4

Poincaré Algebra

In this course, we use the convention that $\eta^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$.

Coordinate systems of 2 arbitrary SR inertial frames are related by an inhomogeneous linear transformation: $x^\mu \longrightarrow \underbrace{\Lambda^\mu{}_\nu}_{\text{Lorentz group: antisymmetric matrix}} x^\nu + \underbrace{a^\mu}_{\text{spacetime translations.}}$

$\sim$ a Poincaré Transformation.

We restrict to $\Lambda^0{}_0 \geq 1$ to preserve time direction
and $\det \Lambda = 1$ spatial orientation (parity) } proper orthochronous Lorentz transformations.

Let's lay out the options for this carefully: five Lorentz groups.

$T: (x^0, x^i) \longrightarrow (-x^0, x^i)$ gives time reversal

both are commuting involutions.

$P: (x^0, x^i) \longrightarrow (x^0, -x^i)$ gives the parity operation.

Let us denote P, T, PT, (P,T) the $\mathbb{Z}_2$, $\mathbb{Z}_2$, $\mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ groups generated by the given quantities.

Then, as well as $O(3,1)$, 4 connected cpts

$SO_0(3,1)$ is connected component of id in $O(3,1)$

$SO(3,1)$ is special Lorentz group: the subgroup of unit determinant 2 connected cpts

$O^\uparrow(3,1)$ is the orthochronous Lorentz group preserving time orientation. 2 connected cpts

$SO^+(3,1)$ as the subgroup preserving space-orientation 2 connected cpts

So

$O(3,1)/SO_0(3,1) = \mathbb{Z}_2 \times \mathbb{Z}_2$ corr to $O(3,1) = (P,T) \ltimes SO_0(3,1)$

$O(3,1)/SO(3,1) = \mathbb{Z}_2$ $\quad O(3,1) = PT \ltimes SO(3,1)$

$O(3,1)/O^\uparrow(3,1) = \mathbb{Z}_2$ $\quad O(3,1) = T \ltimes O^\uparrow(3,1)$

$O(3,1)/SO^+(3,1) = \mathbb{Z}_2$ $\quad O(3,1) = P \ltimes SO^+(3,1)$

N.B. $SO(3,1)$ itself has 2 cpts, so is <u>not connected</u>

Exercise PT alone is an invariant subgroup of $O(3,1)$.

$PT \in Z(O(3,1))$.

→ Is it all of the centre?

Exercise Give short exact sequence interpretations of the five Lorentz groups.

$$1 \longrightarrow (P,T) \xrightarrow{i} O(3,1) \xrightarrow{\pi} SO_0(3,1) \rightarrow 1$$

$$1 \longrightarrow PT \xrightarrow{i} O(3,1) \xrightarrow{\pi} SO(3,1) \rightarrow 1$$

$$1 \longrightarrow T \xrightarrow{i} O(3,1) \xrightarrow{\pi} O^{\uparrow}(3,1) \rightarrow 1$$

$$1 \longrightarrow P \xrightarrow{i} O(3,1) \xrightarrow{\pi} SO^{+}(3,1) \rightarrow 1$$

$$1 \xrightarrow{\text{triv}} 1 \xrightarrow{i} O(3,1) \xrightarrow{\pi\ \text{triv}} O(3,1) \rightarrow 1$$

Exercise Give principal bundles interpretations of the five Lorentz groups.

$$\underbrace{O(3,1)}_{\text{principal bundle}} = P(\underbrace{SO_0(3,1)}_{\text{base space}}, \underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2}_{\text{fibre = structure group}})$$

$SO^{+}(3,1) = P(SO_0(3,1), \mathbb{Z}_2)$

$O^{\uparrow}(3,1) = P(SO_0(3,1), \mathbb{Z}_2)$ → with a different significance in each case.

$SO(3,1) = P(SO_0(3,1), \mathbb{Z}_2)$

$SO_0(3,1) = P(SO_0(3,1), id) \leftrightarrow$ just the base space $SO_0(3,1)$.

The union of all the Poincaré transformations form a Lie group under

$(\Lambda_2, a_2) \times (\Lambda_1, a_1) = (\Lambda_2\Lambda_1,\ a_2 + \Lambda_2 a_1)$

the sign of a semi-direct product.

Infinitesimally, $\Lambda^{\mu}{}_{\nu}$ has generator $M^{\mu\nu}$
a^{μ} has generator P^{μ}

scalar function $\varphi(x) \to (e^{a^{\mu}\partial_{\mu}})\varphi$ translation
$\varphi(x) \to (e^{i\omega_{\mu\nu}M^{\mu\nu}})\varphi$ Lorentz transformation. $\omega_{\mu\nu} = -\omega_{\nu\mu}$ antisymmetric.

do this for Eucl Aff... $\longrightarrow$

Claim P^{μ} is represented by $i\partial^{\mu}$
and $M^{\mu\nu}$ $\qquad x^{\mu}p^{\nu} - x^{\nu}p^{\mu}$ ie $i\{x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu}\}$.

I.e. deduce infinitesimal generators by expanding out their transformations.

Derivation $e^{-i\alpha P}\phi(x) = \phi(x + a)$ translation

$\Rightarrow (1 - i\alpha p + \dots)\phi(x) = \phi(x) + \alpha\frac{\partial}{\partial x}\phi(x)$ (Taylor expansion).

$\Rightarrow$ equating coefs of α^1, $p^{\mu} \sim i\partial^{\mu}$.

$e^{-i\omega_{\mu\nu}M^{\mu\nu}}\phi(x) = \phi(\Lambda x)$ Lorentz transformation

$\Rightarrow (1 - \omega_{\mu\nu}M^{\mu\nu})\phi(x^{\lambda}) = \phi((\delta^{\lambda}{}_{\nu} + \omega^{\lambda}{}_{\nu})x^{\nu}) = \phi(x^{\lambda}) + x_{\nu}\omega^{\lambda\nu}\frac{\partial}{\partial x^{\lambda}}\phi(x^{\lambda})$.
Taylor

$\Rightarrow -i\omega^{\mu\nu}M_{\mu\nu} = \omega^{\mu\nu}x^{\nu}\frac{\partial}{\partial x^{\mu}}$ we cannot cancel off $\omega^{\mu\nu}$ as it is antisymmetric

But $\omega^{\mu\nu}M_{\mu\nu} = \omega^{\mu\nu}i(x_{\mu}\frac{\partial}{\partial x^{\nu}} - x_{\nu}\frac{\partial}{\partial x^{\mu}})$ Now can add $S^{\mu\nu}_{sym}$ to $\omega^{\mu\nu}$ to have an arbitrary tensor

$\Rightarrow M_{\mu\nu} = x_{\mu}P_{\nu} - x_{\nu}P_{\mu}$. □

Claim $p^\mu, M^{\mu\nu}$ obey the Poincaré Algebra

$[p^\mu, p^\nu] = 0$ so spacetime translations are an Abelian subalgebra.

$[M^{\mu\nu}, p^\rho] = i(\eta^{\nu\rho}p^\mu - \eta^{\mu\rho}p^\nu)$ so spacetime translations are a spacetime vector.

$[M^{\mu\nu}, M^{\rho\sigma}] = i(\eta^{\nu\rho}M^{\mu\sigma} + \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho})$ the Lorentz Algebra.

Derivation

$$[p^\mu, p^\nu] = i^2(\cancel{\partial^\mu\partial^\nu - \partial^\nu\partial^\mu}) \overset{\text{mpe}}{=} 0$$

$$[M^{\mu\nu}, p^\rho] = (x^\mu p^\nu - x^\nu p^\mu)p^\rho - p^\rho(x^\mu p^\nu - x^\nu p^\mu)$$

product rule and mpe

$$= i^2(\cancel{x^\mu\partial^\nu\partial^\rho} - \cancel{x^\nu\partial^\mu\partial^\rho} - \cancel{x^\mu\partial^\rho\partial^\nu} - \eta^{\rho\mu}\partial^\nu + \cancel{x^\nu\partial^\rho\partial^\mu} + \eta^{\rho\nu}\partial^\mu)$$

$$= i(\eta^{\nu\rho}p^\mu - \eta^{\mu\rho}p^\nu).$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = (x^\mu p^\nu - x^\nu p^\mu)(x^\rho p^\sigma - x^\sigma p^\rho) - (x^\rho p^\sigma - x^\sigma p^\rho)(x^\mu p^\nu - x^\nu p^\mu)$$

$$= i^2\begin{pmatrix} \cancel{x^\mu x^\rho\partial^\nu\partial^\sigma} + x^\mu\eta^{\nu\rho}\partial^\sigma - \cancel{x^\mu x^\sigma\partial^\nu\partial^\rho} - x^\mu\eta^{\nu\sigma}\partial^\rho \\ -\cancel{x^\nu x^\rho\partial^\mu\partial^\sigma} - x^\nu\eta^{\mu\rho}\partial^\sigma + \cancel{x^\nu x^\sigma\partial^\mu\partial^\rho} + x^\nu\eta^{\mu\sigma}\partial^\rho \\ -\cancel{x^\rho x^\mu\partial^\sigma\partial^\nu} - x^\rho\eta^{\sigma\mu}\partial^\nu + \cancel{x^\rho x^\nu\partial^\sigma\partial^\mu} + x^\rho\eta^{\sigma\nu}\partial^\mu \\ +\cancel{x^\sigma x^\mu\partial^\rho\partial^\nu} + x^\sigma\eta^{\rho\mu}\partial^\nu - \cancel{x^\sigma x^\nu\partial^\rho\partial^\mu} - x^\sigma\eta^{\rho\nu}\partial^\mu \end{pmatrix}$$

$$= i\begin{pmatrix} \eta^{\nu\rho}(x^\mu p^\sigma - x^\sigma p^\mu) + \eta^{\mu\sigma}(x^\rho p^\nu - x^\nu p^\rho) \\ -\eta^{\mu\rho}(x^\nu p^\sigma - x^\sigma p^\nu) - \eta^{\nu\sigma}(x^\mu p^\rho - x^\rho p^\mu) \end{pmatrix}$$

$$= i(\eta^{\nu\rho}M^{\mu\sigma} + \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho}). \quad \square$$

Split space-time form of Poincaré Algebra

Define $J_i = \frac{1}{2}\varepsilon_{ijk} M_{jk}$ spatial rotations $i,j,k = 1,2,3$

$K_i = M_{0i}$ Lorentz boosts.

$P_i = P_i$ spatial momenta.

$E = P_0$ temporal momentum = energy.

These obey

$[J_i, J_j] = i\varepsilon_{ijk} J_k$ spatial rotation subalgebra

$[K_i, K_j] = -i\varepsilon_{ijk} J_k$ Thomas Precession

$[J_i, K_j] = i\varepsilon_{ijk} K_k$.

$[P_i, P_j] = 0$

$[P_i, E] = 0$

$[E, E] = 0$

$[J_i, P_j] = i\varepsilon_{ij}{}^{k} P_k$

$[J_i, E] = 0$

$[K_i, P_j] = -iE$

$[K_i, E] = -iP_i$

Derivation

1) $[J_i, J_j] = -\frac{1}{4}\varepsilon_{ilm}\varepsilon_{jpq}[M_{pq}, M_{lm}]$

$= \frac{i}{4}\varepsilon_{ilm}\varepsilon_{jpq}(\delta_{ql}M_{pm} + \delta_{pm}M_{ql} - \delta_{pl}M_{qm} - \delta_{qm}M_{pl})$

$= \frac{i}{4}(\varepsilon_{ilm}\varepsilon_{jpl}M_{pm} + \varepsilon_{ilm}\varepsilon_{jmq}M_{ql} - \varepsilon_{ilm}\varepsilon_{jlq}M_{qm} - \varepsilon_{ilm}\varepsilon_{jpm}M_{pl})$

$= \frac{i}{4}2(\varepsilon_{mil}\varepsilon_{ljp} + \varepsilon_{pil}\varepsilon_{lmj})M_{pm}$ by M antisymmetric

$= \frac{i}{2}(\varepsilon_{mil}\varepsilon_{ljp} + \varepsilon_{jml}\varepsilon_{lip})M_{pm}$.

Jacobi identity: $0 = \mathfrak{J}[J_i, J_j, J_m] = \varepsilon_{ijl}[J_l, J_m] + \varepsilon_{mil}[J_l, J_j] + \varepsilon_{jml}[J_l, J_i]$

$= \{\varepsilon_{ijl}\varepsilon_{lmp} + \varepsilon_{mil}\varepsilon_{ljp} + \varepsilon_{jml}\varepsilon_{lip}\}J_p$

$\Rightarrow [J_i, J_j] = \frac{i}{2}\varepsilon_{ijl}\varepsilon_{lpm}M_{pm} = i\varepsilon_{ijk}\left(\frac{1}{2}\varepsilon_{kpm}M_{pm}\right) = i\varepsilon_{ijk}J_k$.

2) $[K_i, K_j] = [M_{0i}, M_{0j}] = i(\underbrace{\cancel{\eta_{i0}M_{0j}} + \cancel{\eta_{0j}M_{i0}}}_{\text{diag}} - \eta_{00}M_{ij} - \eta_{ij}\cancel{M_{00}}) = -iM_{ij}$

Trick: but $\varepsilon_{lpi}J_i = \frac{1}{2}\varepsilon_{lpi}\varepsilon_{ijk}M_{jk} = \frac{1}{2}(\delta_{lj}\delta_{pk} - \delta_{lk}\delta_{jp})M_{jk} \overset{M \text{ antisym}}{=} \frac{1}{2}(M_{lp} - M_{pl}) \overset{\text{antisym}}{=} M_{lp}$

$\Rightarrow [K_i, K_j] = -i\varepsilon_{ijk}J_k$.

3) $[J_i, K_j] = \frac{1}{2}\varepsilon_{ilk}[M_{lk}, M_{0j}] = \frac{1}{2}\varepsilon_{ilk}\, i\{\eta_{k0}\cancel{M_{lj}} + \overbrace{\eta_{lj}M_{k0} - \cancel{M_{l0}}M_{kj} - \eta_{kj}M_{l0}}^{\text{diagonal}}$

$= \frac{i}{2}\varepsilon_{ilk}(-\delta_{lj})M_{k0}\cdot 2$

$= i\varepsilon_{ijk}M_{0k} = i\varepsilon_{ijk}K$.

4)-6) The next three, as the split-up of an Abelian block, are trivially immediate.

7) $[J_i, P_j] = \frac{1}{2}\varepsilon_{ilk}[M^{lk}, P_j] = \frac{1}{2}\varepsilon_{ilk}(\eta^{l}_{j}p^k - \eta^{k}_{j}p^l)$

$= \frac{1}{2}\varepsilon_{ilk}\eta^{l}_{j}p^k - \frac{1}{2}\varepsilon_{ilk}\eta^{k}_{j}p^l$

$= \frac{1}{2}(\varepsilon_{ijk}p^k + \underbrace{-\varepsilon_{ikj}p^k}_{\varepsilon_{ijk}p^k}) = \varepsilon_{ijk}p^k.$

8) $[J_i, E] = [\frac{1}{2}\varepsilon_{ijk}M^{jk}, P^0] = \frac{1}{2}\varepsilon_{ijk}[M^{jk}, P^0] = \frac{1}{2}i(\underbrace{\cancel{\eta^{k0}p^j - \eta^{j0}p^k}}_{\eta \text{ is diagonal}}) = 0$

9) $[K_i, P_j] = [M_{0i}, P_j] = i(\eta_{ij}P_0 - \underbrace{\cancel{\eta_{j0}P_i}}_{\text{diagonal}}) = -iE.$

10) $[K_i, E] = [M_{0i}, P_0] = i(\underbrace{\cancel{\eta_{i0}P_0}}_{\text{diagonal}} - \eta_{00}P_i) = -iP_i.$ $\square$

4

A different choice of generators permits a direct product blockwise form:

$$A_i = (J_i + iK_i)/2$$
$$B_i = (J_i - iK_i)/2$$

Then
$$[A_i, A_j] = i\varepsilon_{ijk} A_k$$
$$[A_i, B_j] = 0$$
$$[B_i, B_j] = i\varepsilon_{ijk} B_k$$

Thus the Lorentz group $SO(3,1) \cong SU(2) \times SU(2)$

with states labelled by $(A, B) = (\text{spin } A, \text{spin } B)$

Derivation

$$[A_i, A_j] = \frac{1}{4}[J_i + iK_i, J_j + iK_j]$$

(linearity) $$= \frac{1}{4}\left([J_i, J_j] + i[J_i, K_j] + i[K_i, J_j] - [K_i, K_j]\right)$$
$$= \frac{1}{4}\left(i\varepsilon_{ijk} J_k + i^2\varepsilon_{ijk} K_k - i^2\varepsilon_{ijk} K_k + i\varepsilon_{ijk} J_k\right)$$
$$= i\varepsilon_{ijk}\frac{1}{2}(J_k + iK_k) = i\varepsilon_{ijk} A_k.$$

$$[A_i, B_j] = i\varepsilon_{ijk}(J_k - K_k + K_k - J_k) = 0$$

$$[B_i, B_j] = \frac{1}{4}[J_i - iK_i, J_j - iK_j]$$

(linearity) $$= \frac{1}{4}\left([J_i, J_j] - i[K_i, J_j] - i[J_i, K_j] + i^2[K_i, K_j]\right)$$
$$= \frac{1}{4}\left(i\varepsilon_{ijk} J_k - i^2\varepsilon_{ijk} K_k - i^2\varepsilon_{ijk} K_k - i^2\varepsilon_{ijk} J_k\right)$$
$$= i\varepsilon_{ijk}(J_k - iK_k) = i\varepsilon_{ijk} B_k. \qquad \square$$

Note if one did not know this would happen, one would probe with linear combinations
$$\lambda_i = J_i + \lambda K_i$$
$$\mu_i = J_i + \mu K_i$$

Then demand $[\lambda_i, \mu_j] = i\varepsilon_{ijk}\{J_k + (\lambda+\mu)K_k + \lambda\mu J_k\} = i\varepsilon_{ijk}\big((1-\lambda\mu)J_k + i(\lambda+\mu)\big)$

So we need $\lambda = -\mu$ and $\lambda\mu = 1 \Rightarrow \lambda^2 = -1$ so $\lambda = \pm i$ $\mu = \mp i$.

We then get the normalization factors from demanding the std scaling for $[A_i, A_j]$, $[B_i, B_j]$ to have the std $SU(2)$ commutation relations.

Remark This accidental relation means there's a proper invariant Lie sub algebra
$\Rightarrow SO(3,1)$ is not simple ; is it semi-simple ? (no Abelian subalgebra) yes.
OTOH, Poin(3,1) contains an abelian subalgebra too, so is not semisimple.

Casimir Operators for Lorentz, Poincaré group

Recall that a Casimir operator C of a Lie group G with generators $\{T^i \; i = 1 \text{ to } n\}$ is a (nontrivial) operator C such that $[C, T_i] = 0 \quad i = 1 \text{ to } n$.

So for the Poincaré group, Casimirs require $[C, M^{\mu\nu}] = 0 = [C, P^\mu]$.

Claim: The Casimir operators are $C_1 = P_\mu P^\mu$ readily amenable with mass

and $C_2 = W_\mu W^\mu$ (which we'll show to be related to the spin)

Where $W_\mu := \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} P^\nu M^{\rho\sigma}$ is the Pauli–Ljubánski vector

Derivation

- $[C_1, P^\mu] = [P_\lambda P^\lambda, P^\mu] \overset{\text{product bracket}}{=} P_\lambda [P^\lambda, P_\mu] + [P^\lambda, P^\mu] P_\lambda \overset{\text{P's are Abelian subalgebra}}{=} 0$

$$[C_1, M^{\mu\nu}] = [P_\lambda P^\lambda, M^{\mu\nu}] \overset{\text{product bracket}}{=} P_\lambda [P^\lambda, M^{\mu\nu}] + [P^\lambda, M^{\mu\nu}] P_\lambda$$
$$= P_\lambda\, i(-\eta^{\lambda\mu} P^\nu + \eta^{\lambda\nu} P^\mu) + i(-\eta^{\nu\lambda} P^\mu + \eta^{\mu\lambda} P^\nu) P_\lambda$$
$$= i(-P^\nu P^\mu + P^\mu P^\nu - P^\nu P^\mu + P^\mu P^\nu) = 0.$$

- $[W_\mu, P_\nu] = \frac{1}{2}\varepsilon_{\mu\rho\sigma\delta}[P^\rho M^{\sigma\delta}, P_\nu] = \frac{1}{2}\varepsilon_{\mu\rho\sigma\delta}(P^\rho [M^{\sigma\delta}, P_\nu] + \underbrace{\cancel{[P^\rho, P_\nu]}}_{\text{Abelian}} M^{\sigma\delta})$

$$= \underbrace{\frac{1}{2}\varepsilon_{\mu\rho\sigma\delta}}_{\text{totally antisym}} (P^\rho i (\underbrace{\eta^\delta{}_\nu P^\sigma}_{\text{sym in } \rho,\sigma} - \underbrace{\eta^\sigma{}_\nu P^\delta}_{\text{sym in } \rho,\delta})) = 0$$

Hence $[C_2, P_\nu] = [W_\mu W^\mu, P_\nu] = 2W^\mu [W_\mu, P_\nu] = 2W^\mu \cdot 0 = 0$.

- $[W_\mu, M_{\alpha\beta}] = \frac{1}{2}\varepsilon_{\mu\rho\sigma\delta}[P^\rho M^{\sigma\delta}, M_{\alpha\beta}] = \frac{1}{2}\varepsilon_{\mu\rho\sigma\delta}(P^\rho [M^{\sigma\delta}, M_{\alpha\beta}] + [P^\rho, M_{\alpha\beta}] M^{\sigma\delta})$

$$= \frac{1}{2}\varepsilon_{\mu\rho\sigma\delta}\{P^\rho i (\eta^\delta{}_\alpha M^\sigma{}_\beta - \eta^\sigma{}_\alpha M^\delta{}_\beta - \eta^\delta{}_\beta M^\sigma{}_\alpha + \eta^\sigma{}_\beta M^\delta{}_\alpha) + i(\eta^\rho{}_\alpha P_\beta - \eta^\rho{}_\beta P_\alpha) M^{\sigma\delta}$$

$$= \frac{i}{2}\varepsilon_{\mu\rho\sigma\delta}\{-\eta^{\sigma\alpha} P^\rho M^{\delta\beta} \quad \eta^{\rho\alpha} P^\beta M^{\sigma\delta} + P^\rho \eta^{\delta\alpha} M^{\sigma\beta} \quad - \alpha \leftrightarrow \beta\}$$

$$= \frac{i}{2}\varepsilon_{\mu\rho\sigma}{}^{\alpha}\{\varepsilon^{\rho\sigma\beta\lambda} W_\lambda - \alpha \leftrightarrow \beta\} \quad \text{incl use of } M^{\alpha\beta} \text{ antisym}$$

$$= \frac{i}{\cancel{2}} \cancel{2} (\eta^{\alpha\beta}\eta_{\mu\lambda} - \eta^{\alpha\lambda}\eta_\mu{}^\beta) W_\lambda - \alpha \leftrightarrow \beta$$

$$= i\{\cancel{\eta^{\alpha\beta} W^\mu} - \eta^{\mu\beta} W^\alpha - \cancel{\eta^{\alpha\beta} W^\mu} + \eta^{\nu\alpha} W^\beta\}$$

$$= i\{\eta^{\nu\alpha} W^\beta - \eta^{\mu\beta} W^\alpha\}$$ ie W^μ transforms as a Lorentz vector.

Hence $[W_\mu W^\mu, M^{\alpha\beta}] \overset{\text{bracket of product}}{=} W_\mu [W^\mu, M^{\alpha\beta}] + [W^\mu, M^{\alpha\beta}] W_\mu$

$$= W_\mu (\eta^{\nu\alpha} W^\beta - \eta^{\mu\beta} W^\alpha) + (\eta^{\nu\alpha} W^\beta - \eta^{\mu\beta} W^\alpha) W_\mu$$
$$= \cancel{W^\alpha W^\beta} - \cancel{W^\beta W^\alpha} + \cancel{W^\beta W^\alpha} - \cancel{W^\alpha W^\beta} = 0.$$

A fast argument : W^2 is a Lorentz scalar as $P^\mu, M^{\mu\nu}, \epsilon_{\alpha\beta\gamma\delta}$ are Lorentz tensors.

This must be invariant under Lorentz transformations $\Rightarrow$ commutes with $M_{\mu\nu}$.

Since we are going to use Casimirs for the Rep Theory, it is important to ascertain whether we have found all the (quadratic) casimirs.

prove elsewhere. Casimir CSCO's 1 label per casimir gives some intuition of why this's rank(G) of them

__Claim__ rank of group = # quadratic casimirs

Specifically rank(SO(2N)) = N

rank(SU(N)) = N − 1

Thus Rank(Pcin(3,1)) = Rank(Lcr(3,1)) = Rank(SU(2) x SU(2)) = 2·Rank(SU(2)) = 2·1 = 2;

so there are 2 casimirs, so we have indeed found all of them.

by the Abelian part of a group contributing nothing to its rank.

A further ex : Show that $[W_\mu, W_\nu] = i\epsilon_{\mu\nu\rho\sigma} P^\rho W^\sigma$.

Derivation
$$[W_\mu, W^\nu] = \frac{1}{2}\epsilon_{\mu\rho\sigma\delta}[P^\rho M^{\sigma\delta}, W^\nu]$$
$$= \frac{1}{2}\epsilon_{\mu\rho\sigma\delta}\{[P^\rho, W^\nu]M^{\sigma\delta} + P^\rho[M^{\sigma\delta}, W^\nu]\}$$
$$= \frac{1}{2}\epsilon_{\mu\rho\sigma\delta}P^\rho i(\eta^{\delta\nu}W^\sigma - \eta^{\sigma\nu}W^\delta)$$
$$= \cancel{\frac{i}{2}}\epsilon_{\mu\rho\sigma}{}^{\nu}P^\rho \cancel{2}W^\sigma. \quad \square$$

<u>Remark</u> $A_i = \frac{1}{2}(J_i + k_i)$, $B_i = \frac{1}{2}(J_i - k_i)$

s.t. $[A_i, A_j] = i\epsilon_{ijk} A_k$

$[A_i, B_j] = 0$

$[B_i, B_j] = i\epsilon_{ijk} B_k$

is locally isomorphic to $SU(2) \times SU(2) = SO(4)$

<u>But</u> $A_i = \frac{1}{2}(J_i + ik_i)$, $B_i = \frac{1}{2}(J_i - ik_i)$

is locally isomorphic to $SL(2, \mathbb{C})$.

[There are 2 different ways that two $SU(2)$'s can be put together].

$SO(3,1) \cong SL(2,\mathbb{C})$

Consider the transformation : $X \to X' = x'^m \sigma_m$

from 4-vectors $x^\mu e^\nu{}_\mu$ to 2×2 matrices, for $\sigma_\mu = \begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}0&-i\\i&0\end{pmatrix}, \begin{pmatrix}1&0\\0&-1\end{pmatrix}$

so that $X = \begin{pmatrix} x^0+x^3 & x^1+ix^2 \\ x^1-ix^2 & x^0-x^3\end{pmatrix}$

$$X \to X' = N X N^T \qquad [\, N \in SL(2,\mathbb{C}) \,].$$

- Then $\det X' = \det(NXN^T) = \det N \cdot \det X \cdot \det N^T = (\det N)^2 \det X$.

so if $\det N = 1$ [$N \in SL(2,\mathbb{C})$], $\det X' = \det X$

- Note that $\det X = (x^0+x^3)(x^0-x^3) - (x^1+ix^2)(x^1-ix^2) = (x^0)^2-(x^1)^2-(x^2)^2-(x^3)^2 = \eta_{\mu\nu}x^\mu x^\nu$

preserves $\eta_{\mu\nu}$, i.e. $X \in$ (full) Lorentz group.

$\Rightarrow x'^m = (\Lambda(N))^m{}_n \, x^n$ for $\Lambda(N) \in O(3,1)$.

It remains to be shown that this gives rise to the whole of the Lorentz group.

- $SO(3,1)$: $X^\mu \to \Lambda^\mu{}_\nu X^\nu$
  $SL(2,\mathbb{C})$: $X \to N X N^T$

so for 2 $SL(2,\mathbb{C})$ transformations $\pm N$, we have just the one $SO(3,1)$ transformation.

So what we have is a 2:1 mapping : a double covering.

$\Pi : SL(2,\mathbb{C}) \to O(3,1)$ is an analytic homomorphism

$N \mapsto \Lambda(N)$

We have noted that $\ker \Pi = \pm\begin{pmatrix}1&0\\0&1\end{pmatrix}$, so $SO^{\uparrow}(3,1) \cong SL(2,\mathbb{C})/\mathbb{Z}_2$ (1st Isomorphism Theorem)

by $\Pi^{-1}\begin{pmatrix}R_x(\varphi)\\ \Lambda_x(\psi)\end{pmatrix} = \begin{pmatrix}\pm N(\varphi)\\ \pm M(\psi)\end{pmatrix}$ so all of $SL(2,\mathbb{C})$ expressible as $R\Lambda R'$ is mapped into $SO^{\uparrow}(3,1)$

The 2:1 homomorphism is then $U \to \Lambda(U) = \frac{1}{2}\,\mathrm{tr}(\sigma_i U \sigma_j U^\dagger)$.