§1.15 Notions of distance for graphs

Recall that the length $\ell(\Gamma)$ of a path graph Γ is # of constituent edges.

Definition 1 The distance $d(x,y)$ between vertices x and y in a graph G is given by

$$d(x,y) = \min_{\substack{\text{paths } \Gamma(x\to y) \\ x \text{ to } y}} \left(\ell(\Gamma) \right) \quad (1);$$

in the event of there being no such path, $d(x,y) := \infty$.

Remark Paths of maximal length are sometimes useful too, c.f. Veblen's Thm in §1.1(

Definition 2 The eccentricity $e(v)$ of a vertex v is

$$e(v) = \max_{w\in V(G)} \left(d(v,w) \right) = \max_{w\in V(G)} \left(\min_{\text{paths } \Gamma(x\to y)} \left(\ell(\Gamma) \right) \right)$$

Definition 3 The radius $r(G)$ of a graph G is

$$r(G) = \min_{v\in V(G)} \left(e(v) \right),$$

i.e. $$r(G) = \min_{v\in V(G)} \left(\max_{w\in V(G)} \left(d(v,w) \right) \right)$$

ie $$r(G) = \min_{v\in V(G)} \left(\max_{w\in V(G)} \left(\min_{\text{paths } \Gamma(x\to y)} \left(\ell(\Gamma) \right) \right) \right)$$

cf Hausdorff dist & Gromov-Hausdorff dist in metric space. Things for compactly long such nestings of max/mins or inf/sups ...

Definition 4 A vertex v is central if $e(v) = r(G)$.

The centre $C(G)$ of a graph $= \{ v \mid e(v) = r(G) \}$: the set of central vertices.

Definition 5 The diameter $d(G)$ of a graph is the opposite extreme to the radius:

$$d(G) = \max_{v\in V(G)} \left(e(v) \right)$$

$$= \max_{v\in V(G)} \left(\max_{w\in V(G)} \left(d(v,w) \right) \right)$$

$$= \max_{v\in V(G)} \left(\max_{w\in V(G)} \left(\min_{\text{paths } \Gamma(x\to y)} \left(\ell(\Gamma) \right) \right) \right)$$

Lemma $r(G) \underset{①}{\leq} d(G) \underset{②}{\leq} 2r(G)$.

Proof

(1) follows from the basic inequality $\min \leq \max$.

For (2) : $\exists\, x \in G$ st $d(x, v) \leq r(G)$ ① by the minimality of radius.

This holds for any v.

In particular then it holds for a y, z pair which realize the diameter

$$d(y, z) = d(G). \quad (*)$$

so (1) $\Rightarrow \begin{cases} d(y, v) \leq r(G) & ② \\ d(x, v) \leq r(G) & ③ \end{cases}$.

But $d(\cdot, \cdot)$ is a distance, so the Δ inequality holds for it :

$$d(y, z) \leq d(x, v) + \underbrace{d(v, x)}_{d(x,v)} \leq r(G) + r(G) = 2r(G)$$

$(*) \, \|$ distance is symmetric

$d(G)$

$\square$

Definition 6 The girth $g(G)$ of a graph is

$$g(G) = \min_{\text{cycles } C \subseteq G} (\ell(C))$$

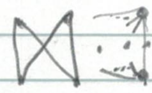
For technical convenience, if G contains no cycles, $g(G) := \infty$

i.e. $G(\text{acyclic graph}) = G(\text{forest}) := \infty$

Examples

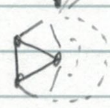
$G(C_n) = n$

$G(K_n) = 3$ 

$G(K_{n,n}) = 4$ 

$G(n\text{-pan}) = G(C_n \# P_1) = \ell(C_n) = n$

$G(\underset{n\text{-fan}}{F_n}) = 3$ 

$G(\underset{n\text{-wheel}}{W_n}) = 3$ 

Remark Girth feeds into the concept of cage, on which there is an Ex in §1.16.