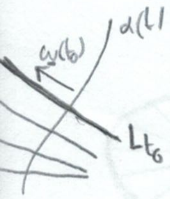
3.A RULED SURFACES

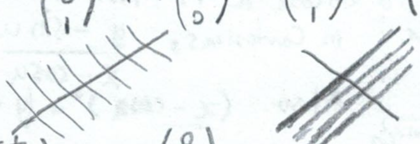
These are surfaces which can be swept out by moving a line through space.
This is modelled as follows:
A (differentiable) 1-parameter family of straight lines $\{\alpha(t), w(t)\}$
is a correspondence assigning to each $t \in I$ a point $\alpha(t) \in \mathbb{R}^3$ and a nonzero vector $w(t) \in \mathbb{R}^3$,
such that both $\alpha(t)$ and $w(t)$ depend differentiably on t.
For each $t \in I$, the line L_t passing through $\alpha(t)$ and parallel to $w(t)$ is the line of the family at t.
Given a 1-parameter family of lines $\{\alpha(t), w(t)\}$, the parametrized surface $x(t,v) = \alpha(t) + v\,w(t)$, $t \in I$, $v \in \mathbb{R}$
is the ruled surface generated by the family $\{\alpha(t), w(t)\}$, and the curve $\alpha(t)$ is called a directrix (or director curve).
The lines L_t are called the rulings of the surface $\underline{x}$.



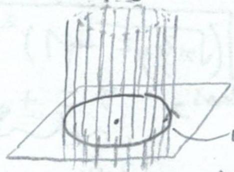
[sometimes we use ruled surface to mean tr x; sometimes we allow $\underline{x}$ to have singular pts: s.t. $x_t \wedge x_v = 0$]
This is conceptually a projective notion.
It also readily turns out to be a significant notion in extrinsic geometry.

indeed the ruled surfaces ... in Algebraic Geometry...

Example 1 $\underline{x}(t,v) = \begin{pmatrix} t \\ v \\ 1 \end{pmatrix}$ plane $= \begin{pmatrix} t \\ 0 \\ 1 \end{pmatrix} + v\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ v \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$: a double-ruled example.



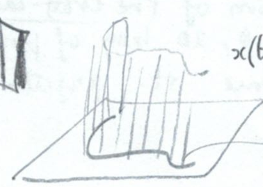
Example 2 the cylinder $\underline{x}(t,v) = \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} + v\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.



Example 3 the (double) cone $\underline{x}(t,z) = z\begin{pmatrix} \cot\alpha \cos t \\ \cot\alpha \sin t \\ 1 \end{pmatrix}$ ← is singular

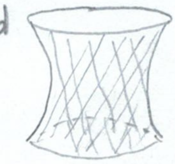
Example 4 The tangent surfaces to any regular curve
$\underline{x}(t,v) = \alpha(t) + v\,\alpha'(t)$ ← can be singular.

Example 5) A generalized notion of cylinder: $x(t,v) = \underline{\alpha}(t) + v\,\hat{\underline{n}}$ ⊥r to plane.



Example 6) A generalized notion of cone

Example 7) The elliptic hyperboloid of one sheet is double-ruled



some bins are of this shape

$$\begin{pmatrix} a(\cos u \mp v\sin u) \\ b(\sin u \pm v\cos u) \\ \pm cv \end{pmatrix} = \begin{pmatrix} a\cos u \\ b\sin u \\ 0 \end{pmatrix} \pm v\begin{pmatrix} -a\sin u \\ b\cos u \\ c \end{pmatrix}$$

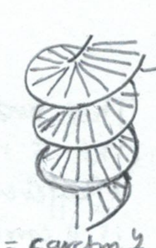
In cartesians, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 u \mp \cancel{2v\cos u \sin u} + v^2\sin^2 u + \sin^2 u \pm \cancel{2v\cos u \sin u} + v^2\cos^2 u = 1 + v^2 = 1 + \frac{z^2}{c^2}$, so $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

Example 8) The hyperbolic paraboloid is a double-ruled surface.

$$\begin{pmatrix} a(u+v) \\ \pm bv \\ u^2 + 2uv \end{pmatrix} = \begin{pmatrix} a u \\ 0 \\ u^2 \end{pmatrix} + v\begin{pmatrix} a \\ \pm b \\ 2u \end{pmatrix}$$

In cartesians, $\frac{x^2}{a^2} = (u^2 + v^2 + 2uv) = z + \frac{y^2}{b^2}$, so $\frac{x^2}{a^2} - \frac{y^2}{b^2} = z$

Example 9) The helicoid $z = c\theta = c\arctan\frac{y}{x}$



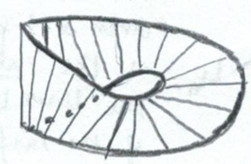
can be parametrized by

$$\begin{pmatrix} a\sinh v\cos u \\ a\sinh v\sin u \\ au \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ au \end{pmatrix} + y\begin{pmatrix} a\cos u \\ a\sin u \\ 0 \end{pmatrix} \quad \text{for } y = \sinh v$$

Check: $\frac{y}{x} = \cancel{\frac{a\sinh v\sin u}{\sinh v\cos u}} = \tan u \Rightarrow \arctan\frac{y}{x} = u = \frac{z}{a}$ so $z = c\arctan\frac{y}{x}$ for $c = a$.

Try to prove using PDEs → Claim The only ruled minimal surfaces are the helicoid and the plane.

Example 10) The Möbius strip is clearly also a ruled surface ~ a nontrivial bundle over S^1 with $\mathbb{R}$ as fibres.



This can be parametrized by

$$\begin{pmatrix} \cos u + v\cos\frac{u}{2}\cos u \\ \sin u + v\cos\frac{u}{2}\sin u \\ v\sin\frac{u}{2} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos u \\ \sin u \\ 0 \end{pmatrix}}_{\text{the circle base}} + v\underbrace{\begin{pmatrix} \cos\frac{u}{2}\cos u \\ \cos\frac{u}{2}\sin u \\ \sin\frac{u}{2} \end{pmatrix}}_{\text{the line fibres.}}$$

to encode a 4π twist.

Check: in Cartesians, $\frac{y - \sin u}{x - \cos u} = \frac{\cancel{v\cos\frac{u}{2}}\sin u}{\cancel{v\cos\frac{u}{2}}\cos u} = \tan u \Rightarrow \cancel{y - \sin u = x\tan u - \sin u}$
$\Rightarrow u = \arctan y/x$ std polar $\angle$.

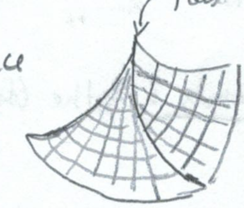
Also $(x-\cos u)^2 + (y-\sin u)^2 + z^2 = v^2\left(\cos^2\frac{u}{2}(\cos^2 u + \sin^2 u) + \sin^2\frac{u}{2}\right) = v^2$.

so by [triangle: $\sqrt{x^2+y^2}$, u, y, x] $\sin u = \frac{y}{\sqrt{x^2+y^2}}$, $\cos u = \frac{x}{\sqrt{x^2+y^2}}$, $v^2 - z^2 = \left(x - \frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(y - \frac{y}{\sqrt{x^2+y^2}}\right)^2 = \cancel{\frac{x^2+y^2}{x^2+y^2}}\left(\sqrt{x^2+y^2}-1\right)^2$

$$\Rightarrow v = \pm\sqrt{\left(\sqrt{x^2+y^2}-1\right)^2 + z^2}.$$

More neatly in cylindrical polars, $v = \pm\sqrt{(r-1)^2+z^2}$.

Example 11) Plücker's conoid $z = \frac{2xy}{x^2+y^2}$ is a ruled surface



Parametrize as $\begin{pmatrix} r\cos\theta \\ r\sin\theta \\ \sin 2\theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \sin 2\theta \end{pmatrix} + r\begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$

Check: $\sin 2\theta = z = \frac{2xy}{x^2+y^2} = 2\frac{(r\cos\theta)(r\sin\theta)}{r^2}$ ✓

Claim This has 2 projectively significant features.
1) It's the inversion of the cross-cap (model for $\mathbb{RP}^2$).
This is clear from θ, 2θ being of periodicity 2π and azimuthal range $\frac{\pi}{2}$.
Dembowski P2 Vol 1 p97 2) It's ! surface s.t. projection of any point $p \in \mathbb{R}^3$ onto the rules forms a plane curve.