BACKGROUND READING FOR

TOPOLOGICAL SPACES §0 Collections of Subsets

Sets containing sets can cause confusion, so we use "collections of sets" nomenclature.

Collections of subsets : topological spaces are a particular subcase of this, and there are others (partitions, covers, filters and σ-algebras are other fairly basic examples).
Collections of subsets are, however, simpler than any of these through having less restrictions.
It is also important not to confuse ϕ and $\{\phi\}$: the collection containing ϕ.
the empty collection

$\#$(subsets of a set of n elements) = (size of power set of n elements) $= 2^n$

$\#$(collections of subsets of a set of n elements) = (size of power set of 2^n elements) $= 2^{2^n}$.

However, almost everything else has no known closed combinatorical formula.

Another useful notion are the inequivalent collections $=$ Collect$(n)/S(n)$ Permutation group.

# elements n	# subsets	# collections	# inequivalent subsets	# inequivalent collections
0	1	2	1	2
1	2	4	2	4
2	4	16	3	12
3	8	256	4	80

$\#$(inequivalent subsets) $= \#$(of elements) $+ 1$ to allow for ϕ as well.

$\#$(inequivalent collections) is more involved.

For $n = 0$, $\{\phi\}$ or ϕ are all the options, and are all distinct

For $n = 1$, $\{\phi, X\}$, $\{X\}$, $\{\phi\}$, ϕ

the whole set

For $n = 2$ use inclusions versus exclusions to simplify the count (also ordering the subsets $\phi, a, \ldots, X$.

# elements	0	1	2	3	4
	ϕ	$\{\phi\}$ a 2 X	$\{\phi, a\}$ 2 $\{\phi, X\}$ $\{a, b\}$ $\{a, X\}$ 2		

Same counts as other ∇ by inclusions $\leftrightarrow$ exclusions

count as $2\times(3\times1 + 1\times2) + 2\times1 + 2\times2 = 16$ collections (check ✓)

the two ∇'s the middle column

$2\times(3 + 1) + 2 + 2 = 12$ inequivalent collections

For $n = 3$

# elements	0	1	2	3	4
	ϕ	$\{\phi\}$ a 3 ab 3 X	$\{\phi, a\}$ 3 $\{\phi, ab\}$ 3 $\{\phi, X\}$ $\{a, b\}$ 3 $\{a, ab\}$ 6 $\{a, bc\}$ 3 $\{a, X\}$ 3 $\{ab, bc\}$ 3 $\{ab, X\}$ 3	$\{\phi, \boxed{\ }\}$ $\{a, b, c\}$ $\{a, b, ab\}$ 3 $\{a, b, bc\}$ 6 $\{a, b, X\}$ 3 $\{a, ab, ac\}$ 3 $\{a, ab, bc\}$ 6 $\{a, ab, X\}$ 6 $\{a, bc, X\}$ 3 $\{ab, bc, ca\}$ $\{ab, bc, X\}$ 3	$\{\phi, \boxed{\ }\}$ $\{a, b, c, ab\}$ 3 $\{a, b, c, X\}$ $\{a, b, ab, bc\}$ 6 $\{a, b, ac, cb\}$ 3 $\{a, b, ab, X\}$ 3 $\{a, b, bc, X\}$ 3 $\{a, ab, ac, bc\}$ 3 $\{a, ab, ac, X\}$ 3 $\{a, ab, bc, X\}$ 6 $\{ab, bc, ca, X\}$

count as
$2\times(6\times1 + 19\times3 + 5\times6) + 4\times1 + 10\times3 + 6\times6 = 186 + 70 = 256$ collections (check ✓)
$2\times(1 + 19 + 5) + 4 + 10 + 6 = 60 + 20 = 80$.

<u>Notation</u>: $\{X, \{\cdot\}, \{\cdot\}, \{\cdot\ \cdot\}\}$ is cumbersome and does not encode vs . so use for $\{X, \phi, \{\cdot\}, \{\cdot, \cdot\}$ with $\{\}$ nested in $\}; \cdot\}$.

One simple type of collection is the <u>partition</u> : i) exhaustive, ii) non-overlapping, iii) no empty piece.

$\bigcup P_i = X$ $\qquad P_i \cap P_j = \phi \ \forall i,j$ $\qquad P_i \neq \phi \ \forall i$

# elements	# partitions	# indistinguishable partitions
0	0	0
1	1	1
2	2	2
3	5	3
4	18	5

by iii)

{a}

{ab} {a|b}

{abc} {a|bc} 3 {a|b|c}

{abcd} {a|bcd} 4 {ab|cd} $\binom{4}{2}=6$

{a|b|cd} $\binom{4}{2}=6$ {a|b|c|d}

doesn't have a closed form (IA Discrete)

$\overrightarrow{\text{recheck}}$, & name it.

(we later point to these as the top spaces which are not connected).

Lattices of subsets (ordered by subset inclusion)

Bottom is empty set $\emptyset$, top is whole set X.

distinguishable elements | indistinguishable elements

0 elements

$\emptyset$ $\qquad\qquad$ $\emptyset$

1 element

$$\begin{tikzcd} X=\{a\} \\ \emptyset \arrow[u] \end{tikzcd} \qquad\qquad \begin{tikzcd} X \\ \emptyset \arrow[u] \end{tikzcd}$$

2 elements

$$\begin{tikzcd} & X=\{a,b\} & \\ \{a\} \arrow[ur] & & \{b\} \arrow[ul] \\ & \emptyset \arrow[ul] \arrow[ur] & \end{tikzcd} \qquad\qquad \begin{tikzcd} X \\ \{\cdot\} \arrow[u] \\ \emptyset \arrow[u] \end{tikzcd}$$

singleton

3 elements

$$\begin{tikzcd} & X=\{a,b,c\} & \\ \{a,b\} \arrow[ur] & \{a,c\} \arrow[u] & \{b,c\} \arrow[ul] \\ \{a\} \arrow[u] \arrow[ur] & \{b\} \arrow[ul] \arrow[ur] & \{c\} \arrow[ul] \arrow[u] \\ & \emptyset \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd} \qquad\qquad \begin{tikzcd} X \\ \{\cdot,\cdot\} \arrow[u] \\ \{\cdot\} \arrow[u] \\ \emptyset \arrow[u] \end{tikzcd}$$

are totally ordered by # of elements.

Note: these are all symmetric about the middle.

They are of Pascal triangle form:

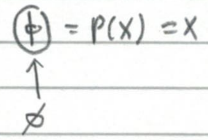
$$\begin{aligned} 2^n = {} & 1 & = {} & \binom{n}{0} \\ & + n & & + \binom{n}{1} \\ & + & & + \binom{n}{2} \\ & \vdots & & \vdots \\ & + n & & + \binom{n}{n-1} = \binom{n}{1} \\ & + 1 & & + \binom{n}{n} = \binom{n}{0} \end{aligned}$$

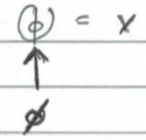
Lattice of collections of subsets (ordered by collection inclusion)

The bottom element is now the empty collection, whereas the top element is the power set.

distinguishable elements | indistinguishable elements

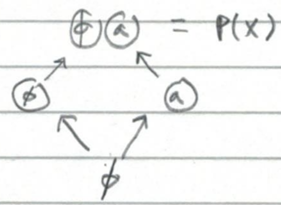
0 elements

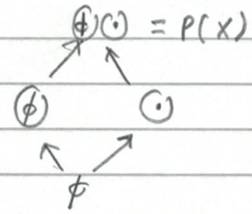




are the same on each row

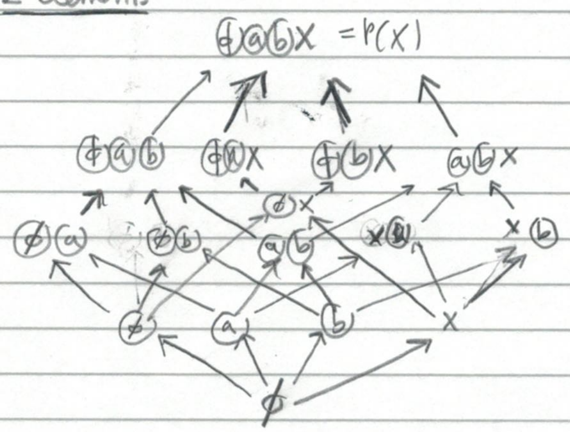
1 element

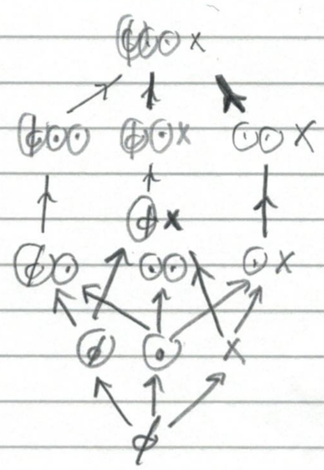




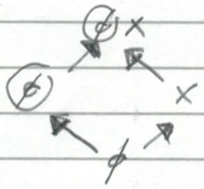
labelling is trivial with 0 or 1 els!

2 elements

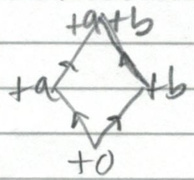




View as



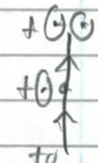
with the same



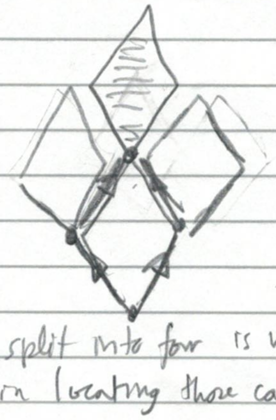
mounted on each

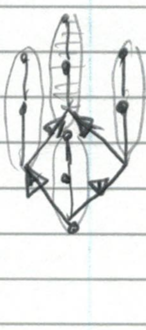
|| ||

with the same chain



mounted on each:





This split into four is useful for >2 elements as well, and in locating those collections which are topologies.