7) In components, and keeping the higher-d signature ambiguous ($\epsilon = 1$ Riemannian, -1 semi-Riemannian)

- $\mathcal{R}_{abcd} = R_{abcd} - 2\epsilon K_{a[c} K_{b|d]}$ Gauss equation
- $\mathcal{R}_{\perp abc} = \qquad - 2\epsilon D_{[c} K_{b]a}$ Codazzi equation
- $\mathcal{R}_{\perp a \perp b} = \dfrac{\delta_{\vec\beta} K_{ab} - \epsilon D_b D_a \alpha}{\alpha} + K_a{}^c K_{cb}$ Ricci equation

This uses the (ADM) split of the higher d metric $g_{\alpha\beta} = \begin{pmatrix} \beta^i\beta_i + \epsilon\alpha & \beta_a \\ \beta_b & h_{ab} \end{pmatrix}$

Then $K_{ab} = -\dfrac{1}{2\alpha}\, \delta_{\vec\beta} h_{ab}$.

$\delta_{\vec\beta} h_{ab} = \dfrac{\partial}{\partial s} h_{ab} + \pounds_{\vec\beta} h_{ab}$ for s the extra dimension's coord (t or a spatial w)

'hypersurface derivative'.

Contracted versions

- $\mathcal{R}_{bd} - \epsilon \mathcal{R}_{\perp b \perp d} = R_{bd} - \epsilon\left(K K_{bd} - K_b{}^c K_{cd}\right)$ singly contracted Gauss eq
- $\mathcal{R}_{a\perp} = - \epsilon\left(D_b K^b{}_a - D_a K\right)$ contracted Codazzi eq
- $\mathcal{R}_{\perp\perp} = \dfrac{\delta_{\vec\beta} K - \epsilon \Delta \alpha}{\alpha} - K \circ K$ contracted Ricci eq

The Gauss equation can be contracted a second time:

- $2\mathcal{R}_{\perp\perp} - \epsilon \mathcal{R} = -\epsilon R + K^2 - K \circ K$ doubly contracted Gauss eq

Also (singly contracted Gauss eq) in (Ricci eq) eliminates $\mathcal{R}_{\perp a \perp b}$, yielding

$$\frac{\delta_{\vec\beta} K_{bd} - \epsilon D_d D_b \alpha}{\alpha} + K_a{}^c K_{cb} = \frac{\mathcal{R}_{bd} - R_{bd}}{\epsilon} + K K_{bd} - K_b{}^c K_{cd}$$

- $\Rightarrow \delta_{\vec\beta} K_{bd} = \alpha\left(\epsilon\left(\mathcal{R}_{bd} - R_{bd}\right) + K K_{bd} - 2K_b{}^c K_{cd}\right) + \epsilon D_d D_b \alpha$ (1)

(2nd order deriv wrt $\frac{\partial}{\partial s}$) 1st & 0th derivs wrt $\frac{\partial}{\partial s}$ only.

Additionally, we can eliminate $\mathcal{R}_{\perp\perp}$ between (contracted Ricci eq) and (doubly contracted Gauss eq)

$$\frac{\delta_{\vec\beta} K - \epsilon\Delta\alpha}{\alpha} - K \circ K = \frac{\epsilon(\mathcal{R} - R) + K^2 - K \circ K}{2}$$

- $\Rightarrow \delta_{\vec\beta} K = \alpha\, \dfrac{\epsilon(\mathcal{R} - R) + K^2 + K \circ K}{2} + \epsilon \Delta \alpha$ (2)

Finally, Trace (1) does not coincide with (2), so we have one more eq

- $\delta_{\vec\beta} K = \alpha\left(\epsilon(\mathcal{R} - R) - \mathcal{R}_{\perp\perp} + K^2\right) + \epsilon \Delta \alpha$

Projections of the Einstein tensor

So these turn out to be

$$2\mathcal{G}_{\perp\perp} = 2R_{\perp\perp} - \epsilon R \doteq -\epsilon R + K^2 - K\circ K .$$ doubly-contracted Gauss eq

$$\mathcal{G}_{a\perp} = R_{a\perp} - \cancel{\frac{R}{2} g_{a\perp}} = -\epsilon\left(D_b K^b{}_a - D_a K\right)$$ contracted Codazzi eq

and they are the K_{ab} formulation versions of the GR Hamiltonian & momentum constraints respectively.

$$
\begin{aligned}
G_{ab} &= R_{ab} - \frac{R}{2} g_{ab} \\
&= \frac{1}{\alpha}\left\{ \epsilon\left(\delta_{\vec{\beta}} K_{ab} - h_{ab}\,\delta_{\vec{\beta}} K\right) - D_b D_a \alpha + h_{ab}\Delta\alpha \right\} \\
&\quad + \epsilon\left(2K_a{}^c K_{bc} - K K_{ab} + \frac{K\circ K + K^2}{2} h_{ab}\right) + G_{ab} .
\end{aligned}
$$

which is a formulation of vefe's if set equal to 0
or of full efe's if equated to $8\pi G T_{ab}$.

Though one can put the trace part into a $\tilde{T}$,
which simplifies the geometrical portion of the equation.

(p9) Projections of R^T_{ab}

$$R^T_{ab} = R_{ab} - \frac{R}{p} g_{ab} .$$

$$= \frac{\epsilon \delta_{\vec\beta} K_{ab} - \epsilon D_b D_a \alpha}{\alpha} - \epsilon\{K K_{ab} - 2K_a{}^c K_{cb}\} + R_{ab}$$

$$- \frac{1}{p} h_{ab}\left(2\epsilon \frac{\delta_{\vec\beta} K - \epsilon \Delta\alpha}{\alpha} - \epsilon(K^2 + K\circ K) + R\right)$$

$$R^T_{a\perp} = R_{a\perp} - \cancel{\frac{R}{p} g_{a\perp}} = -\epsilon(D_b K^b{}_a - D_a K)$$

$$R^T_{\perp\perp} = R_{\perp\perp} - \frac{R}{p} g_{\perp\perp}$$

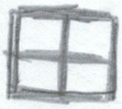
$$= \frac{\delta_{\vec\beta} K - \epsilon\Delta\alpha}{\alpha} - K\circ K + \frac{1}{p}\left(2\frac{(\delta_{\vec\beta} K - \epsilon\Delta\alpha)}{\alpha} - K^2 + K\circ K) + \epsilon R\right)$$

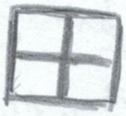
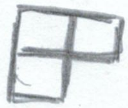
$$= \frac{p-2}{p} \frac{\delta_{\vec\beta} K - \epsilon\Delta\alpha}{\alpha} + \frac{K^2}{p^2} + \frac{p-1}{p} K\circ K - \frac{\epsilon R}{p} .$$

Another common variant is $\delta_{\vec\beta} K^T_{ab}$ equation, $\delta_\beta K$ eq. into K and $K_{ab}{}^T$.
This case is worked out in "conformal GR IVP".

In representation-theoretic terms,

$R_{ABCD} \longrightarrow R_{abcd}, \; R_{\perp abc}, \; R_{\perp a \perp b}$

is a  of SU(n)

is a SU(n) to SU(n−1) Res $=$  $\oplus$  $\oplus$  of SU(n−1)

$$\frac{n^2(n^2-1)}{12} = \frac{(n-1)^2((n-1)^2-1)}{12} + \frac{n(n-1)(n-2)}{3} + \frac{n(n-1)}{2}$$

$$\frac{(n-1)^2((n-1)^2-1)}{12} = \frac{n(n-1)^2(n-2)}{12}$$

Aside 1) The above permits a proof that $\# R_{abcd} = \frac{n^2(n^2-1)}{12}$ by induction, since  and  are easier to work out from combinatorial 1st principles

Note 1) dimension by dimension,

dimension	$\# R_{ABCD}$ =	$\# R_{abcd}$ +	$\# R_{\perp abc}$ +	$\# R_{\perp a \perp b}$
1 R_{ABCD} not defined				
2 R suffices, and	1	0	0	1
3 R_{AB} suffices, and	6	1	2	3
4 All of R_{ABCD} needed	20	6	8	6
5	50	20	20	10