§1.11 Paths, Trails, Walks
Cycles, Ciruits and Tours

<u>Definition 1</u> A <u>path</u> is a graph P of the form $V(P) = \{x_0, x_1, \dots, x_\ell\}$ all distinct.
$E(P) = \{x_0x_1, x_1x_2, \dots, x_{\ell-1}x_\ell\}$.

This is usually denoted by $x_0x_1 \dots x_\ell$.
The vertices x_0 and x_ℓ are the <u>endvertices</u> of P.
$\ell = e(P)$ is the <u>length</u> of P.
edge number

we say that P is a path from x_0 to x_ℓ alias $x_0 - x_\ell$ path
[alias ——— $x_\ell - x_0$ —— $x_\ell - x_0$ —— .]
If a direction – e.g. x_0 to x_ℓ – matters, we call x_0 <u>initial vertex</u> of P and x_ℓ <u>terminal vert</u>

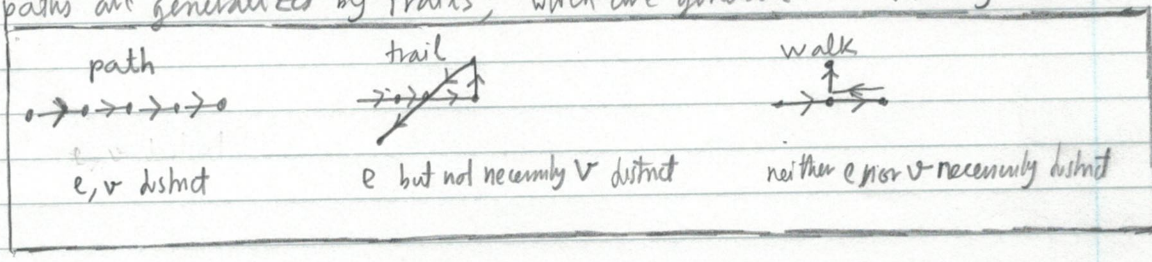
<u>Definition 2</u> A <u>trail</u> is a graph T, which generalize a path by vertices not needing to be distinct.
Its edges, however, remain distinct.
This is an $x_0 - x_\ell$ trail, of length $\ell = e(T)$.
<u>Definition 3</u> A <u>walk</u> W in a graph is a (finite) alternating sequence of vertices and edges

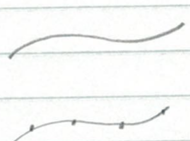
$$x_0, \underset{x_0x_1}{\underset{\shortparallel}{e_1}}, x_1, \underset{x_1x_2}{\underset{\shortparallel}{e_2}}, \dots, \underset{x_{\ell-1}x_\ell}{\underset{\shortparallel}{e_\ell}}, x_\ell$$

This is an $x_0 - x_\ell$ walk, of length $\ell = e(W)$.

A walk's edges are not necessarily distinct.

So paths are generalized by trails, which are generalized in turn by walks





'with milestones'

refers to following distinctive marks on trees, or breadcrumbs!

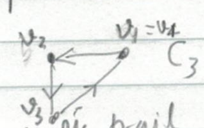
walk, as in <u>random walk</u>: edges and vertices can recur.

but colloquial uses need not match the graph-theoretic terminology

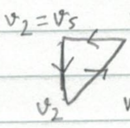
P , T , W
order of increasing generality matches alphabetical order.

<u>Remark</u> It's not possible for a path to not be a trail: sharing an edge ⇒ sharing a vertex

<u>Small examples</u> $\circ\!\!\to\!\!\circ$ P_2
walk, trail and path



C_3
walk, trail but not path: $v_1 = v_4$



walk but not a trail: $v_1v_2 = v_4v_5$
nor a path $v_1 = v_4$, $v_2 = v_5$

(p3)

Definition 4 A cycle is a walk $W = x_0 x_1 \cdots x_\ell$

such that 1) $\ell \geq 3$

2) $x_0 = x_\ell$

3) x_i $0 < i < \ell$ all distinct from each other and from x_0.

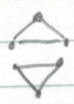
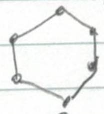
Example By 1), C_3  is the smallest cycle.

Remark 1 A simple notation for the cycle $x_1 x_2 \cdots x_\ell$ is an ℓ-cycle (triangle for $l=3$, quadrilateral for $l=4$)

Remark 2 A cycle has neither a preferred starting vertex nor a preferred direction.
Thus $x_1 x_2 \cdots x_\ell = x_2 x_3 \cdots x_\ell x_1 = $ [indeed 'and cycles'!]

Remark 3 There can be significant technical differences between ℓ even (even cycles) and ℓ odd (odd cycles).

Remark 4 Cycles have all vertices of valency 2 [i.e. they are 2-regular].
This is not however a unique characterization, since disjoint unions of cycles have this feature too.

 and  are both 2-regular 6-vertex graphs

$2C_3$ C_6

Remark 7 cycles are not paths because of the $x_\ell x_0$ edge ending at x_0, which has been used before.
Cycles do however fit the description of trails, but aren't the most general closed trail.

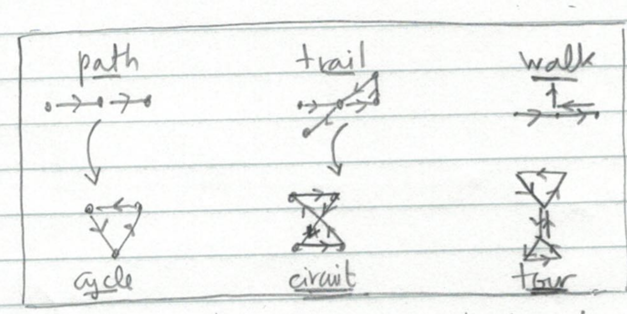
Definition 5 A circuit is a closed trail (i.e. one with coincident endvertices).

Example of a circuit which is not a cycle: butterfly alias hourglass graph  upon being given edge-directions gives  circuit
(if vertices are labelled, multiple choices of such: Ex!

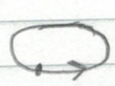
Definition 6 A tour is a closed walk

This definition completes the current page being closed versions of everything on previous page
(sadly quite a few textbooks omit the definition of tour.

its closed version



tourists start and end at same place: home! Helps repeat return flight

perm. cycles |  | [electric circuit. only helps in being closed] | [tour de France isn't always closed, and may be a cycle]

topological cycles | running one lap. | | [Knight's tour is actually a cycle!]

dual to cocycles | many simple processes are/contain | [circuit training: can visit a machine twice | so it's really a tour.