Lie Theory suffices for Local Classical Resolution of Problem of Time
0. Preliminary Relationalism as implemented by Lie Derivatives.

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Abstract

The Problem of Time is due to conceptual gaps between General Relativity and the other observationally-confirmed theories of Physics; it is a major foundational issue in Quantum Gravity. The Problem of Time's multiple facets were mostly pointed out over 50 years ago by Wheeler, DeWitt and Dirac. These facets were subsequently classified by Kuchař and Isham, who argued that the lion's share of the problem consists of interferences between facets and also posed the question of in which order the facets should be approached. By further considering the nature of each facet at the local classical level, I show the facets to be two copies of Lie Theory – spacetime and canonical – with a Wheelerian 2-way route therebetween. This solves the facet ordering question. The resulting mathematical framework turns out moreover to be consistent enough to smooth out all local classical facet interferences as well. Among the Background Independence aspects resolving the Problem of Time facets, closure is central and is modelled by the generalized Lie algorithm: a broadening of Dirac's algorithm. Lie derivatives model relational aspects, Lie brackets algebra commutants cover observables aspects, and Lie brackets algebra deformation leading to Lie rigidity modelling constructability (e.g. the spacetime reconstruction problem facet). Having seen this, it is furthermore straightforward to model the local quantum Problem of Time using a more general bracket algebra's analogous set of structures. The current article additionally details the relational part of Background Independence, in both the spacetime and canonical settings.

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1 Introduction

Each observationally-established physical paradigm has a distinct conceptualization of time. The resulting gap is principally between Newtonian Physics, Special Relativity (SR), Quantum Mechanics (QM), and Quantum Field Theory (QFT) on the one side, and General Relativity (GR) on the other.[1] Newtonian Mechanics and Special Relativity (SR) each have a different background notion of time. General Relativity (GR), however, has coordinate time, and upon assuming a canonical formulation, at least apparently no time. Both SR and GR admit spacetime geometrization. While SR spacetime continues to play a major role in Quantum Field Theory, spacetime may well be but emergent rather than primary for quantum GR.

This gap leads to the *Problem of Time* [116, 69, 70, 106], meant here a multi-faceted sense since multiple differences in conceptualization of time are involved. Most of these facets were first envisaged over 50 years ago by Wheeler [34], DeWitt [35], or Dirac [18, 19, 24, 30]. It took 25 further years for the Problem of Time's full conceptual content to be assembled into Kuchař's and Isham's [69, 70] classification of facets (also summarized in [102]). Numerous observations of attempting to extend one Problem of Time facets' resolution to include a second facet has a strong tendency to interfere with the first resolution. Due to this, they argued for the lion's share of the Problem of Time to consist of facet interferences. Because of this, it is worth according the notation (A, B) for pairwise interference between facets A and B, with obvious extension to n-tuples. In which order the facets should be approached has also been a longstanding problem [69, 70, 73].

A Local Resolution of the Problem of Time (ALRoPoT) has recently been given [116, 123, 124, 125, 126, 127, 128, 129, 130, 131, 133, 132] , alongside reformulation as [116, 120, 122, 135] A Local Theory of Background Independence.[2] The classical part of this can be viewed as [134, 132] requiring just Lie's Mathematics [101, 8, 49, 108, 32, 47, 132], which for around a century has been entrenched in Topology [98, 56, 81, 103] and Differential Geometry [54, 108] while more recently being applied to the setting of contemporary Physics' state spaces [50, 68, 116, 118].

1.1 Spacetime versus canonical perspectives

Let us first mention this distinction in perspective (we will consider the Problem of Time for both, as well as regards a two-way route between the two). GR is not only a theory of 4-d spacetime [85, 57, 46] $\mathfrak{M}$ as equipped with an indefinite 4-metric $\boldsymbol{\gamma}$ with components $\gamma_{\mu\nu}$. It is also a dynamics [25, 44, 88, 105, 116] of 3-d space modelled by a fixed topological manifold $\boldsymbol{\Sigma}$ carrying an evolving positive-definite 3-metric $\mathbf{h}$ with components h_{ij}.[3] Metrics being

[1] See Part I of [116] for details, and also for smaller differences in time and space concepts between the first three.

[2] See e.g. [28, 33, 94] for earlier accounts of Background Independence, and e.g. [2, 7, 63, 76] for previous 'absolute versus relational motion debate' considerations.

[3] We denote det(h) by h, the associated covariant derivative by $\underline{\mathrm{D}}$, and the corresponding Ricci scalar curvature by $\mathcal{R}$. Also the det($\boldsymbol{\gamma}$) is denoted by γ, and its Ricci scalar curvature by $\mathcal{R}[\boldsymbol{\gamma}]$.

symmetric, a priori this has $3 \times 4/2 = 6$ degrees of freedom per space point. The $\mathbf{h}$ form the configuration space $\mathfrak{Riem}(\mathbf{\Sigma})$ [35, 40]. The spacetime formulation brings to the fore [57, 46, 116] e.g. simultaneity, causality, coordinate time and proper time . In contrast, the dynamics of space approach [44, 116] emphasizes constraints, evolution, and canonical machinery such as Poisson brackets and Hamiltonians that is useful in Quantum Theory. [Spacetime approaches my use instead path-integral approaches to quantization.]

A spatial slice within spacetime is characterized not only by its induced metric $\mathbf{h}$ but also by its extrinsic curvature $\boldsymbol{\mathcal{K}}$ with components $\mathcal{K}_{ij}$ and trace $\mathcal{K}$. The spacetime approach's Einstein–Hilbert action (with cosmological constant

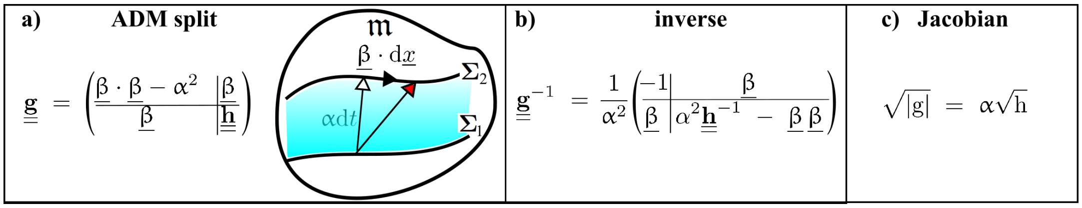

Figure 1: ADM split of a region of spacetime into space-time, with lapse α and shift $\underline{\beta}$ strutting as indicated. White, black and red arrows indicate time elapsed, diffeomorphism shifts within a given spatial slice, and point-identification maps between adjacent spatial slices respectively

term $\mathit{\Lambda}$ included)

$$\mathcal{S}^{\mathrm{GR}}_{\mathrm{EH}} \propto \int_{\mathfrak{M}} \mathrm{d}^4 X \sqrt{|\gamma|}\,(\,\mathcal{R}[\boldsymbol{\gamma}] - 2\,\mathit{\Lambda}\,) \tag{1}$$

then splits to the also well-known ADM action [25][4]

$$\mathcal{S}^{\mathrm{ADM}}_{\mathrm{GR}} \propto \int \mathrm{dt} \int_{\mathbf{\Sigma}} \mathrm{d}^3 x \sqrt{\mathrm{h}}\,\alpha\,(\,\boldsymbol{\mathcal{K}} \bullet \boldsymbol{\mathcal{K}} - \mathcal{K}^2 + \mathcal{R} - 2\,\mathit{\Lambda}\,)\;. \tag{2}$$

In terms of Lagrangian variables $(\mathbf{h},\ \mathbf{h}')$, this becomes

$$\mathcal{S}^{\mathrm{ADM-L}}_{\mathrm{GR}} \propto \int \mathrm{dt} \int_{\mathbf{\Sigma}} \mathrm{d}^3 x\, \mathcal{L}^{\mathrm{ADM-L}}_{\mathrm{GR}} = \int \mathrm{dt} \int_{\mathbf{\Sigma}} \mathrm{d}^3 x\, \alpha \left[\frac{\mathcal{T}^{\mathrm{ADM-L}}_{\mathrm{GR}}}{4\,\alpha^2} + \sqrt{\mathrm{h}}\,(\,\mathcal{R} - 2\,\mathit{\Lambda}) \right]\;. \tag{3}$$

The *GR kinetic term* here is

$$\mathcal{T}^{\mathrm{ADM-L}}_{\mathrm{GR}} := ||\delta_{\underline{\beta}}\mathbf{h}||_{\mathbf{M}}{}^2 = ||(' - \pounds_{\underline{\beta}})\mathbf{h}||_{\mathbf{M}}{}^2 = (\,' - \pounds_{\underline{\beta}}\,)\,\mathbf{h} \bullet \mathbf{M} \bullet (\,' - \pounds_{\underline{\beta}}\,)\,\mathbf{h}\;. \tag{4}$$

$\mathbf{M}$ is here the configuration space metric that GR places on $\mathfrak{Riem}(\mathbf{\Sigma})$, with components

$$\mathrm{M}^{ijkl} := \sqrt{\mathrm{h}}\,(\,\mathrm{h}^{ik}\mathrm{h}^{jl} - \mathrm{h}^{ij}\mathrm{h}^{kl}\,)\;. \tag{5}$$

GR's momenta turn out [25] to be closely related to extrinsic curvature:

$$\mathbf{p} = \sqrt{\mathrm{h}}\,(\,\boldsymbol{\mathcal{K}} - \mathcal{K}\,\mathbf{h}\,)\;: \tag{6}$$

a densitized version of $\boldsymbol{\mathcal{K}}$ with a trace term subtracted; we denote the momenta's components by p^{ij} and trace by p.

Action (3) provides two constraints.

A) From variation with respect to the shift $\underline{\beta}$, the *momentum constraint*

$$\underline{\mathcal{M}} := -\,2\,\underline{\mathrm{D}}\cdot\underline{\underline{\mathbf{p}}} = 0\;. \tag{7}$$

B) From variation with respect to the lapse α, the *Hamiltonian constraint*

$$\mathcal{H} := \frac{1}{\sqrt{\mathrm{h}}}\mathbf{p}\bullet\mathbf{p} - \frac{\mathrm{p}^2}{2} - \sqrt{\mathrm{h}}\,(\,\mathcal{R} - 2\,\mathit{\Lambda}\,) = ||\mathbf{p}||_{\mathbf{N}}{}^2 - \sqrt{\mathrm{h}}\,(\,\mathcal{R} - 2\,\mathit{\Lambda}\,) = 0\;, \tag{8}$$

[4] $\vec{X}$ with components X^μ are spacetime coordinates, $\underline{x}$ with coordinates x^i are spatial coordinates, and t is a coordinate time. The current article emphasizes coordinate-free formulation. $' := \partial/\partial\mathrm{t}$ $\bullet$ is double contraction (= single contraction $\cdot$ if DeWitt's 2-index to 1 index map [35] is declared), e.g. $\boldsymbol{\mathcal{K}} \bullet \boldsymbol{\mathcal{K}} = \mathcal{K}_{ij}\mathcal{K}^{ij}$.

where $\mathbf{N}$ is (the inverse of GR's configuration space metric) alias (DeWitt's supermetric) [35], with components

$$\mathrm{N}_{ijkl} \;=\; \frac{1}{\sqrt{\mathrm{h}}}\left(\mathrm{h}_{ik}\mathrm{h}_{jl} \;-\; \frac{1}{2}\mathrm{h}_{ij}\mathrm{h}_{kl}\right) \;. \tag{9}$$

Interpretation of these constraints is of great significance in conceiving of, and (at least locally) resolving the Problem of Time as manifested between GR and QM. Firstly, the momentum constraint's interpretation is straightforward. At the level of unequipped Differential Geometry, it corresponds to diffeomorphism invariance of the spatial geometry.[5] I.e. to $Diff(\mathbf{\Sigma})$ playing the role of a physically-redundant group of transformations (compare gauge groups' role in Gauge Theory.) By this, $Diff(\mathbf{\Sigma})$-invariant information in spatial 3-metrics – spatial 3-*geometries* – constitutes less redundant configurations for GR. This is why Wheeler coined the more specific term *Geometrodynamics* [34, 44] for GR's dynamical formulation. Spatial 3-geometries have $6 - 3 = 3$ degrees of freedom per space point; the corresponding configuration space is Wheeler's [29, 34]

$$\mathfrak{S}\mathrm{uperspace}(\mathbf{\Sigma}) \;:=\; \frac{\mathfrak{R}\mathrm{iem}(\mathbf{\Sigma})}{Diff(\mathbf{\Sigma})} \;, \tag{10}$$

as further studied e.g. in [40, 77, 99, 112].[6] At the level of more structured metric and extrinsic geometry, the $\boldsymbol{\mathcal{K}}$ form of the GR momentum constraint

$$-2\,(\,\underline{\mathrm{D}}\cdot\underline{\underline{\boldsymbol{\mathcal{K}}}} \;-\; \underline{\mathrm{D}}\mathcal{K}\,) \;=\; 0 \;. \tag{11}$$

carries the further geometrical interpretation [12, 57] of being the contraction of Codazzi's embedding equation for a spatial slice into spacetime.

While the $\boldsymbol{\mathcal{K}}$ form of the undensitized version of the $\Lambda = 0$ GR Hamiltonian constraint is [12, 57] the doubly-contracted Gauss embedding equation for a spatial slice into spacetime,[7]

$$\mathcal{K}^2 \;-\; \boldsymbol{\mathcal{K}} \bullet \boldsymbol{\mathcal{K}} \;+\; \mathcal{R} \;=\; 0 \;, \tag{12}$$

further interpretation of this constraint is tougher than that of $\underline{\boldsymbol{\mathcal{M}}}$ (see Sec 4 and [136]).

Quantizing GR's constraints gives something like

$$\widehat{\mathcal{M}}_i\,\Psi \;=\; 2\,i\,\hbar\,\mathrm{h}_{ik}\mathrm{D}_j\frac{\delta}{\delta \mathrm{h}_{jk}}\Psi \;=\; 0 \tag{13}$$

and the *Wheeler–DeWitt equation*

$$\widehat{\mathcal{H}} \;=\; \hbar^2\,`\left[\frac{1}{\sqrt{|\mathrm{M}|}}\frac{\delta}{\delta \mathbf{h}} \bullet \left(\sqrt{|\mathrm{M}|}\,\mathbf{N} \bullet \frac{\delta}{\delta \mathbf{h}}\right)\right]'\,\Psi \;+\; (\,\mathcal{R} \;-\; 2\,\Lambda\,)\Psi \;=\; 0 \;. \tag{14}$$

More specifically, ' ' here refers to caveats with choice of kinematical quantization [59], need for regularization, and operator ordering ambiguities; the operator ordering displayed is Laplacian operator ordering [23]. Once again, the momentum constraint is more straightforward to handle; we present it with momentum operators ordered to the right [43, 116].

1.2 Problem of time facets' original names

i) *Frozen Formalism Problem.* This refers to the quantum Hamiltonian constraint's left-hand side being zero where one would be expecting some time-derivative term (for some notion of time t). E.g. an $i\,\partial\Psi/\partial t$ term parallel to that in Schrödinger's equation, or a $\partial^2\Psi/\partial t^2$ term parallel to that in the Klein–Gordon equation... The zero renders the wave equation for the universe stationary, i.e. timeless or frozen, hence this facet's traditional name.

ii) *Thin Sandwich Problem.* This facet is, in double contrast, a classical issue with the momentum constraint. Fig 2.a)'s obvious thick sandwich between two spatial hypersurfaces failed to be well-posed as a PDE problem. So Wheeler [26, 29] next considered its thin-sandwich Cauchy data limit (Fig 2.b), for which some theorems have appeared [38, 72]. Its relation to time follows for instance from [26]'s title, and from its end-product of constructing a local slab of GR spacetime immediately adjacent to $\mathbf{\Sigma}$.

[5] Diffeomorphism generators, closing as the infinite-dimensional diffeomerphism Lie algebra, play a significant role [70] in both the spacetime context and in the spatial context in the dynamical formulation. In presenting the Problem of Time, it is moreover important to keep spacetime and space diffeomorphisms clearly distinct.

[6] 'Super' having picked up other uses since, we prefer to call this *space of spaces on* $\mathbf{\Sigma}$, denoted by $\mathfrak{S}\mathrm{pace}(\mathbf{\Sigma})$. This uses the convention that spaces of things are picked out and clearly distinguished from the things themselves by having bold fraktur font leading letters. In reading and speaking, having leading letters in this bold fraktur font is to be pronounced 'space of', reading out the letters, and then the pluralizing 's', by which e.g. $\mathfrak{S}\mathrm{pace}$ reads 'space of spaces'. This provides a means of immediately avoiding confusion between objects of a given type and the spaces thereof. If the space of things in question has brackets dependence on another space, the pluralization is followed by 'on' and then the contents of the bracket, hence 'on $\mathbf{\Sigma}$'.

[7] Λ and minimally-coupled matter do not substantially alter either piece of the above embedding mathematics.

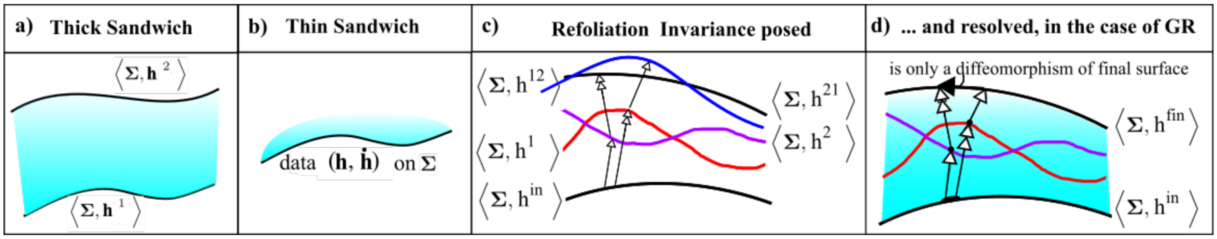

Figure 2: a) Thick and b) Thin Sandwich set-ups, and Refoliation Invariance posed in c) and resolved in d).

iii) *Functional Evolution Problem.* This is the further quantum-level issue that given a set of quantum constraint equations $\widehat{\mathcal{C}}$ that annihilate the system's wavefunction. Schematically,

$$\widehat{\mathcal{C}}\,\Psi \;=\; 0 \;\;\not\Rightarrow\;\; [\,\widehat{\mathcal{C}}, \widehat{\mathcal{C}}'\,]\,\Psi \;=\; 0\;. \tag{15}$$

In the context of the Problem of Time for GR, the $\widehat{\mathcal{C}}$ run over $\widehat{\mathcal{H}}$ and $\widehat{\underline{\mathcal{M}}}$.

iv) *(Canonical) Problem of Observables* At the classical level, this concerns finding quantities that Poisson-brackets-commute with a theory's constraints,

$$\{\,\mathcal{F}, \mathbf{O}\,\}\ \text{`='}\ 0\;. \tag{16}$$

More precisely, a theory's first-class constraints $\mathcal{F}$ are involved at this level (and for quantization). Any other constraints – second-class constraints – can in principle be eliminated prior to this point by passage to Dirac brackets. [These are in turn geometrically a more reduced formulation's Poisson brackets.] The expression '=' [110] covers both Dirac's [30] weak equality $\approx$ – up to linear combinations of our remaining first-class constraints – and the usual (strong) equality $=$, giving corresponding notions of weak and strong classical observables respectively. If commuting with *all* first-class constraints, (16) are also more specifically *Dirac observables* [30]. Finally

$$[\,\widehat{\mathcal{F}}, \widehat{\mathbf{O}}\,]\,\Psi\ \text{`='}\ 0\;. \tag{17}$$

provides a quantum counterpart of these notions of observables. At this level, the Problem of Observables facet is that, for Gravitational Theory in general, observables are rather hard to find.

Wheelerian 2-way routes [34, 44] start with 2-way passage between spacetime and ADM canonical formulations. Let us first consider a change of perspective from spacetime to space.

v) *Foliation Dependence Problem.* In evolving from one spatial slice $\langle\Sigma, \mathbf{h}_1\rangle$ to another $\langle\Sigma, \mathbf{h}_2\rangle$, does going via the red slice or via the purple slice (Fig 2.c) make a difference? This corresponds to whether a fleet of observers moving in two different ways between $\langle\Sigma, \mathbf{h}_1\rangle$ and $\langle\Sigma, \mathbf{h}_2\rangle$ make the same observations upon attaining their common final state.[8] In a sufficiently GR-like theory, general covariance renders this a desirable property. Indeed, Teitelboim [45] observed that the Poisson bracket of two Hamiltonian constraints classically guarantees that this property holds for GR, the two being out by just a diffeomorphism of the final spatial surface (Fig 2.d): Refoliation Invariance.

vi) *Spacetime Construction Problem* constitutes a reverse route, changing perspective from space to spacetime. Fluctuations of the dynamical entities are inevitable at the quantum level by the General Uncertainty Principle. This amounts to fluctuations of the 3-metric and the extrinsic curvature. As Wheeler pointed out [34], these fluctuations are far too numerous to be embeddable within a single spacetime. Schematically,

$$\left(\begin{array}{c}\text{metric-level geometry}\\ \text{embedding data } \mathbf{h}, \mathbf{K} \text{ or } \mathbf{h}, \mathbf{p}\end{array}\right) \;\rightarrow\; \left(\begin{array}{c}\text{operators } \widehat{\mathbf{h}}, \widehat{\mathbf{p}} \text{ subject to}\\ \text{Generalized Uncertainty Principle}\end{array}\right)\;. \tag{18}$$

Due to this, Geometrodynamics (or some other canonical formulation) would be expected to take over from spacetime formulations at the quantum level. At the primary level, it not then clear what becomes of notions that are strongly associated with classical GR spacetime, such as (micro)causality [70], dimensionality, or a continuum. Such would arise, rather, as emergent properties in such as semiclassical or lower-energy regimes; such calculations remain hard to complete. A further consequence of this is that the beautiful geometrical manner in which classical GR manages to be Refoliation Invariant is not expected to carry over to attempted quantum formulations in general.

[8] I.e. their distribution over space alongside their extrinsic curvature values and thus gravitational momentum.

1.3 Lie Theory of Background Independence resolution

As above, many of the facets were conceived of at the quantum level. A Local Problem of Time's facets – i.e. the consistent portion of facets that are neither global nor involving non-uniqueness – however all have classical precursors [106, 116].

Most of the spacetime versions of the Background Independence aspects are simpler to discuss. So we do this first, prior to returning to reconceptualized versions of the previous subsection's four canonical facets. We do not treat underlying Lie-Theoretic Background Independence aspects in the same order of presentation as is convenient for briefly presenting the historical context of the Problem of Time facets as per above. This should not be surprising but rather *expected* as part of the progress made by this reconceptualization.

In this Series, we proceed to generalize these facets to notions that are all of classical-and-quantum, finite-or-field-theoretic and theory-independent. This is a change of conception from Problem of Time facets to underlying Background Independence aspects. The major insight to be developed is that for a mathematically-sharp implementation of this, Lie's mathematics turns out to suffice.

Two copies of a four-aspect structure are thereby found: the *dynamical or canonical copy* and the *spacetime copy*. This doubling is rooted in) GR being not only a theory of spacetime but also a theory of evolving 3-geometries: geometrodynamics (as Sec 1.1 already explained).

The four Lie-theoretic aspects are as follows.

0) *Relationalism*, as implemented by Lie derivatives [21, 39], which is the subject of the current article.

1) *Closure*, as implemented by Lie brackets [49, 108] and a Generalized Lie Algorithm [8, 30, 68, 116, 125, 129, 132]: the subject of [136]. Its output are Lie algebras [49, 27] or Lie algebroids [89, 93]. While Relationalism provides some necessary preliminaries, it is moreover Closure that is held to be central (see the next subsection). Some readers will be familiar with the Dirac Algorithm for Poisson brackets of constraints. The above-mentioned Lie Algorithm is the generalization [134, 132] of this to the far broader arena of Lie brackets of Lie generators. While for us in this series this permits parallel spacetime treatment of canonical constraint closure, [119, 138] explains how this also for instance provides a new approach to Geometry.

2) *Observables* [18, 73, 110, 137], as implemented by zero Lie brackets commutants, which can be rephrased as [129] a first order PDE system flow problem [8, 53, 108]. More specifically, this universal view on what observables are reduces to a slight variant of *Lie's Integral Approach to Geometrical Invariants* in the case of classical observables.

3) *Constructability* [84, 109, 116, 131, 138], as implemented by deformation [31] of Lie Theory's generators [32].

There is still a Wheelerian 2-way route, though in one direction it is another Constructability – of spacetime from space – whereas in the other direction it is the already classically resolved Foliation Independence.

In setting up this subsection's material, one can draw on classical work of Lie, Killing and Cartan in this field, on the one hand. On the other hand, one can draw on the modern resurgence of, and expansion on, Lie Theory from the 1960's onward (Order Theory [49], Lie algebroids [89, 93], deformations and rigidity [31, 32]). Even foliations can be reformulated in Lie-theoretic terms [89].

1.4 Commentary and outline of the current article

Dirac link. There is moreover a 'piece of middle ground' that was largely developed in a more restricted context than general Lie Theory, which has hitherto largely not been connected with early or modern Lie Theory. Namely, Dirac's work on constrained systems [30] in the canonical and canonical-quantization contexts. This is of course a rather obvious bridge to the Problem of Time and its literature. It however remained to the current author to find that 'Dirac magic' extends to 'Lie magic' in a much broader domain of applicability. For some of Dirac's insights in treating constraints so happen to extend to Lie Theory, qualitatively upgrading Lie's own Algorithm and generating substantial new results [132, 119, 131, 136].

Outline In resolving the Problem of Time at the local classical level,[9] starting with Spacetime Relationalism (Sec

[9]If not in conceiving of it, philosophizing about it, or in the historical order of its literature. By this e.g. [116] and [123, 124, 125, 126, 127, 128, 129, 130, 131, 132] are ordered differently, with Temporal Relationalism then Configurational Relationalism and only then

2) makes good sense. For it is simpler than the complex of Configurational and Temporal Relationalisms of the space-time split/dynamical/canonical approach, among which pure Configurational Relationalism strongly resembles Spacetime Relationalism. Spacetime Relationalism entails presenting spacetime redundantly, with this redundancy encoded by a group G_{S}, in particular GR's group of spacetime diffeomorphisms $Diff(\mathfrak{M})$. Configurational Relationalism likewise for space, now with a group G_{C} such as GR's group of spatial diffeomorphisms $Diff(\mathbf{\Sigma})$. Now varying with respect to G_{C}'s auxiliary variables in the action provides constraints linear in the momenta, encoding the corresponding local Lie algebra. The above similarity motivates presenting Configurational Relationalism second (Sec 3), so as to start with two similar sets of material, prior to presenting Temporal Relationalism in isolation (Sec 4). We give two large and cumulative generalizations of thin sandwiches: *Best Matching* [52, 104] and the G-Act G-All Method [104, 116]. Temporal Relationalism implements primary-level timelessness for the universe as a whole, pre-empting the Wheeler–DeWitt equation of GR's frozenness as an already classically present phenomenon. This produces a further constraint, which is to be interpreted as an equation of time. This resolves primary-level timelessness with a concrete emergent time realization of Mach's [7] 'time is to be abstracted from change'. How Temporal and Configurational Relationalism entwine – interpretable as resolving the Frozen Formalism Problem's facet interference with the Thin Sandwich Problem once the both of them are sufficiently generalized – is then the subject of Sec 5.

Shortening of Exposition 0) to 4) seals resolution of spacetime primality's Problem of Time facets in close – Lie!– parallel to canonical primality's (Dirac subcase of Lie). The 2-copy simplification of the 11 aspect model, clarity from keeping the 2 copies distinct, and the next subsection's Lie claw, alongside humanity's careful study of Lie Theory for between 50 and 140 years permits substantial shortening of ALRoPoT's expostion. Identifying and exploiting the underlying Lie nature of the structures involved shortens this exposition by a factor of around 3 from [123, 124, 125, 126, 127, 128, 129, 130, 131, 132] to the current Series. The previous account is in turn a factor of 4 shorter that [116] (though that does include more background, the quantum version and the mathematical methods used). Both of these reductions are contingent on the Lie nature of the mathematics being invariant under TRi (Temporal Relationalism implementing) reformulations, by which presenting these ceases to be as pressing. Readers can consult [123, 124, 125, 126, 127, 128, 129, 130, 131, 132]'s longer account if interested in how TRi works out. Much of the Problem of Time and Background Independence – an exciting field of study – is hereby now prised open to a very accessible level for the very first time.

The Lie Claw. The above four Lie aspects are moreover interlinked in the form of the Lie claw (Fig 3). This explains many 'facet interferences' of the genuine kind (as opposed to those resulting from disjoint mathematizations, or confusing or conflating spacetime and canonical versions of aspects).

a) Closure is central (Fig 3, as the centre and nexus of the Lie claw digraph af Background Independence aspects. Each of Relationalism, Observables and Constructability has ties to Closure (the first two-way, the other two one-way), rather than directly between each other. This describes the anatomy of where 'facet interferences' may occur, by which mastering Closure is essential for overcoming facet interferences. This is to be contrasted with prior literature's misplaced over-emphasis on Relationalism, which this article and especially [136] redress. Zero-commutant and deformation moves – giving Observables and Constructability respectively – are separate add-ons to the Relationalism–Closure problem, which must in general be jointly solved due to the two-way arrow of generator provision and encodement between them.

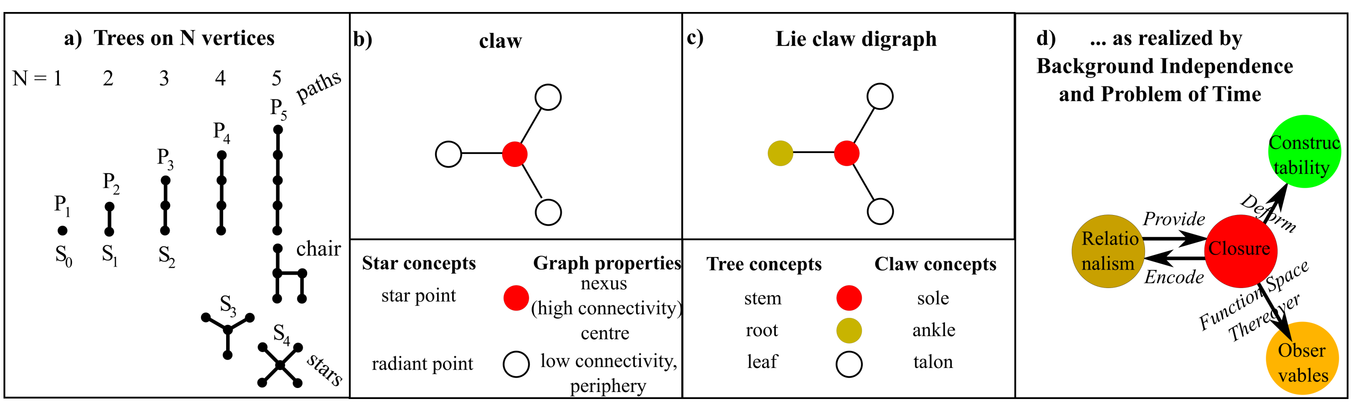

Figure 3: a) The claw graph is the smallest tree graph that is not just a path, and also the smallest nontrivial star graph. b) describes this graph as a star, while c) specializes to our digraph [83] of interest, providing rooted tree and claw nomenclature. d) shows the specific way how Background Independence – as used to resolve the Problem of Time – manifests the Lie claw digraph.

Spactime Relationalism in each of these accounts.

b) The claw inter-relations are structurally general enough to transcend Metric Differential Geometry, or even Differential Geometry in general. This is both as regards other levels of structure for classical modelling, and those suitable for quantum operators. One idea then to eventually concentrate on what can differ between categories: the Comparative Theory of Background Independence. Self-constructions do not necessarily work. The Wheelerian 2-way route – completing ALRoPoT at the Metric Differential Geometry level of classical GR – does not necessarily work for each level of structure either. For Differential-Geometric theories, this does still however lie within the previous four items' standard list of Lie structures. Thereby, *the local classical Problem of Time reduces to just Lie Theory*. It also becomes clear that the four Lie aspects involved are categorically meaningful, thus possessing in particular quantum counterparts.

b) points to what these substantial further questions are; we do not however further explore what answers these questions lead to in the current Article and [136].

Accounting for facet interferences By the above analysis, difficulties these used to cause rest on combinations of

I) Lie Theory's parts possessing much better-known inter-relations that facet interferences reduce to.

II) Lie Theory's relevant local parts are not ordered linearly but rather in the form of the claw digraph (Fig 3.d). Thereby, attempting to linearly order Problem of Time facets was an impasse.

III) Some Problem of Time literature did not notice, or take into account, that facets come in two copies: canonical and spacetime.

IV) Work prior to [104, 113, 116] treated each facet in a mathematical formulation that is not immediately combineable with the others'.

2 Spacetime Relationalism

2.1 Principles

Structure 1 *Spacetime* is conventionally modelled by [57] a 4-*d* topological manifold $\mathfrak{M}$ equipped with a $(-+++)$ metric $\boldsymbol{\gamma}$: a pseudo-Riemannian alias semi-Riemannian metric [62]. This corresponds to having 3 spatial directions and 1 time direction, modelling a 3-*d* notion of space and a 1-*d* notion of time.

Structure 2 The *space of spacetimes* on a given $\mathfrak{M}$ is denoted, following Isham [61], as

$$\mathfrak{P}\mathfrak{R}\text{iem}(\mathfrak{M}) \ . \tag{19}$$

This stands for 'space of pseudo-Riemannian metrics' on our given $\mathfrak{M}$;[10] and had its geometrical structure worked out in [60].

Spacetime Relationalism is mathematically sharply implemented by the following postulates.

STR-a) Background Independence precludes a *background* spacetime metric. So no absolute or background-dependent properties are to be attributed to a such.

Remark 1 See Secs 2.4-2.6 for discussion of which physical paradigms do and do not depend on such a structure, by which some are Background Dependent and some are Background Independent (at least in this sense).

Remark 2 In practice, Physics often requires a mathematically-convenient description involving physically-redundant states subject to a physically-irrelevant group of transformations.

STR-b) Allow for a symmetry group of physically-redundant transformations G_{S} to act on spacetime $\langle\mathfrak{M}, \boldsymbol{\gamma}\rangle$.

[10] The current article, and [136, 137, 138], do not go as far as considering topological-level Background Independence; see e.g. [67, 71, 135] and Epilogues II.C and III.C of [116] for work in this direction.

Example 1 For GR in vacuo,[11] this symmetry group consists of the diffeomorphisms of spacetime,

$$G_{\mathrm{S}} \;=\; Aut(\mathfrak{M}) \;=\; Diff(\mathfrak{M}) \;. \tag{20}$$

Infinitesimally implementing diffeomorphisms by the Lie derivative is a well-known application of the Lie derivative. For GR in vacuo, the idea then is that Physics is not on spacetime $\langle \mathfrak{M}, \boldsymbol{\gamma} \rangle$ or even on $\mathfrak{PR}$iem($\mathfrak{M}$), but rather on the quotient

$$\mathfrak{S}\text{pacetime}(\mathfrak{M}) \;=\; \frac{\mathfrak{PR}\text{iem}(\mathfrak{M})}{Diff(\mathfrak{M})} \;. \tag{21}$$

I.e. on the *space of equivalence classes of spacetimes on* $\mathfrak{M}$ *modulo the corresponding diffeomorphisms* on $\mathfrak{M}$ [40, 61, 116] (occasionally previously called 'Superspacetime'). This being hard to work on in many ways in practise is why we work with $\langle \mathfrak{M}, \boldsymbol{\gamma} \rangle$ and $\mathfrak{PR}$iem($\mathfrak{M}$) while making ab initio allowance for $Diff(\mathfrak{M})$'s physical rendundancy.

Example 2 In the presence of matter fields on spacetime, $\mathfrak{PR}$iem($\mathfrak{M}$) needs to be enlarged to include the space of these matter fields as well. For footnote 11's issue, $G_{\mathrm{S}} \;=\; Diff(\mathfrak{M})$ persists.

Example 3 For Electromagnetism and Yang–Mills Theory, however, their internal gauge groups need to be further adjoined to spacetime's $Diff(\mathfrak{M})$. This prompts extending the above notion of Spacetime Relationalism to the following 'Relationalism of Spacetime-and-Fields-Thereupon'.

E-STR-a) Background Independence precludes a *background* spacetime metric and any background internal structures pertaining to fields $\boldsymbol{\Phi}$ on spacetime.

E-STR-b) Allow for a symmetry group of physically-redundant transformations G_{S} to act on spacetime and the fields thereupon, $\langle \mathfrak{M}, \boldsymbol{\gamma}, \boldsymbol{\Phi} \rangle$.

Remark 3 We shall see in Sec 3 that there is also a configurational-level parallel of the above four principles. We use C superscripts for the group in this case to S superscripts in the spacetime case, and no superscripts for considerations that apply just as well in each case. Joint conceptualization of these C and S cases is in fact our first application of there being two copies of the Lie claw: 'C' and 'S' are claw copy labels. Such joint conceptualization begins with the notion of state, covering both spacetime state and space state, in each case with internal extensions allowed. The corresponding *carrier space* [117] (e.g. $\mathfrak{M}$ for spacetime or the spatial topological manifold $\boldsymbol{\Sigma}$ in the space-or-configuration-space approach) is denoted by $\mathfrak{C}$. The corresponding *state space* (covering e.g. all of spaces of spacetimes, configuration space, or phase space) is denoted by $\mathfrak{S}$. We use *base objects* B forming the space $\mathfrak{B}$ to refer to the corresponding constituent objects. Some $G \;=\; Aut(\mathfrak{C}, \boldsymbol{\sigma})$ then acts, by which

$$\widetilde{\mathfrak{S}} \;:=\; \frac{\mathfrak{S}}{Aut(\mathfrak{C}, \boldsymbol{\sigma})} \tag{22}$$

constitutes a furtherly relevant reduced arena. Lie mathematics *on the state space arena* is to be updated to generators algebraic structure, observables algebraic structure and space of deformed generator algebraic structures in the subsequent articles [136, 137, 138] respectively.

2.2 Direct implementation

It may be that one has a sufficient set of G-invariant objects with which to construct a Principles of Dynamics action for one's theory, by which a direct approach is possible. In this setting, 'a)' Principles suffice. This setting is rather more likely in the current section's spacetime case than it is for Sec 3's space-or-configuration-space case.

Example 1 Indeed, for GR the Ricci 4-scalar is $Diff(\mathfrak{M})$-invariant and serves as the integrand of the Einstein–Hilbert action for vacuum GR, (1). This is a major realization of the assertion at the end of the previous subsection's Example 1.

Example 2 With all the above-mentioned matter fields being minimally coupled, ordinary-looking matter term contributions to the Lagrangian for these are also spacetime scalars, so the Direct Approach continues to apply.

Remark 1 Spacetime Relationalism does however become a nontrivial matter at the quantum level, giving rise e.g. to a Measure Problem in the path integral formulation [61, 70, 116].

[11]This extends to inclusion of effective matter, scalar field matter, and many sufficiently GR-like alternative theories of Gravitation. Mathematicians would say 'automorphism group' [41] rather than symmetry group. The current article, and [136, 137, 138], only consider continuous automorphisms; more global works show how to model discrete automorphisms as well. With discrete 'large diffeomorphisms' being not so well known [78], we point to spatial reflections and time-reversal as more widely known discrete transformations of relevance in flat spacetime.

2.3 Indirect implementation

In many cases, however, working directly is not possible since reduced state space's geometry is unknown, complicated, or highly pathological [121, 122]. This is covered by a distinct indirect approach [104, 111, 116] which, at least formally, is *universal*.

If some object O is not G-invariant, it can be subjected to a group action

$$O \longrightarrow \overrightarrow{G_{\mathbf{g}}}\, O \, . \tag{23}$$

This sends $\mathfrak{O}$ – the space of objects O – locally to $\mathfrak{O} \times G$. While this looks to be a step in the wrong direction as regards freedom from G, this is doubly compensated for by following up by a move that uses all of G,

$$\mathrm{S}_{\mathbf{g}} \, . \tag{24}$$

This sends us to the quotient space

$$\widetilde{\mathfrak{O}} \;=\; \frac{\mathfrak{O}}{G} \, . \tag{25}$$

Such 'all' moves include the following.

Example 0) The familiar *group averaging*, which goes back to Cauchy,[12]

$$\mathrm{S} \;=\; \frac{1}{|G|} \sum_{g \,\in\, G} \, , \tag{26}$$

in the finite case, or

$$\mathrm{S} \;=\; \frac{1}{\int_{\mathbf{g} \,\in\, G} \mathrm{d}\mathbf{g}_{\mathrm{H}}} \int_{\mathbf{g} \,\in\, G} \mathrm{d}\mathbf{g}_{\mathrm{H}} \;\times \tag{27}$$

in the compact Lie group case, where the H subscripts indicate Haar measure.

Example 1) The next section makes considerable use of *extremization*

$$\mathrm{E}_{\boldsymbol{g} \,\in\, G} \, . \tag{28}$$

(we pick out G-All operations using oversized capital letters). The G-Act G-All Method can moreover be further generalized by inserting any well-defined G-independent *Maps* between the G-Act and G-All operations:

$$\boldsymbol{O}_{G\text{-inv}} \;:=\; \mathrm{S}_{\mathbf{g} \,\in\, G} \circ \mathit{Maps} \circ \overrightarrow{G_{\mathbf{g}}}\, \boldsymbol{O} \, . \tag{29}$$

Structure 1 For Classical Physics with G continuous, the infinitesimal group action takes the form of a Lie derivative [11, 14, 16, 21, 39] correction [104, 132]

$$\boldsymbol{O} \longrightarrow \boldsymbol{O} \;-\; \pounds_{\mathbf{g}}\boldsymbol{O} \, . \tag{30}$$

Structure 2 If STR-a) holds, these encode all infinitesimal continuous diffeomorphisms of spacetime. If instead just STR-b) holds, the Lie derivative corrections involved additionally need to preserve some further level of geometrical structure $\boldsymbol{\sigma}$, such as a metric $\mathbf{m}$.

A) In the case of $\mathbf{m}$, a systematic prescription for obtaining these is provided by solving *Killing's equation* [9, 21, 39, 57]

$$\pounds_{\underline{\xi}}\mathbf{m} \;=\; 0 \;\Rightarrow\; 0 \;=\; 2\,\mathscr{D}_{(A}\xi_{B)} \;=\; \partial_A\xi_B \;+\; \partial_B\xi_A \;-\; 2\,\Gamma^{C}{}_{AB}\xi_C \, . \tag{31}$$

N.B. that the first, defining, form for this equation is itself in terms of the Lie derivative. The other forms included are more useful for differential-geometric and PDE considerations.[13] Killing's equation is to be solved for the *Killing vectors* $\underline{\xi} \;=\; \underline{\boldsymbol{g}}$ that generate the isometries that form the Lie algebra $\mathfrak{g}$ associated with the Lie group G in this case. These take the infinitesimal form

$$\epsilon_{AB} \;\longrightarrow\; \epsilon_{AB} \;-\; \pounds_{\underline{\xi}}\mathrm{m}_{AB} \;=\; \epsilon_{AB} \;-\; 2\,\mathscr{D}_{(A}\xi_{B)} \tag{32}$$

[12] He used it in 1845 [6] to prove a version of what is now known as Burnside's Lemma. It was only with Burnside's own later work [10], however, that group averaging entered common knowledge among mathematicians; see [49, 90, 75] for some basic modern uses in Group Theory and Representation Theory.

[13] $\mathbf{m}$ is our notation for a metric free of context, covering e.g. $\boldsymbol{\gamma}$ for spacetime and $\mathbf{h}$ for space. $\mathscr{D}$ and $\boldsymbol{\Gamma}$ denote the corresponding context-free covariant derivative and Christoffel symbol of the second kind. The current article uses capital Latin indices in the general context-free case.

B) For other $\boldsymbol{\sigma}$, such as similarity, conformal, affine or projective structure, there is a parallel treatment involving a generalized Killing equation

$$\pounds_{\underline{\xi}}\boldsymbol{\sigma} \ = \ 0 \ , \tag{33}$$

solutions of which – generalized Killing vectors – forming $aut(\mathfrak{C}, \boldsymbol{\sigma})$ can be arrived at. In this way, we have a *continuous automorphism Lie algebra finding equation* that is solved by *continuous automorphism Lie algebra generators*. See e.g. [21, 39] for geometrical-level exposition of this, and [120] for applications to Configurational and Spacetime Relationalisms.

Remark 1 For use in [136] the infinitesimal-generator form of automorphism vectors is as per Fig 4. Also the $aut(\mathfrak{C}, \boldsymbol{\sigma})$ are usually finite [15, 21] subalgebras of $diff(\mathfrak{M})$, in the sense of usually being a finite count of independent generators.

$$\boxed{\underline{\xi} \ = \ \underline{\boldsymbol{T}}(\mathbf{B}) \cdot \underline{\boldsymbol{\nabla}}}$$

Figure 4: Solutions of generalized Killing equations interconvert base object tensors to generator tensors. In the Background Independence setting, base object tensors play 0) Relational roles, while generator tensors play 1) Closure roles, so this is a (0, 1) interaction. Also with these aspects being 0) and 1), we use bold zero-turn underlines for the first's coordinate-free notation, and one-turn-undelines for the second. We continue this pattern with aspect 2)'s Observables having two-turn underlines and aspect 3)'s deformations having three-turn underlines. Keeping all these Tensor Calculi distinct (and further subvariants like spacetime, space, configuration space, phase space Tensor Calculi distinct) is crucial to clear understanding of, and exposition about, Background Independence and the Problem of Time. $\boldsymbol{\nabla}$ is the base objects' gradient, and $\boldsymbol{T}$ is the 2-tensor straddling the two different tensor calculi involved, which machinates (more precisely *solders*) the Tensor Calculus interconversion.

Structure 3 All in all, we have a *G-Act*, *G-All procedure* [104] for producing G-invariant versions of objects (29); this includes an S-superscripted version for use in the current section's Spacetime Relationalism context. This is moreover general enough for use at any level of formulation. Our objects $\boldsymbol{O}$ can be not only action principles but also e.g. notions of distance [87, 80], information [65, 116], correlation [82, 116], estimator [86, 79, 115], or of quantum operator [48, 116]. In this way, all Problem of Time facet interferences that involved what has become Configurational, or Spacetime, Relationalism, can in principle be banished for good.

2.4 Minkowski spacetime

Let us now take a brief look at this most common case of spacetime Background Dependence [in the sense of STR-a) not holding] as occurs in SR. [Thus it also occurs as a local approximation to GR in everyday familiar weakly-gravitating regimes such as on Earth.] $\langle \mathfrak{M}, \boldsymbol{\gamma} \rangle \ = \ \langle \mathbb{M}^4, \boldsymbol{\eta} \rangle$ here, for $\boldsymbol{\eta}$ the flat Minkowski spacetime metric which can be put into the form diag$(-1, 1, 1, 1)$.

STR-b) continues to hold, now for automorphisms that respect not only $\mathfrak{M} \ = \ \mathbb{M}^4$ but also the specific background metric $\boldsymbol{\gamma} \ = \ \boldsymbol{\eta}$. For this, Killing's equation returns the spacetime translations alongside the Lorentz transformations which form the Poincaré group, $Poin(3, 1)$. One standard interpretation for this is that

$$Aut(\langle \mathbb{M}^4, \boldsymbol{\eta} \rangle) \ = \ Isom(\langle \mathbb{M}^4, \boldsymbol{\eta} \rangle) \ = \ Poin(3, 1) \ . \tag{34}$$

Many technically useful features follow.

a) The timelike Killing vector – the one associated with the translation in a timelike direction – provides a background notion of time.

b) With this in place, such as QFT's familiar split into positive and negative modes is possible.

c) Also particles in SR can be taken to be representations of $Poin(3, 1)$. Such have two labels, which can be interpreted as the mass and spin of the particle in question.

d) All of the associated wave equations have a temporal part rather than being stationary, i.e. frozen (and well-defined inner products [69] to accompany each are clear).

2.5 Perturbations thereabout

With massless spin-2 particles corresponding to gravitons modelled by a perturbation about Minkowski spacetime, some kind of Theory of Gravitation is included in the above scheme. This comes with technical problems, however, such as nonrenormalizability, without which its UV regime becomes intractable.

A classical counterpart also merits a brief mention, as follows. At the level of perturbations [92] about a given spacetime, a *point identification map* is used, which also implements Lie derivative corrections [66].

2.6 Conceptual problems

In addition to technical problems such as the above, there are conceptual problems with this general kind of framework as well.

Firstly, suppose that one attempts to re-run QFT's success about other backgrounds than SR's Minkowski spacetime $\mathbb{M}^4$. Then one finds that almost none of these are anywhere nearly as amenable to SR QFT's techniques. This is since these rely on Killing vectors in general, and quite often on one or both of there being specifically a timelike Killing vector or a high number of Killing vectors. But the generic GR spacetime has *no* Killing vectors. Furthermore, even in other spacetimes that possess quite a lot of Killing vectors (relative to the maximum number [15, 21] that a given dimensionality supports), SR QFT type workings, and notions, quite rapidly break down. In other words, QFTiCS ('in curved spacetime') [55, 100] is a rather difficult and case-by-case subject.

a) For instance, the concept of uniquely-defined vacuum – which serves as ground state – breaks down. Without this, expanding in modes ceases to be unique (different expansions are interrelated by *Bogoliubov transformations*) [100, 55].

b) In $\mathbb{M}^4$, the modes that simplify QFT are eigenfunctions of the $\partial/\partial t$ operator, which arises as the timelike Killing vector. In particular, QFT's construction of Fock space is based upon the *split into positive and negative modes*. Standard QFT's 'natural modes' are tied to $\mathbb{M}^4$'s natural rectangular coordinates t, x, y, z, which indeed rest in turn upon $Poin(4)$. Unfortunately, even in other spacetimes that have (near-)maximal number of Killing vectors, not all the beneficial properties of these natural modes are recovered.

c) The labelling of particle types in $\mathbb{M}^4$ by $Poin(4)$ representations is also in general a lost commodity once one passes to QFTiCS. $Diff(\mathfrak{m})$'s own Representation Theory is *much* harder.

d) *Time and energy are conjugate*. So in spacetimes without a timelike Killing vector total energy *E is not conserved*. This leads to some further technical difficulties in attempting to carry QFT over to nonstationary spacetimes.

Secondly, in the specific case of Gravitation, *splitting a metric into a background part and a perturbation part is not a well-defined prescription*. Suppose one just does this for some convenient background. Then, on the one hand, properties of this background (such as a timelike Killing vector or there being numerous Killing vectors) can be used to study the perturbation to some extent. On the other hand, all of this is taken away when the background is not particularly close to any metric with such Killing vectors. This is as a combination of the perturbation having to be small, leaving one unable to rest on an approximate spacetime with many Killing vectors.

Remark 1 One function of the previous two subsections is to account for the limited extent to which Background Dependent calculations hold in strongly-gravitating regimes. Another is that some of it is useful through (generalized) Killing equations recurring in models of Configurational Relationalism.

Remark 2 As well as starting from a spacetime perspective, Spacetime Relationalism is also relevant when spacetime is constructed from space [138].

3 Configurational Relationalism

3.1 Zeroth principles

A starting point for considerations of Background Independence is as follows (this subsection can be taken to underly whichever of this article's three Relationalisms).

Relationalism-A) *Physics is to solely concern relations between tangible entities.*[14]

Two key diagnostics for these are as follows.

[14]These are not 'just matter', and are named thus, via Isham, along the lines of Heidegger, with reference to the philosophy of 'what is a thing?'; c.f. the title of [97].

Relationalism-B) Tangible entities *act testably and are actable upon.*

Things which do not act testably or cannot be acted upon are held to be *physical* non-entities. These can still be held to be a type of entity as regards being able to *philosophize about* them or *mathematically represent* them. Absolute space [1] $\mathfrak{A}\mbox{bs}(d)$ (usually taken to be modelled by 3-d flat space $\mathbb{R}^3$) is an obvious archetypal example of such a non-entity. Relational intuition is that imperceptible objects should not be playing causal roles influencing the motions of actual bodies. As a first sharpening of this, James L. Anderson [33] posited that *"the dynamical quantities depend on the absolute elements but not vice versa"*, and also that an absolute object *"affects the behavior of other objects but is not affected by these objects in turn"* [36].

Relationalism-C) [Leibniz's Identity of Indiscernibles] [2] *Any entities indiscernible from each other are held to be identical.*

Remark 1 This is a statement of physical indiscernibility trumping multiplicity of mathematical representation. Such multiplicity still exists mathematically. The mathematical entity corresponding to the *true* physics in question, however, is the equivalence class spanning that multiplicity. One would then only wish to attribute physical significance to calculations of tangible entities which are independent of the choice of representative of the equivalence class. By this, e.g. our Universe and a copy in which all material objects are collectively displaced by a fixed distance surely share all observable properties, and so are one and the same. An archetype of such an approach in Theoretical Physics is Gauge Theory. This additionally factors in the major insight that a mixture of tangible entities and non-entities is often far more straightforward to represent mathematically.

Remark 2 In Secs 4 and 3, we consider separate treatments of time on the one hand, and space, configurations, dynamics and canonical formulation on the other. This befits the great conceptual heterogeneity between these (see Part I of [116] for a detailed exposition). Relational postulates can then be stated for each separately, and a coherent subset of these are sharply mathematically implementable. Further relational postulates have already been presented for spacetime (Sec 2) Between them, these postulates amount to rejecting all of absolute time, absolute space and absolute spacetime.

3.2 Leibniz and Mach's space principles

The current section's Background Independence aspect is grounded on the following relational considerations of space.

Leibniz's Space Principle is that *space is the order of coexisting things* [2].

Mach's Space Principle is that[15] [7] *"No one is competent to predicate things about absolute space and absolute motion. These are pure things of thought, pure mental constructs that cannot be produced in experience. All our principles of mechanics are, as we have shown in detail, experimental knowledge concerning the relative positions of motions and bodies."*

Remark 1 In the current article, we do not dwell on this and the previous subsection's generalities, developing rather the above-mentioned sharply mathematically implementable subset.

3.3 Sharply mathematically implementable principles

One's objects are now *configurations*: [17, 50]

$$\boldsymbol{Q} \ \text{ with components } \ Q^A \ : \tag{35}$$

instantaneous snapshots of the state of a system $\mathfrak{S}$.

Definition 1 The space of all possible configurations $\boldsymbol{Q}$ for a given system $\mathfrak{S}$ is the corresponding *configuration space* [17, 50, 79, 116, 118],

$$\mathfrak{q}(\mathfrak{S}) \ . \tag{36}$$

Notation 1 In the current Series, we use slanted font for finite-dimensional entities and straight font for field entities.

[15] This is not to be confused with Mach's Principle for the Origin of Inertia ([76], Section 3 of [116]). Or with anything else called Mach's Principle by some author or other (e.g. [76, 91] having an extensive selection of other such uses).

Example 1 For N particles in the usual flat-space $\mathbb{R}^d$ absolute space model of Newtonian Mechanics, we denote the incipient configurations – *N-point constellations* – by

$$\boldsymbol{q} \text{ with components } q^{aI} \ , \tag{37}$$

a is here a $\mathbb{R}^d$ vector index running from 1 to d (such vectors are also denoted by underlining) and I a particle label running from 1 to N. The corresponding configuration spaces are *constellationspaces*

$$\mathfrak{q}(d, N) \ := \ \mathfrak{q}(\mathbb{R}^d, N) \ := \ \times_{I=1}^{N}\mathbb{R}^d \ = \ (\mathbb{R}^d)^N \ = \ \mathbb{R}^{d\,N} \ . \tag{38}$$

(We start with a background notion of space, which we subsequently undermine.)

Example 2 $\mathfrak{Riem}(\boldsymbol{\Sigma})$ for GR, as per Sec 1.1.

Configurations are 'what is left' to build upon in primarily timeless pictures, so we had better focus on these next...

Configurational Relationalism consists of Spatial Relationalism [52] and Internal Relationalism (referring to spatial gauge fields) according to the following postulates.

CR-a) One is to include no extraneous configurational structures (spatial or internal-spatial metric geometry variables of a fixed-background rather than dynamical nature) [104].

CR-b) Physics in general involves not only a $\mathfrak{q}$ but also a group G_{C} of transformations acting upon $\mathfrak{q}$ that are taken to be physically redundant [6, 10, 52, 58, 79, 95, 104, 116, 120].

Remark 1 Again, the 1-postscript case is a matter of practical convenience: often redundant $\mathfrak{q}$ are simpler to envisage and calculate with.

Example 1 Modelling translations and rotations – jointly the continuous part of the Euclidean group $Eucl(d)$ – relative to a preliminary flat absolute space $\mathfrak{A}\text{bs}(d) = \mathbb{R}^d$ that is thus rendered physically irrelevant.

Example 2 Modelling spatial diffeomorphisms $Diff(\boldsymbol{\Sigma})$ as redundant in GR.

Quotient spaces then enter consideration as less redundant configuration spaces

$$\widetilde{\mathfrak{q}} \ := \ \frac{\mathfrak{q}}{G} \ , \tag{39}$$

for instance *relational space* [104, 116, 117]

$$\mathfrak{R}(N, d) \ := \ \frac{\mathfrak{q}(N, d)}{Eucl(d)} \ , \tag{40}$$

for Mechanics, or Sec 1.1's $\mathfrak{S}\text{pace}(\boldsymbol{\Sigma})$ [alias $\mathfrak{S}\text{uperspace}(\boldsymbol{\Sigma})$] for GR.

Some useful limitations on the choice of $(\mathfrak{q}, G)$ pairings are as follows.

Criterion C) *Nontriviality.* G cannot be too large, i.e. a bounding criterion on the count of degrees of freedom. Using $k \ := \ \dim(\mathfrak{q})$ and $l \ := \ \dim(G)$, a theory on $\widetilde{\mathfrak{q}}$ counts out as

$$\textit{inconsistent} \text{ if } \ l \ > \ k \ , \tag{41}$$

$$\textit{utterly trivial} \text{ if } \ l \ = \ k \ , \tag{42}$$

$$\textit{relationally trivial} \text{ if } \ l \ = \ k \ - \ 1 \ , \quad \text{or} \tag{43}$$

$$\textit{nontrivial} \text{ if } \ l \ < \ k \ - \ 1 \ . \tag{44}$$

Remark 2 Relational nontriviality is meant here in the sense of requiring at least two degrees of freedom. Then at least one of these can be expressed in terms of at least one other such. This is to be contrasted with the idea of degrees of freedom being 'meaningfully expressed' in terms of some external or otherwise unphysical 'time parameter'.

Criterion B) Further *structural compatibility* is required.

For instance, if one is considering d-dimensional particle configurations, then G is to involve the same d (or smaller, but certainly not larger).

Criterion A) A more general structural compatibility criterion is for G is to admit a group action on $\mathfrak{q}$. A group action's credibility may further be enhanced through its being 'natural'. Some further mathematical advantages are conferred from group actions being one or both of faithful or free, with the combination of free and proper conferring yet further advantages.

One might additionally wish to choose G for a given $\mathfrak{q}$ so as to eliminate *all* trace of extraneous background entities. The automorphism group $Aut(\mathfrak{a}\text{bs})$ of absolute space $\mathfrak{a}\text{bs}$ is an obvious possibility for G. Some subgroup of $Aut(\mathfrak{a}\text{bs})$ [41] might however also be desirable. For instance, this could be since the inclusion of some automorphisms depends on which level of mathematical structure $\boldsymbol{\sigma}$ is to be taken to be physically realized. In this way, the corresponding

$$G \ \leq \ Aut(\langle \mathfrak{a}\text{bs}, \sigma \rangle) \tag{45}$$

is a more general possibility.

Remark 3 The next two sections cover one distinct implementation of Configurational Relationalism each. Such seeking can be either by direct formulation on reduced configuration spaces [95, 104]: 'relational spaces'. or indirect by applying a correctory [52, 104] group action on unreduced configuration spaces

3.4 Configurational Relationalism correcting Lie derivatives

Configurational Relationalism, unlike Spacetime Relationalism, only occasionally has a feasible direct implementation. This is due to velocities picking up Lie derivative correction terms [66],

$$\boldsymbol{Q}' \ - \ \pounds_{\underline{\boldsymbol{g}}}\boldsymbol{Q} \tag{46}$$

A particular realization of this is then *Best Matching* [52, 74, 104, 116, 124, 127, 128] (and Fig 5.d). This counterbalances the above corrections by extremization over the G auxiliary variables entering these corrections,

$$\mathrm{E}_{\boldsymbol{g} \, \in \, G} \, \mathcal{S}(\boldsymbol{Q} \ , \ \ \boldsymbol{Q}' \ - \ \pounds_{\underline{\boldsymbol{g}}}\boldsymbol{Q}) \ . \tag{47}$$

In this way, infinitesimally distinct pairs of configurations are brought into minimum incongruence with each other by application of G's group action. Once again, the form taken by the Lie derivative corrections can in principle be derived by solving the corresponding generalized Killing equation. The extremization involved amounts to obtaining the first-class linear constraints corresponding to G by varying with respect to $\boldsymbol{g}$, and then solving the Lagrangian form of these for the $\boldsymbol{g}$ themselves.

Example 1 Against a historical backdrop lacking in viable relational alternatives to Newton's absolute Mechanics, we provide a spatially-relational Mechanics in this section, a temporally-relational Mechanics in the next section and their combination [52, 104] in Sec 5. For a $Eucl(d)$ spatially-relational particle mechanics preliminarily on $\mathbb{R}^d$, it suffices to correct the standard Euler–Lagrange action's velocities with respect to translations and rotations

$$\underline{\boldsymbol{q}}' \ \longrightarrow \ \underline{\boldsymbol{q}}' \ - \ \underline{A} \ - \ \underline{B} \times \underline{\boldsymbol{q}} \tag{48}$$

followed by varying with respect to the $Eucl(d)$ auxiliaries $\underline{a}$ and $\underline{b}$. The momentum–velocity relations are

$$\underline{\boldsymbol{p}} \ = \ \underline{\dot{\boldsymbol{q}}} \ - \ \underline{A} \ - \ \underline{B} \times \underline{\boldsymbol{q}} \ . \tag{49}$$

This produces the secondary [30] constraints

$$\underline{\mathcal{P}} \ := \ \sum\nolimits_{I=1}^{N} \underline{p}_I \ = \ 0 \quad \text{(zero total momentum constraint)} \ \ , \tag{50}$$

$$\underline{\mathcal{L}} \ := \ \sum\nolimits_{I=1}^{N} \underline{q}^I \ \times \ \underline{p}_I \ = \ 0 \quad \text{(zero total angular momentum constraint)} \ \ . \tag{51}$$

The Lagrangian versions of these constraints are obtained by substituting in (49) to convert from momenta to velocities.

Example 2 Best Matching applied to GR returns the Thin Sandwich when applied to a more or less Temporal Relationalism implementing product-type action (see Sec 5.2) and a partial precursor when applied to a difference-type action like (3) Best Matching is thus the arbitrary theory and arbitrary G generalization of one of Isham and

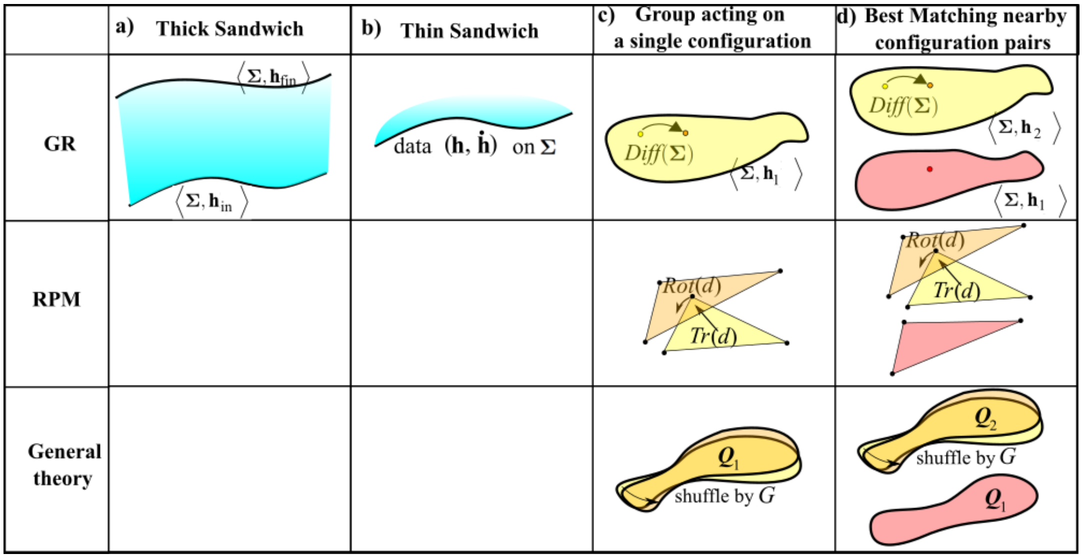

Figure 5: Expanding from a) and b)'s Sandwiches to GR non-specific active group actions c) and Best Matchings d).

Kuchař's listed Problem of Time facets [69, 70]. It is still specific to the classical Lagrangian level, though the G-Act G-All method constitutes a freeing from this restriction as well.

Remark 1 As a further part of keeping track of the multitude of objects entering Background Independence implementations alias Problem of Time resolutions, we jointly denote constraints (and generators more generally) – a sizeable subclass of such objects – by the undersized calligraphic font. Some such are provided by 0) Relationalism and subsequently need checking as regards 1) closure, and so are (0, 1) objects. For now, what we have shown is that Configurational Relationalism thus acts as a *Constraint Provider*: an underlying principle that produces constraints [34, 74].[16]

4 Temporal Relationalism

Adopting a split space-time approach requires further consideration of Temporal Relationalism.

Leibnizian Time(lessness) Principle There is no time at the primary level for the Universe as a whole [2, 74, 116].

The following two-part selection principles give a mathematically-sharp implementation [52, 104] at the level of Principles of Dynamics actions, according to the following postulates.

TR-a) Include no extraneous times or extraneous time-like variables [104].

TR-b) Include no label times either [104].

Remark 1 If time is not primary, moreover, we need to study whatever other entities that are still regarded as primary. Configurations $\boldsymbol{Q}$ and configuration spaces $\mathfrak{q}$ constitute a starting point for this. A further reason to consider Configurational Relationalism prior to Temporal Relationalism thus arises.

Remark 2 Since each of external time and label time can be viewed as 1-d time metrics, each Relationalism contains some element of freedom from fixed-background Metric Geometry.

Remark 3 By TR-b), one cannot however make use of the usual difference-type Euler–Lagrange action [3, 4, 51, 17] since that depends on Newton's extraneous notion of absolute time [1].

[16]This is in opposition to the 'Applied Mathematics' point of view that constraints just are, with no questions asked. Which opposition is made, in particular, in the context of investigating origins for *fundamental theories' constraints*. This is furthermore an example of Wheeler asking for 'zeroth principles' [34] whenever presented with 'first principles'.

4.1 Jacobi-type actions

The product-type action

$$\mathcal{S} \;=\; 2\int\sqrt{W\,\frac{1}{2}\left|\left|\frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}\lambda}\right|\right|^2_{\boldsymbol{M}}}\,\mathrm{d}\lambda \;=\; \sqrt{2}\int\sqrt{W}\,||\mathrm{d}\boldsymbol{Q}||_{\boldsymbol{M}} \;=\; \sqrt{2}\int\sqrt{W}\,||\pounds_{\mathrm{d}}\boldsymbol{Q}||_{\boldsymbol{M}}\ . \tag{52}$$

complies with Temporal Relationalism, encompassing four progressively superior formulations in this regard.[17]

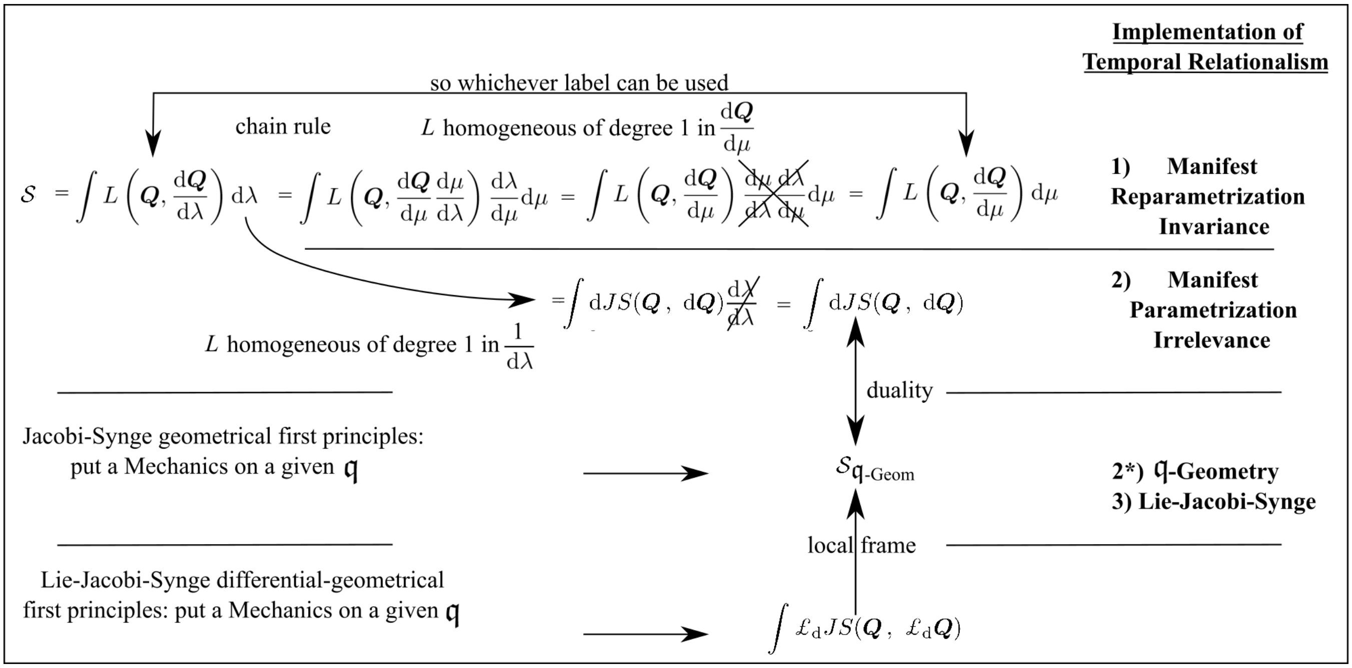

Figure 6: Inter-relation of the current article's four implementations of Temporal Relationalism at the level of actions a-c).

Form 1) Comply with TR-a) by involving not an external time but a label time λ. Velocities are then parametrized: built from $\dot{} := \frac{\mathrm{d}}{\mathrm{d}\lambda}$. Label time is furthermore meaningless by the interchange[18] in the first row of Fig 6. This implementation is therefore called *Manifest Reparametrization Invariance (MRI)* [30, 104].

Form 2) Comply with TR-b) as well, by effectuating the cancellation in row 2 of Fig 6, so that the meaningless label λ does not feature at all in the structure of our action. This implementation is thus termed *Manifest Parametrization Irrelevance (MPI)* [104]. An immediate consequence of there being no time at the primary level is that velocities are not primarily meaningful either. These are to be replaced by *change(-in-configuration) variables* $\mathrm{d}\boldsymbol{Q}$; *configuration–change variables* $(\boldsymbol{Q}, \mathrm{d}\boldsymbol{Q})$ then replace parametrized Lagrangian variables $(\boldsymbol{Q}, \dot{\boldsymbol{Q}})$. This means moreover that parametrized kinetic terms $T(\boldsymbol{Q}, \dot{\boldsymbol{Q}}) := \frac{1}{2}||\partial\boldsymbol{Q}/\partial\lambda||_{\boldsymbol{M}}$ need replacing by *kinetic arc elements*

$$\mathrm{d}s(\boldsymbol{Q}, \mathrm{d}\boldsymbol{Q}) \;:=\; ||\mathrm{d}\boldsymbol{Q}||_{\boldsymbol{M}}\ , \tag{53}$$

and parametetrized Lagrangians $L(\boldsymbol{Q}, \dot{\boldsymbol{Q}})$ by *physical* alias *Jacobi arc elements* $\mathrm{d}J(\boldsymbol{Q}, \mathrm{d}\boldsymbol{Q})$.[19]

Form 2*) is to give the same formula its dual interpretation as a geometrical action on configuration space. This $\mathfrak{q}$-*geometry action* [116] alias 'geodesic principle' [42, 74] has the further nominative advantage of *not referring to* the meaningless parameter that does not feature in its formulation. For it is *very much* the business of Background Independent theorization to cease to use notions, or names, of physically irrelevant or Background Dependent entities.

Remark 1 Both the first and second formulae date back to Jacobi [5] and so can be and are usually referred to as forms of *Jacobi's action principle.* He and most successors – writing in the Dynamics literature [17, 50] – interpreted this in the dual geometric manner without reference to Temporal Relationalism.

[17] The *potential factor* $W = W(\boldsymbol{Q}) := E - V(\boldsymbol{Q})$ for E the total energy and $V = V(\boldsymbol{Q})$ the potential itself.
[18] This λ to μ transformation is furthermore monotonic, to ensure that no zero factors sneak in under the guise of $d\mu/\mathrm{d}\lambda$ terms.
[19] $\boldsymbol{M}$ is configuration space's kinetic metric and $||\ ||_{\boldsymbol{M}}$ is the corresponding inner product.

Form 3) Further cast form 2*) in Lie-theoretic terms using how d can be viewed as the Lie derivative $\pounds_{\mathrm{d}}$ in a particular frame (the MPI counterpart of a result in e.g. [66]). This places all of Relationalism on a common mathematical footing, with all three parts of it implemented by Lie derivatives. Let us refer to form 3 as *Lie–Jacobi action principle* (row 4 of Fig 6).

Remark 2 Jacobi's action principle is homogeneous linear in its velocities (form 1), changes (form 2), or Lie derivatives (form 3), by being the square root of a square in each of these. *Jacobi–Synge actions* [13, 17] generalize this to arbitrarily-attained homogeneous linearity. Fig 6's moves are given for this extension. The last of these – the *Lie–Jacobi–Synge action*: most general implementation of Leibnizian timelessness by Lie's mathematics at the level of action principles – is new to the current article, though we subsequently almost always stick to the Jacobi case.

Example 1 Temporally Relational Particle Mechanics [52, 74, 104, 114, 116] has action

$$\mathcal{S} \;=\; \sqrt{2}\int \mathrm{d}s\,\sqrt{W}\;. \tag{54}$$

for kinetic arc element built from a quadratic metric (53).

Example 2 Minisuperspace GR [42] has action

$$\mathcal{S} \;=\; \sqrt{2}\int \mathrm{d}s^{\mathrm{GR}}_{\mathrm{MSS}}\,\sqrt{R \;-\; 2\,\Lambda} \tag{55}$$

for $\mathrm{d}s^{\mathrm{GR}}_{\mathrm{MSS}}$ minisuperspace GR's own kinetic arc element built out of a quadratic metric, up to some weighting factor.

4.2 Conjugate momenta

For MRI, we can still use a familiar-looking defining formula for generalized momentum, albeit now involving velocities with respect to label time,

$$\underline{\boldsymbol{P}} \;=\; \frac{\partial L_{\mathrm{JS}}}{\partial \underline{\dot{\boldsymbol{Q}}}} \;\overset{\text{Jacobi subcase}}{=}\; \sqrt{\frac{W}{T}}\,\underline{\underline{\boldsymbol{M}}}\cdot\underline{\dot{\boldsymbol{Q}}}\;; \tag{56}$$

the second, computational, formula has a new prefactor relative to the more familiar Euler–Lagrange action version. For MPI or its $\mathfrak{q}$-geometry dual, however, neither velocities nor Lagrangians remain meaningful, so we need the first equality below's double substitution

$$\underline{\boldsymbol{P}} \;=\; \frac{\partial \mathrm{d}JS}{\partial \mathrm{d}\underline{\boldsymbol{Q}}} \;\overset{\text{Jacobi subcase}}{=}\; \frac{\sqrt{2\,W}}{\mathrm{d}s}\,\underline{\underline{\boldsymbol{M}}}\cdot\mathrm{d}\underline{\boldsymbol{Q}}\;. \tag{57}$$

The first equality here is equivalent to (56)'s by the 'cancellation of the dots' Lemma [51]. Also, equating the first and third expression in (57) is now a *momentum–change* relation in place of a momentum–velocity one. We can finally write

$$\underline{\boldsymbol{P}} \;=\; \frac{\partial \pounds_{\mathrm{d}}JS}{\partial \pounds_{\mathrm{d}}\boldsymbol{Q}} \;\overset{\text{Jacobi subcase}}{=}\; \frac{\sqrt{2\,W}}{\pounds_{\mathrm{d}}s}\,\underline{\underline{\boldsymbol{M}}}\cdot\pounds_{\mathrm{d}}\boldsymbol{Q} \tag{58}$$

so as to have a Lie formulation of generalized momentum giving rise to a momentum–Lie change relation.

4.3 Equations of motion

In MPI form or its $\mathfrak{q}$-geoemetrical form, the equations of motion are

$$\frac{\sqrt{2\,W}}{||\mathrm{d}\boldsymbol{Q}||_{\boldsymbol{M}}}\,\mathrm{d}\left(\frac{\sqrt{2\,W}\,\mathrm{d}\underline{\boldsymbol{Q}}}{||\mathrm{d}\boldsymbol{Q}||_{\boldsymbol{M}}}\right) \;+\; \frac{\sqrt{2\,W}\,\mathrm{d}\underline{\boldsymbol{Q}}}{||\mathrm{d}\boldsymbol{Q}||_{\boldsymbol{M}}}\cdot\underline{\underline{\boldsymbol{\Gamma}}}\cdot\frac{\sqrt{2\,W}\,\mathrm{d}\underline{\boldsymbol{Q}}}{||\mathrm{d}\boldsymbol{Q}||_{\boldsymbol{M}}} \;=\; -\,\underline{\underline{\boldsymbol{N}}}\cdot\frac{\partial V}{\partial \underline{\boldsymbol{Q}}}\;. \tag{59}$$

4.4 Temporal Relationalism provides at least one primary constraint

Definition 1 In the Hamiltonian formulation's configuration–momentum variables $(\boldsymbol{Q}, \boldsymbol{P})$, quite a general notion of *constraint* consists of relations

$$\boldsymbol{\mathcal{C}}(\boldsymbol{Q}, \boldsymbol{P}) \;=\; 0 \tag{60}$$

between the momenta $\boldsymbol{P}$, by which these are not independent. A useful classification of constraints is into the following.

1) *Primary constraints* [30, 68], which arise from noninvertibility of the Legendre (transformation) matrix

$$\underline{\underline{\boldsymbol{Leg}}} \ := \ \frac{\partial^2 L}{\partial \underline{\dot{\boldsymbol{Q}}} \partial \underline{\dot{\boldsymbol{Q}}}} \tag{61}$$

without use of variation. When this occurs, the momenta $\boldsymbol{P}$ cannot be independent functions of the velocities $\dot{\boldsymbol{Q}}$.

2) Constraints furthermore requiring input from the variational equations of motion are termed *secondary* [30, 68].

Remark 1 Dirac [30] gave an elementary proof that primary constraints [30, 68] must arise for reparametrization invariant actions. This proof straightforwardly extends to [104, 116] actions of form 2), 2*) and 3) as well. *Temporal Relationalism is thus necessarily also a constraint provider.*

For Temporally Relational Particle Mechanics, using $\boldsymbol{N} \ := \ \boldsymbol{M}^{-1}$, the primary constraint arises as follows [52, 74, 104, 116].

$$||\boldsymbol{P}||_{\boldsymbol{N}} \ = \ \left|\left|\frac{\sqrt{2\,W}}{\mathrm{d}s}\,\boldsymbol{M}\cdot\mathrm{d}\boldsymbol{Q}\right|\right|_{\boldsymbol{N}} \ = \ \frac{\sqrt{2\,W}}{\mathrm{d}s}||\mathrm{d}\boldsymbol{Q}||_{\boldsymbol{MNM}} \ = \ \frac{\sqrt{2\,W}}{\mathrm{d}s}||\mathrm{d}\boldsymbol{Q}||_{\boldsymbol{M}} \ = \ \frac{\sqrt{2\,W}}{\mathrm{d}s}\mathrm{d}s \ = \ \sqrt{2\,W} \ . \tag{62}$$

I.e.

$$\varepsilon \ := \ \tfrac{1}{2}||\boldsymbol{P}||_{\boldsymbol{N}}{}^2 \ + \ V(\boldsymbol{Q}) \ = \ \tfrac{1}{2}\,N^{AB}P_AP_B \ + \ V(\boldsymbol{Q}) \ = \ E \ . \tag{63}$$

Remark 2 The above derivation can be envisaged as a 'Pythagorean' or 'direction-cosines' working. By this, the MRI L's quadraticness in its parametrized velocities induces ε's quadraticness in its momenta.

Example 1 Temporally Relational and yet Spatially Absolute Mechanics is of the above form

Example 2 By a parallel – if now indefinite-metric – working, minisuperspace's Hamiltonian constraint is also of the above form (with $R \ - \ 2\Lambda$ in place of $E \ - \ V$).

4.5 Equation of time, alias Chronos, interpretation

A natural question to is whether there is a paradox between the Leibnizian Time(lessness) Principle, and our appearing to 'experience time', which moreover features in many Laws of Physics that appear to apply in the Universe. Two discrepancies between these contexts are however as follows. Firstly, everyday experience concerns subsystems rather than the whole-Universe setting of this Leibnizian Principle. Secondly, whereas 'time' is a useful concept for everyday experience, the nature of 'time' itself is in general less clear.

Mach's Time Principle is that [7] "*It is utterly beyond our power to measure the changes of things by time. Quite the contrary, time is an abstraction at which we arrive through the changes of things.*" I.e. '*time is to be abstracted from change*'.

Remark 1 Indeed, it is change that we directly experience, and temporal notions are merely an abstraction from that, albeit a very practically useful abstraction if carefully chosen. One idea then is that primary-level Leibnizian Timelessness is *resolved* by Mach's Time Principle providing a *secondary*, i.e. *emergent* notion of time.[20]

Remark 2 ε, $\mathcal{H}_{\mathrm{mini}}$ (and full GR's $\mathcal{H}$) usually receive energy equation interpretations. In the present context of Temporal Relationalism at the primary level for the universe as a whole, however, the conjugate interpretation as an 'equation of time' [22, 74, 116] is more enlightening. We thus collectively denote these constraints by $\mathcal{C}$hronos. Indeed, e.g. ε rearranges to give the expressions in Fig (7).

Remark 3 In this way, by understanding the Frozen Formalism Problem's origin, a natural resolution becomes apparent. The constant $t^{\mathrm{em}}(0)$ above encodes the freedom of choice of 'calendar year zero'; it is not too hard to exhibit the freedom in choice of 'tick-length' as well [116]. Schematically, the above is a formula for an emergent timefunction as an explicit *functional* of change.

$$t^{\mathrm{em}} \ = \ \mathcal{F}[\boldsymbol{Q}, \mathrm{d}\boldsymbol{Q}] \ = \ \mathcal{F}[\boldsymbol{Q}, \pounds_{\mathrm{d}}\boldsymbol{Q}] \ . \tag{64}$$

Following Mach, this is ab initio a highly dependent variable rather than the independent variable that time is usually taken to be. This is because while the 'usual' situation assumes that one knows beforehand what the notion

[20] See in particular [69, 70, 116] for discussion of alternative strategies for handling Temporal Relationalism. This covers, for instance, primary-level time (e.g. hidden or from appended matter), adhering to timelessness, or supplanting time by a notion of history.

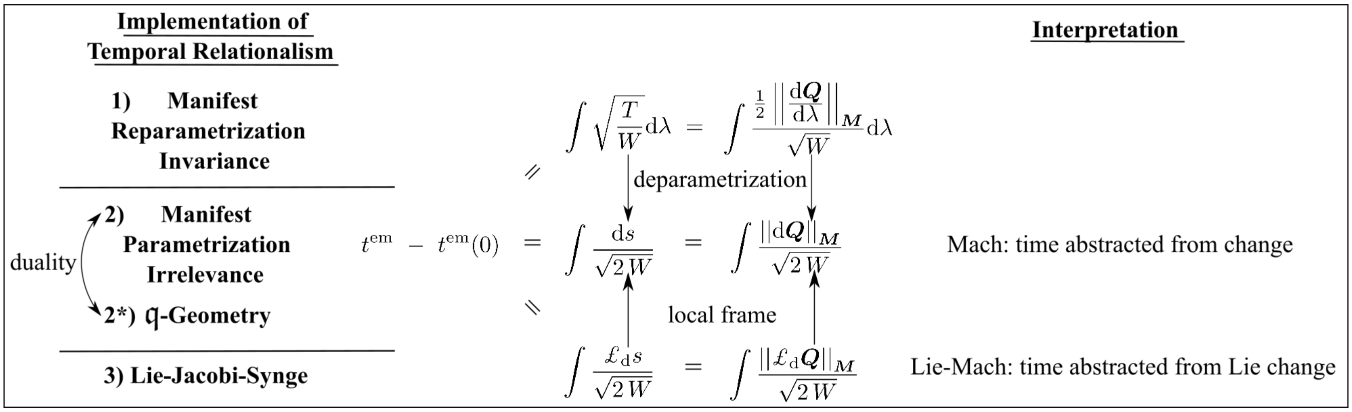

Figure 7: Each Temporal Relationalism implementation's recovery of emergent time.

of time to use, the current position instead involves operationally establishing this notion. Finally, by form 3 in Fig 7, classical Machian emergent time itself possesses a Lie derivative characterization: '*time is to be abstracted from Lie-derivative-change of configuration*'.

Remark 4 (64) moreover also implements the further principle [44] of 'choose time so that motion is simplest'. For, via

$$* \ := \ \frac{\partial}{\partial t^{\mathrm{em}}} \ := \ \sqrt{\frac{W}{T}}\frac{\partial}{\partial \lambda} \ , \tag{65}$$

it is also distinguished by its simplification of the momentum-velocity relations and equations of motion from (56) and (59).

$$\underline{\boldsymbol{P}} \ = \ \underline{\underline{\boldsymbol{M}}} \cdot *\underline{\boldsymbol{Q}} \ , \tag{66}$$

$$**\underline{\boldsymbol{Q}} \ + \ *\underline{\boldsymbol{Q}} \cdot \underline{\underline{\underline{\boldsymbol{\Gamma}}}} \cdot *\underline{\boldsymbol{Q}} \ = \ -\underline{\underline{\boldsymbol{N}}} \cdot \frac{\partial V}{\partial \underline{\boldsymbol{Q}}} \ . \tag{67}$$

One can finally posit the Euler–Lagrange principle, in terms of t^{em} as an action principle encoding these simplest-form equations.

Remark 5 The quantum-level frozenness of the Wheeler–DeWitt equation $\widehat{\mathcal{H}}\,\Psi \ = \ 0$ and its simpler mechanical analogue $\widehat{\mathcal{E}}\,\Psi \ = \ E\,\Psi$ can then be traced back to their classical counterparts being being quadratic in the momenta without linear pieces. These follow as the primary constraint encoded by whichever of forms 1, 2 or 3 for our action, which in turn originate with Leibniz's a priori timelessness. Thus the Frozen Formalism Problem [34, 69, 70] is identified with Temporal Relationalism [106, 116]; this moreover points to an emergent Machian time resolution by which time is abstracted from change.

Remark 6 We can furthermore split $\boldsymbol{Q}$ into heavy slow $\boldsymbol{h}$ and light fast $\boldsymbol{l}$ parts, leading to an expansion of t^{em} with $\boldsymbol{h}$ part as leading term. This is a significant move to make as regards making contact with classical and quantum cosmological modelling [116]. [107, 116] argue that the situation in hand is best described as 'generalized local ephemeris time (GLET) being abstracted from a 'sufficient totality of locally relevant change' (STLRC). This is to be contrasted with the simpler and yet less verisimilitudinous possibilities of 'any change' or 'all change' being involved. The realities of timekeeping are such that all changes have *an opportunity* to contribute, yet some changes are negligible (and are better omitted, especially when they come with large error bars). 'Ephemeris time' is itself a feature of astronomical timekeeping [22]. While time has since come to be read off atomic clocks, these are still to be interpreted as clock hands that are in regular need of calibration checks against Solar System observations.

Remark 7 Whereas such an ephemeris time has long been in use, its Machian character has only relatively recently been remarked upon [74, 107, 116].

Remark 8 This emergent time is *provided by* the system, in which way it complies with Mach's Time Principle. In contrast, the notion of time usually assumed as an independent variable is absolute. Yet once the above emergent time has been abstracted from change, it *is* a convenient choice for (emergent) independent variable.

Remark 9 Classical emergent time will moreover not do at the quantum level, for the furtherly Machian reason that now quantum change has to be given the opportunity to contribute.

4.6 TRiPoD

Structure 1 Temporal Relationalism can moreover be applied at all levels of the Principles of Dynamics (and various sequels: foliations, canonical quantization, path integrals...) [104, 109, 113, 116]. We refer to these new formulations *TRi*, standing for *Temporal Relationalism implementing*. Using TRi versions is essential as regards avoiding 'facet interferences' as one gradually moves away from formulations in terms of actions. Using these modifications does not however alter *the Lie content* of the formulation, saving us from having to present them here. TRi is just a recategorization that does not alter physical content or Lie-Theoretic methodology. See [116] if interested in seeing how these wipe out all (Frozen Formalism, arbitrary) facet interferences for our Lie Background Independence program.

Remark 1 It turns out that combining this resolution with those of all the other facets requires reworking around half of the Principles of Dynamics [17, 51] into Temporally Relational implementing (TRi) form. This can be thought of as 'taking Jacobi's Principle more seriously than Jacobi himself did', and thus reworking the rest of Principles of Dynamics to follow from this in place of Euler–Lagrange's.

Remark 2 We will generally not continue to explain the development of the four implementations of Temporal Relationalism, since passing between these does not alter the Lie mathematical content of the classical ALRoPoT. A few key formulae are presented in Lie derivative explicit form, since these have not previously featured in the literature.

5 (Temporal Relationalism, Configurational Relationalism)

At the level of the action, the Lagrange multiplier formulation of the auxiliaries used so far to encode Configurational Relationalism breaks Temporal Relationalism's implementation. The TRi resolution of this is to use cyclic velocities (MRI) or cyclic differentials (MPI), in each case with free end point value variation.[21]

Using our most advanced forms 2*) and 3), our changes or Lie changes are in general to be corrected to

$$\mathrm{d}\boldsymbol{Q} \;-\; \pounds_{\mathrm{d}\boldsymbol{g}}\boldsymbol{Q} \;=\; (\pounds_{\mathrm{d}} \;-\; \pounds_{\mathrm{d}\boldsymbol{g}})\boldsymbol{Q} \tag{68}$$

in place of (46). This is built up into the kinetic term T by a first use of $Maps$, then multiplied by $\sqrt{2\,W}$ and integrated to form a Jacobi-type action in a second use of $Maps$. Finally, this action is extremized over the group's auxiliaries g, so G-Act G-All is implemented. This encodes linear G-constraints as secondary constraints. The action also being MPI, its $\mathfrak{q}$-geometry dual, or the Lie realization thereof, Temporal Relationalism is implemented and a quadratic Chronos constraint is guaranteed. Thus Configurational and Temporal Relationalism are jointly implemented.

5.1 Example 1) Euclidean Relational Particle Mechanics

This can be formulated [104] to jointly obey TR-a), TR-b) and CR-b). Aside from Sec 3.4's Configurational-Relationalism-only version of this, from which $\mathfrak{q}(N,d) \;=\; \mathbb{R}^{Nd}$ and $G \;=\; Eucl(d)$ can be read off, [52] gave an oppositely-ordered 'Temporal Relationalism first with Spatial Relationalism then breaking the Temporal Relationalism' antecedent. We have a theory in which just relative separations and relative angles are meaningful.

Now the corrected change are

$$\mathrm{d}\underline{\boldsymbol{q}} \;\longrightarrow\; \mathrm{d}\underline{\boldsymbol{q}} \;-\; \mathrm{d}\underline{a} \;-\; \mathrm{d}\underline{b} \,\times\, \underline{\boldsymbol{q}}\;, \quad \text{or} \quad \pounds_{\mathrm{d}}\underline{\boldsymbol{q}} \;\longrightarrow\; \pounds_{\mathrm{d}}\underline{\boldsymbol{q}} \;-\; \pounds_{\mathrm{d}}\underline{a} \;-\; \pounds_{\mathrm{d}}\underline{b} \,\times\, \underline{\boldsymbol{q}}\;, \tag{69}$$

the kinetic arc element is

$$\mathrm{d}s_{\mathrm{RPM}} \;=\; ||\,\mathrm{d}\underline{\boldsymbol{q}} \;-\; \mathrm{d}\underline{a} \;-\; \mathrm{d}\underline{b} \,\times\, \underline{\boldsymbol{q}}\,|| \;=\; ||\,\pounds_{\mathrm{d}}\underline{\boldsymbol{q}} \;-\; \pounds_{\mathrm{d}}\underline{a} \;-\; \pounds_{\mathrm{d}}\underline{b} \,\times\, \underline{\boldsymbol{q}}\,|| \;=\; \pounds_{\mathrm{d}}s_{\mathrm{RPM}}\;, \tag{70}$$

and the (Lie–)Jacobi-type action principle is

$$\mathcal{S}_{\mathrm{RPM}} \;=\; \sqrt{2}\int\sqrt{W}\mathrm{d}s_{\mathrm{RPM}} \;=\; \sqrt{2}\int\sqrt{W}\pounds_{\mathrm{d}}s_{\mathrm{RPM}} \tag{71}$$

for potentials of the form [104]

$$V(\boldsymbol{q}) \;=\; V\left(\underline{q}^I \cdot \underline{q}^J \text{ alone}\right)\;. \tag{72}$$

[21] Since the auxiliaries are physically meaningless everywhere, their values are in particular free at the end point (or more generarally end notion of space) of variation [96, 104, 116]. Thus such free, alias natural [20] variation is the appropriate kind. This turns the cyclic equation's constant into a zero just like the multiplier equation's.

Remark 1 This model is motivated in particular by the elsewise more commonly used minisuperspace model failing to exhibit Configurational Relationalism (and thus also interplay between this and other Background Independence aspects). For GR, having nontrivial Configurational Relationalism involves inhomogeneity (which renders the modelling much harder). The two cannot be isolated as both stem from spatial covariant derivatives $\mathbf{D}$ being nontrivial. On the other hand, for RPMs having Configurationl Relationalism and its linear constraints and having inhomogeneities are *independent features*, permitting isolation of Configurational Relationalism from the complicating effects of inhomogeneity. More advanced works could use e.g. perturbatively inhomogenous GR (e.g. [116] and references therein).

Remark 2 CR-a) fails since the usual absolutey-interpreted flat Euclidean metric $\underline{\underline{\delta}}$ clearly features in the construction of the kinetic term.

Remark 3 Our action works as follows. The momenta conjugate to the $\boldsymbol{q}$ are

$$\boldsymbol{p} \;=\; \frac{\sqrt{2\,W}}{\mathrm{d}s}\, m_I \delta_{IJ} \left(\mathrm{d}\boldsymbol{q} \;-\; \mathrm{d}\underline{a} \;-\; \mathrm{d}\underline{b} \,\times\, \boldsymbol{q} \right) . \tag{73}$$

Due to Temporal Relationalism, these obey a primary constraint that is purely quadratic in the momenta ($\boldsymbol{n} := \boldsymbol{m}^{-1}$),

$$\varepsilon \;:=\; \tfrac{1}{2}||\boldsymbol{p}||_{\boldsymbol{n}}{}^2 \;+\; V(\boldsymbol{q}) \;=\; E\;. \tag{74}$$

Due to Configurational Relationalism, constraints homogeneous-linear in the momenta $\underline{\mathcal{P}}$ and $\underline{\mathcal{L}}$ as per (50, 51) also hold.

Remark 4 Finishing off the Best Matching procedure, the constraints (50, 51), rewritten in Lagrangian configuration–velocity variables are to be solved for the auxiliary variables $\mathrm{d}\underline{a}$ and $\mathrm{d}\underline{b}$ themselves. This solution is then substituted back into the action, so as to produce a final $Tr(d)$- and $Rot(d)$-independent expression that *directly* implements Configurational Relationalism. One has the good fortune of being able to solve Best Matching explicitly for a wide range of RPMs. Firstly, eliminating translations is very straightforward; $\underline{\mathcal{P}}$ can be interpreted as passing to, or factoring out, centre of mass motion. All the tangible physics resides in the remaining relative vectors between particles, which are most conveniently treated in Jacobi coordinates [64] (a diagonalized presentation). Secondly, in 1-d one is done, while in 2-d the rotations are straightforward to eliminate as well [95, 104]. The ensuing reduced actions moreover coincide with the direct formulation's 1- and 2-d relational actions [79, 95, 104].

Remark 5 Explicit such actions can be found in [104] alongside their coincidence with those arrived at by the direct approach; see e.g. [104, 114, 116] for RPMs with respect to further G.

Remark 6 The emergent time clearly exhibits (Temporal Relationalism, Configurational Relationalism) interaction:

$$t^{\mathrm{em}}_{\mathrm{Ri}} \;:=\; t^{\mathrm{em}}_{G\text{-free}} \;:=\; \mathrm{E}^{(2)}_{\boldsymbol{g}\,\in\,G} \int \frac{||\mathrm{d}_{\boldsymbol{g}}\boldsymbol{Q}||_{\boldsymbol{M}}}{\sqrt{2\,W(\boldsymbol{Q})}}\;. \tag{75}$$

The extremization here takes an implicit form [the meaning of the (2) index: it is not of t^{em} itself, but of a second functional, $\mathcal{S}$ [116]].

5.2 Example 2: GR

The ADM action (3) fails TR-a) since the lapse, signifying 'time elapsed', is an extraneous timelike variable. The Baierlein–Sharp–Wheeler (BSW) [26] action (with Λ included), however,

$$\mathcal{S}^{\mathrm{BSW}}_{\mathrm{GR}} \;=\; \int \mathrm{d}\lambda \int_{\boldsymbol{\Sigma}} \sqrt{\mathcal{T}^{\mathrm{BSW}}_{\mathrm{GR}}\;\overline{\mathcal{R} \;-\; 2\,\Lambda}}\;\mathrm{d}^3x\;, \tag{76}$$

using overline to denote desitization, and with kinetic term

$$\mathcal{T}^{\mathrm{GR}}_{\mathrm{BSW}} \;:=\; ||\,(\,\dot{}\; - \;\pounds_{\underline{\upbeta}}\,)\,\mathbf{h}||_{\mathbf{M}}{}^2\;. \tag{77}$$

This succeeds in being lapse-free, but fails to be MRI because its shift corrections spoil this property of the uncorrected velocities. So, while the BSW action is the one that the Thin Sandwich originally referred to, we use rather our conceptually preferred (Lie–)Jacobi $\mathfrak{q}$-geometry implementation of TRi GR action is

$$\mathcal{S} \;=\; \int\int_{\boldsymbol{\Sigma}} \sqrt{\overline{\overline{\mathcal{R} \;-\; 2\Lambda}}}\mathrm{d}s\,\mathrm{d}^3x\;, \quad \mathrm{d}s \;:=\; ||\mathrm{d} \;-\; \pounds_{\mathrm{dF}}||_{\mathrm{M}}{}^2 \tag{78}$$

Remark 1 The Misner action can then be viewed as the minisuperspace truncation of either of (76, 78).]

GR's relational momenta are

$$\mathrm{p}^{ij} \;=\; \frac{\sqrt{\mathcal{R} \,-\, 2\,\Lambda}}{\mathrm{d}s}\mathrm{M}^{ijkl}(\,\mathrm{dh}_{ij} \;-\; \pounds_{\mathrm{d}\underline{\mathrm{F}}}\mathrm{h}_{kl}\,) \;=\; \frac{\sqrt{\mathcal{R} \,-\, 2\,\Lambda}}{\pounds_{\mathrm{d}}s}\mathrm{M}^{ijkl}(\,\pounds_{\mathrm{d}} \;-\; \pounds_{\mathrm{d}\underline{\mathrm{F}}}\,)\mathrm{h}_{kl}\;. \tag{79}$$

The relational action's encodement of the momentum constraint $\underline{\mathcal{M}}$ is fairly standard. I.e. the free end spatial hypersurface value variation with respect to the frame variable $\underline{\mathrm{F}}$, which works out equivalently to varying the ADM or BSW actions with respect to the shift $\underline{\beta}$. In both cases this just encodes that $Diff(\boldsymbol{\Sigma})$-invariance is to be respected.

Full GR's $\mathcal{H}$ now arises as a primary constraint, by a working paralleling Sec 4's, and using overlines to denote densitization

$$||\mathbf{p}||_{\mathbf{N}} \;=\; \left|\left|\frac{\sqrt{\mathcal{R}-2\Lambda}}{\mathrm{d}s^{\mathrm{Rel}}_{\mathrm{GR}}}\underline{\underline{\underline{\overline{\mathbf{M}}}}}\cdot\mathrm{d}_{\underline{\mathrm{F}}}\underline{\mathbf{h}}\right|\right|_{\mathbf{N}} \;=\; \frac{\sqrt{\mathcal{R}-2\Lambda}}{\mathrm{d}s^{\mathrm{Rel}}_{\mathrm{GR}}}||\mathrm{d}_{\underline{\mathrm{F}}}\mathbf{h}||_{\mathbf{MNM}} \;=\; \frac{\sqrt{\mathcal{R}-2\Lambda}}{\mathrm{d}s^{\mathrm{Rel}}_{\mathrm{GR}}}||\mathrm{d}_{\underline{\mathrm{F}}}\mathbf{h}||_{\mathbf{M}} \;=\; \frac{\sqrt{\mathcal{R}-2\Lambda}}{\mathrm{d}s^{\mathrm{Rel}}_{\mathrm{GR}}}\mathrm{d}s^{\mathrm{Rel}}_{\mathrm{GR}} \;=\; \sqrt{\mathcal{R}-2\Lambda}\;. \tag{80}$$

In the traditional Thin Sandwich [29, 38, 72], one solves (the Lagrangian form of the GR momentum constraint $\underline{\mathcal{M}}$ with $\mathcal{H}$ used to eliminate its emergent lapse) = (the thin sandwich equation) with thin sandwich data

$$(\mathbf{h},\ \mathrm{d}\mathbf{h}) \tag{81}$$

for the shift variable $\underline{\beta}$, giving the *Thin Sandwich Problem.* In the TRi Thin Sandwich [116, 128], one solves (the Jacobi–Mach form of $\underline{\mathcal{M}}$ with $\mathcal{H}$ used to eliminate its emergent differential of the instant [116]) = (the TRi thin sandwich equation) with (Lie–)TRi thin sandwich data

$$(\mathbf{h},\ \mathrm{d}\mathbf{h}) \;=\; (\mathbf{h},\ \pounds_{\mathrm{d}}\mathbf{h}) \tag{82}$$

for the (Lie-)change of frame variable $\mathrm{d}\underline{\mathrm{F}}$ with components $\mathrm{d}\underline{\mathrm{F}}$, giving the *TRi Thin Sandwich Problem.* The TRi thin sandwich equation is

$$\mathcal{D}_j\left[\frac{\sqrt{\mathcal{R} \,-\, 2\,\Lambda}}{||\pounds_{\mathrm{d}-\mathrm{d}\underline{\mathrm{F}}}\mathbf{h}||_{\mathbf{M}}}(\mathrm{h}^{jk}\delta^{l}{}_{i} \;-\; \delta^{j}{}_{i}\mathrm{h}^{kl})(\delta_{\mathrm{d}-\mathrm{d}\underline{\mathrm{F}}}\mathrm{h}_{kl})\right] \;=\; 0\;, \tag{83}$$

or, as an explicit PDE,

$$\mathcal{D}_j\left[\sqrt{\frac{\mathcal{R} \,-\, 2\,\Lambda}{(\mathrm{h}^{ac}\mathrm{h}^{bd}-\mathrm{h}^{ab}\mathrm{h}^{cd})(\mathrm{dh}_{ab} \;-\; 2\,\mathcal{D}_{(a}\mathrm{dF}_{b)})(\mathrm{dh}_{cd} \;-\; 2\,\mathcal{D}_{(a}\mathrm{dF}_{b)})}}(\mathrm{h}^{jk}\delta^{l}{}_{i} \;-\; \delta^{j}{}_{i}\mathrm{h}^{kl})(\mathrm{dh}_{kl} \;-\; 2\mathcal{D}_{(k}\mathrm{dF}_{l)})\right] \;=\; 0\;. \tag{84}$$

[Making the Thin Sandwich TRi in no way alters its properties as a PDE [38, 72], so everything known about the traditional Thin Sandwich carries over].

The (TRi) thin sandwich does not end here, including also then a) solving for the corresponding lapse or differential of the instant. b) Then forming the reduced action, and expression for the extrinsic curvature giving a local coating of spacetime to the future of our thin sandwich data slice. In the TRI case specifically, the instant is moreover identified as the Machian emergent time (neat in that instants are then labelled by values of this time!) By this, step a) is furthermore re-envisaged as recovery of emergent time. Including this step interpreted in this way amounts to joint resolution of two Problem of Time facets at the classical level.

Remark 1 The title of BSW's paper, "*Three-dimensional geometry as carrier of information about time*", supports the above instant–time duality. Upon subsequently passing to the relational GR action, this can moreover be rephrased in the temporally Machian form 'geometry and change of geometry as carrier of information about time'. GR's spatial geometries are moreover but an example of $\mathfrak{q}$ geometry. This can thus be further generalized as regards range of theories, to '*Configuration and change of configuration as carrier of information about time*'.

Remark 2 See [128] for Best Matching in Electromagnetism, Yang–Mills Theory and for each of these coupled to GR.

Remark 3 Accommodating fermions requires a square root of a square plus linear term implementation of homogeneous-linear action. For this the Randers action subcase of the Jacobi–Synge action provides a simple model [116].

6 Conclusion

Implementing each of Configurational and Spacetime Relationalism is covered by group-invariant constructs, be these direct or indirect: using a G-Act, G-All move [104]. The group's generators involved (linear constraints in the canonical case) close under a corresponding brackets operation [136]. The canonical case involves three further steps that are less usual. Firstly, Best Matching [52, 104]: solving the Lagrangian form of the linear constraints for the group's auxiliaries, and substituting back into the action (a type of reduction). Secondly, Temporal Relationalism[52, 74, 116], which captures Leibniz's a priori timelessness, must by an argument of Dirac's, give rise to at least on primary constraint, so a second constraint provider enters. This gives the quadratic constraint we interpret as an equation of time, $\mathcal{C}$hronos; in GR, this is the Hamiltonian constraint $\mathcal{H}$. This explains why the root to nexus arrow is labelled 'provide' [116]: meaning generator provision, including constraint provision. Thirdly, $\mathcal{C}$hronos can be rearranged to produce a classical Machian emergent time [74, 116]: realizes Mach's 'time is to be abstracted from change'. When configurational Relationalism is also nontrivially present, this and Temporal Relationalism interact in providing this emergent time.

Once armed with the above account, that quantum GR's Wheeler–DeWitt equation (14) is frozen − exhibiting primary-level timelessness − is *expected*, rather than *surprising*.

Emergent Machian time amounts to a relational recovery of t^{Newton} for Mechanics, and of t^{proper} for GR (and of an approximation to t^{comic} in cosmological settings).

Let us also distinguish between:

A) little method: for a piecemeal facet.

B) A large method: suitable for combination to form A Local Resolution of the Problem of Time.

Example 1 For Temporal Relationalism, the little method is to use a Jacobi(–Synge) geometrical action.

In contrast, our large method is to

$$\textit{take the Jacobi(–Synge) action principle and rederive the rest of Physics concordantly} \ . \tag{85}$$

Doing this [116, 127, 128, 129, 130, 131] carries a guarantee of *remaining within* Temporal Relationalism as one successively deals with each further local facet. It thus prevents the Frozen Formalism Problem inadvertently re-entering while one is subsequently addressing further facets.

Example 2 For Configurational Relationalism, the little method is Best Matching and the large method is the general G-Act G-All method.

Example 3 For Spacetime Relationalism, the little method is use of invariant actions, whereas the large method is another use of the G-Act, G-All method.

G automatically closing by virtue of being a group, is no longer sufficient in the canonical case once we have a further constraint provider in Temporal Relationalism. Article [136] explains how to analyze this situation, both in general and for the current article's examples of Relational Particle Mechanics and GR.

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