envisaged the need to include *inconsistencies* – i.e. equations reducing to $0 = 1$ among the output of the algorithm. This imbues the Algorithm with the capacity to act as a selection principle. This points to using the language of 'candidate sets', which only become theoretically bona fide sets of constraints if they pass the test set by the algorithm; see Sec 5.6 for details. Dirac's considerably successful use of c) is thus set loose on a much larger arena of mathematics; see Article 3 for some first consequences. That the Lagrangian

$$L = \dot{q} + q \, , \tag{35}$$

gives as its Euler–Lagrange equations

$$0 = 1 \tag{36}$$

suffices to show that inconsistencies are possible in the Principles of Dynamics. See Article 3 for vindication of inconsistency as a general Lie, rather than just Dirac, feature.

Definition 1 Lie's Little Algorithm is generalization of Dirac's Little Algorithm: the opening run of a smaller algorithm in his book prior to introducing a rebracketing procedure to factor in the possibility of second-class constraints. This permits outcomes a), b), c) only.

5.4 Outcome d) specifier equations

In the presence of an appending procedure – which Dirac's treatment of constraints has – '*specifier equations*' are also possible.

E.g. Dirac's [14, 19, 25, 47] appending of constraints to Hamiltonians [2] H using Lagrange multipliers $\boldsymbol{\phi}$, to make total Hamiltonian type objects,

$$H \longrightarrow H_{\text{Total}} + \boldsymbol{\phi} \cdot \boldsymbol{\mathcal{C}} \, . \tag{37}$$

In the TRi-Dirac counterpart [68, 72, 77, 87], this appending is to the bare differential Hamiltonian dH of constraints using cyclic differentials $\mathrm{d}\boldsymbol{\lambda}$ so as to form total differential-almost-Hamiltonian type objects,

$$\mathrm{d}H \longrightarrow \mathrm{d}A_{\text{Total}} = \mathrm{d}H + \mathrm{d}\boldsymbol{\lambda} \cdot \boldsymbol{\mathcal{C}} \, . \tag{38}$$

Specifier equations are then indeed are equations that specify what forms a priori free appending variables can take.

Definition 1 The **Extended Lie's Little Algorithm** is for cases that come with an appending procedure: a), b), c), d) only.

5.5 Outcome e) rebracketing

A further classification of constraints is as follows [14, 20, 25].

Definition 1 *First-class constraints* [14, 25, 47] $\mathcal{F}$ are those that close among themselves under Poisson brackets.

Definition 2 *Second-class constraints* [14, 25, 47] are defined by exclusion to be those that are not first-class.

Diagnostic For the purpose of counting degrees of freedom, first-class constraints use up 2 each whereas second-class constraints use up only 1 [47].

Remark 2 First-class constraints are not necessarily gauge constraints. For now we give the canonical example of Dirac's Conjecture [25] failing [47, 87] as a counterexample.

Second-class constraints are to be dealt with by rebracketing: passage from the Poisson bracket to the Dirac bracket [14, 25],

$$\{A, B\}^{*} := \{A, B\} - \{A, \underline{\mathcal{I}}\} \cdot \{\underline{\mathcal{I}}, \underline{\mathcal{I}}\}^{-1} \cdot \{\underline{\mathcal{I}}, B\} . \tag{39}$$

so as to remove irreducible [47] second-class constraints $\mathcal{I}$. The −1 here denotes the inverse of the given matrix, and each · contracts the underlined objects immediately adjacent to it. Geometrically, however, this is still a Poisson bracket [37] (and TRi-invariant [77, 87]). We do now need to stipulate that first class constraints are to close under the final Poisson bracket involved. (Second-class constraints having the capacity [47] of arising at each iteration of the Algorithm, by which a sequence of intermediary brackets may be needed.)

Structure 2 The preceding can moreover happen on subsequent iterations of the TRi-Dirac Algorithm, were these to reveal more second-class constraints. I.e. while still in the process of investigating a physical theory's constraints, one does not yet know which are first-class. This is because a given constraint may close with all the constraints found so far but *not* close with some constraint still awaiting discovery. Thus one's characterization of constraints needs to be updated step by step until either of the following apply.

The notion of *final classical bracket* alias *maximal Dirac bracket* thus also carries over as already-TRi.

Next consider the corresponding subdivision into *first* and *second-class generators* according to whether they weakly Lie brackets commute, and the subsequent Lie-Dirac bracket

$$|[A, B]|^{*} := |[A, B]| - |[A, \underline{\mathcal{I}}]| \cdot |[\underline{\mathcal{I}}, \underline{\mathcal{I}}]|^{-1} \cdot |[\underline{\mathcal{I}}, B]| . \tag{40}$$

to eliminate irreducible [90] Lie second class objects

Remark 1 Second-class generators (or constraints) can be *self-second-class*, meaning that brackets between some new objects do not close. Or *mutually-second-class*, signifying that some bracket between a new object and a previously found object does not close. Thereby, this previously prescribed or found object can just be viewed as *hitherto* first-class. I.e. first-classness of a given generator (or constraint) can be lost whenever new generators (or constraints) are discovered.

Remark 2 a), b), c), e) is a final significant combination: including rebracketing while in a context possessing no appending procedure.

5.6 Termination conditions

Given some initial candidate set of generators $\mathcal{G}$, we assess them under Lie brackets. The Generalized Lie Algorithm proceeds iteration by iteration until one of the following *termination conditions* hold [91, 90].

0) *Hitting an immediate inconsistency*, by at least one inconsistent equation arising [25].

I) *Combinatorially critical cascade.* Here iterations of the Lie Algorithm produce a cascade of new objects down to the point of leaving the candidate with no degrees of freedom. This is a combinatorial triviality condition, and was envisaged by both Lie [7] and Dirac [25].

II) *Cascade to inconsistency.* This transcends the point of no degrees of freedom into inconsistency.

III) *Arriving at an iteration that produces no new objects* while retaining some degrees of freedom. This concluding iteration of the Lie Algorithm produces no new generators, indicating that all such have been found. Re-running the algorithm past this point cannot find any new equations.

Remark 1 In the canonical case, Dirac-type Algorithms have first-class constraints and specifier equations as objects. In the general case, this role is played by first class generators alongside specifiers equations (if an appending process is supported in the given context).

Remark 2 It is III) that is the termination condition which renders a candidate theory successful at this stage: consistent and nontrivial. In this case, the final output is a Lie algebraic structure.

5.7 Comment on synthesis of Relationalism and Closure

The Dirac Algorithm fits our program well, with Temporal and Configurational Relationalism providing candidate sets of constraints for it to assess. While not itself Temporally Relational, a TRi Dirac-type algorithm is available [68, 72, 77]. Its classification of local outcomes is in one-to-one correspondence with a)-e) and so has the same termination conditions 0)-III). The Generalized Lie Algorithm covers both this and candidate spacetime generators assessed by the separate spacetime Lie bracket.

A reverse operation to Constraint Provision is the *encoding of constraints*. I.e. upon finding constraints, one aims to subsequently build auxiliary variables into the theory's action. Suppose that a model's constraints arising from Relationalism are demonstrated by the Dirac Algorithm to require further constraints in order to close. Then it may be possible to revisit Relationalism so as to further encode these integrability constraints. So in general the two-way arrow in Fig 1 is to be interpreted as a loop. This continues until either Relationalism and Closure are jointly satistied – a 3-aspect synthesis – or the candidate theory is discarded as trivial or inconsistent.

5.8 Each iteration's problem is a linear system

Remark 1 If further first-class generators arise, these are fed into the subsequent iteration of the algorithm.

a) Define $\mathcal{Q}$ as one's initial $\mathcal{P}$ alongside the subset $\mathcal{R}$ of the candidate theory's formulation's $\mathcal{S}$ that have been discovered so far, indexed by $\mathsf{Q} \ = \ \mathsf{P} \coprod \mathsf{R}$.

b) Form a system

$$0 \ \approx \ \dot{\mathcal{Q}} \ = \ \{\, \mathcal{Q}, H_{\boldsymbol{u}\mathcal{Q}} \,\} \ = \ \{\, \mathcal{Q}, H \,\} \ + \ \{\, \mathcal{Q}, \underline{\mathcal{Q}} \,\} \cdot \underline{\boldsymbol{u}} \ \approx \ 0 \ , \tag{41}$$

in coordinate-free notation. In the case of completion being attained, (the final $\mathcal{R}$) $=$ $\mathcal{S}$ itself, whereas $\mathcal{Q} \ = \ \mathcal{F}$. (For now under the assumption explained below that all constraints involved are first-class: $\mathcal{F}$.)

Remark 2 (41) is a linear system. Its general solution thus splits according to

$$\boldsymbol{u} \ = \ \boldsymbol{p} \ + \ \boldsymbol{C} \ , \tag{42}$$

for particular solution $\boldsymbol{p}$ and complementary function $\boldsymbol{C}$. By definition, $\boldsymbol{C}$ solves the corresponding homogeneous equation

$$\underline{\boldsymbol{C}} \cdot \{\, \underline{\mathcal{C}}, \mathcal{P} \,\} \ \approx \ 0 \ , \tag{43}$$

where underbrackets are coordinate-free notation for constraint vector.

Furthermore, $\boldsymbol{C}$ has the structure

$$\underline{\boldsymbol{C}} \ = \ \underline{\boldsymbol{s}} \cdot \underline{\underline{\boldsymbol{R}}} \ , \tag{44}$$

where the $\boldsymbol{s}$ are the totally arbitrary coefficients of the independent solutions. $\boldsymbol{R}$ is a mixed-index (and thus in general rectangular rather than square) matrix. Its second index runs over primary constraints while its first index runs over the generally-distinct independent solutions (hence the subscript S). Our general solution is next to be substituted into the total Hamiltonian, updating it.

Remark 3 The TRi version counterpart of (41) are also linear problems; see [87] for details.

6 Generator algebraic structures

Structure 1 The end product of a successful candidate theory's passage through the Generalized Lie Algorithm is a *generators algebraic structure*. This consists solely of Lie-first-class generators closing under Lie (or in a sense more generally Lie–Dirac) brackets. [t could however more generally be a *generators-and-specifiers algebraic structure*.]

Structure 2 For now, assuming that no tertiary complications explained below occur, schematically,

$$|[\mathcal{F}, \mathcal{F}]| \text{ `='} \ 0 \ . \tag{45}$$

This is a portmanteau for the strong version

$$|[\mathcal{F}, \mathcal{F}]| \ = \ 0 \ , \tag{46}$$

and the weak version:

$$|[\underline{\mathcal{F}}, \underline{\mathcal{F}}]| \ = \ \underline{\underline{\boldsymbol{F}}} \cdot \underline{\mathcal{F}} \ . \tag{47}$$

The $\boldsymbol{F}$ here can be the structure constants of a Lie algebra, or a Lie algebroid's phase space functions, $\boldsymbol{F}(\boldsymbol{Q}, \boldsymbol{P})$.

Remark 1 Generator algebraic structures (including constraints algebraic structures) have comparable significance to base spaces $\mathfrak{B}$ (such as spaces of spacetimes $\mathfrak{S}$, configuration space $\mathfrak{q}$ or phase spaces $\mathfrak{p}$hase) as regards the study of the nature of Physical Law. Generator algebraic structures' detailed features are moreover needed to understand any given theory. This refers in particular to the topological, differential and higher-level geometric structures of observables algebraic structures supported [94]; both function space and algebraic levels of structure are relevant to these. We consequently need to pay attention to the Tensor Calculus on constraint algebraic structures as well. This justifies our use of one-turn underbrackets to keep generator-tensors distinct from base objects' zero-turn underlined ones [93].

Structure 3 The *full space of classical first-class generators* is[2]

$$\mathfrak{F} \ . \tag{48}$$

The *space of absence of generators* is just *id*. The *space of classical first-class linear generators* is

$$\mathfrak{F}\text{lin} \ , \tag{49}$$

the *space of classical gauge generators* is[3]

$$\mathfrak{G}\text{auge} \ , \tag{50}$$

In the canonical case, one can have also the *space of Chronos constraints*,

$$\mathfrak{C}\text{hronos}(\mathfrak{p}\text{hase}(\mathfrak{S})) \ . \tag{51}$$

6.1 Lattices from the generalized Lie Algorithm

Structure 1 The totality of generator subalgebraic structures for a given formalism of a given theory form [76] a

$$\text{bounded lattice} \ \ \mathfrak{L}_{\mathfrak{G}} \ . \tag{52}$$

The identity algebraic structure is the bottom alias zero element, and the full algebraic structure of first-class generators is the top alias unit element. All other elements are middle elements: the *Z generator algebraic structures*, denoted by

$$\mathfrak{Z} \ \text{ with each type indexed by } \ \mathsf{Z} \ . \tag{53}$$

See row 2 of Fig 4 for a schematic sketch. Thus $\mathfrak{G}$ comprises *id*, $\mathfrak{Z}_{\mathsf{Z}}$ and $\mathfrak{F}$, arranged to form $\mathfrak{L}_{\mathfrak{B}}$.

Remark 1 We take lattices - and Order Theory [55] more generally - to be a standard occurrence in the theory of Lie algebras and Lie groups at least since Serre [26].

[2] Suppressing less notation, this is $\mathfrak{F}(\mathfrak{B}(\mathfrak{S}))$ and likewise for subsequent generator algebraic structures listed here.
[3] In the canonical case, $\mathcal{F}$lin and $\mathcal{G}$auge need not coincide, by examples given in [47, 87].

6.2 Finite theory simplification

Lemma 1 Temporal Relationalism self-consistency is automatic for finite quadratic theories, in the form of a 1-d Abelian algebra.

This is since these only have one single-component constraint. But any single component object strongly commutes with itself. This enforces $\boldsymbol{W} = 0$, $X = 0$ in this case.

Example 0 Minisuperspace (spatially homogeneous GR) [30, 33] has a single finite constraint, $\mathcal{C}$hronos $= \mathcal{H}_{\text{mini}}$, so Lemma 1 applies:

$$\{ \mathcal{H}_{\text{mini}} \, , \, \mathcal{H}_{\text{mini}} \} = 0 \, . \tag{54}$$

Remark 1 As we shall see in Section 8, however, Lemma 1 can break down upon passing to Field Theory.

6.3 Example 1) Electromagnetism

Internal $G = U(1)$ gauge symmetry provides the Gauss constraint $\mathcal{G}$auss$(\underline{x})$. This is not TRi by itself. Standard smearing with scalar functions $\zeta(\underline{x}), \omega(\underline{x})$ will thus do. The resulting constraint algebra is

$$\{ (\mathcal{G}\text{auss} \,|\, \zeta), (\mathcal{G}\text{auss} \,|\, \omega) \} = 0 \, . \tag{55}$$

This of course reflects the underlying gauge group. (|) is here the integral-over-flat-space functional inner product. The final Hamiltonian for Electromagnetism is (using Π_0 as momentum conjugate to A_0)

$$H_{\mathcal{F}} = H + \Lambda \, \mathcal{G}\text{auss} \quad (\text{strictly} \quad + \, \lambda \cdot \Pi_0) \, . \tag{56}$$

6.4 Example 2) Yang–Mills Theory

Internal more general $\mathfrak{g}$ gauge symmetry provides a more general Gauss constraint $\mathcal{G}$auss$_I(\underline{x})$. Electromagnetism's lack of TRi and standard smearing follow suit, now with internal-vector smearing functions $\zeta^I(\underline{x}), \omega^I(\underline{x})$. The resulting constraint algebra is

$$\{ (\mathcal{G}\text{auss}_I \,|\, \zeta^I), (\mathcal{G}\text{auss}_J \,|\, \omega^J) \} = G^K{}_{IJ} (\mathcal{G}\text{auss}_K \,|\, [\boldsymbol{\zeta}, \boldsymbol{\omega}]^K) \, . \tag{57}$$

for structure constants $\boldsymbol{G}$ and internal-index commutator Lie bracket [,]. Once again, this represents the corresponding gauge group. The final Hamiltonian for Yang–Mills Theory is

$$H_{\mathcal{F}} = H + \Lambda^I \cdot \mathcal{G}\text{auss}_I \quad (\text{strictly} \quad + \, \lambda^I \cdot \Pi_{0I}) \, . \tag{58}$$

6.5 Example 4) Angular Momenta

If we have angular mementum components J_x and J_y, then the angular momentum component J_z is also implied by

$$[J_x \, , \, J_y] = J_z \, . \tag{59}$$

This is a well-known instance of discovering a generator as an integrability.

7 Split Lie algebraic structures

7.1 Motivation

Next suppose that a hypothesis is made about some subset $\mathcal{J}$ of the generators being significant; we denote the remaining independent generators by $\boldsymbol{\kappa}$. Gilmore [36] is the recommended reference for learning about such splits, including as regards each of the following applications by which splits of this general kind are already quite widely known in the literature. Firstly, the $\mathcal{J}$ could be distinguished by forming a '*little group*' (alias stabilizer or isotropy subgroup) [12, 64]. Secondly, the $\mathcal{J}$ could form a subalgebra corresponding to a *Lie algebra contraction* [16, 21, 27]. Thirdly, the split could be into qualitatively distinct blocks as per *Cartan's useful decomposition of Lie algebras* [18, 26, 62].

This article's specific motivation is moreover split space-time's split Relationalism, as a part of Background Independence and the Problem of Time. In particular, this split results in constraint provision itself being split, with constraints provided by Configurational Relationalism, $\mathcal{J} = \mathcal{S}$**huffle**, and constraints provided by Temporal Relationalism, $\boldsymbol{\kappa} = \mathcal{C}$hronos. In this context, Constraint Closure is a *necessary test* for *candidate* Temporally and

Configurationally Relational theories to be ratified as actually being *consistent*. It is additionally only after succeeding with closure that we are entitled to refer to the $\mathcal{S}$**huffle** as first-class linear constraints, $\mathcal{F}$lin. This is since some constraints being first-class entails their being known to brackets-close with all other first-class constraints that a given system possesses. A fair amount of subsequent '*physical*' or '*philosophical*' interpretations in the literature [48, 49, 50, 77] then treats $\mathcal{F}$**lin** and $\mathcal{C}$hronos distinctly, possibly by reference to their linearity and quadraticity in momenta, or under GR or some other Gravitational Theory's name for each constraint. Such ideas however only make sense when detailed analysis of Closure confirms the split to be *algebraically* meaningful. Some consistent and mainstream theories algebraically support such interpretations while others do not ([75, 77] and below).

7.2 2-piece split of a given Lie algebraic structure

This takes the form

$$|[\, \underline{\mathcal{J}}, \underline{\mathcal{J}}' \,]| \;=\; \underline{\underline{\underline{\boldsymbol{A}}}} \cdot \underline{\mathcal{J}}'' \;+\; \underline{\underline{\underline{\boldsymbol{B}}}} \cdot \underline{\mathcal{K}} \;, \tag{60}$$

$$|[\, \underline{\mathcal{J}}, \underline{\mathcal{K}} \,]| \;=\; \underline{\underline{\underline{\boldsymbol{C}}}} \cdot \underline{\mathcal{J}}' \;+\; \underline{\underline{\underline{\boldsymbol{D}}}} \cdot \underline{\mathcal{K}}' \;, \tag{61}$$

$$|[\, \underline{\mathcal{K}}, \underline{\mathcal{K}}' \,]| \;=\; \underline{\underline{\underline{\boldsymbol{E}}}} \cdot \underline{\mathcal{J}} \;+\; \underline{\underline{\underline{\boldsymbol{F}}}} \cdot \underline{\mathcal{K}}'' \;. \tag{62}$$

This is taken to extend [77] Gilmore's split [36] to include Lie algebroids as well as Lie algebras.

7.3 Closure posed in the split-Relationalism context

The split space-time perspective's split of Relationalism containing two separate constraint providers means that, at least ab initio, that Closure splits into three checks

1) Configurational Relationalism self-consistency. Whether our candidate constraints $\mathcal{S}$**huffle** provided by Configurational Relationalism self-close under classical brackets,

$$\{\, \underline{\mathcal{S}\textbf{huffle}}, \underline{\mathcal{S}\textbf{huffle}} \,\} \;=\; \underline{\underline{\underline{\boldsymbol{S}}}} \cdot \underline{\mathcal{S}\textbf{huffle}} \;+\; \underline{\underline{\boldsymbol{T}}}\, \mathcal{C}\text{hronos} \;, \tag{63}$$

for structure constants-or-functions $\boldsymbol{S}$ and $\boldsymbol{T}$.

2) Mutual consistency between Configurational and Temporal Relationalisms. Whether $\mathcal{S}$**huffle** and $\mathcal{C}$hronos mutually close:

$$\{\, \underline{\mathcal{S}\textbf{huffle}}, \mathcal{C}\text{hronos} \,\} \;=\; \underline{\underline{\boldsymbol{U}}} \cdot \underline{\mathcal{S}\textbf{huffle}} \;+\; \underline{\boldsymbol{V}}\, \mathcal{C}\text{hronos} \;. \tag{64}$$

for structure constants-or-functions $\boldsymbol{U}$, $\boldsymbol{V}$.

3) Temporal Relationalism self-consistency. Whether $\mathcal{C}$hronos self-closes:

$$\{\, \mathcal{C}\text{hronos}, \mathcal{C}\text{hronos} \,\} \;=\; \underline{\boldsymbol{W}} \cdot \underline{\mathcal{S}\textbf{huffle}} \;+\; X\, \mathcal{C}\text{hronos} \;, \tag{65}$$

for structure constants-or-functions $\boldsymbol{W}$ and X. Our six structure constants-or-functions as ordered above are to be taken to be split Relationalism's specific realizations of the general Lie algebraic split's $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}, \boldsymbol{E}, \boldsymbol{F}$ respectively.

7.4 Significant subcases of split Lie algebraic structures

Remark 1 $\boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}, \boldsymbol{E} \;=\; 0$ are non-interaction conditions, the first and fourth of which render $\mathfrak{j}$ and $\mathfrak{k}$ subalgebraic structures respectively. If the first is accompanied by $\boldsymbol{A} \;=\; 0$, $\mathfrak{j}$ is an abelian Lie algebra; the same applies to $\mathfrak{k}$ if the fourth is accompanied by $\boldsymbol{F} \;=\; 0$.

The following further particular cases are realized in this Series of Articles. Each step down the ladder from I) to IV) represents a large increase in complexity and generality. One now needs to check, however, the extent to which the algebraic structure actually complies with splits' assignation of significance. Such checks place limitations on the generality of intuitions and concepts which only hold for some simple examples of algebraic structures.

Case i) *Direct product* [58]. Suppose that $\boldsymbol{B} \;=\; \boldsymbol{C} \;=\; \boldsymbol{D} \;=\; \boldsymbol{E} \;=\; 0$. Then

$$\mathfrak{g} \;=\; \mathfrak{j} \;\times\; \mathfrak{k} \;. \tag{66}$$

Example 1 Rotations and dilations form a direct product $\times$ [68].

Case ii) *Semidirect product* [58, 24, 45]. If solely $\boldsymbol{C} \neq 0$, then

$$\mathfrak{g} = \mathfrak{j} \rtimes \mathfrak{K} \, . \tag{67}$$

Example 2 Translations and rotations form a $\rtimes$ [68], e.g. in 3-d

$$\{ \underline{\mathcal{L}} , \underline{\mathcal{L}} \} = \underline{\underline{\underline{\epsilon}}} \cdot \underline{\mathcal{L}} \, , \tag{68}$$

$$\{ \underline{\mathcal{P}} , \underline{\mathcal{L}} \} = \underline{\underline{\underline{\epsilon}}} \cdot \underline{\mathcal{P}} \, . \tag{69}$$

for ϵ the alternating tensor supported by 3-d space. The first of these means that the $\underline{\mathcal{L}}$ close as a Lie algebra: a Lie subalgebra of the full Euclidean Lie algebra. The second signifies that $\underline{\mathcal{P}}$ is a 'good object' – in this case a vector – under the rotations generated by the $\underline{\mathcal{L}}$.

Example 3 Taking the above $\mathcal{P}$, $\mathcal{L}$ as a single $\mathcal{S}$**huffle** block, and then adding $\mathcal{E}$ as a separate $\mathcal{C}$hronos block, unreduced Euclidean RPM is $\times$. For Lemma 1 applies, and $\mathcal{E}$ commutes with $\mathcal{S}$**huffle** as well; the latter can be traced back to building Euclidean RPM's action as a good $Eucl(d)$ object. $\mathcal{S}$**huffle** can thus be accorded the name $\mathcal{F}$**lin** in this case. This $\times$ can be interpreted as Mechanics on Euclidean space supporting each of Temporal and Configurational Relationalism independently of the other. This underlies why [68, 77, 93] were able to entertain Temporally and yet not Spatially Relational Particle Mechanics and vice versa. So

$$\text{Euclidean RPM realizes the } \{\mathcal{E}\} \times Eucl(d) \text{ subcase of } \mathfrak{Chronos} \times \mathfrak{F}\text{lin} \, . \tag{70}$$

The final d-almost-Hamiltonian for unreduced Euclidean RPM is

$$\mathrm{dA}_{\mathcal{F}} = \mathrm{d}I \, \mathcal{E} + \mathrm{d}\underline{A} \cdot \underline{\mathcal{P}} + \mathrm{d}\underline{B} \cdot \underline{\mathcal{L}} \ (\text{strictly } + \mathrm{d}\underline{\lambda} \cdot \underline{P}^B) \, . \tag{71}$$

The reduced formulation of Euclidean RPM also attains closure, now by Lemma 1 applying to the sole remaining constraint $\widetilde{\mathcal{E}}$. In this case,

$$\mathrm{dA}_{\mathcal{F}} = \mathrm{d}I \, \widetilde{\mathcal{E}} \, . \tag{72}$$

Relative to our Minisuperspace example, this has the added merit that Configurational Relationalism has been incorporated. Generalizing what we have learned above gives the following two simplifications within the arena of split Relationalism.

Simplification 1 Insisting on a group input for Configurational Relationalism guarantees $\mathfrak{g}$-subalgebra closure of the first bracket. This is by forcing $\boldsymbol{B} = 0$ (closed) and $\boldsymbol{A} = const$ (subalgebra, not subalgebroid).

Simplification 2 Insisting on a good $\mathfrak{g}$-object TRi action induces a $\mathcal{C}$hronos that is itself a good $\mathfrak{g}$-object. This is by forcing $\boldsymbol{D} = 0$ and $\boldsymbol{C} = const$.

Remark 2 The combination of the above two simplifications is *so far* consistent with our residing within the $\rtimes$ case. For a finite theory with only one $\mathcal{C}$hronos, moreover, we know the final bracket is Abelian, confirming closure and vindicating $\rtimes$ status. In this way, our ERPM example above extends to a large class of finite theories.

Remark 3 If $\boldsymbol{T} = \boldsymbol{U} = \boldsymbol{V} = \boldsymbol{W} = 0$, then Temporal and Configurational Relationalism are totally decoupled from each other. In fact, the first and last of these suffice, so $\rtimes$, $\ltimes$ and the general second bracket also give such decouplings. This is reflected by Fig 4.c)'s subalgebras.

Case iii) *One-way integrability* [75, 77] Suppose $\boldsymbol{E} \neq 0$. Then $\mathfrak{K}$ is not a subalgebraic structure. Attempting to close this leads to some $\mathcal{J}$ being discovered to be integrabilities. Let us denote this by

$$\mathfrak{K} \ominus \mathfrak{j} \, . \tag{73}$$

Example 4 A simple example of this occurs in splitting SR spacetime's Lorentz group's generators up into rotations $\underline{J}$ and boosts $\underline{K}$, schematically

$$|[\underline{J}, \underline{J}]| \sim \underline{J} \, , \quad |[\underline{J}, \underline{K}]| \sim \underline{K} \, , \quad |[\underline{K}, \underline{K}]| \sim \underline{K} + \underline{J} \, . \tag{74}$$

The last bracket is key, since by this the boosts $\underline{K}$ do not constitute a subalgebra. This is the group-theoretic underpinning [36] of *Thomas precession*, referring to the rotation arising in this manner from a combination of boosts. 'Thomas integrability' is thus an alternative name for one-way integrability.

Remark 2 Yet in this example, linearly recombining the two blocks reveals a simpler split form, in accord with the well-known accidental relation [36]

$$so(3,1) \;\cong\; so(3) \;\times\; so(3) \; . \tag{75}$$

This does however amount to abandoning one's originally declared partition of generators. This does not matter much in the current example's SR spacetime context.

Example 5 We shall see in the next section, moreover, that GR's constraints realize $\oplus\!\!\!\!\rightarrow$ as well, and now without anything in Remark 1 applying.

Case iv) *Two-way integrability* [75, 77]. Suppose $\boldsymbol{B}, \boldsymbol{E} \;\neq\; 0$. Then neither $\mathfrak{j}$ nor $\mathfrak{K}$ are subalgebraic structures. This is due to their imposing integrabilities on each other. Let us denote this by

$$\mathfrak{j} \;\; \ominus \;\; \mathfrak{K} \; . \tag{76}$$

In this case, attempting to attribute a solo role to $\mathfrak{j}$ is almost certainly dashed by detailed consideration of the underlying algebraic structure. *By this stage, the tentative split into 2 blocks has ceased to be algebraically meaningful. Indeed,* $\ominus$ *is not 'a more complicated structure' but a red herring. It is Algebra's way of telling us we had no right to assume that particular block decomposition in the first place.*

Example 6 We shall see in the next Section hat attempting a Temporal to Configurational Relationalism split in Supergravity is struck by such a rebuttal.

8 GR's Constraint Closure

8.1 Full vacuum GR

At the classical level, GR's Constraint Closure Problem is *solved* by its constraints closing in the form of the *Dirac algebroid* [15, 20, 25, 35]

$$\mathfrak{D}\text{irac}(\boldsymbol{\Sigma}) \; . \tag{77}$$

The TRi-smeared[4] [72, 77] and furthermore coordinate-independent [87] form for this is

$$\boldsymbol{\{}\, (\, \underline{\boldsymbol{\mathcal{M}}} \,|\, \partial\underline{\mathrm{L}}\,), (\, \underline{\boldsymbol{\mathcal{M}}} \,|\, \partial\underline{\mathrm{M}}\,)\, \boldsymbol{\}} \;=\; (\, \underline{\boldsymbol{\mathcal{M}}} \,|\, \pounds_{\partial\underline{\mathrm{L}}}\partial\underline{\mathrm{M}}\,) \;=\; (\, \underline{\boldsymbol{\mathcal{M}}} \,|\, \underline{|[\,\partial\mathrm{L}, \partial\mathrm{M}\,]|}\,) \; , \tag{78}$$

$$\boldsymbol{\{}\, (\, \mathcal{H} \,|\, \partial\mathrm{K}\,), (\, \underline{\mathcal{M}} \,|\, \partial\underline{\mathrm{L}}\,)\, \boldsymbol{\}} \;=\; (\, \pounds_{\partial\underline{\mathrm{L}}}\mathcal{H} \,|\, \partial\mathrm{K}\,) \; , \tag{79}$$

$$\boldsymbol{\{}\, (\, \mathcal{H} \,|\, \partial\mathrm{J}\,), (\, \mathcal{H} \,|\, \partial\mathrm{K}\,)\, \boldsymbol{\}} \;=\; (\, \underline{\mathcal{M}} \cdot \underline{\underline{\mathrm{h}}}^{-1} \cdot \,|\, \partial\mathrm{J} \, \overleftrightarrow{\underline{\partial}} \, \partial\mathrm{K}\,) \; . \tag{80}$$

(|) is here the integral-over-curved-space functional inner product, [,] the differential-geometric commutator Lie bracket, and $\partial\underline{\mathrm{L}}$ and $\partial\underline{\mathrm{M}}$, $\partial\mathrm{J}$ and $\partial\mathrm{K}$ are TRi-smearing functions. Thus GR's ***shuffle*** – the momentum constraint – is confirmed to be ***$\mathcal{F}$lin***. Finally, GR's total differential-almost-Hamiltonian is

$$\mathrm{d}A_{\mathcal{F}} \;=\; \mathrm{dI}\,H \;+\; \partial\underline{\mathrm{F}} \cdot \underline{\mathcal{M}} \;\; (\text{strictly} \;\; + \; \partial\underline{\lambda} \cdot \underline{\mathrm{p}}^{\beta}) \; . \tag{81}$$

Main Result The above 4 equations in TRi form amount to joint incorporation of (Temporal Relationalism, Configurational Relationalism, Constaint Closure) aspects of Background Independence in the case of GR. This gives a conceptual and formal resolution of the local classical Problem of Time's versions of Frozen Formalism, Thin Sandwich, Functional Evolution facets, including all interferences between these. It is not complete as a technical solution without solving each patch of each spatial topological manifold's Thin Sandwich Problem explicitly (for which we have local existence and uniqueness theorems but not general solution in closed analytic form).

Remark 1 Of course, we also need to *provide interpretation* for what these equations mean. This best proceeds via noting that, aside from the TRi form enabling our main achievement, using TRi form does not change the nature of other previous commentary in the literature as regards the Dirac algebroid formed by GR's constraints (pace one exception detailed below). In other words, we can largely rely on standing on the shoulders of giants in procuring the below commentary.

Remark 2 The first bracket closes as a subalgebra [35, 38] by Simplification 1; it is moreover $diff(\boldsymbol{\Sigma})$: an infinite-$d$ Lie algebra in the sense of having an infinity of generators. The second bracket signifies that $\mathcal{H}$ is a good $Diff(\boldsymbol{\Sigma})$-object – in this case a scalar density [38] – by virtue of Simplification 2. While both of the above are kinematical, (80) is dynamical. This is so much more complicated in both form and meaning [35] that every other Remark below is dedicated to it. The geometrical significance of each bracket is depicted in Fig 3 [35, 38, 72, 77].

[4] Smearing's 'multiplication by a test function' serves to render rigourous a wider range of 'distributional' manipulations [43], provided that these occur under an integral sign.

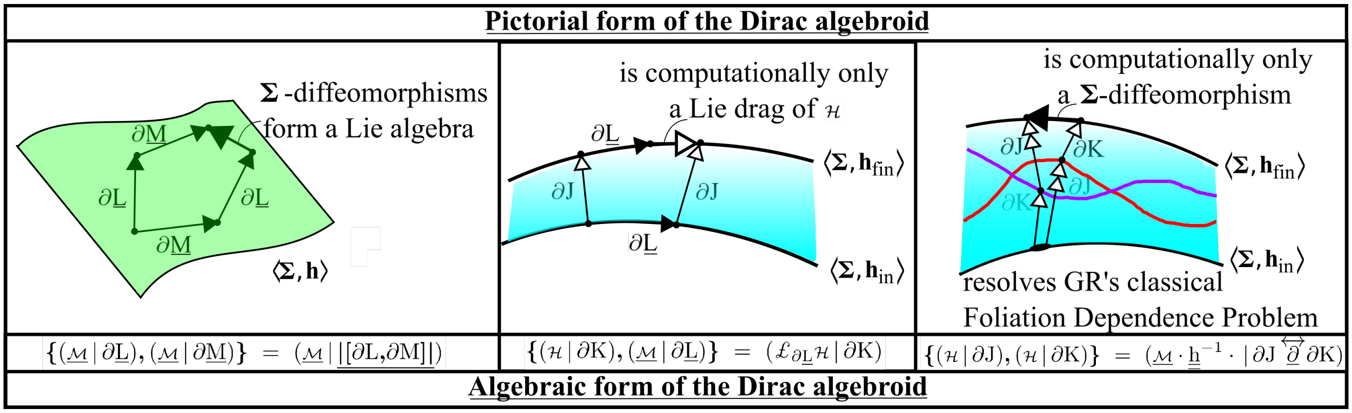

Figure 3: TRi-dressed Dirac algebroid of GR constraints' a) geometrical significance and b) algebraic structure.

8.2 GR's third bracket

Remark 1 Moncrief and Teitelboim [34] pointed out that this means that $\underline{\mathcal{M}}$ is an *integrability* of $\mathcal{H}$. Suppose furthermore that $Diff(\mathbf{\Sigma})$-Relationalism were not initially entertained [59, 60]. Then Dirac's Algorithm would enforce it anyway [72, 77, 89, 95], via this $\oplus$ and then an example of Sec 5.7's encoding. Thereby, neither GR's $\mathcal{H}$, nor its underlying Temporal Relationalism, can be entertained without $\underline{\boldsymbol{\mathcal{M}}}$ or its underlying Configurational Relationalism; see Fig 4.c) for the subalgebraic structures supported. This indicates a greater amount of interaction between Temporal and Configurational Relationalism in GR than the RPM model exhibited, the two Relationalisms being realizable piecemeal there.

Remark 2) (GR–Thomas analogy) [75, 77].

$$\text{GR manifests the } \{\mathcal{H}\} \ominus \{\underline{\mathcal{M}}\} \text{ subcase of } \mathfrak{Chronos} \ominus \mathfrak{Gauge}\,. \quad (82)$$

There is thus a parallel between the following.

a) Composing two boosts producing a rotation: *Thomas precession*.

b) Composing two time evolutions producing a spatial diffeomorphism: *Moncrief–Teitelboim on-slice Lie dragging* [34].

Remark 3 One limitation on this analogy is that the GR case's integrability cannot be undone by linearly combining constraints. This is counterbalanced by the Temporal-to-Configurational Relationalism split playing a meaningful role in GR. All in all, GR's Dirac algebroid is a mathematically stronger realization of $\oplus$ than the Thomas split of the Lorentz group.

Remark 4 (80)'s right-hand-side contains *structure functions* $\mathbf{h}^{-1}(\mathbf{h}(\underline{x}))$. So it is this bracket by which GR's constraint closure indeed forms an algebroid. In the Theoretical Physics literature, Bojowald raised awareness of this subtlety [69].[5] This gives a second limitation on the Thomas precession analogy: GR's version is a Lie algebroid effect whereas Thomas' is merely a Lie algebra effect.

Remark 5 Consequently, the transformation in question itself depends on the object acted upon, in contrast with the familiar case of the rotations. Teitelboim [35] poetically phrased as being able to speak of rotations without saying whether it is a black cube, or a yellow cat that is being acted upon by the rotation, whereas the Dirac algebroid *does* act differently on each such object.

Remark 6 By not forming a Lie algebra, the constraints – and $Diff(\mathfrak{M}, \mathfrak{Fol})$ – clearly form a structure other than $Diff(\mathfrak{M})$. Indeed, Dirac algebroids are vastly larger than such diffeomorphism algebras, with the difference in size reflecting [61] the variety of possible foliations [71] within an evolving spacetime.

Remark 7 The Dirac algebroid already features in Minkowski spacetime $\mathbb{M}^n$ in *general coordinates*. I.e. when this is split up with respect to an arbitrary spatial surface rather than a necessarily flat spatial surface. This is so as to

[5] This structure had previously been referred to as 'Dirac algebra', though 'Dirac algebroid' is not only more mathematically correct but also not open to confusion with fermionic theory's Dirac algebra.

model fleets of observers [66] undergoing acceleration therein. This is in fact the context in which Dirac first found this algebroid [15], though he subsequently also considered the GR case in [20].

Remark 8 The Dirac algebroid admits a *deformation algebroid* interpretation [38], with the Hamiltonian constraint playing the role of a pure deformation therein. The above-mentioned exception to being able to uplift to TRi is that the TRi version does not necessarily admit a primary level hypersurface deformation interpretation, by now not necessarily presupposing spacetime [72, 77]. This is also a bonus, since it enables [72, 77, 89, 95] instead for Spacetime Reconstruction from Space to be posed!

Remark 9 The GR–Thomas analogy's first limitation can now be taken to rest on GR splits (or SR splits with respect to arbitrary spatial surfaces) having theoretical significance in excess of that of splitting SR with respect to flat spatial surfaces. This is down to spatial deformations being much more general than boosts (local versus global, and generic versus in possession of many Killing vectors). Variety of deformations furthermore encodes variety of possible foliations. This requiring a Lie algebroid to encode accounts for the GR–Thomas analogy's second limitation. The two limitations are thus bridged by fundamental mathematical differences between deformations and boosts: a point new to the current article.

Remark 10 In minisuperspace, the Dirac algebroid collapses to just one equation due to the spatial covariant derivative $\boldsymbol{\mathcal{D}}$ now annihilating everything by homogeneity. This includes there not being a momentum constraint in the first place. Whichever of this derivative condition and Lemma 1 also ensures that the surviving third bracket has collapsed from a Lie algebroid to a 1-d abelian Lie algebra. Only full and trivial subalgebras of constraints are supported in this case (Fig 4.b).

Remark 11 Strong Gravity [39] demonstrates [60] a smaller a collapse in which both the integrability and the algebroid nature are lost; this is covered in Article 3.

Remark 12 For $\boldsymbol{\Sigma} = \mathbb{S}^1$, the Dirac algebroid collapses to a Lie algebra (albeit infinite-d)that is well known: the Witt algebra, or, with central extension, the Virasoro algebra [46].

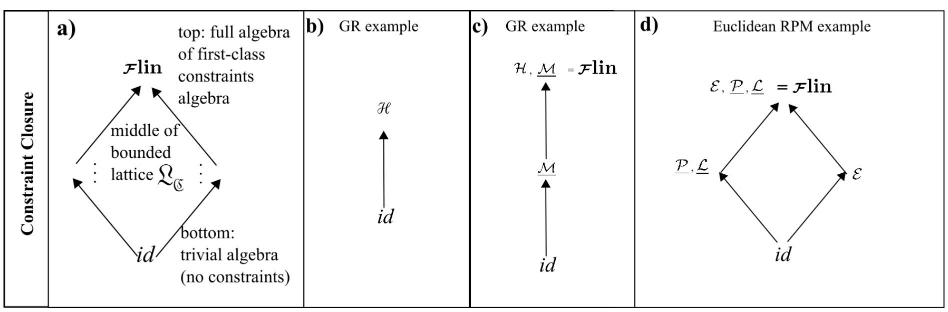

Figure 4: Lattices of notions of constraints (row 2) and of constraint subalgebraic structures (row 1).

a) in general, schematically.

b) For minisuperspace GR.

c) for full GR: a first arena with a nontrivial middle.

d) for Euclidean RPM: a first arena with a nontrivial-poset middle rather than just a chain.

8.3 Inclusion of matter

Remark 1 Upon including minimally-coupled matter (including no curvature couplings), one has the *Teitelboim split* [42] for minimally-coupled matter

$$\mathcal{H} = \mathcal{H}^{\mathrm{g}} + \mathcal{H}^{\Psi} , \tag{83}$$

$$\underline{\boldsymbol{\mathcal{M}}} = \underline{\boldsymbol{\mathcal{M}}}^{\mathrm{g}} + \underline{\boldsymbol{\mathcal{M}}}^{\Psi} . \tag{84}$$

There is no difficulty with extending this approach to Einstein–Maxwell or Einstein–Yang–Mills theories (the $\mathcal{F}$lin block is enlarged by whatever Gauss constraints are present). See [42] for a traditional account, or [77, 86] for the TRi version. This immediately extends to scalar Gauge Theories as well. For fermionic gauge theories, one needs to work with beins (or similar), by which frame constraints enter at the secondary level. This does not however change the integrability structure or algebroid nature of the subsequent algebraic structure.

Remark 2 In contrast, in Supergravity – a much more subtle and not minimally-coupled theory of gravitation with matter – the constraint algebraic structure exhibits 'two-way integraility'. This amounts to Temporal and Configurational Relationalisms being fused enough here that a split cannot be made along these exact lines. The constraints can moreover be meaningfully split into three blocks: the supersymmetric constraints, the non-susupersymmetric linear constraints, and $\mathcal{C}$hronos. [77, 95] even argue that Temporal Relationalism is *redundant* in Supergravity for reasons already known to Teitelboim [40].

9 Including the possibility of discoveries

We next attempt to *maintain* one set of generators' $\mathcal{J}$ Brackets Closure in the presence of a further disjoint set $\mathcal{K}$, while now making allowance for new generators to be discovered. We do this to cover the case of loose transformations, or, more specifically, of deforming known groups while not yet knowing whether any further groups will ensue from these deformations ([72, 79, 89, 95] contain examples).

$$|[\, \underline{\mathcal{J}}, \underline{\mathcal{J}} \,]| \;=\; \underline{\underline{\underline{\boldsymbol{A}}}} \cdot \underline{\mathcal{J}} \;+\; \underline{\underline{\underline{\boldsymbol{B}}}} \cdot \underline{\mathcal{K}} \;+\; \underline{\underline{\underline{\boldsymbol{H}}}} \cdot \underline{\mathcal{J}}^{\text{new}} \;+\; \underline{\underline{\underline{\boldsymbol{I}}}} \cdot \underline{\mathcal{K}}^{\text{new}} \,, \tag{85}$$

$$|[\, \underline{\mathcal{J}}, \underline{\mathcal{K}} \,]| \;=\; \underline{\underline{\underline{\boldsymbol{C}}}} \cdot \underline{\mathcal{J}} \;+\; \underline{\underline{\underline{\boldsymbol{D}}}} \cdot \underline{\mathcal{K}} \;+\; \underline{\underline{\underline{\boldsymbol{J}}}} \cdot \underline{\mathcal{J}}^{\text{new}} \;+\; \underline{\underline{\underline{\boldsymbol{K}}}} \cdot \underline{\mathcal{K}}^{\text{new}} \,, \tag{86}$$

$$|[\, \underline{\mathcal{K}}, \underline{\mathcal{K}} \,]| \;=\; \underline{\underline{\underline{\boldsymbol{E}}}} \cdot \underline{\mathcal{J}} \;+\; \underline{\underline{\underline{\boldsymbol{F}}}} \cdot \underline{\mathcal{K}} \;+\; \underline{\underline{\underline{\boldsymbol{L}}}} \cdot \underline{\mathcal{J}}^{\text{new}} \;+\; \underline{\underline{\underline{\boldsymbol{M}}}} \cdot \underline{\mathcal{K}}^{\text{new}} \,. \tag{87}$$

Remark 1 Within the more general ansatz above, we return to the case with no discoveries has

$$\boldsymbol{H} \;=\; \boldsymbol{I} \;=\; \boldsymbol{J} \;=\; \boldsymbol{K} \;=\; \boldsymbol{L} \;=\; \boldsymbol{M} \;=\; 0 \,. \tag{88}$$

Case i′) The direct product case now has

$$\boldsymbol{B} \;=\; \boldsymbol{C} \;=\; \boldsymbol{D} \;=\; \boldsymbol{E} \;=\; \boldsymbol{I} \;=\; \boldsymbol{J} \;=\; \boldsymbol{K} \;=\; \boldsymbol{L} \;=\; 0 \,. \tag{89}$$

Case ii′) The orientation of semidirect product which respects the $\mathcal{J}$'s self-closure generalizes (89) further allowing for $\boldsymbol{D} \neq 0$.

Remark 2 $\boldsymbol{B}$ or $\boldsymbol{I} \neq 0$ means that the class of $\mathfrak{j}$-objects does not close as a subalgebraic structure.

Remark 3 $\boldsymbol{C} \neq 0$ signifies that our algebraic structure was chosen too small for $\mathfrak{j}$ to represent it.

Remark 4 If we require that the $\mathcal{J}$ represent the purported $\mathfrak{g}$ prior to bringing in the $\mathcal{K}$, however, all of the above are moot.

Remark 5 If $\boldsymbol{K} \neq 0$, this may indicate that the $\mathcal{J}$ are incompatible with the $\mathcal{K}$'s $\mathfrak{g}$-invariance. This is to be resolved by the same methods as in Case iii) but now treating the $\mathcal{J}$ and $\mathcal{K}$ together.

Remark 6 $\boldsymbol{J}$ or $\boldsymbol{L} \neq 0$ indicate that adjoining the $\mathcal{K}$ to the $\mathcal{J}$ forces $\mathfrak{g}$ to be extended.

Example 1) Correcting one's action with respect to just the combination of translations $\underline{P}$ and special conformal transformations $\underline{K}$ fails [77]. This is because the ensuing secondary constraints $\underline{\mathcal{P}}$, $\underline{\mathcal{K}}$ do not form a group without both scaling $\mathcal{D}$ and rotations $\underline{\mathcal{L}}$. I.e. schematically,

$$\{\underline{\mathcal{P}}, \underline{\mathcal{K}}\} \;\sim\; \mathcal{D} \;+\; \underline{\mathcal{L}} \,. \tag{90}$$

This additionally serves as an example of mutual integrabilities.

Remark 7 The type of block structure the generator algebraic structure has is the main determiner of how a theory implenents Relationalism. Of which Closures are possible, and, as we shall see in subsequent articles, which notions of observables and which Constructabilities a theory possesses. This suggests that it is formulations or theories with qualitatively distinct block structure that constitute interesting variety between how theories can implement

Background Independence. This is one part of the Comparative Theory of Background Independence; see [92] for others.

Remark 8 Tertiary constraints can appear at any stage. So can further second-class constraints [87] and specifier equations [89]. Henneaux and Teitelboim emphasize and largely illustrate [47] how all combinations of first and second class, and primary and secondary, constraints are possible in whichever steps of the Dirac Algorithm. This is not however a relationally rooted account; it would be interesting to see if restriction to rr can reproduce this diversity.

10 Closure Problems

10.1 More specific Closure Problems

Generator-and-Specifier Closure Problem This allows for specifier equations as well as generators arising from the Lie Algorithm. In this way, the generators category of our ab initio Closure Problem is itself incomplete.

Generator Closure Problem by Sufficient-Cascade Inconsistency This arises if a sufficient cascade leaves us with no degrees of freedom, or inconsistency. This is a 'death by 1000 cuts' type phenomenon, though the role of the 1000 is played, more precisely, by the number of iterations in the Dirac Algorithm. As such, 'sufficient cascade' is a truer name, and we subsequently use **Constraint Closure Problem by Sufficient-Cascade Inconsistency**.

Enforced Group Extension Problem The initially considered $\mathfrak{g}$ may require extending due to further $\mathcal{F}$**lin** arising as tertiary constraints.

Rebracketing Problem Second-class generators arising as tertiary generators require reassessing all other generators using a redefined Lie-Dirac bracket.

10.2 Strategies

Strategic Element 1) Abandon one's candidate theory.

Strategic Element 2) Avoid specifiers, sufficient cascades, algebroids, integrabilities, or even any tertiaries at all when required, in those case that all such are accompanied by factors that can **strongly vanish**. This means fixing the constants $\boldsymbol{c}$ can be fixed so that that no new $\mathcal{N}$ feature. This is termed *strong avoidance of integrabilities.*

Strategic Element 3 Avoidance by removing or adding terms to one's action principle gives another alternative; adding can work by cancelling contributions. This works provided that, firstly, we are left with *some* terms in the action principle. Secondly, that this does not run contrary to any 'commensurate or higher' principles we require.

Remark 1) One consequence of adopting strategies permitting extension or reduction of $\mathfrak{q}$ or $\mathfrak{p}$hase is as follows. Formulations with second-class constraints are ultimately seen as half-way houses to further formulations which are free thereof. This is largely the context in which both the effective formulation and the Dirac bracket formulation were developed. $\mathfrak{p}$hase is extended in the former and reduced in the latter.

Remark 2 With reference to the preceding subsection's classification of Closure Problems, on the one hand whichever of cascades, specifiers, and algebraic interference can be addressed by any of these strategies.

On the other hand, Enforced Group Extension and Enforced Group Reduction require one of strategies 3) or 5-7).

Remark 3 Going full circle, we remind the reader that 'cascade' includes each of relational triviality, triviality, or inconsistency as worst-scenario bounding subcases. The last two of these, we called jointly a 'sufficient cascade', so let us use 'relationally sufficient cascade' for the three cases together.

One idea is then that a set of whichever of the preceding may imply further such under Lie brackets, or may, rather (and perhaps eventually) close.

Structure 1 Strongly vanishing brackets are clearly universal. There is moreover some motivation to extend Dirac's notion of weakly vanishing from Poisson or Dirac brackets of constraints

If a severe form of the Constraint Closure Problem strikes, one may have to entirely abandon the candidate theory's triple $\langle \mathfrak{B}, \mathfrak{g}, \mathcal{S} \rangle$. I.e. the Machian variables [93], a group acting thereupon and the Jacobi–Synge [4, 10, 13] geometrical action.[6] In some cases, however, modifying one or more of these may suffice to attain consistency. This gives the cubic lattice of strategies of Fig 5.

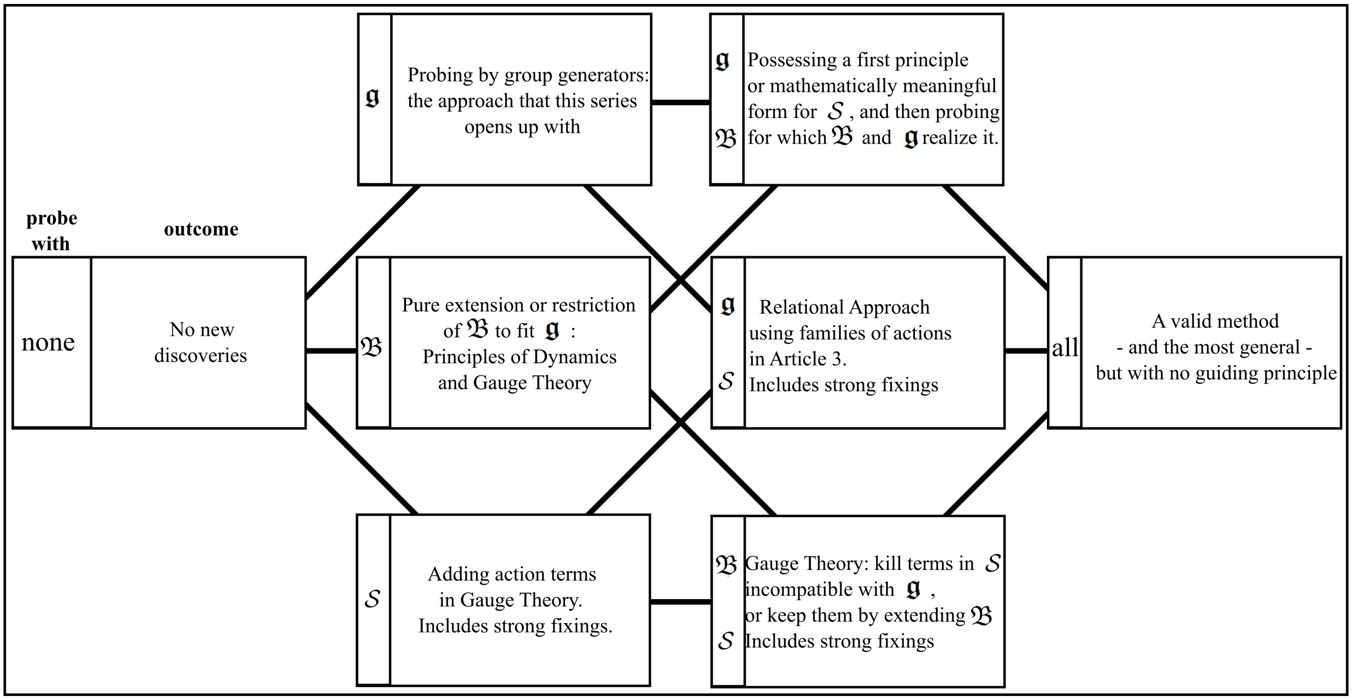

Figure 5: Seven strategies with some capacity for generating new theories from what is allowed by Generator Closure.

Remark 4 Fig 5's strategic diversity continues to apply if $\mathfrak{p}$hase and an integrated (d-A-)Hamiltonian – or its constituent set of constraints in whole-universe theories – are considered in place of $\mathfrak{q}$ and $\mathcal{S}$. Similar considerations apply in 1) the spacetime formulations of $\mathcal{S}$ with $\mathfrak{g}_{\mathrm{S}}$ acting thereupon. 2) At the quantum level (further extending the Hamiltonian presentation).

Remark 5 Preserving a particular $\mathfrak{g}$ in Particle Physics includes insisting on a particular internal gauge group, or on the Poincaré group of SR spacetime.

10.3 Further ties back to Relationalism

Using TRi circumvents some facet interferences. The split Constraint Provider input has, on the one hand, Temporal Relationalism provides a constraint $\mathcal{C}$hronos that is quadratic and so is also denoted by $\mathcal{Q}$uad. On the other hand, Configurational Relationalism provides candidate $\boldsymbol{\mathcal{S}}$**huffle** constraints that are linear and so are also denoted by $\boldsymbol{\mathcal{L}}$**in**. One is then to use the Dirac Algorithm on this combined incipient set of constraints, so as to see whether Constraint Closure is met or the Constraint Closure Problem arises. This split induces a further split consideration of Constraint Closure. I.e. whether each of $\mathcal{C}$hronos and $\boldsymbol{\mathcal{S}}$**huffle** are self first-class, and whether they are mutually first-class. The self and mutual behaviour of $\boldsymbol{\mathcal{S}}$**huffle** determines whether Configurational Relationalism has succeeded. Supporting Principles of Dynamics for TRi Constraint Closure is provided in [87]'s Appendix.

11 Conclusion

We give a Lie algorithm upgrade in generality of the Dirac Algorithm for Constraint Closure, now including also in particular assessment of Spacetime Closure as well.

When successful starting from spacetime's or dynamics' relational input, two Lie aspects of Background Independence are jointly implemented. This amounts to two Problem of Time facets being resolved in the spacetime case, or three in the canonical case The Dirac Algorithm itself also needs to be rendered TRi; this does not in any way alter its

[6] One might augment this to a quadruple by considering varying the type of group action of $\mathfrak{g}$ on the tangent space $\mathfrak{T}(\mathfrak{q})$.

function or outcome. See [87] for a full combined implementation of the first three facets of the Problem of Time's details.

This account tightens presentation of the Generalized Lie Algorithm as a generalization of the Dirac Algorithm relative to [83, 87, 90]. While I have presented the subcases of the 2-block split before [68, 75, 77, 83, 87] that was separated out into layers and with lesser use of principles to cut down on the options. These improvements represent a major change in the overall shape of the subject, and to the content, clarity and efficiency of its exposition.

This two Lie aspect implementation – of Relationalism and Closure – works out for RPM and GR. It does not work out for Gauge Theory in isolation, since this does not possess Temporal Relationalism; upon coupling Gauge Theory to GR, however, success returns. Pure GR's constraints (including with minimally coupled matter) moreover close not as a Lie algebra but as a Lie algebroid: the Dirac algebroid.

This position reached, each of Assignment of Observables and Spacetime Construction can be considered as separate extensions. I.e. the two leaves of the Lie claw digraph, as covered in [94, 95] respectively.

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