(p1) §8.4 **Bel decomposition of Riemann tensor (Electric, Magnetic and Topic parts)**

$\mathbb{E}^{\alpha}{}_{\beta} := R^{\alpha}{}_{\mu\beta\nu} U^{\mu} U^{\nu}$ is the 'electric part' of the Riemann tensor. alias 'gravitoelectric'

$\mathbb{H}^{\alpha}{}_{\beta} := * R^{\alpha}{}_{\mu\beta\nu} U^{\mu} U^{\nu}$ is the 'magnetic part' alias 'gravitomagnetic'. Approached this way, this is an exclusively 4-d split.

- These have some analogies with the Maxwellian electric and magnetic parts $\underline{E}$ and $\underline{B}$

- In fact, the analogy is better with the electric tidal tensor $E_{\alpha\beta} := F_{\alpha\gamma;\beta} U^{\gamma}$, $B_{\alpha\beta} := * F_{\alpha\gamma;\beta} U^{\gamma}$

'Tidal tensor' here refers to 'electromagnetic worldline deviation'.
These corr to expressions 'as measured by the particle whose 4-velocity is U'

- Moreover, whereas $\underline{E}$ and $\underline{B}$ (or $E_{\alpha\beta}$ and $B_{\alpha\beta}$) are exhaustive decompositions, in GR there is a third piece.
This can be seen from counting components.
Let us first establish what the symmetries of $\mathbb{E}$ and $\mathbb{H}$ are.

Lemma i) $\mathbb{E}_{\alpha\beta}$ is symmetric and spatial
ii) $\mathbb{H}_{\alpha\beta}$ is traceless and spatial

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Proof

i) $\mathbb{E}_{\beta\alpha} = \underset{\text{1st 12nd pair sym}}{R_{\beta\mu\alpha\nu} U^{\mu} U^{\nu}} = R_{\alpha\nu\beta\mu} U^{\mu} U^{\nu} \underset{\text{dummy index change, symmetry in } U^{\mu}U^{\nu}}{=} R_{\alpha\mu\beta\nu} U^{\mu} U^{\nu} = \mathbb{E}_{\alpha\beta}$

with U^{μ} timelike and $R_{\alpha\beta\gamma\delta}$ antisymmetric in each of 1st, 2nd pairs, this leaves α and β having to be spatial indices.

ii) $\mathbb{H} = \frac{1}{2} \epsilon^{\alpha\mu\rho\pi} R_{\pi\rho\alpha\nu} U_{\mu} U^{\nu} \underset{\text{R: 1st, 2nd pair sym}}{=} \frac{1}{2} \epsilon^{\alpha\mu\rho\pi} R_{\alpha\nu\pi\rho} U_{\mu} U^{\nu} \underset{\text{R: antisym in 1st pair.}}{=} -\frac{1}{2} \epsilon^{\alpha\mu\rho\pi} R_{\nu\alpha\pi\rho} U_{\mu} U^{\nu}$ ①

① serves to place the 3 indices of R contracted into ϵ where we can use the 1st Bianchi id.

$$= \frac{1}{2} \epsilon^{\alpha\mu\rho\pi} (R_{\nu\pi\rho\alpha} + R_{\nu\rho\alpha\pi}) U_{\mu} U^{\nu}$$

dummy index changes

$$= \frac{1}{2} (\epsilon^{\rho\mu\pi\alpha} + \epsilon^{\pi\mu\alpha\rho}) R_{\nu\alpha\pi\rho} U_{\mu} U^{\nu}$$

by $\epsilon^{\rho\mu\pi\alpha} = -\epsilon^{\mu\rho\pi\alpha} = \epsilon^{\mu\pi\rho\alpha} = -\epsilon^{\mu\pi\alpha\rho} = \epsilon^{\pi\mu\alpha\rho}$ (single swaps)

$$= \epsilon^{\mu\pi\rho\alpha} R_{\nu\alpha\pi\rho} U_{\mu} U^{\nu}$$

by $\epsilon^{\mu\pi\rho\alpha} \underset{\text{cycle}}{=} \epsilon^{\alpha\mu\pi\rho} \underset{\text{1 swap}}{=} -\epsilon^{\alpha\mu\rho\pi}$.

$$= -\epsilon^{\alpha\mu\rho\pi} R_{\nu\alpha\pi\rho} U_{\mu} U^{\nu} \quad ②$$

So (1) and (2) $\Rightarrow \frac{3}{2} \mathbb{H} = 0 \Rightarrow \mathbb{H} = 0$. □

Corollary 1) $\# \mathbb{E}^{\alpha}{}_{\beta} = \frac{n(n-1)}{2}$ for spacetime dimension n, giving 6 in 4-d.
2) $\# \mathbb{H}^{\alpha}{}_{\beta} = 8$ in the 4-d case for which this is defined.

Proof This implicitly rests on $\mathbb{E}$, $\mathbb{H}$ having no more symmetries.
Then 1) $\# E^{\alpha}{}_{\beta} = \frac{(n-1)(n-1+1)}{2} = \frac{n(n-1)}{2}$ since it is purely spatial (dimension n − 1), symmetric.
So setting n = 4, $\# E^{\alpha}{}_{\beta} = \frac{4.3}{2} = 6$.

2) $\# \mathbb{H}^{\alpha}{}_{\beta} = \underset{\text{purely spatial}}{(4-1)^2} - \underset{\text{traceless}}{1} = 3^2 - 1 = 8$

□

ⓟ2

Dual formulation of 'magnetic' part

$\mathbb{B}_{abc} := R_{abc\delta} U^{\delta}$ admits a (~~n~~ implicitly-spatial) dual

$$-\frac{1}{2} U^{\alpha} \underset{\sim}{\epsilon}_{\alpha p}{}^{bc} \mathbb{B}_{bcq} = -\frac{1}{2} \epsilon_{\alpha p}{}^{bc} \underset{R \text{ antisym in 1st pair, def of } *R}{R_{bcq\delta} U^{\alpha} U^{\delta}} = \underset{\text{defn of } \mathbb{H}}{* R_{pq\alpha\delta} U^{\alpha} U^{\delta}} = \mathbb{H}_{pq}$$

forced to be timelike by total antisymmetry and the other 3 indices being spatial

so $\mathbb{H}_{pq}$ itself has a dual, $\mathbb{B}_{abc}$, which is free ~~of~~ of ϵ's, and thus generalizes to arbitrary dimension.

$\mathbb{B}_{abc}$ can, moreover, be thought of as a space^3 time projection of $R_{\alpha\beta\gamma\delta}$.

Bel split

- Everything is contracted with 2 U^{μ}'s.
- The number of ϵ's used varies for each piece:

0 in $\mathbb{E}$
1 in $\mathbb{H}$
2 in $\mathbb{L}$.

This renders this split 4-d specific.

Projective split

- The number of contractions with U^{μ} varies for each piece:

2 in $\mathbb{E}$
1 in $\mathbb{B}$
0 in $\mathbb{R}$

- No ϵ's are used.

This renders this split arbitrary-d.

$\mathbb{R}_{abcd} := R_{abcd}$: n-d Riemann tensor restricted to (n-1)-d indices.

Finally, $\mathbb{H}$ and $\mathbb{B}$ are related by duality. Both are in a sense 'magnetic'.

They have the same info content.

Thus $\mathbb{L}$ and $\mathbb{R}$ also have the same info content, by each split being exhaustive.

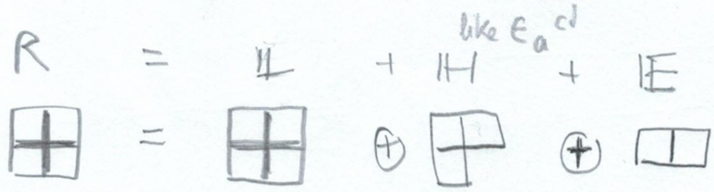
Indeed, $\mathbb{L}$ and $\mathbb{R}$ are related by a 'two-sided implicitly spatial duality'

$$\mathbb{L}_{ab} = *R*_{a\mu b\nu} U^{\mu} U^{\nu} = \frac{1}{2} \epsilon_{\mu a}{}^{cd} R_{cdef} \frac{1}{2} \epsilon^{ef}{}_{b\nu} U^{\mu} U^{\nu}$$

$$= -\frac{1}{4} \underset{\text{like } \epsilon_a{}^{cd}}{\underbrace{(U^{\mu} \epsilon_{\mu a}{}^{cd})}} R_{cdef} \underset{\text{like } \epsilon^{ef}{}_b .}{\underbrace{(\epsilon^{ef}{}_{b\nu} U^{\nu})}}$$

Hence $R = \mathbb{L} + \mathbb{H} + \mathbb{E}$

is just another version of



SU(n)

reduction in Representation-theoretic terms

And $\# \mathbb{E} = \frac{n(n-1)}{2}$, $\# \mathbb{H} = n(n-1)(n-2)$, $\# \mathbb{L} = (n^2-1)^2((n-1)^2-1) = n(n-1)^2(n-2)$

(p 3)

§8.5 'Electric' and 'magnetic' pieces of the Weyl tensor

C_{abcd} Not zero because of extrinsic pieces.

$$E_{ab} := C_{\perp a \perp b}$$

$B_{abc} := C_{\perp abc}$, which admits a dual in the 4-d case:

$H_{ab} := \frac{1}{2} \epsilon_{\perp bef} B_a{}^{ef}$ Is a 3-contraction so that there are really on-hypersurfaces $\epsilon^{abc}{}_{s}$

Check the duality:

$$\epsilon^{\perp qrb} H_{ab} = -\frac{1}{2} \epsilon^{\perp qrb} \epsilon_{b\perp ef} B_a{}^{ef} = -\frac{1}{2}(\delta^q_e \delta^r_f - \delta^q_f \delta^r_e) B_a{}^{ef} = \frac{1}{2}(B_a{}^{rq} - B_a{}^{qr}) = -B_a{}^{qr}$$

Then
$$|C_4|^2 = C_{ABCD}C^{ABCD} = C_{abcd}C^{abcd} + 4C_{\perp abc}C^{\perp abc} + 6C_{\perp a \perp b}C^{\perp a \perp b}$$
$$= |\bar{C}_3|^2 + 4|B|^2 + 6|E|^2.$$

In components,

$\frac{(n^2-1)(n-4)}{12}$ cpts $\neq$ Weyl$(n-1)$)

$$\check{C}_{abcd} = R_{abcd} - 4 g_{[a[c} S_{d]} {}_{b]}$$

$$= \boxed{R_{abcd} - 4h_{[a[c} S_{d]b]}} - K_{ac}K_{bd} + K_{bc}K_{ad} - \frac{4h_{[a[c}}{p-2}\left(\frac{-\delta_{\vec{\beta}} K_{d]b]} - D_{b]}D_{d]}\alpha}{\alpha} + KK_{d]b]} - 2K_{d]}{}^{c}K_{c b]} - \frac{h_{b]d]}}{2(p-1)}\left\{R - K^2 - K\circ K + \frac{2\epsilon}{\alpha}(\delta_{\vec{\beta}} K - \epsilon \Delta\alpha)\right\}\right)$$

But this still is not C_{abcd}, because this S has the p-d cpts not the p+1-d ones.

$\frac{(n+1)(n-1)(n-3)}{3}$ cpts.

$$B_{abc} = R_{\perp abc} - \underbrace{4 g_{[\perp[b} S_{c]a]}}_{2(g_{\perp[b} \cancel{S_{c]a}}_{0} - g_{a[b} S_{c]\perp})}$$

$$= 2D_{[c|} K_{a|b]} + 2\frac{h_{a[c}}{p-2}\{D_e K^e{}_{c]} - D_{c]} K\} h_{c]b}$$

$\frac{(n+1)}{2} - 1$
$\parallel$
$\frac{(n+1)(n-2)}{2}$ cpts

$$E_{ab} = R_{\perp a \perp b} - \underbrace{4 g_{[\perp[a} S_{\perp]b]}}_{g_{\perp a} \cancel{S_{\perp b}}_{0} - g_{\perp\perp} S_{ab} + g_{ba} S_{\perp\perp}) - \cancel{g_{b\perp}} S_{a\perp}}$$

$$= \frac{\delta_{\vec{\beta}} K_{ab} - \epsilon D_b D_a \alpha}{\alpha} + K_a{}^c K_c{}^b - \frac{1}{p-2}\left(\frac{-\delta_{\vec{\beta}} K_{ab} - D_b D_a \alpha}{\alpha} + R_{ab} + KK_{ab} - 2K_a{}^c K_{cb} \boxed{- \frac{h_{ab}}{2(p-1)}}(\cancel{R} - \cancel{K^2} - K\circ K + \frac{2\epsilon}{\alpha}(\delta_{\vec{\beta}} K - \epsilon \Delta\alpha)) + \frac{h_{ab}}{p-1}\left(\frac{\delta_{\vec{\beta}} K + \epsilon\Delta\alpha}{\alpha} + \frac{\epsilon}{2}\frac{\cancel{R} - \cancel{K^2} + (1-2p) K\circ K}{2(p-1)} \right)\right)$$

$\parallel$

$$\Rightarrow E_{ab} = \frac{p-1}{p-2} \cdot \frac{\delta_{\vec{\beta}} K_{ab} - \epsilon D_b D_a \alpha}{\alpha} + \frac{p}{p-2} K_a{}^c K_{cb} - \frac{1}{p-2} k K_{ab}$$

$$\frac{h_{ab}}{(p-1)(p-2)} \left((1-p) K \circ K + \frac{\epsilon}{2\alpha} \left(\delta_{\vec{\beta}} K - \epsilon \Delta \alpha \right) \right)$$

(not carefully checked yet!)

Representation-theoretic interpretation

$n \geq 4$

		Ricci split		
nd Riemann tensor	$\frac{n^2(n^2-1)}{12}$	S .. 1	E .. $\frac{(n-1)(n+2)}{2}$	C .. $\frac{n(n+1)(n+2)(n-3)}{12}$
projective or Bel split $\mathbb{E}$:	$\frac{n(n-1)}{2}$	1	0	$\frac{(n+2)(n-1)}{2}$
$\mathbb{H} = \mathbb{B}$:	$\frac{n(n-1)(n-2)}{2}$	0	$n-1$	$\frac{(n+1)(n-1)(n-3)}{3}$
$\mathbb{L} = \mathbb{R}$:	$\frac{n(n-1)^2(n-2)}{12}$	0	$\frac{n(n-1)}{2}$	$\frac{(n+1)n(n-1)(n-4)}{12}$

lower-d cases

1-d: nothing to split

2-d

R_1 =	S_1
$\mathbb{E}_1$	1

3-d

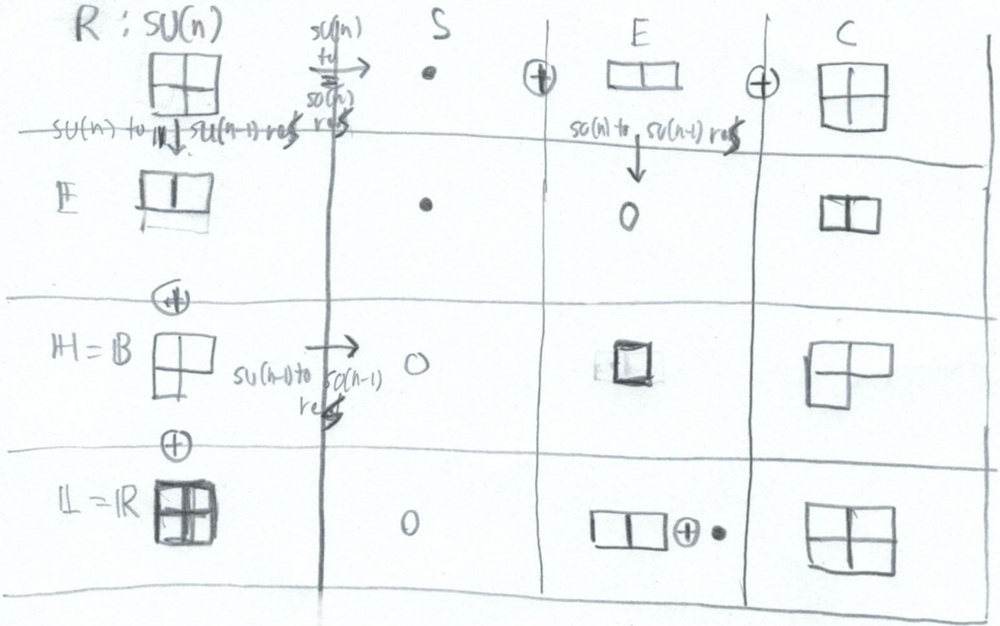
R = 6	S + 1	E 5
$\mathbb{E}$	1	②
$\mathbb{H} = \mathbb{B}$	0	2
$\mathbb{L} = \mathbb{R}$	0	1

② — Exceptional

4-d

20	1	9	10
6	1	0	5
8	0	3	5
6	0	6	0

$\| $
$5 \oplus 1$



[except in 2d it's •⊕□□⊕•]

SU(n) rep	SO(n) reps
SU(n-1) reps	SO(n-1) reps

↕

R_{ABCD}	S_{ABCD} contains R	E_{ABCD} contains $\overset{\top}{R}_{AB}$	C_{ABCD}
$\mathbb{E}_{ab} = R_{\perp a \perp b}$	R	0	Electric Weyl $C_{\perp a \perp b}$
$\mathbb{H}_{ab}$ or $\mathbb{B}_{abc} = R_{\perp abc}$	0	$R_{\perp a}$	Magnetic Weyl $C_{\perp abc}$
$\mathbb{L}_{abcd}$ or $\mathbb{R}_{abcd}$	0	$\overset{\top}{R}_{ab} \oplus R_{\perp\perp}$	C_{abcd}