$GL(2, \mathbb{R})$ unsplit

$$[G_{AB}, G_{CD}] = [x_A \partial_B, x_C \partial_B] = x_A \partial_B x_C \partial_D - x_C \partial_D x_A \partial_B$$

$$= x_A \delta_{BC} \partial_D - x_C \delta_{AD} \partial_B$$

$$= \delta_{BC} x_A \partial_D - \delta_{AD} x_C \partial_B$$

$$= \delta_{BC} G_{AD} - \delta_{AD} G_{CB} \quad (1)$$

$GL(1, \mathbb{R})$'s triviality

$$[G, G] = \left[x \frac{d}{dx}, x \frac{d}{dx}\right] = 0$$ (anything commutes with itself.

At level of (1), this is $\delta_{11} G - \delta_{11} G = 1 \cdot G - 1 \cdot G = 0$.

$GL(2,\mathbb{R})$ in terms of R, D, S, P

$$[R,D] = \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) - \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

$$= \cancel{x\frac{\partial}{\partial y}} - \cancel{y\frac{\partial}{\partial x}} - \cancel{x\frac{\partial}{\partial y}} + \cancel{y\frac{\partial}{\partial x}} = 0 \quad \text{; Rot \& Dil are indep.}$$

$$[R,S] = \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\left(x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}\right) - \left(x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}\right)\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

$$= x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$$

$$= 2\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right) = 2P \qquad (R,S) \to P$$

$$[R,P] = \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right) - \left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

$$= -x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

$$= -2\left(x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}\right) = -2S \quad : (R,P) \to S$$

$$[S,D] = \left(x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}\right)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) - \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}\right)$$

$$= \cancel{x\frac{\partial}{\partial y}} + \cancel{y\frac{\partial}{\partial x}} - \cancel{x\frac{\partial}{\partial y}} - \cancel{y\frac{\partial}{\partial x}} = 0 \text{ : S \& Dil indep.}$$

$$[P,D] = \left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) - \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)$$

$$= \cancel{x\frac{\partial}{\partial x}} - \cancel{y\frac{\partial}{\partial y}} - \cancel{x\frac{\partial}{\partial x}} + \cancel{y\frac{\partial}{\partial y}} = 0 \text{ : P \& Dil indep.}$$

$$[S,P] = \left(x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}\right)\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right) - \left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)\left(x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}\right)$$

$$= -x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}$$

$$= -2\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = -2R \quad : (S,P) \to R$$

So we have $D \times (R, S, P)$
and R, S, P can be massaged into $\mathfrak{sl}(2,\mathbb{R})$ form.
so $GL(2,\mathbb{R}) = \mathbb{R}_+ \times SL(2,\mathbb{R})$.

Check the recasting into $\mathfrak{sl}(2,\mathbb{R})$ form:
Use $\underset{\substack{\shortparallel \\ A}}{x\dfrac{\partial}{\partial y}}$ and $\underset{\substack{\shortparallel \\ B}}{y\dfrac{\partial}{\partial x}}$ in place of their sym, antisym parts S, R:

$$\text{Then} \quad [A,B] = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} = P$$

$$[P,A] = x\frac{\partial}{\partial y} + x\frac{\partial}{\partial y} = 2A$$

$$[P,B] = -y\frac{\partial}{\partial x} - y\frac{\partial}{\partial x} = -2B \quad ✓$$

GL(n, ℝ) unsplit (is immediate generalization of 2-d case.

$$
\begin{aligned}
[G_{AB}, G_{CD}] &= \\
[x_A \partial_B, x_C \partial_D] &= x_A \partial_B x_C \partial_D - x_C \partial_D x_A \partial_B \\
&= x_A \delta_{BC} \partial_D - x_C \delta_{AD} \partial_B \\
&= \delta_{BC} x_A \partial_D - \delta_{AD} x_C \partial_B \\
&= \delta_{BC} G_{AD} - \delta_{AD} G_{CB}
\end{aligned}
$$

$GL(n,\mathbb{R})$ in irreducible pieces

$$A_{AB} = 2x_{[A}\partial_{B]}, \quad S^T_{AB} = 2x_{(A}\partial_{B)} - \frac{2}{n}\delta_{AB}x_C\partial_C, \quad D = x_C\partial_C$$

$$\begin{aligned}
[A_{AB}, A_{CD}] &= (x_A\partial_B - x_B\partial_A)(x_C\partial_D - x_D\partial_C) - (x_C\partial_D - x_D\partial_C)(x_A\partial_B - x_B\partial_A) \\
&= x_A\partial_B x_C\partial_D - x_A\partial_B x_D\partial_C - x_C\partial_D x_A\partial_B + x_C\partial_D x_B\partial_A \\
&\quad - x_B\partial_A x_C\partial_D + x_B\partial_A x_D\partial_C + x_D\partial_C x_A\partial_B - x_D\partial_C x_B\partial_A \\
&= \delta_{BC}(x_A\partial_D - x_D\partial_A) - \delta_{BD}(x_A\partial_C - x_C\partial_A) \\
&\quad + \delta_{AD}(x_B\partial_C - x_C\partial_B) - \delta_{AC}(x_B\partial_D - x_D\partial_B) \\
&= \underbrace{\delta_{BC}A_{AD} - \delta_{BD}A_{AC} + \delta_{AD}A_{BC} - \delta_{AC}A_{BD}}_{} \quad \text{$\cdot$ std rotations.}
\end{aligned}$$

$$\begin{aligned}
[S^T_{AB}, S^T_{CD}] &= \left(x_A\partial_B + x_B\partial_A - \frac{1}{n}\delta_{AB}x_P\partial_P\right)\left(x_C\partial_D + x_D\partial_C - \frac{1}{n}\delta_{CD}x_Q\partial_Q\right) \\
&\quad - \left(x_C\partial_D + x_D\partial_C - \frac{1}{n}\delta_{CD}x_Q\partial_Q\right)\left(x_A\partial_B + x_B\partial_A - \frac{2}{n}\delta_{AB}x_P\partial_P\right) \\
&= \delta_{BC}\begin{pmatrix}x_A\partial_D \\ -x_D\partial_A\end{pmatrix} + \delta_{BD}\begin{pmatrix}x_A\partial_C \\ -x_C\partial_A\end{pmatrix} + \delta_{AC}\begin{pmatrix}x_B\partial_D \\ -x_D\partial_B\end{pmatrix} + \delta_{AD}\begin{pmatrix}x_B\partial_C \\ -x_C\partial_B\end{pmatrix} \\
&\quad + \frac{2}{n}\left(\cancel{-\delta_{CD}\begin{pmatrix}x_A\partial_B \\ +x_B\partial_A\end{pmatrix}} - \delta_{AB}\cancel{\begin{pmatrix}x_C\partial_D \\ +x_D\partial_C\end{pmatrix}} + \cancel{\delta_{CD}\begin{pmatrix}x_A\partial_B \\ +x_B\partial_A\end{pmatrix}} + \delta_{AB}\cancel{\begin{pmatrix}x_C\partial_D \\ +x_D\partial_C\end{pmatrix}}\right) \\
&= \underbrace{\delta_{BC}A_{AD} + \delta_{BD}A_{AC} + \delta_{AC}A_{BD} + \delta_{AD}A_{BC}}_{} \quad (= [S_{AB}, S_{CD}])
\end{aligned}$$

clear by D indep of shape.

$$[D, A_{AB}] = 0 = [D, S^T_{AB}] \quad (= [D, S_{AB}]) \quad \text{D indep of shape.}$$

$$\begin{aligned}
[A_{AB}, S^T_{CD}] &= [A_{AB}, S_{CD}] = (x_A\partial_B - x_B\partial_A)(x_C\partial_D + x_D\partial_C) - (x_C\partial_D + x_D\partial_C)(x_A\partial_B - x_B\partial_A) \\
&= \delta_{BC}\begin{pmatrix}x_A\partial_D \\ +x_D\partial_A\end{pmatrix} + \delta_{BD}\begin{pmatrix}x_A\partial_C \\ +x_C\partial_A\end{pmatrix} - \delta_{AC}\begin{pmatrix}x_B\partial_D \\ +x_D\partial_B\end{pmatrix} - \delta_{AD}\begin{pmatrix}x_B\partial_C \\ +x_C\partial_B\end{pmatrix} \\
&= \delta_{BC}S_{AD} + \delta_{BD}S_{AC} - \delta_{AC}S_{BD} - \delta_{AD}S_{BC}. \\
&= \underbrace{\delta_{BC}S^T_{AD} + \delta_{BD}S^T_{AC} - \delta_{AC}S^T_{BD} - \delta_{AD}S_{BC}}_{} \\
&\quad + \frac{2}{n}\left(\delta_{BC}\cancel{\delta_{AD}} + \delta_{BD}\cancel{\delta_{AC}} - \cancel{\delta_{AC}\delta_{BD}} - \cancel{\delta_{AD}\delta_{BC}}\right)x_Cx_C
\end{aligned}$$

D indep of shape once more

So the A's close but the S's (or S^T's) do not. So $GL(n,\mathbb{R})$ has $SO(n)$ as a subalgebra. and the form of the algebra is $D \times (A, S^T)$

$$GL(n,\mathbb{R}) = D \times SL(n,\mathbb{R}).$$

What is the geometrical meaning of the $SL(2,\mathbb{R})$ Casimir?

$$C = P^2 + 1 + 2(AB + BA)$$

$$= \left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right) + 1 + 2\left(x\frac{\partial}{\partial y}\, y\frac{\partial}{\partial x} + y\frac{\partial}{\partial x}\, x\frac{\partial}{\partial y}\right)$$

$$= x\frac{\partial}{\partial x} - 2xy\frac{\partial^2}{\partial x\partial y} + y\frac{\partial}{\partial y} + 1 + 2\, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 4xy\frac{\partial^2}{\partial x\partial y}$$

$$= 1 + 3\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) + 2xy\frac{\partial^2}{\partial x\partial y}$$

$$\rightarrow \boxed{C = 1 + 3D + 2xy\frac{\partial^2}{\partial x\partial y}}.$$

so to nontrivial leading order it's dilation pointing to D indep of R, S, P

another 4th op that commutes with everything

Also $A + B = S$, $A - B = R$ so $A = \frac{R+S}{2}$, $B = \frac{S-R}{2}$.

$$\text{so } C = P^2 + 1 + 2\left(\left(\frac{R+S}{2}\right)\left(\frac{S-R}{2}\right) + \left(\frac{S-R}{2}\right)\left(\frac{R+S}{2}\right)\right)$$

$$= 1 + P^2 + \frac{1}{2}\left(\cancel{RS} + S^2 - \cancel{SR} - R^2 + \cancel{SR} - R^2 - \cancel{RS} + S^2\right)$$

$$\boxed{\begin{aligned} C &= 1 + P^2 + S^2 - R^2 \\ &= 1 + (\text{pressure})^2 + (\text{shear})^2 - (\text{rotation})^2 \end{aligned}}$$

check that $P^2 + S^2 - R^2 \sim 3D$ to leading order.

$$\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)^2 + \left(x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}\right)^2 - \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)^2$$

$$\approx x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} = 3D \checkmark$$

§17 Affine Lie algebra

$$[B_a{}^b, B_c{}^d] = 2\delta_c{}^{(b} B_a{}^{d)}$$ 2-tensor

$$[B_a{}^b, a_c] = \delta_c{}^b v_a$$ vector

Check std generators obey this.
Check split-up form of this (i.t.o. T, S^T, A pieces).

Affine algebra unsplit

$G_{AB} = x_A \partial_B$, $P_C = \partial_C$

① $[G_{AB}, G_{CD}] = \delta_{BC} G_{AD} - \delta_{AD} G_{BC}$ GL(n, ℝ) algebra

② $[P_C, P_D] = 0$ commute

③ $[P_C, G_{AB}] = -x_A \partial_B \partial_C + \partial_C x_A \partial_B = -\delta_{AB} P_C$

so P_C is a GL(n, ℝ) vector.

so we have $\underbrace{Aff(n) = \mathbb{R}^n \rtimes GL(n,\mathbb{R})}$

Trace–tracefree split affine algebra

$G^T_{AB} = x_A \partial_B - \frac{1}{n} \delta_{AB} x_C \partial_C$, $D = x_C \partial_C$, $P_C = \partial_C$.

$[G_{AB}, D] = x_A \partial_B x_C \partial_C - x_C \partial_C x_A \partial_B = \delta_{BC} x_A \partial_C - \delta_{AC} x_C \partial_B = x_A \partial_B - x_A \partial_B = 0$

so D commutes with the whole of G, and thus is part of $Z(GL(n,\mathbb{R}))$.

Hence

(I) $[G^T_{AB}, D] = 0$ as well

(II) $[G^T_{AB}, G^T_{CD}] = [G_{AB}, G_{CD}] = \delta_{BC} G_{AD} - \delta_{AD} G_{BC} = \delta_{BC} G^T_{AD} - \delta_{AD} G^T_{BC}$ $\cancel{+ \frac{1}{n} \delta_{BC} \delta_{AD} (1-1) \ldots}$

(III) $[P_A, D] = \partial_A x_C \partial_C - x_C \partial_C \partial_A = \delta_{AC} \partial_C = \partial_A = P_A$

so P is a dilational vector.

(IV) $[P_C, P_D] = 0$, as were,

(V) $[P_C, G^T_{AB}] = [P_C, G_{AB}] - \frac{1}{n} \delta_{AB} [P_C, D]$

$\underset{③}{=} \delta_{AB} P_C \underset{(III)}{-} \frac{1}{n} \delta_{AB} P_C = \frac{n-1}{n} \delta_{AB} P_C$

so P is also a SL(n, ℝ) vector.

Integrability structure: the P's close by themselves
D closes by itself
G^T close by themselves
(P, G) produce P. $\underbrace{Aff(n) = \mathbb{R}^n \rtimes GL(n,\mathbb{R})}$

Also (II, IV, V) close by themselves,

giving the Equi-n-voluminal algebra $\underbrace{Equi(n) = \mathbb{R}^n \rtimes SL(n,\mathbb{R})}$

$\mathbb{R}^n$ is an Abelian subalgebra of both Aff(n) and Equi(n).
GL(n, ℝ), SL(n, ℝ) and Dil are subalgebras of Aff(n)
SL(n, ℝ) is a subalgebra of Equi(n)
$\mathbb{R}^n \rtimes$ Dil is a subalgebra of Aff(n)

We can furthermore split G^T_{AB} into A_{AB} and S^T_{AB}.
This reveals that: $SO(n)$ is a subalgebra of $SL(n,\mathbb{R})$, $GL(n,\mathbb{R})$, $Equi(n)$ & $Aff(n)$.
(obvious from it being a subalgebra of $SL(n,\mathbb{R})$.
Also that P_A is an $SO(n)$-vector.

We can now build a lattice of subalgebras: