TRAILER

Gauss-Bonnet Theorem

This concerns Gaussian curvature's influence on the global shape of a surface S.

We have already met the Euler characteristic $\chi(S)$ in IB geometry.

Any advantage of this over triangulation?

Theorem 1 i) Every cpct surface S has a rectangular decomposition

ii) $v+f-e=\chi(S)$: the same $\forall$ rectangular decompositions of S.

Thm 1 is covered in basic Topology.

iii) For S with g holes, $\chi(S) = 2(1-g)$.
(the genus? [or handles])

iv) $\chi(S)$ is a topological invariant.

We have already defined total Gaussian curvature as $\iint_S K\,dS$.

Definition 1 For $\gamma : [a,b] \to M$ a regular curve segment in an oriented geometric surface M, the total geodesic curvature of γ is $\int_{s(a)}^{s(b)} k_g(s)\,ds$

for $k_g(s)$ the geodesic curvature of a unit-speed reparametrization of γ.
[this is the total change of angle as one moves along that curve segment]

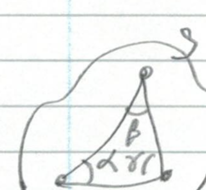
Some forms of Gauss-Bonnet are then as follows.

Theorem (Gauss-Bonnet) i) Let S be a cpct orientable geometric surface (no bdy).
Then (total Gaussian curvature) $= 2\pi\chi(S)$

i.e. $$\iint_S K\,dS = 2\pi\chi(S) = 4\pi(1-g) \quad ①$$
for S with g handles.

Is an easy subcase of (Atiyah-Singer) Index theorem [8]

Note that this is of the form
(global (geometrical entity) = (topological entity).



Can view this as an 'local element' calculation of IB Geometry.

ii) For Δ a triangle in a geometric surface S, then
$$\iint_\Delta K\,dS + \int_{\partial\Delta} k_g\,ds = (\text{interior angle sum}) - \pi = \alpha+\beta+\gamma-\pi \quad ②$$

iii) This may be extended to [triangular patch by patch] to the case of P an oriented polygonal region in a geometric surface S, giving the form
$$\iint_P K\,dS + \int_{\partial P} k_g\,ds + (\text{exterior angle sum}) = 2\pi\chi(P). \quad ③$$

Diff Geom [5] covers case with bdry.

Note: iii) goes a long way toward modelling surfaces with bdry.

Reconciling ② and ③: Let $P=\Delta$. Then $\chi(\Delta) = v+f-e = 3+1-3 = 1$.
Then use $\theta_{ext} := \pi - \theta_{int}$, so $2\pi\chi(\Delta) - \sum\theta_{ext} = 2\pi\cdot 1 - \sum(\pi-\theta_{int}) = 2\pi - 3\pi + \sum\theta_{int}$
$= \sum\theta_{int} - \pi$

$\pi-\alpha$